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PRACTICAL THEORY OF THREE-PARTICLE STATES.

II. RELATIVISTIC, SPIN ZERO.

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A B S T R A C T

We obtain covariant equations for the scattering of composite particles. They are coupled linear integral equations in one variable. The solutions satisfy three-particle unitarity, and all observables in three-particle systems can be expressed in terms of them. The equations are derived from field theory, the basic approximation being that each two-particle subsystem is dominated by its bound states and resonances. However, the final equations involve only the wave functions of the composite particles, and not the original Lagrangian. Overlapping resonances are correctly taken into account, and some three-body forces are also included. The "potential" is essentially the Peierls mechanism, and its imaginary part gives the interference effect between overlapping resonances. Our equations are different from and simpler than those of Alessandrini and Omnès, because we eliminate the relative energies in a way compatible with the Landau-Cutkosky rules. The present paper only gives the equations when the elementary particles are spinless (unequal masses).

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1. PRELIMINARIES

1a INTRODUCTION

We would understand strong interactions better if we had an adequate dynamical theory of three-particle processes. Many models exist, but they all suffer from serious and obvious drawbacks. To provide the background of the present work, we will start with a list of such drawbacks. Of course, no one model suffers from all of them.

i) Many previous theories lack three-particle unitarity.

ii) Most of them do not apply to overlapping resonances. By this we mean the situation where we have three particles in a final state, and any two of them are known to resonate.

iii) Some contain numerous arbitrary constants.

iv) Some have no clear theoretical foundation, the equations appearing conjured out of thin air.

v) Almost all would be invalid near a three-particle resonance, of which many have recently been found experimentally.

vi) Some assume that unstable particles can be approximated by stable ones.

vii) Mass-shell approaches, such as the Khuri-Treiman equation, diverge for P wave particles unless *ad hoc* cut-offs are introduced.

By writing down this list of bad points, we have specified the properties of an ideal theory. We must also add, of course, that it be numerically soluble, and most important that it agree with experiment.

Recently, one of us showed [1)] that, for three non-relativistic particles interacting by pair potentials, an approximation can be formulated which has none of these disadvantages, yet is still simple enough to solve numerically.

There are two basic ideas. Firstly, we used the Faddeev formulation [2] of three-particle scattering, in which the original potentials have disappeared and been replaced by the solutions of the three two-particle subsystems. Secondly, we supposed that each of these two-particle subsystems could be approximated by a finite number of bound states and resonances, to be taken from experiment. For the three-nucleon system, a special case of these equations had been obtained earlier by various authors [3]. Several people have now solved them and compared the results with experiment [3],[4]. It seems fair to say that one arbitrary constant is required to represent the short-range part of the two-nucleon interaction. A considerable number of three-nucleon observables can then be correctly predicted, all other constants being determined from low-energy two-nucleon scattering, which is assumed known. This compares favourably with two-particle dispersion theory, where a number of phenomenological parameters usually have to be introduced.

The present paper contains the relativistic extension of this theory. Instead of a potential, we now start from a three-particle Bethe-Salpeter equation. This can be rearranged into Faddeev-type equations, containing the two-particle solutions as the kernel. Our basic approximation is that these two-particle solutions (for each of the three possible subsystems) are dominated by a finite number of bound states and resonances. We then end up with equations describing the scattering of these composite particles, and giving all observables of the three-particle system in terms of them. Though derived from Lagrangian field theory, these final equations do not assume any particular Lagrangian. The Bethe-Salpeter equation is used merely as a convenient scaffolding. Instead, they require a knowledge of the bound states and resonances of the various two-particle subsystems. True, we need to know not only their observable positions and widths, but also their wave functions. However, non-relativistic calculations have shown that wave functions do not depend much on the shapes of their potentials. It seems a good approximation to assume a generalized Hulthén wave function, and adjust the constants to give the experimental position and width [5]. We shall hope the same is true relativistically.

This brings us to one of the fundamental difficulties of the relativistic theory. In three-particle potential scattering, all results can now be proved rigorously [6], even by mathematical standards. For the Bethe-Salpeter equation this is not yet possible, even in the two-particle case, because of the unsolved problem of justifying the Wick transformation. If we assume that the metric can be made Euclidean by the Wick rotation, then most of the non-relativistic proofs carry over with slight alteration. However, this basic gap makes such partial proofs rather pointless. The present paper therefore will not make any attempt at complete mathematical rigour, though our arguments should appear satisfactory to most physicists.

The second difficulty of the relativistic theory, which is related to the first, is the presence of extra variables of integration - the relative energies. These seem to have no physical meaning, and must be eliminated if the equations are to be tractable numerically. This problem we have solved. It is a rather remarkable fact that our equations for composite particle scattering, though they appear to be two-particle equations, nevertheless satisfy three-particle unitarity. By taking the equations to pieces, we can isolate the mechanism that ensures three-particle unitarity. This we maintain to be the vital part of the equations, so we put them together again, with the unitarity mechanism intact, but the relative energies fixed at their mass-shell values.

Since our equations describe the scattering and disintegration of composite particles, you may think they only apply to processes like nd scattering, which are not of any fundamental importance. The application to elementary processes, such as $N + \pi \rightarrow N + \pi + \pi$, comes by assuming that all particles are composite - in this case that the initial nucleon can be regarded as an $N\pi$ bound state. Nearly everyone nowadays has their own meaning for "elementary" and "composite". The sense in which we shall use them is that a particle is elementary if its propagator can be well approximated by a pole, and its vertex function by a constant. Conventional mass-shell S matrix theory is then a theory in which all particles are "elementary". Unstable particles, however, do not have mass-shells - the Lehmann representation for their propagators contains no pole term, and they are experimentally observed as continuous distributions of mass. The usual answer is to give them a complex mass-shell, corresponding to a pole on the unphysical sheet. However, this representation leads to serious difficulties with unitarity and threshold behaviour, which seems to be the basic reason why S matrix approaches to three-particle states have never been able to satisfy three-particle unitarity. Also, the pole on the unphysical sheet is much further removed from experiment than the concept of a continuous distribution of real masses. For unstable particles mass-shell theory is actually less physical than off-shell theory, contrary to the usual belief. Similarly, we consider a stable particle to be composite, when the continuum part of the Lehmann representation cannot be neglected. This definition has the consequence that a particle may be "elementary" at low energies, and "composite" at high. So there is really no paradox when the nucleon appears in our equations for $N + \pi \rightarrow N + \pi + \pi$ as composite in the initial state, and elementary in the final one, since the energies available are different.

The aim of the present work is to obtain numerically soluble equations for processes like $N + \pi \rightarrow N + \pi + \pi$ at energies where four-particle production can be neglected. We hope to use them to investigate such questions as the dynamics of the $N\pi$ P_{11} object, and the validity of the evidence for the σ particle in the $N(\pi\pi)$ final state [7]. Our equations will be covariant, satisfy three-particle unitarity, take correct account of overlapping resonances such as the final N^* and ρ in this process, and include most previous models as approximations. In the present paper we shall only give them for three spinless particles (with unequal masses). This is because we need a rather elaborate notation, mainly for the reduction of the relative energies, and did not want to complicate it unbearably. A paper giving the equations for any spin is at present in preparation.

Though the derivation is long and complicated, the final equations are very simple. We have therefore tried to make Section 8, where they are given, readable independently of the rest of the paper. We would also like the general reader to look at Section 5b, which we are rather proud of. The rest is for specialists, by whom we mean anyone with a special interest in production processes and many-particle unitarity.

Section 1b expounds our notation. We are sorry there is so much of it, but we think the reader would be sorrier still if he had to face three-particle off-shell relativistic kinematics unarmed. In Section 2a we give the three-particle Bethe-Salpeter equations that we start with and discuss the various Green's functions (Feynman amplitudes) involved. Section 2b is an outline of scattering theory for composite particles. It explains how to obtain the S matrices for bound-state processes, knowing the Feynman amplitudes (off-shell) for the elementary particles. Section 3 recapitulates some two-particle theory: 3a considers especially the structure of the bound-state poles of the Bethe-Salpeter equation, 3b the relation to separable interactions, 3c the elimination of the relative energy by the method of Blankenbecler and Sugar, and 3d the relation to the experimental S matrix.

Section 4 starts the three-particle theory proper: 4a gives the relativistic Faddeev equations, 4b discusses their unitarity properties and gives the explicit formulae for all S matrices - those involving bound states as well as the three-particle ones. So far no approximations have been introduced, but the integral equations are in eight variables. In Section 5, they are reduced to four, by assuming that each of the two-particle subsystems is dominated by a limited number of bound states and resonances. The equations are first obtained in Section 5a under the additional approximation that the three-particle irreducible graphs (three-body forces) can be neglected, but this restriction is partly removed in Section 5c. 5b is the core of the present paper, and shows how these apparently two-particle equations are able to satisfy three-particle unitarity. We feel that the proof gives considerable physical insight into the nature of the isobar and optical models. In Section 6 this result is used to eliminate the relative energies, thereby obtaining an integral equation in three variables. A firm grasp of our notations is needed here. Section 7 performs the angular momentum separation using Wick's helicity formalism [8], which appears ideally suited to the Faddeev equations. The integral equations are now in one variable, and contain one channel for each composite-particle scattering process. The potential is an integral over Wick's recoupling coefficient, and the only remaining complexity is in the formulae for the Wick angles, which express the effect of the Lorentz transformation from the two-particle to the three-particle c.m. frames. The effects of isospin and identity of particles have already been adequately discussed in the non-relativistic paper [1]. Section 8a gives our final equations, and Section 8b some concluding remarks.

After the second draft of the present paper was complete, a preprint by Alessandrini and Omnès arrived [9], claiming to do the same. It appears to us that the crucial step - the elimination of the relative energies - is wrong in their work. This is discussed in detail in Sections 3c and 6 below, but briefly the mistake seems to be that they take the Landau-Cutkosky rules in a specialized form applying only to the singularities in the total energy, and use them to write dispersion relations in the external masses for which even the normal thresholds appear incorrect. In extenuation we should record that it took us six months to get this (hopefully) right.

1b KINEMATICS AND NOTATION

The complexities of relativistic kinematics for three-particles off the energy and mass shells require a rather elaborate notation. This is explained in the present subsection. We hope that the conscientious reader will henceforth have an effortless mastery of relativistic kinematics.

