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PRACTICAL STABILITY AND LYAPUNOV FUNCTIONS*

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. The notion of "practical stability" was discussed in the monograph by LaSalle and Lefschetz [6] in which they point out that stability investigations may not assure "practical stability" and vice versa. For example an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Motivated by this, Weiss and Infante introduced the concept of finite time stability [7]. They were interested in the behavior of systems contained within specified bounds during a fixed time interval. Many problems fall into this category including the travel of a space vehicle between two points and the problem, in a chemical process, of keeping the temperature within certain bounds.

In particular, Weiss and Infante [7] provided sufficient conditions for finite time stability in terms of Lyapunov functions. Moreover, Weiss [9] provided necessary and sufficient conditions for uniform finite time stability and exponential contractive stability. These results were extended by Kayande [3] who obtained necessary and sufficient conditions for contractive stability (without requiring the exponential behavior assumed in [9]).

The sufficiency part of the above results were extended by Kayande and Wong [4], and Gunderson [1], who applied the comparison principle. Moreover Hallam and Komkov [2] generalized the concept of the finite time stability of the zero solution to that of arbitrary closed sets.

In this paper we analyze a more general notion of practical stability than is provided for by finite time stability considerations. Our state space includes finite as well as infinite dimensional Banach spaces. The sets upon which we impose our stability conditions are not restricted to balls containing the origin as is done by the others. This leads to interesting implications. We first present necessary and sufficient conditions for generalized practical stability, in a more meaningful setting than that of Kayande [3] for finite time stability. We then apply our

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results to a discussion of perturbations on the flow. In particular we study the amount of change incurred upon the initial set, target set, and constraint set under the influence of these perturbations. Such analysis is very important when modeling real world problems. This perturbation study is new even in the case of finite time stability, where Weiss and Infante [8] have discussed results on stability under small disturbances. They do not discuss the important relationship between the practical stability of the unperturbed system and that of the perturbed system.

Since many physical models can be realized as ordinary differential equations in Banach spaces we feel that it is important to assume the flow is in a finite or infinite dimensional Banach space.

2. **Notation and preliminaries.** Let $(B, \|\cdot\|)$ be a Banach space (either finite or infinite dimensional), and let $J = [t_0, t_0 + T]$ for some $T > 0$, $t_0 \geqq 0$. We consider the following system

$$x' = f(t, x) \,, \tag{E}$$

where f is defined and continuous on $J \times B$ and satisfies the Lipschitz condition: for each bounded set A and all $t \in J$, there exists $\lambda_A(t)$ which is in $L^1(J)$ such that

$$|f(t, x) - f(t, y)| = \lambda_A(t)\|x - y\| \tag{2.1}$$

for any two points $x, y \in A$.

Let M, N, and Γ be three bounded sets in B such that $M \cup N \subseteqq \Gamma$, the closure of M is contained in Γ, and Γ is open and connected. We shall refer to M, N, and Γ as our initial set, target set, and constraint set respectively.

DEFINITION. The system (E) is (M, N, Γ, T) practically stable if $x_0 \in M$ implies that

$$x(t, t_0, x_0) \in \Gamma \quad \text{for} \quad t \in J \tag{2.2}$$

and

$$x(t_0 + T, t_0, x_0) \in N \tag{2.3}$$

In case $N \subset M \subset \Gamma$, where N, M, and Γ are neighborhoods of the origin then we have contractive stability [7]. If $M \subset N \subset \Gamma$, where M, N, Γ are neighborhoods of the origin then we have expansive stability [4]. Our notion of practical stability which includes finite time stability as a special case, offers a reasonable mechanism in analyzing the question of stability under perturbations.

We say $V: J \times B \to R$ is a Lyapunov function if $V(t, x)$ is continuous in (t, x), bounded on bounded subsets of B, i.e. for each bounded set $A \subset B$ there exists Q such that

$$\sup_{x \in A, t \in J} |V(t, x)| \leqq Q \tag{2.4}$$

(notice this will always be true if B is finite dimensional), and satisfies the Lipschitz condition: for each bounded set $A \subset B$ there exists $\lambda_A(t)$ which is integrable on J such that for each $t \in J$ and $x, y \in A$

$$|V(t, x) - V(t, y)| \leqq \lambda_A(t) \|x - y\| . \tag{2.5}$$

We shall need the following known fact [10]: if V is a Lyapunov function then

$$V'(t, x) \overset{\text{def}}{\equiv} \lim_{h \to 0+} \{V(t + h, x(t + h)) - V(t, x)\}/h \tag{2.6}$$

