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Practical Methods of Measuring the Generalized Dimension and the Largest Lyapunov Exponent in High Dimensional Chaotic Systems

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Practical methods to extract the generalized dimension D_q and the largest Lyapunov exponent from experimental data are proposed and tested on examples. The measured values agree well with known values. In applications to chaotic signals, convergence of dimension is investigated for varying the delay time and the embedding dimension.

The quantitative characterization of chaotic irregular motion has been achieved by measurements of the metric entropy, the dimensions and the spectrum of Lyapunov exponents. Recently much progress has been made in understanding the infinite hierarchical series of dimension D_q through the fluctuation of scaling properties ("f-α spectrum"),[1)~3)] and the same formalism for the series of Renyi entropy K_q and q-order Lyapunov exponent ("h-γ spectrum")[4),5)] has been proposed. Therefore, it is important to develop efficient methods which can extract these dynamical invariants from experimental data. However, the straightforward way of extracting the generalized dimensions from the original definition is not practical in general. In this paper we present practical methods of extracting the series of dimensions D_q and the largest Lyapunov exponent from experimental data, which are applicable to high-dimensional systems. We also discuss the dependence of convergence of the attractor's dimension on parameters of reconstructing phase space, i.e., the delay time and the dimension of phase space.

We describe the dynamical system by a set of differential equations $dX/dt=F(X)$ where X is a d-dimensional vector obtained from a single scalar time series by using a delay time T; $X(t)=(x(t), x(t+T),\cdots, x(t+(d-1)T))$.[6)] Suppose now that d-dimensional phase space is uniformly partitioned to boxes of size ε, and N points $\{X_i\}_{i=1}^N$ in a time sequence are given by sampling from $X(t)$ every Δt. One can estimate the invariant probability measure p_i associated with box i by N_i/N (where N_i is the number of points falling within box i) provided N is large enough. The q-order dimension is given by $D_q=1/(q-1)\lim_{\varepsilon\to 0}[\log(\sum_i p_i^q)/\log\varepsilon]$. However, determining N_i from the box counting algorithm is ineffective in computing especially for high-dimensional systems. To bypass this difficulty we introduce the individual correlation function proposed by Cohen et al.[7)] The individual correlation function at data point X_i is

$$C_{X_i}(r)=\frac{1}{N}\sum_j \theta(r-|X_j-X_i|)\,, \tag{1}$$

where θ is the Heaviside function. Let us consider the assumption[7)]

$$[\prod_{k=1}^{N_i} C_{Z_{i,k}}(r)]^{1/N_i} = C_{Z_i}(r)\,, \tag{2}$$

where Z_i is the center of box i and $Z_{i,k}$ are all the points falling within box i. This means that the individual correlation function of the center of the box is the geometric mean of the individual correlation functions of $Z_{i,k}$. Hence,

$$\{C_{Z_i}(r)\}^q = [\prod_{k=1}^{N_i} C_{Z_{i,k}}(r)^q]^{1/N_i} \leq \frac{1}{N_i}\sum_{k=1}^{N_i} C_{Z_{i,k}}(r)^q\,. \tag{3}$$

The equality holds $C_{Z_{i,k}}(r) = C_{Z_i}(r)$ for all k. For sufficiently small r, using the relation (3),

$$\sum_i p_i{}^q \simeq \sum_i p_i C_{Z_i}(r)^{q-1} \leq \frac{1}{N}\sum_i \sum_{k=1}^{N_i} C_{Z_{i,k}}(r)^{q-1}$$

$$= \frac{1}{N}\sum_n C_{Z_n}(r)^{q-1}\,. \tag{4}$$

We thus obtain the exponent

$$\nu_q = \frac{1}{q-1}\lim_{r\to 0}\frac{\log\langle C_Z(r)^{q-1}\rangle}{\log r}\,, \tag{5}$$

where angular brackets are the ensemble average for all possible points. Equation (5) gives the upper bound of D_q. We expect that ν_q is a good approximation for D_q in most cases which we are interested in. For $q=1$, it is easy to see that $\nu_1 = \lim_{r\to 0}[\langle \log C_Z(r)\rangle / \log r]$. Furthermore, one can obtain the f-α spectrum via the pair of formulas (in Ref. 1))

$$\alpha(q) = \frac{d}{dq}[(q-1)D_q]\,,$$

$$f(q) = q\alpha(q) - (q-1)D_q\,. \tag{6}$$

Our method was tested on some dynamical systems to see the consistency with true values. The numerical results are listed in Table I. The accuracy of our method was always within a few percent for positive q's. For negative q's the accuracy was

Table I. Generalized dimensions. All examples were calculated with 20000 points and 2000 reference points.