We use a metric in which the time component is positive, and denote four-vectors by hats, $\hat{k}$, and three-vectors by bold face type, $\mathbf{k}$. $k = |\mathbf{k}|$ is the magnitude of the spacelike part. Most of our equations will be written in the momentum representation, in which the three spinless particles of mass m_1, m_2 and m_3 have four-momenta $\hat{k}_1$, $\hat{k}_2$ and $\hat{k}_3$. The total momentum will be denoted by

$$\hat{P} = \hat{k}_1 + \hat{k}_2 + \hat{k}_3 . \tag{1.1}$$

In the three-particle centre-of-mass frame

$$\hat{P} = (\mathbf{0}, W). \tag{1.2}$$

We shall always use W to denote the total mass of the three-particle system. We shall also need

$$\hat{p}_3 = \frac{1}{m_1 + m_2}\left[m_2 \hat{k}_1 - m_1 \hat{k}_2\right], \tag{1.3}$$

the relative momentum in the (1,2) subsystem, and

$$\hat{q}_3 = \frac{1}{m_1 + m_2 + m_3}\left[m_3(\hat{k}_1 + \hat{k}_2) - \right.$$
$$\left. - (m_1 + m_2)\hat{k}_3\right], \tag{1.4}$$

the momentum of particle 3 relative to this subsystem. The definitions of $\hat{p}_1$, $\hat{p}_2$ and $\hat{q}_1$, $\hat{q}_2$ are obtained from (1.3) and (1.4) by a cyclic permutation of the indices. From these we obtain the useful relations

$$\hat{q}_\alpha = \frac{m_\alpha}{m_1 + m_2 + m_3}\hat{P} - \hat{k}_\alpha \tag{1.5}$$

and

$$\hat{q}_1 + \hat{q}_2 + \hat{q}_3 = 0. \tag{1.6}$$

It is easily verified that the Jacobians

$$\left|\frac{d^4P\, d^4p_\alpha\, d^4q_\alpha}{d^4k_1\, d^4k_2\, d^4k_3}\right| =$$
$$= \left|\frac{d^4P\, d^4q_\alpha\, d^4q_\beta}{d^4k_1\, d^4k_2\, d^4k_3}\right| = 1, \text{ for } \alpha \neq \beta. \tag{1.7}$$

Hence, instead of the three independent vectors $\hat{k}_\gamma$, we can just as well use $\hat{P}$, $\hat{p}_\alpha$, $\hat{q}_\alpha$ for any α, in which case the $\hat{k}_\gamma$ are given by

$$\begin{aligned}
\hat{k}_1 &= \frac{m_1}{m_1+m_2+m_3}\hat{P} + \frac{m_1}{m_1+m_2}\hat{q}_3 + \hat{p}_3 \,, \\
\hat{k}_2 &= \frac{m_2}{m_1+m_2+m_3}\hat{P} + \frac{m_2}{m_1+m_2}\hat{q}_3 - \hat{p}_3 \,, \qquad (1.8) \\
\hat{k}_3 &= \frac{m_3}{m_1+m_2+m_3}\hat{P} - \hat{q}_3 \,,
\end{aligned}$$

or by the relations obtained from these by cyclic permutation of the indices. By (1.7), we can also use $\hat{P}$, $\hat{q}_1$, $\hat{q}_2$ or any other pair of $\hat{q}$'s, in which case

$$\begin{aligned}
\hat{p}_1 &= -\frac{m_2}{m_2+m_3}\hat{q}_1 - \hat{q}_2 \,, \\
\hat{p}_2 &= \hat{q}_1 + \frac{m_1}{m_1+m_3}\hat{q}_2 \,. \qquad (1.9)
\end{aligned}$$

As in the non-relativistic paper [1], we shall make the convention that, whenever more than two of the six vectors $\hat{p}_\alpha$, $\hat{q}_\alpha$ appear in the same equation, only two of them are linearly independent, and the others are related to these by the formulae we have given. If we wish to distinguish $\hat{p}$'s and $\hat{q}$'s belonging to different sets of momenta for the three particles, we will always denote them by a different number of primes. Thus $\hat{p}_1$ is implicitly supposed to be linearly dependent on $\hat{q}_1$ and $\hat{q}_2$, but $\hat{p}_1'$ is not.

When considering the (1,2) subsystem, it is convenient to use the momenta $\hat{p}_3$ and the total subsystem momentum

$$\hat{K}_3 = \hat{k}_1 + \hat{k}_2 , \qquad (1.10)$$

for which

$$\left| \frac{d^4 K_3 \, d^4 p_3}{d^4 k_1 \, d^4 k_2} \right| = \left| \frac{d^4 P \, d^4 q_3}{d^4 K_3 \, d^4 k_3} \right| = 1 , \qquad (1.11)$$

and

$$\begin{aligned} \hat{k}_1 &= \frac{m_1}{m_1 + m_2} \hat{K}_3 + \hat{p}_3 , \\ \hat{k}_2 &= \frac{m_2}{m_1 + m_2} \hat{K}_3 - \hat{p}_3 . \end{aligned} \qquad (1.12)$$

In the centre-of-mass frame of the (1,2) subsystem

$$\hat{K}_3 = (\underset{\sim}{0} , w_3) \qquad (1.13)$$

w_α will always be used to denote the total mass of the (β, γ) subsystem, where $\alpha \neq \beta \neq \gamma \neq \alpha$. In Section 3, where we consider a particular two-particle subsystem, we will omit the subscripts from $\hat{K}$, $\hat{p}$, w.

Whenever we refer to components $\underset{\sim}{q}_\alpha$, q^0_α, or $\underset{\sim}{k}_\alpha$, k^0_α we shall assume that they are evaluated in the centre-of-mass frame of the three-particle system. Thus

$$\underset{\sim}{q}_\alpha = -\underset{\sim}{k}_\alpha \tag{1.14}$$

On the other hand, whenever we refer to components $\underset{\sim}{p}_\alpha$, p^0_α, we shall assume that they are evaluated in the centre-of-mass frame of the (β, γ) subsystem. If we want to speak of k^0_β evaluated in this subsystem frame, we shall call it $E_{\beta\alpha}$. This use of different Lorentz frames for the different momentum vectors results in a considerable simplification of the final equations, though of course it necessitates extra care in the derivation.

When we use radial co-ordinates in these frames, we shall denote those of $\underset{\sim}{p}_\alpha$ by p_α, ϑ_α, φ_α, and those of $\underset{\sim}{q}_\alpha$ by q_α, Θ_α, Φ_α. The two angles ϑ_α and φ_α will be denoted collectively by $\underset{\sim}{\omega}_\alpha$, with

$$d^2\omega_\alpha \equiv d\cos\vartheta_\alpha \, d\varphi_\alpha . \tag{1.15}$$

In general, our momentum four-vectors will be off the mass shell, so that there is no connection between k^0_α and k_α. However, we can define certain functions of the spacelike components which will coincide with the timelike components on the mass shell. We shall denote these by placing a bar over them. Thus

$$\bar{k}^0_\alpha \equiv +\sqrt{k^2_\alpha + m^2_\alpha} \tag{1.16}$$

($k_\alpha = |\underset{\sim}{k}_\alpha|$). Remembering the particular Lorentz frames in which the timelike part of each momentum vector is defined, we have

$$\bar{k}^0_\alpha = \sqrt{m_\alpha^2 + q_\alpha^2}\,, \tag{1.17}$$

$$\begin{aligned} \bar{q}^0_3 &= \frac{m_3}{m_1+m_2+m_3}\left[\bar{k}^0_1 + \bar{k}^0_2\right] - \frac{m_1+m_2}{m_1+m_2+m_3}\,\bar{k}^0_3 \\ &= \frac{m_3}{m_1+m_2+m_3}\left[\sqrt{m_1^2+q_1^2} + \sqrt{m_2^2+q_2^2}\right] - \\ &\qquad - \frac{m_1+m_2}{m_1+m_2+m_3}\sqrt{m_3^2+q_3^2}\,, \end{aligned} \tag{1.18}$$

$$\bar{W} = \bar{P}^0 = \sqrt{m_1^2+q_1^2} + \sqrt{m_2^2+q_2^2} + \sqrt{m_3^2+q_3^2}\,, \tag{1.19}$$

$$\bar{E}_{\beta\alpha} = \sqrt{m_\beta^2 + p_\alpha^2}\,, \tag{1.20}$$

$$\bar{p}^0_3 = \frac{m_2}{m_1+m_2}\sqrt{m_1^2+p_3^2} - \frac{m_1}{m_1+m_2}\sqrt{m_2^2+p_3^2}\,, \tag{1.21}$$

$$\bar{w}_3 = \bar{K}^0_3 = \sqrt{m_1^2+p_3^2} + \sqrt{m_2^2+p_3^2}\,. \tag{1.22}$$

This notation will be mainly used in the sections dealing with the elimination of the relative energies, a process which we shall call "reduction". This is easily the most difficult problem in the relativistic extension of the Faddeev equations, and its correct solution would hardly have been possible without this bar notation.

The (1,2) subsystem in its centre-of-mass frame and on the mass shells of both particles has three independent variables. We may take these to be either p_3, or else $\bar{w}_3$ and the angles ϑ_3, φ_3. The Jacobian connecting them is

$$\left| \frac{d\bar{w}_3 \; d^2\underset{\sim}{\omega}_3}{d^3 \underset{\sim}{p}_3} \right| = \frac{\bar{w}_3}{p_3 \bar{E}_{13} \bar{E}_{23}} . \qquad (1.23)$$

We shall denote the five variables $\Theta_\alpha, \Phi_\alpha, \bar{w}_\alpha, \vartheta_\alpha, \varphi_\alpha$ collectively by $\underset{\sim}{\Omega}_\alpha$, with

$$d^5\Omega_\alpha \equiv d\cos\Theta_\alpha . d\Phi_\alpha . d\bar{w}_\alpha . d\cos\vartheta_\alpha . d\varphi_\alpha . \qquad (1.24)$$

Together with $\bar{W}$, they form a complete set of independent variables for the three-particle system in its centre-of-mass frame with all three particles on the mass shell. The Jacobian connecting them to the other set $\underset{\sim}{p}_\alpha, \underset{\sim}{q}_\alpha$ is

$$\left| \frac{d\bar{W} . d^5\Omega_\alpha}{d^3 p_\alpha \; d^3 q_\alpha} \right| = \frac{\bar{W}}{q_\alpha p_\alpha \bar{k}_1^0 \bar{k}_2^0 \bar{k}_3^0} . \qquad (1.25)$$

Here we have allowed for our convention that $\underset{\sim}{p}_\alpha$ and $\underset{\sim}{q}_\alpha$ are defined in different Lorentz frames. This formula can be derived by using the fact that

$$d^3 k_\gamma / (2 \bar{k}_\gamma^0) \tag{1.26}$$

is Lorentz invariant, together with (1.7), (1.11) and (1.23).

The Lorentz transformation taking the three-particle centre-of-mass system into that of the (1,2) subsystem has velocity

$$\underset{\sim}{q}_3 / \left(\sqrt{m_1^2 + q_1^2} + \sqrt{m_2^2 + q_2^2} \right). \tag{1.27}$$

The relationships between quantities defined in these different Lorentz frames are most conveniently calculated by hyperbolic trigonometry [8),10)]. In particular,

$$\bar{w}_3 = \left\{ \left[\sqrt{m_1^2 + q_1^2} + \sqrt{m_2^2 + q_2^2} \right]^2 - q_3^2 \right\}^{1/2}, \tag{1.28}$$

$$\begin{aligned} p_3 = \Big\{ \Big[& q_1^2 + q_2^2 - q_3^2 + \\ & + 2\sqrt{(m_1^2 + q_1^2)(m_2^2 + q_2^2)} \Big]^2 - \\ & - 4 m_1^2 m_2^2 \Big\}^{1/2} / (2 \bar{w}_3). \end{aligned} \tag{1.29}$$

Our angular momentum notation is that of Wick [8], except that we sometimes add subscripts, to make it clear which subsystem we are referring to. Also we replace M, m by H, h to avoid confusion with masses. Thus J is the total angular momentum in the three-particle centre-of-mass frame, and H is its z component referred to some space-fixed axis. j is the total angular momentum of an initial two-body resonance, and h is its total helicity. It will always be clear from the rest of the equations, which subsystem j refers to. In the subsystem rest frame, h is the component of $\underset{\sim}{j}$ in the direction of $\underset{\sim}{q}_\alpha$. Similarly j' and h' are the total angular momentum and helicity of a two-body resonance in the final state. s_α are the spins of the three particles, and λ_α their helicities. In the present paper s_α and λ_α are all zero.