$$\equiv \lim_{h \to 0+} \{V(t + h, x + hf(t, x)) - V(t, x)\}/h . \tag{2.7}$$

3. Characterization of practical stability. Before stating our characterization theorem, we need the following quantities:

$$a_M = \sup_{x \in M} V(t_0, x) , \quad b_\Gamma = \inf_{t \in J, x \in \partial\Gamma} V(t, x) , \quad b_{\Gamma N} = \inf_{x \in \bar{\Gamma} - N} V(t_0 + T, x) . \tag{3.1}$$

where $\bar{\Gamma}$ is the closure of Γ, and $\partial\Gamma$ is the boundary of Γ. Notice these quantities always exist in finite dimensional space since V is continuous. However these quantities may not exist in infinite dimensional space unless we assume (2.4). We now have the following characterization theorem.

THEOREM 3.1. *A necessary and sufficient condition for* (E) *to be* (M, N, Γ, T) *practically stable is that there exists a Lyapunov function such that*

(a) $V'(t, x) \leqq g(t, V(t, x))$ *for* $(t, x) \in J \times \bar{\Gamma}$, *where* $g: R \times R \to R^+$ *is continuous,*

(b) (i) $r(t, t_0, a_M) < b_\Gamma$, *for* $t \in J$, *and* (ii) $r(t_0 + T, t_0, a_M) < b_{\Gamma N}$, *where* $r(t, t_0, u_0)$ *is the maximal solution of*

$$u' = g(t, u) , \quad u(t_0) = u_0 .$$

PROOF. *Sufficiency.* We first show $x(t, t_0, x_0) \in \Gamma$ for all $t \in [t_0, t_0 + T]$ whenever $x_0 \in M$. Assume there exists $t_1 > t_0$ such that $x(t_1, t_0, x_0) \in \partial\Gamma$ and $x(t, t_0, x_0) \in \Gamma$ for $t \in [t_0, t_1)$. The comparison principle [5] and (a) and (b) (i) imply for all $t \in [t_0, t_1]$

$$V(t, x(t, t_0, x_0)) \leqq r(t, t_0, V(t_0, x_0)) \leqq r(t, t_0, a_M) < b_\Gamma \tag{3.2}$$

However, (3.1) implies $V(t_1, x(t_1, t_0, x_0)) \geqq b_\Gamma$, a contradiction to (3.2) at

$t = t_1$. We now show $x(t_0 + T, t_0, x_0) \in N$ whenever $x_0 \in M$. Since $x(t, t_0, x_0) \in \Gamma$ for $t \in [t_0, t_0 + T]$, then as before $V(t, t_0, x_0) \leqq r(t, t_0, a_M)$. Letting $t = t_0 + T$, we obtain with the aid of (b) (ii)

$$V(t_0 + T, t_0, x_0) \leqq r(t_0 + T, t_0, a_M) < b_{\Gamma N} . \tag{3.3}$$

If $x(t_0 + T, t_0, x_0) \notin N$ then from (3.1) $b_{\Gamma N} \leqq V(t_0 + T, t_0, x_0)$, a contradiction to (3.3). The sufficiency part is proved.

Necessity. Observe, first, that the Lipschitz condition on $f(t, x)$ given by (2.1) yields, with the aid of the Gronwall inequality, the estimate

$$\|x(t, s, x_0) - x(t, s, y_0)\| \leqq \exp\left(\int_{t_0}^{t_0+T} \lambda(s)ds\right)\|x_0 - y_0\| \tag{3.4}$$

for $t, s \in [t_0, t_0 + T]$, $x_0, y_0 \in M$, and $\lambda(t) \overset{\text{def}}{\equiv} \lambda_{\bar{\Gamma}}(t)$. Let $K \equiv \exp\left(\int_{t_0}^{t_0+T} \lambda(s)ds\right)$.

Consider the system

$$x' = F(t, x) , \tag{3.5}$$

where $F: J \times B \to B$ is continuous, Lipschitz in x, and bounded on $J \times B$, such that $F(t, x) = f(t, x)$ on $J \times \Gamma$. It follows from standard theory of differential equations [10] and the previous paragraph that all solutions of (3.5) exist on J for any initial point $(\bar{t}, \bar{x}) \in J \times B$ and depend continuously on initial conditions. Let x^* denote a trajectory of (3.5). Define

$$V(t, x) = d(x^*(t_0, t, x), M) + \|M\| , \tag{3.6}$$

where $\|M\| = \sup_{x \in M} \|x\|$.