System	Other methods	Our method
Logistic map ($a=3.56994$)	$D_0=0.538\cdots$[16)] $D_\infty=0.37775\cdots$[1)] $D_{-\infty}=0.75551\cdots$	$D_0=0.567\pm0.04$ $D_\infty=0.391\pm0.03$ $D_{-\infty}=0.769\pm0.07$
Circle map with golden mean	$D_0=1$[1)] $D_\infty=0.6326\cdots$ $D_{-\infty}=1.898\cdots$	$D_0=0.99\pm0.03$ $D_\infty=0.65\pm0.03$ $D_{-\infty}=1.95\pm0.2$
Henon map ($a=1.4$, $b=0.3$)	$D_0=1.28$ $D_1=1.26$ $D_2=1.22$[9)]	$D_0=1.27\pm0.02$ $D_1=1.26\pm0.02$ $D_2=1.23\pm0.02$

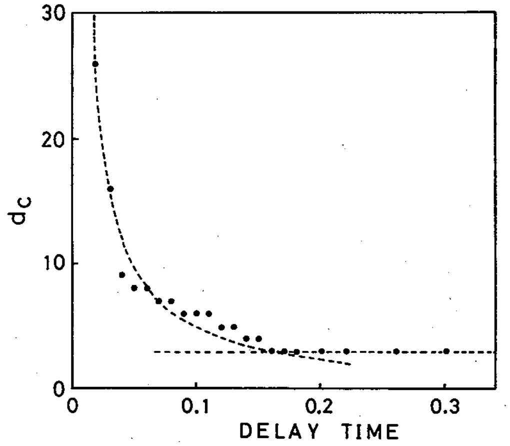

Fig. 1. The embedding dimension d_c as a function of delay time T. The dotted lines are $d_c=3$ and $d_cT=0.485$. The latter was obtained by using least square methods.

less than the positive q's and error bar was $10\sim20\%$ around $D_{-\infty}$ in some cases. But we expect that larger N makes the error much smaller.

When the generalized dimensions are measured from experimental data, it is important to choose the best delay time and the best embedding dimension. Unsuitable choice sometimes leads to incorrect results. However, only a few authors have discussed this point.[8] We studied the dependence of convergence of dimension on the delay time T and embedding dimension d in details for the most familiar dimension D_2.

Let us consider the Lorenz system as an example. With varying d, for $d<d_c$ estimated dimension is smaller than 2.06 (true dimension D_2 of the Lorenz system is 2.06 in Ref. 9)). And for $d\geq d_c$ the dimension converges to a limiting value which is independent of the dimension of phase space. We consider that D_2 converges when the estimated dimension becomes uniform within 2.06 ± 0.05 as d increases. In Fig. 1 we plot d_c vs T for an attractor with 10000 data points and 150 reference points.

There are two factors which determine convergence of D_2. The first is topological properties of a projection of an attractor. In small dimensional phase space the dimension of an image is equal to the dimension of phase space. In order to obtain the true dimension Whitney's embedding theorem shows that one must embed an m-dimensional attmactor to $2m+1$-dimensional phase space (see, e.g., Ref. 10)). But this condition may be too strong. For a Lorenz attractor the minimum d_c is 3, i.e., the minimum integer which is greater than the attractor's dimension. The second is connected with the choice of the delay time. When one chooses too small time delays, two vectors $X(t)$ and $X(t+T)$ are strongly correlated and the projection of an attractor to $X(t)X(t+T)$-plane is suppressed to the diagonal line of the plane. However, even for small time delays, a sufficient number of vectors which are linearly independent of each other can be obtained when the embedding dimension is large enough. Because there exists some characteristic time length T_c at which two vectors $X(t)$ and $X(t+T_c)$ become uncorrelated. If dT is greater than mT_c, then vectors spanning phase space contain $m+1$ independent vectors. Therefore, the curve in Fig. 1 is approximated well by two curves, $d_cT=$constant and $d_c=3$.

Another basic quantity to characterize chaotic behaviors is the Lyapunov exponents which represent the exponential growth rate of nearby orbits. Wolf et al. have proposed a method to estimate the largest exponent.[11] However, the method has a deficiency of orientational problem. Because one has to successively replace nearby orbits, minimizing the orientational change. When the dimension of the attractor is high, it becomes difficult to find such replacement orbits satisfying the orientational condition. Two of the present anthors have proposed the Jacobian estimation method which can estimate a set of Lyapunov exponents.[12] However, the applicability of the method is restricted to relatively low dimensional attractors, e.g., up to 3~5 dimensions. We present here a new method to extract the largest Lyapunov exponent which is easy to implement and has less limitation even for high dimensional attractors.