It turns out that the "potentials" for the scattering of a composite particle in a particular total angular momentum state can be very simply expressed in terms of the Wick recoupling coefficients, which is why the Wick formalism is so useful. It is therefore very important to know explicitly the Wick angles, which occur as the arguments of the d-functions in these recoupling coefficients. When recoupling between all three channels is being considered, Wick's notation for these angles is ambiguous. In terms of Figure 1 of Wick's paper [8], with 0 the central point, and 1, 2, 3 the vertices, we therefore define χ_ν to be the angle $\mu 0 \sigma$, $\beta_{\mu\nu}$ to be the angle $0\mu\nu$, and ρ_ν to be the angle $\mu\nu\sigma$, where $\mu\nu\sigma$ are some cyclic permutation of 1, 2, 3. We define ϑ_3 to be Wick's ϑ, ϑ_1 to be Wick's ϑ', and ϑ_2 to be the similar angle between 13 and 20, pointing in the anticlockwise direction. All of these angles can be expressed in terms of the three independent variables q_1, q_2, q_3 which are the magnitudes of the momenta of the three particles in the total centre-of-mass system. We have already expressed $\bar{W}$, $\bar{w}_\alpha$ and p_α in terms of these variables, in (1.19), (1.28) and (1.29). They appear to be a very convenient choice for the three-particle system after angular momentum decomposition. The explicit formulae for the Wick angles can be obtained by hyperbolic trigonometry. We record them here for convenient reference, though they will be used mainly in other papers of the series.

$$\cos \chi_2 = (q_2^2 - q_1^2 - q_3^2)/2q_1 q_3 \,, \tag{1.30}$$

$$\cos \vartheta_3 = \\ = \frac{\sqrt{m_1^2+q_1^2}\left[q_1^2-q_2^2-q_3^2\right]+\sqrt{m_2^2+q_2^2}\left[q_1^2-q_2^2+q_3^2\right]}{q_3\left\{\left[q_1^2+q_2^2-q_3^2+2\sqrt{(m_1^2+q_1^2)(m_2^2+q_2^2)}\right]^2-4m_1^2m_2^2\right\}^{1/2}} \tag{1.31}$$

$$\cos \beta_{12} = \\ = \frac{\sqrt{m_1^2+q_1^2}\left(q_1^2+q_2^2-q_3^2\right)+\sqrt{m_2^2+q_2^2}\left(2q_1^2\right)}{q_1\left\{\left[q_1^2+q_2^2-q_3^2+2\sqrt{(m_1^2+q_1^2)(m_2^2+q_2^2)}\right]^2-4m_1^2m_2^2\right\}^{1/2}} \tag{1.32}$$

$$\begin{aligned} \cos \rho_2 = {} & \Big\{\Big[\sqrt{m_2^2+q_2^2}\left(q_2^2+q_3^2-q_1^2\right)+\sqrt{m_3^2+q_3^2}\left(2q_2^2\right)\Big]\times \\ & \times\Big[\sqrt{m_2^2+q_2^2}\left(q_2^2+q_1^2-q_3^2\right)+\sqrt{m_1^2+q_1^2}\left(2q_2^2\right)\Big]- \\ & -m_2^2\left[(q_1+q_3)^2-q_2^2\right]\left[q_2^2-(q_1-q_3)^2\right]\Big\}\Big/\Big[q_2^2\Big\{\Big[q_2^2+ \\ & +q_3^2-q_1^2+2\sqrt{(m_2^2+q_2^2)(m_3^2+q_3^2)}\Big]^2-4m_2^2m_3^2\Big\}^{1/2}\times \\ & \times\Big\{\Big[q_2^2+q_1^2-q_3^2+2\sqrt{(m_2^2+q_2^2)(m_1^2+q_1^2)}\Big]^2-4m_2^2m_1^2\Big\}^{1/2}\Big]. \end{aligned} \tag{1.33}$$

The other angles can be obtained from those given here by a cyclic permutation of indices in (1.30), (1.31) and (1.33), and by any permutation in (1.32).

2. FOUNDATIONS

2a EQUATIONS FOR FEYNMAN AMPLITUDES

All Feynman graphs for two-particle scattering can be divided into two classes - two-particle reducible, e.g., Fig. 1a, and two-particle irreducible, e.g., Fig. 1b. The former are obtained by iterating the latter. Formally, a graph is two-particle irreducible if it cannot be cut in two by breaking just two internal lines, in such a way that the two initial external lines lie in one half of the graph, and the two final ones in the other. It is important to note that this definition assumes that we are considering one particular scattering process, so that we know which external lines are initial and which final. The definition of irreducible graphs is therefore not crossing-symmetric.

Graphically we denote the Feynman amplitude for two-particle scattering by a bubble, and that for the irreducible graphs by a bubble with a line through it. Figure 1c then describes the way in which all such graphs are obtained by iterating the irreducible ones. This is the graphical expression of the Bethe-Salpeter integral equation [11]. The integrals are over the momenta of the two internal lines. By separating out the total energy-momentum delta function, we can write it as an integral over the relative four-momentum $\hat{p}_3$ only. (This variable is defined in (1.3). The subscript 3 arises because we suppose that the two particles which are scattering are 1 and 2.) Just as in the non-relativistinc theory [12], we consider this integral equation as an operator equation. The Hilbert space will then consist of functions of the four-vector $\hat{p}_3$. The operators involved will still depend on the total four-momentum $\hat{K}_3$ as external parameters. In the c.m. frame, however, they will only depend on the total mass $w_3 = (\hat{K}_3^2)^{\frac{1}{2}}$, and the operators in the Bethe-Salpeter equation will have the general form

$$A(w_3) \equiv \langle \hat{p}_3 | A(w_3) | \hat{p}_3' \rangle, \tag{2.1}$$

To get the equation explicitly, we define

$$d_\alpha = \left[m_\alpha^2 - i\varepsilon - \hat{k}_\alpha^2 \right]^{-1} \tag{2.2}$$

to be the propagators of the elementary particles, and

$$G_3^0(w_3) = \left[2\pi^2 i \right]^{-1} d_1 d_2 \tag{2.3}$$

to be the free two-particle Green's function for particles 1 and 2. More precisely, we should write

$$\langle \hat{p}_3 | G_3^0(w_3) | \hat{p}_3' \rangle =$$

$$= \frac{\delta_4(\hat{p}_3 - \hat{p}_3')}{2\pi^2 i \left[m_1^2 - i\varepsilon + p_3^2 - \left(p_3^0 + \frac{m_1 w_3}{m_1 + m_2} \right)^2 \right] \times \left[m_2^2 - i\varepsilon + p_3^2 - \left(p_3^0 - \frac{m_2 w_3}{m_1 + m_2} \right)^2 \right]} , \tag{2.4}$$

but we shall not always trouble to distinguish between diagonal operators (in the momentum representation) and the functions representing them. We shall denote by $G_3(w_3)$ the complete two-particle Green's function for particles 1 and 2, i.e., the Feynman amplitude for their scattering, and by I_3 the sum of all two-particle irreducible graphs (without the propagators for their external lines). The two-particle Bethe-Salpeter equation is then

$$G_3(w_3) = G_3^0(w_3) + G_3(w_3)\, I_3\, G_3^0(w_3) . \tag{2.5}$$

In the ladder approximation

$$\langle \hat{p}_3 | I_3 | \hat{p}_3' \rangle =$$

$$= \frac{g^2}{8\pi^2} \left[\mu^2 - (\hat{p}_3 - \hat{p}_3')^2 \right]^{-1}. \tag{2.6}$$

The Bethe-Salpeter equation involves intermediate particles off their mass shells. Its solution will define a Feynman amplitude with the initial and final particles off their mass shells, also. In terms of the notation of the previous section, where $\bar{w}_3$ is a function of p_3, and $\bar{w}_3'$ of p_3', the solution of the Bethe-Salpeter equation (2.5) will define $\langle \hat{p}_3 | G_3(w_3) | \hat{p}_3' \rangle$ for values for which the mass-shell condition

$$\bar{w}_3 = w_3 = \bar{w}_3' \tag{2.7}$$

is not satisfied. This extension off the mass shell is such that all the right-hand singularities of the total energy are in the variable w_3, while the left-hand singularities and anomalous thresholds are in $\bar{w}_3$ and $\bar{w}_3'$ (cf. the non-relativistic case in Refs. [1] and [12]). We made this happen when we defined I_3 to be independent of the <u>off-shell</u> energy w_3. Any physical energy dependence of the interaction will be expressed by its dependence on the physically equivalent on-shell variables $\bar{w}_3$ and $\bar{w}_3'$. The answer to S matrix dogmatists who think this device of distinguishing between on-shell and off-shell energies an artificial one, is that it enables us to do things they have never been able to do.

Off-shell unitarity formulae will be very important in our analysis. They can be derived from the Bethe-Salpeter equation by taking the imaginary part and performing some operator algebra, or alternately by expanding in a perturbation series, using the Cutkosky rules [13], and then summing. In both cases, we require certain mass-shell delta-functions

$$\Delta_\alpha = \vartheta(k_\alpha^0)\,\delta(\hat{k}_\alpha^2 - m_\alpha^2) \tag{2.8}$$

In terms of these, the imaginary part of the <u>free</u> two-particle Green's function (2.3), for positive energies, is given by

$$\frac{1}{2i}\left[G_3^0(w_3 + i\varepsilon) - G_3^0(w_3 - i\varepsilon)\right] \doteq \Delta_1 \Delta_2 \tag{2.9}$$

The $2\pi^2 i$ factor in (2.3) was chosen to simplify (2.9).

Now we consider the three-particle Feynman amplitudes. The three-particle analogue of the Bethe-Salpeter equation is shown graphically in Fig. 2. It can easily be derived by summing graphs, or non-perturbatively by the methods of J.G. Taylor [14]. The sum over α in the second term means that we must take the two-particle irreducible graphs I_α for each of the final pairs (α is the non-interacting final particle). The right-hand bubble in the third term consists of all three-particle irreducible graphs, i.e., those that cannot be cut in two by breaking just three internal lines in such a way that the three initial external lines all lie in one half and the three final lines in the other. Graphs with an intermediate state of one or two particles must be considered as three-particle-irreducible if the intermediate particles occur in the Lagrangian, but not if they are bound states.