Clearly $V(t, x)$ satisfies (2.4) and from (3.4) we notice that

$$|V(t, x) - V(t, y)| \leqq \exp\left(\int_{t_0}^{t_0+T} \lambda(s)ds\right)\|x - y\| ;$$

that is V satisfies (2.5). Moreover from (2.7) $V'(t, x) = 0$; that is V satisfies (a) with $g \equiv 0$. Hence it remains to prove that $a_M < b_\Gamma$ and $a_M < b_{\Gamma N}$ since $r(t, t_0, x_0) \equiv x_0$. But this follows from arguments similar to those in [9, page 1321], or [2, page 498] or [3, page 603]. We leave the details to the reader. (Notice that (3.1) and (3.6) imply that the values of b_Γ and $b_{\Gamma N}$ may depend on the set M.)

REMARK. In (3.6) we include the term $\|M\|$ since then

$$V(t, x) \exp\left(\int_{t_0}^{t_0+T} \lambda(s)ds\right) \geqq \|x\| . \tag{3.7}$$

Condition (3.7) will be used in Section 4.

4. **Perturbation results.** We now apply Theorem 3.1 to the perturbed

equation

$$x' = f(t, x) + h(t, x) . \tag{P}$$

Let us assume the unperturbed equation (E) is (M, N, Γ, T) practically stable. We ask the following two questions: (i) What effect does the "size" of perturbation term $h(t, x)$ have on the stability of (E)? (ii) If a given $h(t, x)$ is prescribed, how do we find new quantities $\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T}$ such that system (P) is $(\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T})$ practically stable?

We proceed to answer these questions. Let us assume

$$||h(t, x)|| \leqq \psi(t)(\phi(||x||)) , \tag{4.1}$$

where $\phi(\cdot)$ is nonnegative and nondecreasing. Now from Theorem 3.1 there exists $V(t, x)$ such that $V'_E(t, x) \leqq 0$. Hence

$$V'_P(t, x) \leqq \exp\Big(\int_{t_0}^{t_0+T} \lambda(s)ds\Big) ||h(t, x)|| \leqq K\psi(t)(\phi(||x||)) , \tag{4.2}$$

where again $K \equiv \exp \int_{t_0}^{t_0+T} \lambda(s)ds$. From (3.7) and the nondecreasing nature of $\phi(\cdot)$ we have from (4.2)

$$V'_P(t, x) \leqq K\psi(t)(\phi(KV(t, x))) . \tag{4.3}$$

The comparison principle suggests we consider the maximal solution of

$$r' = K\psi(t)\phi(Kr) , \qquad r(t_0) = a_M . \tag{4.4}$$

In order for system (P) to be (M, N, Γ, T) stable, Theorem 3.1 requires us to show that the solutions of (4.4) satisfy

$$\begin{cases} r(t, t_0, a_M) < b_\Gamma \\ r(t_0 + T, t_0, a_M) < b_{\Gamma N} , \end{cases} \tag{4.5}$$

where b_Γ and $b_{\Gamma N}$ are defined in (3.1). Consider the system (4.4); then

$$\int_{a_M}^{r} \frac{du}{\phi(Ku)} = K\int_{t_0}^{t} \psi(s)ds , \quad \text{or} \quad \int_{Ka_M}^{Kr} \frac{du}{\phi(u)} = K^2\int_{t_0}^{t} \psi(s)ds .$$

Define

$$G(r) = \int^{r} \frac{du}{\phi(u)} . \tag{4.6}$$

Then

$$G(Kr) = G(Ka_M) + K^2\int_{t_0}^{t} \psi(s)ds ;$$

that is, (4.5) is satisfied if

$$(4.7)\qquad r(t, t_0, a_M) \equiv \frac{1}{K} G^{-1}\Big(G(Ka_M) + K^2\int_{t_0}^{t} \psi(s)ds\Big) < \min(b_\Gamma, b_{\Gamma N}) .$$

This leads us to the following result.