We note the distance of two different points x, y on the attractors by $\mathrm{dis}(x, y)$ and the flow of the system by f^t. If one chooses y as the nearest point of x, then $f^t y$ approaches the most unstable direction with increasing t. Therefore, $\mathrm{dis}(f^t x, f^t y)$ exponentially grows with increasing t, and one obtains

$$\lambda(t)=\frac{1}{t}\left\langle \ln\frac{\mathrm{dis}(f^t x, f^t y)}{\mathrm{dis}(x, y)}\right\rangle, \tag{7}$$

where angular brackets are the ensemble average with respect to x. Equation (7) approximates the largest Lyapunov exponent for suitable t.

However, Eq. (7) converges very slowly since the discrepancy between the position y and unstable manifold of x causes an error for estimating the exponent. In some cases Eq. (7) does not converge because $\mathrm{dis}(f^t x, f^t y)$ cannot be greater than the extent of the attractor. Therefore, we take another equivalent expression of Eq. (7)

$$\lambda(t, \tau)=\frac{1}{\tau}\langle \ln \Lambda(t, \tau)\rangle, \tag{8}$$

where $\Lambda(t, \tau)=\mathrm{dis}(f^{t+\tau}x, f^{t+\tau}y)/\mathrm{dis}(f^t x, f^t y)$, and y is the nearest point of x. In Eq. (8) the ensemble average is taken for all possible points of x. Equation (8) converges more rapidly than Eq. (7). Figure 2 shows an example of Eq. (8) for the Lorenz equations. We plot λ vs t for $\tau=0.3$ with different dimensions of phase space and a numerical result based on Eq. (7) by a dotted line for comparison. In Fig. 2 we can find a plateau which gives an estimation of the largest Lyapunov exponent. The obtained value is in good agreement with the known value. The results of other examples are summarized in Table II. The present algorithm has less limitation as compared with the other methods, thus it is expected to work well even for high

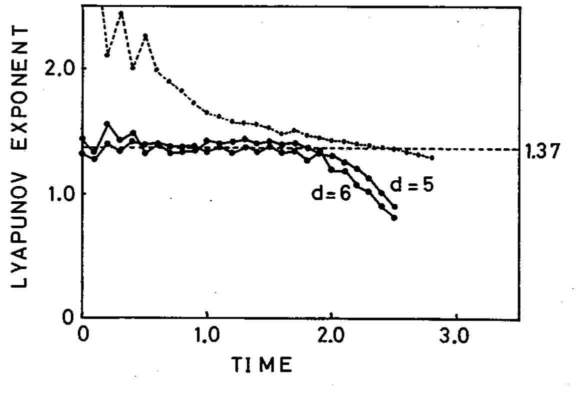

Fig. 2. The largest Lyapunov exponent of the Lorenz system for a special chaotic attractor ($\sigma=16$, $r=40$, $b=4$) with different dimensions of phase space $d=5$, 6. For comparison, estimated values based on Eq. (8) for $d=4$ and the numerical results $\lambda=1.37$ by the original definition are shown by dotted lines.

Table II. The largest Lyapunov exponent. d is the embedding dimension and D_{KY} is the Lyapunov dimension.

System	Other methods	Our method
Henon map ($a=1.4$, $b=0.3$)	$\lambda=0.417$ [17]	$\lambda=0.41\pm0.05$ $d=3$
Lorenz equations ($\sigma=16$, $b=4$, $r=40$)	$\lambda=1.37$ [18]	$\lambda=1.37\pm0.06$ $d=5$
Roessler equations ($a=b=0.2$, $c=5.7$)	$\lambda=0.069$	$\lambda=0.067\pm0.005$ $d=5$
Mackey-Glass equations ($a=0.2$, $b=0.1$, $c=10$, $T=150$)	$\lambda=0.0025\pm1\times10^{-4}$ $D_{KY}=14.45\cdots$	$\lambda=0.0021\pm5\times10^{-4}$ $d=20$

dimensional attractors such as the Mackey-Glass attractor whose Lyapunov dimension[13] is 14.45 as shown in Table II. Moreover, it may be worth noting that the estimation of q-order Lyapunov exponents λ_q[14),15)] is straightforward by simple modification of Eq. (8)

$$\lambda_q(t,\tau)=\frac{1}{\tau}\frac{1}{q}\ln\langle\Lambda(t,\tau)^q\rangle\,. \tag{9}$$

One can obtain h-γ spectrum from the series of λ_q's.

In conclusion, we showed that it is possible to estimate the upper bound of the infinite series of dimensions D_q and the largest Lyapunov exponent from time series by using ensemble average, and investigated the convergence of D_2 for varying time delays. The upper bound ν_q is a good approximation of D_q in most cases. We hope that the new algorithms have wide applicability to extracting experimental values of dimensions and the largest Lyapunov exponent.

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