The integration in the three-particle Bethe-Salpeter equation will be over three internal lines. Removing the energy-momentum delta-function, then gives us integral equations in two four-vectors, which we can take to be $\hat{p}_\alpha$

and $\hat{q}_\alpha$, for any $\alpha = 1,2,3$. (We recall that of the six vectors $\hat{p}_\alpha$ and $\hat{q}_\alpha$, only two are linearly independent.) If we write the equation in terms of operators, then they will act in Hilbert space of functions of two four-momenta. In the c.m. frame they will depend on the total three-particle mass W as an external parameter. Thus our operators have the general form

$$A(W) \equiv \langle \hat{p}_\alpha \hat{q}_\alpha | A(W) | \hat{p}'_\alpha \hat{q}'_\alpha \rangle. \tag{2.10}$$

Figure 2 also involves the two-particle irreducible kernels I_α, for $\alpha = 1,2,3$, so we need to consider these as acting also in the three-particle Hilbert space. In the non-relativistic papers we were very careful to distinguish between a two-particle operator, and the corresponding operator acting in the three-particle Hilbert space. In the present paper, the notation is already too elaborate to make this practical, and we hope also that most of our readers will have some acquaintance with the non-relativistic theory. The usual rule is that a two-particle operator belonging to the α subsystem, with matrix elements

$$\langle \hat{p}_\alpha | \tilde{A}(w_3) | \hat{p}'_\alpha \rangle \tag{2.11}$$

becomes a three-particle operator with matrix elements

$$\langle \hat{p}_\alpha \hat{q}_\alpha | A(W) | \hat{p}'_\alpha \hat{q}'_\alpha \rangle =$$

$$= \langle \hat{p}_\alpha | A(w_\alpha[W, q_\alpha]) | \hat{p}'_\alpha \rangle \, \delta_4(\hat{q}_\alpha - \hat{q}'_\alpha), \tag{2.12}$$

where now

$$w_\alpha[W, q_\alpha] = \left\{ \left[W - \sqrt{m_\alpha^2 + q_\alpha^2} \right]^2 - q_\alpha^2 \right\}^{1/2} . \tag{2.13}$$

As discussed in Ref. [12], (2.13) has the effect of drawing out the bound-state poles into bound-state-scattering cuts.

The three particle system has quite a lot of Green's functions. The complete Feynman amplitude for three-particle scattering we write $G(W)$. Its free field analogue is

$$G^0(W) = \frac{1}{\pi(2\pi i)^2} d_1 d_2 d_3 . \tag{2.14}$$

In between these, there are products of an exact two-particle Green's function G_α with a free propagator for the third particle. These we write as

$$G^\alpha(W) = \frac{1}{2\pi i} G_\alpha d_\alpha . \tag{2.15}$$

We shall also use the two-particle Green's functions G_α without multiplication by the third particle propagator. A superscript will mean that the third particle propagator is present, a subscript that it is not. [G^0 and G_0^0 both mean (2.14).] I_α for $\alpha = 1,2,3$ will denote the sum of all two-particle irreducible graphs for the α subsystem, while I_0 will be the sum of all three-particle irreducible graphs.

Using these quantities, the three-particle Bethe-Salpeter equation of Fig. 2 becomes

$$G = G^{0} + G\Big[\sum_{\alpha=1,2,3} 2\pi i\, d_{\alpha}^{-1} I_{\alpha} + I_{0}\Big] G^{0}$$

$$= G^{0} + G\Big[\sum_{\alpha=0}^{3} I_{\alpha} G_{\alpha}^{0}\Big]. \tag{2.16}$$

This is the relativistic analogue of the three-particle Lippmann-Schwinger equation, and suffers from the same disconnectedness. In Section 4 we will rearrange it into equations of the Faddeev type.

We will also need mass-shell delta-functions for the two-particle bound states

$$\Delta_{\alpha n} = \vartheta(K_{\alpha}^{0})\, \delta(\hat{K}_{\alpha}^{2} - M_{\alpha n}^{2}), \tag{2.17}$$

where $M_{\alpha n}$ is the mass of the n^{th} bound state of the α subsystem. Lastly, we establish a special notation Δ_{0} for the product of the three mass-shell delta-functions occurring in the three-particle unitarity equation. The imaginary part of the free Green's function, for positive energies, is then

$$\frac{1}{2i}\left[G^{0}(W+i\varepsilon) - G^{0}(W-i\varepsilon)\right] =$$

$$= \Delta_{0} = \Delta_{1}\Delta_{2}\Delta_{3}. \tag{2.18}$$

2b SCATTERING THEORY

Non-perturbatively, the Feynman amplitudes are defined as vacuum expectation values of Heisenberg operators. Thus, if $\tilde{A}_\alpha(\hat{k}_\alpha)$ is the Heisenberg field operator for particle α, we have

$$\langle \hat{p}_3 | G_3(\omega_3) | \hat{p}_3' \rangle \, \delta_4(\hat{K}_3 - \hat{K}_3') =$$

$$= \frac{i}{2\pi^2} \langle 0 | T\left(\tilde{A}_1(\hat{k}_1)\tilde{A}_2(\hat{k}_2)\tilde{A}_1(-\hat{k}_1')\tilde{A}_2(-\hat{k}_2')\right) | 0 \rangle \tag{2.19}$$

and $$\langle \hat{p}_\alpha \hat{q}_\alpha | G(W) | \hat{p}_\alpha' \hat{q}_\alpha' \rangle \, \delta_4(\hat{P} - \hat{P}') =$$

$$= \frac{i}{4\pi^3} \langle 0 | T\left(\tilde{A}_1(\hat{k}_1)\tilde{A}_2(\hat{k}_2)\tilde{A}_3(\hat{k}_3) \times \right.$$
$$\left. \times \tilde{A}_1(-\hat{k}_1')\tilde{A}_2(-\hat{k}_2')\tilde{A}_3(-\hat{k}_3')\right) | 0 \rangle \tag{2.20}$$

in the respective c.m. frames. The arguments of these Heisenberg operators are off the mass shell. The position space operators are

$$A_\alpha(\hat{x}_\alpha) = \frac{1}{(2\pi)^2} \int d^4k_\alpha \, e^{i(\hat{k}_\alpha \cdot \hat{x}_\alpha)} \tilde{A}_\alpha(\hat{k}_\alpha). \tag{2.21}$$

Now suppose we have solved the Bethe-Salpeter equations somehow, and know the Feynman amplitudes (2.19) and (2.20), how do we get from them the experimental S matrices? In the case of elementary particles, the answer is simple - we just remove the external line propagators, put all external momenta on the mass shell, and multiply by some kinematic factors. However, the three-particle Feynman amplitude (2.20) must also contain within itself the S matrices for the scattering of two-particle bound states by the third particle. How do we extract these ? The corresponding problem in potential scattering has only been fully solved quite recently (see Ref. [6] and the Appendix of Ref. [1]). Fragments of a theory of composite particle scattering in axiomatic field theory have been constructed by a number of authors [15),16),17)]. In the present section and Section 4b we shall state the result, and try to give a coherent sketch of the steps leading to it. This includes some steps as yet unproven, which are expected to hold on physical grounds and because of the non-relativistic analogy.

The Heisenberg operators themselves describe elementary particles. However, products of them can be used as "quasi-local" field operators for bound states. For example, for a two-particle bound state, we can take the time-ordered product

$$\tilde{B}_3(\hat{K}_3) = T\left(\tilde{A}_1(\hat{k}_1)\,\tilde{A}_2(\hat{k}_2)\right) - \langle 0|T\left(\tilde{A}_1(\hat{k}_1)\,\tilde{A}_2(\hat{k}_2)\right)|0\rangle \tag{2.22}$$

Here $\tilde{B}_3(\hat{K}_3)$ depends also implicitly on the internal momentum $\hat{p}_3$. In position space it will be a bilocal operator. The second term has been subtracted to make

$$\langle 0|\tilde{B}_3(\hat{K}_3)|0\rangle = 0. \tag{2.23}$$

The first step is to discuss the asymptotic limit of these quasi-local operators, as the time goes to infinity. This has been considered by many authors, but most rigorously by Haag [15]. He assumes that products of quasi-local operators with large spacelike separations have a certain cluster decomposition property. This can be proved for products of fields satisfying local commutativity [18], but seems to be much weaker. From this, he shows that the quasi-local fields tend asymptotically to free fields in the sense of weak operator convergence. A precise statement of Haag's result requires that the fields be multiplied by Klein-Gordon wave packets, in order that they exist as unbounded operators. However, the physical meaning becomes much clearer when it is stated for plane waves

$$\begin{aligned} \underset{z^0 \to -\infty}{\text{weak lim}} \Big\{ (-i)(2\pi)^{-3/2} \int d^3 z \, \exp\Big(i\big[\underset{\sim}{K}\cdot\underset{\sim}{z} - \\ - \sqrt{K^2+M^2}\, z^0\big]\Big) \Big[\frac{\partial B(\hat{z})}{\partial z^0} + i\sqrt{K^2+M^2}\, B(\hat{z}) \Big] \Big\} \\ = (2\pi)^{3/2} \langle 0 | B(0) | \underset{\sim}{K}, M \rangle \, b_{in}(\underset{\sim}{K}) \end{aligned} \tag{2.24}$$

$$\begin{aligned} \underset{z^0 \to +\infty}{\text{weak lim}} \Big\{ \text{same expression} \Big\} \\ = (2\pi)^{3/2} \langle 0 | B(0) | \underset{\sim}{K}, M \rangle \, b_{out}(\underset{\sim}{K}) \end{aligned} \tag{2.25}$$

Here $B(\hat{z})$ is any quasi-local field satisfying (2.23). The limit will vanish unless there exists a particle with mass M, and one-particle states $|\underset{\sim}{K}, M\rangle$ for which

$$\langle 0 | B(0) | \underset{\sim}{K}, M \rangle \neq 0 . \tag{2.26}$$

$b_{in}(\underset{\sim}{K})$ and $b_{out}(\underset{\sim}{K})$ will then be free field operators for this particle. If $\tilde{B}(\hat{K}) = \tilde{A}_{\alpha}(\hat{K})$, and $M = m_{\alpha}$, then (2.26) becomes a constant, and we get the free fields for the elementary particles. If $\tilde{B}(\hat{K})$ is the quasi-local operator (2.22), and M is the mass of a two-particle bound-state, then (2.26) is just its Bethe-Salpeter wave function [11].

This convergence property of the field operators appears to originate in the fact that the unitary operators representing Lorentz displacements tend asymptotically to the projection operators for the point spectrum of the generator, in the generalized sense of strong Abel convergence. This projects out the zero and one-particle states from any translation-covariant field operator in the infinite time limit. We hope to discuss this topic more fully elsewhere.

The next step is to show that <u>any</u> quasi-local operator converging asymptotically to a free field is related to the S matrix by a reduction formula. Applying this to the particular quasi-local field (2.22) gives a connection between Feynman amplitudes and bound state scattering.