THEOREM 4.1. *Assume* (E) *is* (M, N, Γ, T) *practically stable. In the perturbed equation* (P) *assume* $h(t, x)$ *satisfies* (4.1) *where* $\phi(\cdot)$ *is nonnegative and nondecreasing and* $\psi(\cdot)$ *is integrable on* $[t_0, t_0 + T]$. *If* a_M *is such that*

$$(4.8)\qquad G^{-1}\Big(G(Ka_M) + K^2\int_{t_0}^{t_0+T} \psi(s)ds\Big) < K\min(b_\Gamma, b_{\Gamma N}) ,$$

where $G(\cdot)$ *satisfies* (4.6), *then* (P) *is* (M, N, Γ, T) *practically stable.*

EXAMPLE 1. Assume the system (E) is (M, N, Γ, T) practically stable. Suppose $\|h(t, x)\| \leqq \|x\|$. Then solutions of (P) are (M, N, Γ, T) practically stable if

$$a_M \exp(K^2T) < \min(b_\Gamma, b_{\Gamma N}) ;$$

That is, we require

$$(4.9)\qquad a_M \leqq \exp(-K^2T)\min(b_\Gamma, b_{N\Gamma}) .$$

Notice that the stability of (P) requires according to Theorem 3.1 that

$$(4.10)\qquad a_M \leqq \min(b_\Gamma, b_{\Gamma N}) .$$

Consequently (4.9) may be an unreasonable restriction in some cases. So suppose we shrink M to some set $\bar{M}$ in which $a_{\bar{M}} = a_M \exp(-K^2T)$. Then (P) is $(\bar{M}, N, \Gamma, T)$ practically stable in view of (4.10).

REMARK. The above analysis shows that in Theorem 4.1 we may modify some or all of the quantities M, N, Γ, T in order to ascertain that (P) is $(\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T})$ practically stable given that (E) is (M, N, Γ, T) practically stable. Here $\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T}$ are modifications of $M, N, \Gamma,$ and T respectively. To do this we require that there exist sets $\bar{M}, \bar{N}, \bar{\Gamma}$ and a time $\bar{T}$ and constants $a_{\bar{M}}, b_{\bar{\Gamma}}, b_{\bar{\Gamma}\bar{N}}$ defined in (3.1) where $V(t, x)$ is constructed using system (E) in Theorem 3.1 (we always assume (E) is (M, N, Γ, T) practically stable). We now state this as our final result which is a generalization of Theorem 4.1. (We included Theorem 4.1 for motivational reasons.)

THEOREM 4.2. *Assume system* (E) *is* (M, N, Γ, T) *practically stable and let* $V(t, x)$ *be the known Lyapunov function satisfying the conditions in the proof of Theorem* 3.1. *Consider the perturbed equation* (P) *and assume* $h(t, x)$ *satisfies* (4.1) *where* $\phi(\cdot)$ *is nonnegative and nondecreasing* *and* $\psi(\cdot)$ *is integrable on* $[t_0, t_0 + T]$. *Let* $\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T}$ *be any modifications of* M, N, Γ, T *respectively (we do allow for any of the four possibilities* $\bar{T} = T$, $\bar{M} = M$, $\bar{N} = N$, $\bar{\Gamma} = \Gamma$). *Define as in* (3.1)

$$a_{\bar{M}} = \sup_{x \in \bar{M}} V(t_0, x) \,, \quad b_{\bar{\Gamma}} = \inf_{t \in J = [t_0, t_0 + \bar{T}], x \in \partial \Gamma} V(t, x) \,, \quad \textit{and} \quad b_{\bar{\Gamma}} = \inf_{x \in \bar{\Gamma} - \bar{N}} V(t_0 + \bar{T}, x) \,.$$

If $a_{\bar{M}}$ *satisfies*

$$G^{-1}\Big(G(Ka_{\bar{M}}) + K^2 \int_{t_0}^{t_0 + \bar{T}} \psi(s) ds\Big) \leqq K \min(b_{\bar{\Gamma}}, b_{\bar{\Gamma}\bar{N}}) \,,$$

where $G(\cdot)$ *satisfies* (4.6), *then* (P) *is* $(\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T})$ *practically stable.*

REMARK. This theorem provides us a relationship between the original quantities M, N, Γ, T and the new quantities $\bar{M}, \bar{N}, \bar{\Gamma}, \bar{T}$ in terms of our perturbation term $h(t, x)$. Of course the assumption (4.1) on $h(t, x)$ can be generalized. Further results in Banach spaces can be obtained assuming accretive type conditions on $f(t, x)$. We will not consider these extensions here.

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