Specifically, the S matrix for the bound-state scattering process $(1,2) + 3 \rightarrow (1,2) + 3$ can be constructed from the in and out fields in the usual way. It is

$$\langle 0 | b_{3\,in}(\underset{\sim}{K}_3)\, a_{3\,in}(\underset{\sim}{k}_3)\, a^{*}_{3\,out}(\underset{\sim}{k}'_3)\, b^{*}_{3\,out}(\underset{\sim}{K}'_3) | 0 \rangle \qquad (2.27)$$

(Note that we put our initial state and creation operators on the left, contrary to the usual convention.) Because of the weak convergence, we can substitute (2.24) and (2.25) into (2.27). Following Zimmermann [16], we now use Stokes' theorem to transform the three-dimensional integral at infinite times, arising from (2.24) and (2.25), into a four-dimensional integral. This involves some assumptions about domains of operators which have never been completely proved, though Hepp has recently obtained a partial result in this direction [19]. In the case of elementary fields this gives the reduction formula of Lehmann, Symanzik and Zimmermann [20] (stated for plane waves)

$$\left[a_{in}(\underset{\sim}{k}), T\left(A_1(\hat{x}_1) A_2(\hat{x}_2) \ldots A_n(\hat{x}_n)\right) S\right] =$$
$$= i(2\pi)^{-3/2} \int d^4z \, \exp\left\{ i\left[\underset{\sim}{k} \cdot \underset{\sim}{z} - \sqrt{k^2 + m_{n+1}^2}\, z^0\right]\right\} \times$$
$$\times \left(\Box_z + m_{n+1}^2\right) T\left(A_1(\hat{x}_1) \ldots A_n(\hat{x}_n) A_{n+1}(\hat{z})\right) S \, . \tag{2.28}$$

Here $\Box_z + m_{n+1}^2$ is the Klein-Gordon operator. For the composite field (2.22) we must use the fact that the relative time remains finite asymptotically, due to the quasi-local nature of the field. This tells us that both constituents will be much earlier than all the other fields, giving the reduction formula

$$(2\pi)^{3/2} \langle 0 | T\left(A_1\left(\frac{m_2}{m_1+m_2}\hat{\xi}\right) A_2\left(\frac{-m_1}{m_1+m_2}\hat{\xi}\right)\right) | \underset{\sim}{K}, M \rangle \times$$
$$\times \left[b_{in}(\underset{\sim}{K}), T\left(A_i(\hat{x}_i) \ldots A_n(\hat{x}_n)\right) S\right] =$$
$$= i(2\pi)^{-3/2} \int d^4z \, \exp\left\{ i\left[\underset{\sim}{K} \cdot \underset{\sim}{z} - \sqrt{K^2 + M^2}\, z^0\right]\right\} \times \tag{2.29}$$
$$\times \left[\Box_z + M^2\right] T\left(A_i(\hat{x}_i) \ldots A_n(\hat{x}_n) A_1\left(\hat{z} + \frac{m_2}{m_1+m_2}\hat{\xi}\right) \times\right.$$
$$\left.\times A_2\left(\hat{z} - \frac{m_1}{m_1+m_2}\hat{\xi}\right)\right) S \, .$$

Substituting these, together with the corresponding formula for the out fields, successively into (2.27), and transforming to momentum space gives the following formula for the bound-state-scattering S matrix

$$
\begin{aligned}
&\langle 0|T(\tilde{A}_1(\hat{k}_1)\tilde{A}_2(\hat{k}_2))|\underset{\sim}{K}_3, M\rangle \{\langle \underset{\sim}{K}_3\, \underset{\sim}{k}_3|S|\underset{\sim}{K}'_3\, \underset{\sim}{k}'_3\rangle - \\
&- 4\bar{K}^0_3\, \bar{k}^0_3\, \delta_3(\underset{\sim}{K}_3 - \underset{\sim}{K}'_3)\, \delta_3(\underset{\sim}{k}_3 - \underset{\sim}{k}'_3)\} \times \\
&\times \langle \underset{\sim}{K}'_3, M|T(\tilde{A}_1(-\hat{k}'_1)\tilde{A}_2(-\hat{k}'_2))|0\rangle = \\
&= -2\pi^2 i\, \delta_4(\hat{P} - \hat{P}') \lim_{\hat{K}^2_3, \hat{K}'^2_3 \to M^2} \{ \lim_{\hat{k}^2_3, \hat{k}'^2_3 \to m^2_3} \times \\
&\times (M^2 - \hat{K}^2_3)(m^2_3 - \hat{k}^2_3)\langle \hat{p}_3 \hat{q}_3|G(W)|\hat{p}'_3 \hat{q}'_3\rangle \times \\
&\times (M^2 - \hat{K}'^2_3)(m^2_3 - \hat{k}'^2_3)\}.
\end{aligned} \tag{2.30}
$$

Here we have used (2.20). This expresses the bound-state-scattering S matrix in terms of the residue of the three-particle Green's function at the pole on the bound state mass shell. This residue involves the S matrix multiplied by the Bethe-Salpeter wave functions of the initial and final bound state. It is therefore very analogous to the non-relativistic case [6)]. However, this residue recipe is not very practical if we only know the three-particle Green's function numerically rather than analytically. The remaining step consists in getting it into a more convenient form. This we shall leave until Section 4b, when we possess the Faddeev equations.

The diagonal term in (2.30) arises from the term

$$\langle 0 | [b_{3in}(\underset{\sim}{K}), b^{*}_{3in}(\underset{\sim}{K}')] \; [a_{3in}(\underset{\sim}{k}), a^{*}_{3in}(\underset{\sim}{k}')] | 0 \rangle \tag{2.31}$$

in the reduction of (2.27). In the case of the S matrix for three elementary particles in the initial and final states, these disconnected terms are less trivial, and lead to

$$\begin{aligned}
\langle \underset{\sim}{k}_1 \underset{\sim}{k}_2 \underset{\sim}{k}_3 | S | \underset{\sim}{k}'_1 \underset{\sim}{k}'_2 \underset{\sim}{k}'_3 \rangle &= \prod_{\alpha=1}^{3} 2\bar{k}^0_\alpha \, \delta_3(\underset{\sim}{k}_\alpha - \underset{\sim}{k}'_\alpha) + \\
&+ \sum_{\alpha=1}^{3} 2\bar{k}^0_\alpha \, \delta_3(\underset{\sim}{k}_\alpha - \underset{\sim}{k}'_\alpha) \Big[\langle \underset{\sim}{k}_\beta \underset{\sim}{k}_\gamma | S | \underset{\sim}{k}'_\beta \underset{\sim}{k}'_\gamma \rangle - \\
&- 4\bar{k}^0_\beta \bar{k}^0_\gamma \, \delta_3(\underset{\sim}{k}_\beta - \underset{\sim}{k}'_\beta) \, \delta_3(\underset{\sim}{k}_\gamma - \underset{\sim}{k}'_\gamma) \Big] + \\
&+ i(2\pi)^5 \pi \, \delta_4(\hat{P} - \hat{P}') \Big[\lim_{\hat{k}^2_\alpha \to m^2_\alpha} \; \lim_{\hat{k}'^2_\alpha \to m^2_\alpha} \Big]_{\alpha=1,2,3} \times \\
&\times (m_1^2 - \hat{k}_1^2)(m_2^2 - \hat{k}_2^2)(m_3^2 - \hat{k}_3^2) \langle \hat{p} \hat{q} | G(W) | \hat{p}' \hat{q}' \rangle \times \\
&\times (m_1^2 - \hat{k}'^2_1)(m_2^2 - \hat{k}'^2_2)(m_3^2 - \hat{k}'^2_3) ,
\end{aligned} \tag{2.32}$$

where

$$\alpha \neq \beta \neq \gamma \neq \alpha .$$

Here the disconnected terms involve the T matrices for the two-particle subsystems.

The axiom of the completeness of the asymptotic states tells us that when we put all these three-particle and bound-state S matrices together, we get a unitary one. This is a physical postulate in field theory. In three-particle potential scattering with finite-range potentials it can be proved [6], but only with great difficulty. It says firstly that we have not forgotten any bound states, and secondly that the particles always emerge as particles (either elementary or composite) and never as "goo".

3. TWO-PARTICLE STATES

3a THE BETHE-SALPETER EQUATION AND ITS BOUND STATES

The two-particle Green's function (four-point τ-function) for the (1,2) subsystem is related to the two-particle transition amplitude by

$$G_3(w) = G_3^0(w) + G_3^0(w)\, T_3(w)\, G_3^0(w). \tag{3.1}$$

The G_3^0 arise from the propagators of the external lines. They cancel the Klein-Gordon operators in the reduction formula [Eq. (2.30)], leading to the following expression for the S matrix

$$\langle \vartheta\varphi | S_3(\bar{w}) | \vartheta'\varphi' \rangle = \delta(\cos\vartheta - \cos\vartheta')\,\delta(\varphi-\varphi') + \tag{3.2}$$
$$+ 2i \langle \vartheta\varphi | \bar{T}_3(\bar{w}+i\varepsilon) | \vartheta'\varphi' \rangle,$$

where

$$\langle \vartheta\varphi | \bar{T}_3(\bar{w}+i\varepsilon) | \vartheta'\varphi' \rangle =$$

$$= \frac{p}{4\bar{w}} \left[\langle \underset{\sim}{p}, \bar{p}^0 | T_3(\bar{w}+i\varepsilon) | \underset{\sim}{p}', \bar{p}'^0 \rangle \right]_{\bar{w}=\bar{w}'}, \tag{3.3}$$

and the kinematic notations are all explained in Section 1b. In this section we shall omit subscripts on such variables as w_3 and p_3, since we are only dealing with one two-particle subsystem.

The transition operator satisfies the Bethe-Salpeter equation [11].

$$T_3(w) = I_3 + T_3(w)\, G_3^0(w)\, I_3 , \tag{3.4}$$

where I_3 is the sum of two-particle-irreducible graphs. Explicitly, in momentum space,

$$\langle \hat{p} | T_3(w) | \hat{p}' \rangle = \langle \hat{p} | I_3 | \hat{p}' \rangle +$$

$$+ \int d^4 p'' \langle \hat{p} | T_3(w) | \hat{p}'' \rangle\, G_3^0(w; \hat{p}'') \langle \hat{p}'' | I_3 | \hat{p}' \rangle , \tag{3.5}$$

where $G_3^0(w;\hat{p})$ is given by (2.4). The off-shell unitarity equation for $T_3(w)$ states that the discontinuity across the two-particle cut is given by

$$T_3(w + i\varepsilon) - T_3(w - i\varepsilon) =$$

$$= 2i\, T_3(w + i\varepsilon)\, \Delta_1 \Delta_2\, T_3(w - i\varepsilon) . \tag{3.6}$$

This may easily be derived, either by resolvent methods [12] (see Section 5b), or from the Cutkosky rules [13].

The two-particle bound states will occur as poles in $T_3(w)$ and $G_3(w)$ [21]

$$\langle \hat{p} | T_3(w) | \hat{p}' \rangle \approx \sum_n \frac{g_n(\hat{p})\, g_n^*(\hat{p}')}{M_n^2 - w^2} , \tag{3.7}$$

$$\langle \hat{p} | G_3(w) | \hat{p}' \rangle \approx \sum_n \frac{\psi_n(\hat{p})\, \psi_n^*(\hat{p}')}{M_n^2 - w^2} , \tag{3.8}$$

where M_n is the bound-state mass, $g_n(\hat{p})$ its form factor and $\psi_n(\hat{p})$ is its wave function. (3.1) gives the relation between them, in the c.m. frame,

$$g_n(\hat{p}) = 2\pi^2 i \left[\left(\frac{m_1 M_n}{m_1 + m_2} + p^0 \right)^2 - \underset{\sim}{p}^2 - m_1^2 \right] \times$$
$$\times \left[\left(\frac{m_2 M_n}{m_1 + m_2} - p^0 \right)^2 - \underset{\sim}{p}^2 - m_2^2 \right] \psi_n(\hat{p}). \tag{3.9}$$

It follows from time-reversal invariance that

$$\langle \underset{\sim}{p}, p^0 | T_3(w) | \underset{\sim}{p}', p'^0 \rangle =$$
$$= \langle \underset{\sim}{p}, -p^0 | T_3(-w) | \underset{\sim}{p}', -p'^0 \rangle. \tag{3.10}$$

This implies that the bound-state form factors have the symmetry property

$$g_n(\underset{\sim}{p}; p^0) = \pm\, g_n(\underset{\sim}{p}, -p^0). \tag{3.11}$$

These relations become more complicated if the elementary particles have spin [22]. g then has to be split into its even and odd parts with respect to p^0.

Applying to (3.7) the same resolvent techniques which give the unitarity equation, leads to a consistency condition for the residue at the pole

$$2 \sum_\nu g^{\mu\nu} K^\nu = \int d^4p \; g_n^*(\hat{p}) \frac{d}{dK^\mu} G_3^0(\hat{K}; \hat{p}) \times$$
$$\times\, g_n(\hat{p}), \qquad \text{if} \quad \hat{K}^2 = M_n^2. \tag{3.12}$$

Substituting (2.4) and (3.11) leads to the normalization conditions for the Bethe-Salpeter wave function [23]

$$
\begin{aligned}
1 = & -2\pi^2 i \int d^4 p\, g_n^*(\hat{p})\, G_3^0(\pm M_n; \underset{\sim}{p}, p^0) \times \\
& \times \Big\{ \frac{1}{(m_1+m_2)^2} \Big[\frac{2 m_1^2 m_2^2}{(m_1+m_2)^2} M_n^2 + (m_1^2 + m_2^2)(p^{02} - \underset{\sim}{p}^2) - \\
& - \frac{2 m_1 m_2 (m_1 - m_2)}{m_1 + m_2} (\pm M_n p^0) - 2 m_1^2 m_2^2 \Big] \pm \\
& \pm \frac{p^0}{M_n (m_1+m_2)} \Big[- \frac{m_1 m_2 (m_1 - m_2)}{(m_1+m_2)^2} M_n^2 \mp \frac{4 m_1 m_2}{m_1+m_2} M_n p^0 + \\
& + (m_1 - m_2)(p^{02} - \underset{\sim}{p}^2) + m_1 m_2 (m_1 - m_2) \Big] \Big\} \times \\
& \times G_3^0(\pm M_n; \underset{\sim}{p}, p^0)\, g_n(\hat{p}).
\end{aligned}
\tag{3.13}
$$

3b THE SEPARABLE APPROXIMATION

In the non-relativistic case, the use of a separable potential is a valid approximation if the two-particle amplitudes are dominated by a finite number of bound states and resonances [1]. This leads to a great simplification in the three-particle equations, and puts them within the range of practical computation. In the present subsection, we shall consider a similar approximation for the relativistic theory. Due to the unsolved mathematical difficulties associated with the Bethe-Salpeter equation, it is not possible at present to give a rigorous justification. Nevertheless, it is extremely plausible physically that this approximation has the same validity as in the non-relativistic theory.

We shall assume that there is not more than one bound state or resonance in any partial wave. Experimentally, this seems to be true so far as purely two-particle resonances are concerned. This assumption could easily be removed by considering a potential matrix in each partial wave, but this would greatly complicate the notation.

If we introduce radial form factors by

$$g_{jh}(\underset{\sim}{p}, p^0) = g_j(p, p^0)\, Y_j^h(\vartheta\varphi), \tag{3.14}$$

where Y are the usual spherical harmonics, we can then write

$$\langle \hat{p} | I_3 | \hat{p}' \rangle \approx \sum_j \frac{(2j+1)}{4\pi} P_j\left((\underset{\sim}{p}\cdot\underset{\sim}{p}')/pp'\right) \times \\ \times f_j\, g_j(p, p^0)\, g_j(p', p'^0), \tag{3.15}$$

or symbolically

$$I_3 \approx \sum_j |j\rangle f_j \langle j| . \tag{3.16}$$

The Bethe-Salpeter equation (3.4) can then be solved in closed form to give

$$\langle \hat{p} | T_3(w) | \hat{p}' \rangle \approx \sum_j \frac{(2j+1)}{4\pi} P_j((\underset{\sim}{p} \cdot \underset{\sim}{p}')/pp') \times$$
$$\times g_j(p, p^0)\, t_j(w^2)\, g_j(p', p^{0\prime}), \tag{3.17}$$

where

$$t_j(w^2) = \left[f_j^{-1} - \langle j | G_3^0(w) | j \rangle \right]^{-1} \tag{3.18}$$

is the propagator of the two-particle bound state or resonance. Explicitly in the c.m. system

$$[t_j(w^2)]^{-1} = f_j^{-1} - \frac{1}{2\pi^2 i} \int_0^\infty dp \int_{-\infty}^{\infty} dp^0 \times$$
$$\times \frac{p^2 |g_j(p, p^0)|^2}{\left[\left(\frac{m_1 w}{m_1 + m_2} + p^0 \right)^2 - p^2 - m_1^2 + i\varepsilon \right] \left[\left(\frac{m_2 w}{m_1 + m_2} - p^0 \right)^2 - p^2 - m_2^2 + i\varepsilon \right]} \tag{3.19}$$

The time-reversal invariance of the form factors, (3.11), implies that $t_j(w^2)$ is a function of w^2 only, and not of w.

The sum of partial waves in (3.17) must be confined to those containing either a bound state, a virtual bound state, or a resonance. It is only for these that the separable approximation is valid [1]. The factors of the separable potential in (3.15) must be chosen so as to give the form factors of the bound state or resonance correctly. We will discuss in Subsection 3d how to do this experimentally. The coupling constant f_j is also not arbitrary, but must be chosen so as to give the correct position of the bound state or resonance, leading to the eigenvalue condition

$$\left[t_j (M_j^2) \right]^{-1} = 0 . \tag{3.20}$$

In the case of a resonance, the integral obtained by substituting (3.19) into (3.20) is to be interpreted as a principal value one, and the "mass" of the resonance, M_j, as the point on the real axis where the phase shift goes through 90^0. This was discussed in the non-relativistic paper. Attempts to represent resonances by single poles on unphysical sheets lead to difficulties with unitarity and threshold behaviour [1].

We also quote the unitarity condition for $t_j(w^2)$, which we shall need later

$$t_j(w^2 + i\varepsilon) - t_j(w^2 - i\varepsilon) =$$
$$= 2i \; t_j(w^2 + i\varepsilon) \langle j | \Delta_1 \Delta_2 | j \rangle \, t_j(w^2 - i\varepsilon) . \tag{3.21}$$

Nothing essential would be changed, if we were to allow f_j to be a polynomial in w, instead of a constant. This corresponds to CDD poles [24].

3c ELIMINATION OF THE RELATIVE ENERGY

Even after angular momentum separation, the Bethe-Salpeter wave function depends on two variables, p and p^o – the magnitude of the relative momentum, and the relative energy. The second does not occur in the non-relativistic theory and has no direct physical meaning. It also makes all the equations integral equations in two variables, instead of one, which is very unpleasant numerically. For this reason, numerous authors have sought to eliminate this variable from the Bethe-Salpeter equation [25]. The neatest technique for doing so is that of Blankenbecler and Sugar [26]. We shall state it in the form needed for application to the three-particle case.

Alessandrini and Omnès in a recent preprint [9] have claimed that this method is non-unique and not Lorentz invariant. It appears to us that both these statements are incorrect. The approximation consists in neglecting the negative energy part of certain <u>mass-shell</u> delta-functions, a procedure which is invariant under all orthochronous Lorentz transformations. Also the ambiguity in the choice of momentum variables to which they refer, is merely a standard Jacobian change of variables, and cannot lead to any physically different results if performed correctly. The mistake appears to lie in Eq. (2.9) of their paper. This cannot be deduced from their Eq. (2.6), since q_1 (= our $\hat{k}_1$) has an implicit dependence on s' arising from total energy-momentum conservation, so that the first delta-function cannot be taken outside the integral. Also their Eq. (2.9) cannot be deduced directly, since it applies the Landau-Cutkosky rules in a specialized form which is only true for the singularities in the total energy, and not for those in an external mass, in particular, even the normal threshold in an external mass must involve the mass of the exchanged particle (the rung of the ladder), which nowhere appears in their equations (see Fig. 3). As for covariance, we have, in fact, verified explicitly that our reduced form factors for the two-particle bound states are the same whether the elimination of the relative energies is performed in the two-particle or three-particle centre-of-mass frames (see Section 6).

According to the Landau-Cutkosky rules [13], we get the singularities of the four-point function arising from the right-hand cut in the total energy by cutting it 2+2 in all possible ways, as in Fig. 4a, and then putting the internal lines which have been cut on the mass shell. If we apply this to the Bethe-Salpeter equation in the ladder approximation, besides the graphs such as Fig. 4a with two-particle intermediate states, we also get graphs such as Fig. 4b and c, with three and four-particle intermediate states. The distinction between them is a Lorentz invariant one, because the positive energy mass-shell delta-functions occurring in the Cutkosky rules are Lorentz invariant. Now the Bethe-Salpeter equation is a two-particle equation, and these higher intermediate states do not really belong in it. There is no reason to suppose that they are any more important than other many-particle graphs which have been omitted. Also they are responsible for all the difficulties in the Bethe-Salpeter equation - the extra relative energy variable, and the overlapping singularities which affect the compactness proof. It is therefore very natural to get rid of these graphs, especially if we plan to include three-particle states in a more full and consistent way later on, as we do. The two-particle unitarity condition (3.6) then becomes valid at all energies. To reconstruct the Bethe-Salpeter equation from its discontinuity by means of a dispersion integral in w, we must now make the replacement

$$G_3^0(w;\hat{p}) \rightarrow \frac{1}{\pi}\int_{m_1+m_2}^{\infty} dw' \frac{\Delta_1(\hat{k}_1')\Delta_2(\hat{k}_2') - \Delta_1(-\hat{k}_1')\Delta_2(-\hat{k}_2')}{w' - w - i\varepsilon}$$

$$= \frac{1}{4\pi \bar{E}_{13}\bar{E}_{23}}\left\{\frac{\delta(\bar{p}^0 - p^0)}{\bar{w} - w - i\varepsilon} + \frac{\delta(\bar{p}^0 + p^0)}{\bar{w} + w + i\varepsilon}\right\}. \qquad (3.22)$$

Here we have made elaborate use of our notation (Section 1b). Only the first pair of delta-functions in (3.22) are actually required by this argument - the second pair have been added to preserve time-reversal invariance. Contrary to appearances, this equation is Lorentz invariant. The reason why w appears in it, is not because we are in the centre-of-mass frame where $w = K^0$, but because the Cutkosky rules in the form in which we have been applying them give the singularities in the Mandelstam variable $s = w^2$, and not in anything else.

When we insert (3.22) into other equations, the relative energy will be put onto its mass shell $p^0 = \bar{p}^0$ by the delta-functions, while the total energy w remains off-shell. Applying it to the bound-state propagator, (3.18) or (3.19), gives us the "reduced" propagator

$$[\bar{E}_j(w^2)]^{-1} = \mathfrak{J}_j^{-1} - \frac{1}{2\pi}\int_0^\infty \frac{dp\, p^2\, \bar{w}\, |\bar{g}_j(p)|^2}{\bar{E}_{13}\bar{E}_{23}(\bar{w}^2 - w^2 - i\varepsilon)}, \qquad (3.23)$$

where

$$\bar{g}_j(\underset{\sim}{p}) \equiv g_j(\underset{\sim}{p}, \bar{p}^0) \qquad (3.24)$$

is the "reduced" bound-state form factor. We shall refer to this process of eliminating the relative energies as "reduction". Note that the derivation of (3.23) made use of the time-reversal invariance of the form factor, (3.11).

The normalization condition for the reduced bound-state form factor is much simpler than (3.13). Inserting (3.22) in (3.12) leads to

$$1 = \frac{1}{2\pi}\int_0^\infty dp\, \frac{p^2\, \bar{w}\, |\bar{g}_j(p)|^2}{\bar{E}_{13}\bar{E}_{23}(\bar{w}^2 - M_j^2)^2}. \qquad (3.25)$$

A suitable reduced wave function can be defined by

$$\bar{\psi}_j(p) = \left[\frac{\bar{w}}{2\pi \bar{E}_{13}\bar{E}_{23}}\right]^{1/2} \frac{\bar{g}_j(p)}{(\bar{w}^2 - M_j^2)} . \tag{3.26}$$

In the case of a bound state, we can use the eigenvalue condition (3.20) and the normalization condition (3.25) to exhibit explicitly the pole contribution to the bound-state propagator (3.23). We obtain

$$\left[\bar{t}_j(w^2)\right]^{-1} = \frac{(M_n^2 - w^2)}{2\pi}\int_0^\infty \frac{dp.\, p^2 \bar{w} \,|\bar{g}_j(p)|^2}{\bar{E}_{13}\bar{E}_{23}(\bar{w}^2 - M_j^2)(\bar{w}^2 - w^2 - i\varepsilon)} \tag{3.27}$$

$$= \frac{1}{M_n^2 - w^2} + \frac{\left[\frac{1}{2\pi}\int_0^\infty \frac{dp\, p^2 \bar{w}\, |\bar{g}_j(p)|^2}{\bar{E}_{13}\bar{E}_{23}(\bar{w}^2 - M_j^2)^2(\bar{w}^2 - w^2 - i\varepsilon)}\right]}{\left[1 + \frac{(w^2 - M_j^2)}{2\pi}\int_0^\infty \frac{dp\, p^2 \bar{w}\, |\bar{g}_j(p)|^2}{\bar{E}_{13}\bar{E}_{23}(\bar{w}^2 - M_j^2)(\bar{w}^2 - w^2 - i\varepsilon)}\right]} \tag{3.28}$$

The second term in (3.28) is the contribution of the two-particle continuum states, starting at $w^2 = (m_1+m_2)^2$. This equation is obviously closely analogous to the Lehmann spectral representation for the propagator of an elementary particle [27], and especially to equations for the inverse propagator derived from it [28]. This justfies us in calling $\bar{t}_j(w^2)$ the bound-state propagator. It also plays a very similar rôle in bound-state scattering, as we shall see.

The reduction process, which eliminates the relative energy, can also be carried out before introducing the separable approximation. However, except in the equal mass case, it does not seem possible to do it in a way which preserves time-reversal invariance, unless we perform the separable approximation first. Also, even in the equal mass case, it is known that the reduction procedure distorts the analytic properties of the Bethe-Salpeter amplitude [29], though the changes are distant from the physical region.

We recommend readers who could not understand the meaning of the formulae in this subsection to consult Subsection 1b.

3d DEDUCTION OF THE FORM FACTORS FROM EXPERIMENT

Equations (3.2), (3.3), (3.17) and (3.23), together with (1.23), imply the following relation between the reduced propagator and the experimental phase shifts for two-particle elastic scattering

$$(p/\bar{\omega})\,|\bar{g}_j(p)|^2 \cot\delta_j =$$

$$= \frac{4}{f_j} - \frac{1}{\pi} P\int_{(m_1+m_2)^2}^{\infty} \frac{d\bar{\omega}'^2\, p'\,|\bar{g}_j(p')|^2}{\bar{\omega}'(\bar{\omega}'^2 - \bar{\omega}^2)} \qquad (3.29)$$

where P means principal value integral. This shows that both the unknowns - the coupling constant f_j and the reduced form factor $\bar{g}_j(p)$ - can be deduced from the experimental phase shifts. However, some caution is required in doing so. Numerical experiments in potential scattering [5] have shown that, if a good approximate formula for the form factor is chosen with <u>one</u> arbitrary parameter, and if this parameter together with the coupling constant f_j are adjusted to fit the phase shift and its first derivative at some point near a resonance, by the non-relativistic analogue of (3.29), then the off-shell scattering amplitude will be obtained reasonably accurately near this point. However, if more parameters are introduced into the form factor, and these are adjusted to fit the higher derivatives of the phase shift, then the predicted off-shell amplitude gets worse rather than better. The reason is that the separable approximation is only valid in the vicinity of a resonance, and in practice this means that only the position and width of the resonance, but not its shape dependence, are determined by the separable term. Thus, Eq. (2.64) of I cannot be used to calculate the off-shell form factor from the experimental shape dependence.

It is obviously very important then to get good one-parameter formulae for form factors, which can be fitted to experiment. In the non-relativistic case, it is known that the Hulthén form factor,

$$g_0(p) = N/[p^2+\mu^2], \tag{3.30}$$

where N is a normalization constant, is a good approximation for the S wave ground state of most finite range potentials if μ is suitably chosen. In particular, it is definitely a better approximation than $N/(p^2+\mu^2)^{\frac{1}{2}}$ for a Yukawa potential, even though the latter reproduces the phase shift and the dependence of the binding energy on the coupling constant better. The right extension of (3.30) to higher partial waves in the non-relativistic case appears to be [30]

$$g_j(p) = N p^j/(p^2+\mu_j^2)^{j+1} \tag{3.31}$$

which has the correct threshold and asymptotic behaviour [9]. N here is not arbitrary, but has to be adjusted to fulfil the normalization condition (3.25) in the case of a bound state, and can be absorbed into f_j in the case of a resonance. There are therefore just two constants, f_j and μ_j, which can be determined by substituting (3.31) into (3.29) and fitting to the experimental position and width of the resonance. The basic requirement here is that the wave function should be numerically insensitive to the <u>shape</u> of the potential, as opposed to its strength and range.

Can the same formula (3.31) be used relativistically ? Our hope that it can is based on the similarity of the analytic properties [31], threshold and asymptotic behaviour [9] between relativistic and non-relativistic wave functions. Numerical tests would of course be desirable.

Note that in the S wave case, with $\bar{g}_0(p)$ = constant (zero-range forces), (3.29) reduces to the Chew-Mandelstam effective range formula. This is the relativistic analogue of the zero-range approximation [32]. In this particular case, on-shell and off-shell theory will be identical.

4. THREE-PARTICLE STATES

4a RELATIVISTIC FADDEEV EQUATIONS

In Subsection 2a we defined various Green's functions for relativistic three-particle systems. In particular G, the complete three-particle Green's function, is the vacuum expectation value of a time-ordered product of six Heisenberg operators, and satisfies the three-particle Bethe-Salpeter equation (1.40), which is expressed graphically in Fig. 2. The kernel consists of the two-particle and three-particle irreducible graphs. This equation can easily be derived by summing graphs, and non-perturbatively by the methods of J.G. Taylor [14].

As in the non-relativistic case [1),2),12)] (1.40) contains disconnected graphs, which lead to undesirable properties if the equation is used as it stands. It is necessary to rearrange it into equations of the Faddeev type. This can be done in complete analogy to the non-relativistic case [1], and we shall therefore leave out the details of the derivations [12),33)]. We shall write them in a form which includes the effect of three-particle irreducible graphs, since there seems to be a false impression that the presence of three-body forces would invalidate the Faddeev approach [34].

All indices go from 0 to 3. Define the 4×4 matrix

$$\mathcal{E}_{\alpha\beta} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \qquad (4.1)$$

Define two-particle transition operators in the three-particle Hilbert space

$$T_3(W) \equiv \langle \hat{p}_3 | T_3(w_3[W, q_3]) | \hat{p}_3' \rangle \, \delta_4(\hat{q}_3 - \hat{q}_3') \tag{4.2}$$

with T_1 and T_2 similar, (see (2.13) for $w_\alpha[W, q_\alpha]$) and

$$T_0 = I_0 \tag{4.3}$$

Note that we do not multiply these by the propagator of the third particle. For notational convenience also put

$$\begin{aligned} V_\alpha &= 2\pi i (d_\alpha)^{-1} I_\alpha \quad , \quad \alpha = 1, 2, 3, \\ V_0 &= I_0 , \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} G_3^0 &= [2\pi^2 i]^{-1} d_1 d_2 , \\ G_0^0 &= G^0 = [\pi(2\pi i)^2]^{-1} d_1 d_2 d_3 , \end{aligned} \tag{4.5}$$

with G_1^0 and G_2^0 similar (cf. (2.15) et seq.).

The transition operators for three-particle and bound-state scattering can now be defined in analogy to the non-relativistic case [1]. They are

$$U_{\alpha\beta}^{(+)} = \sum_\gamma \varepsilon_{\alpha\gamma} V_\gamma + \sum_{\gamma\delta} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} V_\gamma G V_\delta , \tag{4.6}$$

$$U^{(-)}_{\alpha\beta} = \sum_{\delta} \varepsilon_{\beta\delta} V_{\delta} + \sum_{\gamma\delta} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} V_{\gamma} G V_{\delta} . \tag{4.7}$$

By using the relativistic analogues of the second resolvent identities [1)]

$$\begin{aligned} G &= G^{\alpha} + \sum_{\beta} \varepsilon_{\alpha\beta} G^{\alpha} V_{\beta} G , \\ &= G^{\alpha} + \sum_{\beta} \varepsilon_{\alpha\beta} G V_{\beta} G^{\alpha} , \end{aligned} \tag{4.8}$$

we can invert (4.6) and (4.7) to obtain

$$G = G^{\alpha} + G^{\alpha} U^{(+)}_{\alpha\beta} G^{\beta} , \tag{4.9}$$

$$G = G^{\beta} + G^{\alpha} U^{(-)}_{\alpha\beta} G^{\beta} . \tag{4.10}$$

These equations are the motive for the definitions (4.6) and (4.7), since they relate $U^{(\pm)}_{\alpha\beta}$ to the S matrices, as we shall show below. Substituting (4.8) into (4.6) and (4.7), and making some rearrangements, gives the relativistic Faddeev equations

$$U^{(+)}_{\alpha\beta} = \sum_{\gamma} \varepsilon_{\alpha\gamma} V_{\gamma} + \sum_{\delta} \varepsilon_{\beta\delta} U^{(+)}_{\alpha\delta} G^{0}_{\delta} T_{\delta} , \tag{4.11}$$

$$U^{(-)}_{\alpha\beta} = \sum_{\delta} \varepsilon_{\beta\delta} V_{\delta} + \sum_{\gamma} \varepsilon_{\alpha\gamma} T_{\gamma} G^{0}_{\gamma} U^{(-)}_{\gamma\beta} . \tag{4.12}$$

These equations involve the operators

$$< \hat{p}_{\alpha} \hat{q}_{\alpha} | U^{(\pm)}_{\alpha\beta}(W) | \hat{p}'_{\beta} \hat{q}'_{\beta} > \tag{4.13}$$

which act in a Hilbert space of functions of two momentum four-vectors $\hat{p}_{\alpha}$ and $\hat{q}_{\alpha}$. In the centre-of-mass frame, they also depend on W as operator-valued functions (analogous to s in the non-relativistic case). By substituting our various notational definitions, it is easy to write out (4.11) and (4.12) fully in the momentum representation. However, we shall not do so, because they become very complicated, and because we shall never use the equations directly in this general form.

For scattering theory, it is important to separate out the parts of $U^{(\pm)}_{\alpha\beta}$ arising from disconnected graphs. This can be done by iterating (4.11) and (4.12) to obtain

$$U^{(+)}_{\alpha\beta} = (1-\delta_{\alpha\beta})(1-\delta_{\beta 0})V_{\beta} + \sum_{\gamma\neq\alpha,\beta,0} 2\pi i\, d^{-1}_{\gamma} T_{\gamma} + \\ + W_{\alpha\beta} , \tag{4.14}$$

$$U^{(-)}_{\alpha\beta} = (1-\delta_{\alpha\beta})(1-\delta_{\alpha 0})V_{\alpha} + \\ + \sum_{\gamma\neq\alpha,\beta,0} 2\pi i\, d^{-1}_{\gamma} T_{\gamma} + W_{\alpha\beta} , \tag{4.15}$$

where

$$W_{\alpha\beta} = I_0 + I_0 G^0 I_0 + \\ + \sum_{\gamma\neq 0,\alpha} \sum_{\delta\neq 0,\gamma,\beta} V_\gamma G_\delta^0 T_\delta + \sum_\gamma \sum_\delta \times \\ \times \varepsilon_{\beta\delta} \varepsilon_{\delta\gamma} U_{\alpha\gamma}^{(+)} G_\gamma^0 T_\gamma G_\delta^0 T_\delta \tag{4.16}$$

is obviously completely connected.

4b UNITARITY AND S MATRICES

In the non-relativistic case, the operators $\langle \underset{\sim}{p}_\alpha \underset{\sim}{q}_\alpha | U_{\alpha\beta}^{(\pm)}(s) | \underset{\sim}{p}'_\beta \underset{\sim}{q}'_\beta \rangle$ have only right-hand cuts in the variable s, the left-hand and anomalous singularities all being in the variables $\underset{\sim}{p}$ and $\underset{\sim}{q}$ [1]. Due to the mathematical difficulties associated with Bethe-Salpeter type equations, we cannot yet prove this rigorously in the relativistic case, but it nevertheless seems very plausible that it is true. It would certainly be true if we were allowed to make the metric Euclidean by a Wick transformation, or if there were any Banach space in which any iteration of the kernel were compact. We then expect the only singularities in W to be the right-hand cuts associated with three-particle scattering, and with scattering of a two-particle bound state by the third particle.

The discontinuities across these cuts are given by unitarity. The derivation of the unitarity equations satisfied by the $U_{\alpha\beta}^{(\pm)}(W)$ is very similar to the non-relativistic case [12]. The only significant difference is that, due to the factor d_α^{-1} in (4.4), V_α will depend on W. However, it will be non-singular, so that the limits from above and below the cut are the same, and the extra terms cancel in the unitarity formula. The equation obtained is then

$$U^{(\pm)}_{\alpha\beta}(W+i\varepsilon) - U^{(\pm)}_{\alpha\beta}(W-i\varepsilon) =$$
$$= U^{(+)}_{\alpha\rho}(W+i\varepsilon)\left[G^{\rho}(W+i\varepsilon) - G^{\rho}(W-i\varepsilon)\right]U^{(-)}_{\rho\beta}(W-i\varepsilon), \qquad (4.17)$$

where ρ is arbitrary. The discontinuities across the various cuts can then be obtained by the Cutkosky rules [13]. For the three-particle cut, we take $\rho = 0$ in (4.17). Applying (2.18) gives

$$G^{0}(W+i\varepsilon) - G^{0}(W-i\varepsilon) = 2i\Delta_1\Delta_2\Delta_3 \equiv 2i\Delta_0 \qquad (4.18)$$

so that we get the three-particle unitarity formula

$$\text{three-particle discontinuity of } U^{(\pm)}_{\alpha\beta}(W) =$$
$$= 2i\, U^{(+)}_{\alpha 0}(W+i\varepsilon)\Delta_0 U^{(-)}_{0\beta}(W-i\varepsilon), \qquad (4.19)$$

(This formula was the motivation for our peculiar choice of 2π factors in Section 1d.) Of course, this is not a rigorous derivation and all sorts of arguments about Hölder continuity and the inversion of the order of limits would be needed to make it so [6]. Also $U^{(\pm)}_{\alpha\beta}$ contain disconnected terms which vanish on the mass shell and the interpretation of how these combine with the mass-shell delta-functions requires some care. We must in fact substitute (4.14) and (4.15) into (4.17) <u>before</u> applying the Cutkosky rule. The d^{-1}_{γ} in the second terms of (4.14) and (4.15) will then cancel with the d_{γ} in G^{0} to give $G^{0}_{\gamma}(W+i\varepsilon) - G^{0}_{\gamma}(W-i\varepsilon)$, whose discontinuity is given by the <u>two</u>-particle Cutkosky rule to be $\prod_{\delta\neq\gamma,0}\Delta_{\delta}$. This leads to the well-known [35] disconnected terms in the three-particle unitarity relation.

According to (2.15) and (3.8), the bound-state contribution to the two-particle Green's function G^{γ} is

$$\langle \hat{p}_\gamma \hat{q}_\gamma | G^\gamma(W) | \hat{p}'_\gamma \hat{q}'_\gamma \rangle \approx$$
$$\approx \sum_n \frac{\psi_{\gamma n}(\hat{p}_\gamma)\, \psi^*_{\gamma n}(\hat{p}'_\gamma)\, \delta_4(\hat{q}_\gamma - \hat{q}'_\gamma)}{2\pi i\,(M^2_{\gamma n} - i\varepsilon - \hat{K}^2_\gamma)(m^2_\gamma - i\varepsilon - \hat{k}^2_\gamma)} \tag{4.20}$$

Just as in the non-relativistic case [1], the bound-state pole will [12] be drawn out into a cut (in the sense of operator analyticity by the presence of the third particle. The Cutkosky rules applied to (4.20) give for the discontinuity

$$\left[\langle \hat{p}_\gamma \hat{q}_\gamma | G^\gamma(W+i\varepsilon) - G^\gamma(W-i\varepsilon) | \hat{p}'_\gamma \hat{q}'_\gamma \rangle\right]_{\text{bound state scattering}}$$
$$= 2\pi i\, \psi_{\gamma n}(\hat{p}_\gamma)\, \psi^*_{\gamma n}(\hat{p}'_\gamma)\, \Delta_{\gamma n}(\hat{K}_\gamma)\, \Delta_\gamma(\hat{k}_\gamma)\, \delta_4(\hat{q}_\gamma - \hat{q}'_\gamma). \tag{4.21}$$

Here we have used the mass-shell delta-function for the bound state (2.17). Substituting this into (4.17) with $\rho = \gamma$, gives us the generalized unitarity formulae for the discontinuity across the bound-state-scattering cuts

$$\text{bound-state-scattering discontinuity of } \langle \hat{p}_\alpha \hat{q}_\alpha | U^{(\pm)}_{\alpha\beta}(W) | \hat{p}'_\beta \hat{q}'_\beta \rangle$$
$$= 2\pi i \int d^4p''_\gamma\, d^4p'''_\gamma\, d^4q''_\gamma\, \langle \hat{p}_\alpha \hat{q}_\alpha | U^{(+)}_{\alpha\gamma}(W+i\varepsilon) | \hat{p}''_\gamma \hat{q}''_\gamma \rangle \times \tag{4.22}$$
$$\times\, \psi_{\gamma n}(\hat{p}''_\gamma)\, \Delta_{\gamma n}(\hat{K}''_\gamma)\, \Delta_\gamma(\hat{k}''_\gamma)\, \psi^*_{\gamma n}(\hat{p}'''_\gamma) \langle \hat{p}'''_\gamma \hat{q}''_\gamma | U^{(-)}_{\gamma\beta}(W-i\varepsilon) | \hat{p}'_\beta \hat{q}'_\beta \rangle.$$

This shows that $\sqrt{\pi}\, \psi_{\gamma n}(\hat{p}_\gamma)$ is the effective wave function to be used in forming unitary bound-state-scattering amplitudes.

Next we consider the calculation of the various S matrices in terms of the $U^{(\pm)}_{\alpha\beta}(\bar{W})$. We showed in Section 2b that the S matrices are given by residues of the total Green's function G at the poles on the various mass shells. The formulae (4.9) and (4.10) enable us to evaluate these residues. For the three-particle $\rightarrow$ three-particle S matrix, we use (4.9) and (4.10) with $\alpha = \beta = 0$. The Klein-Gordon operators in the last term of (2.32) then cancel with the three free propagators in G^0, and we find that the three-particle S matrix can be expressed in terms of $\langle \hat{p}\hat{q} | U^{(\pm)}_{00}(\bar{W}) | \hat{p}'\hat{q}' \rangle$ evaluated on the mass shell. Similarly for bound-state processes, we use (4.8) and (4.9) with α or $\beta \neq 0$. Equation (4.20) then gives us the bound-state pole contributions, and the denominators cancel with the Klein-Gordon operators in (2.30). The amplitude for the process in which an α subsystem bound-state is scattered by the third particle, ending up as a β subsystem bound state plus third particle, is then given by $U^{(\pm)}_{\alpha\beta}$ multiplied by the appropriate bound-state wave functions, as in the non-relativistic case [12]. The $\pm$ forms again become identical on the mass shell.

To justify this, it is necessary that the $U^{(\pm)}_{\alpha\beta}$ themselves should not have poles on the relevant mass shells. If they did, however, then by (4.9) and (4.10) G would have a double pole, the limits in (2.30) and (2.32) would diverge, and the S matrix would be infinite, contradicting unitarity. An alternative argument, used in the non-relativistic case [4], is based on the fact that the kernel of the Faddeev equations does not contain diagonal terms with $\alpha = \beta$ (except for the three-body force I_0 which must be assumed non-singular). Assuming that the integral in the Faddeev equations is uniformly convergent, we can then argue that the singularities in the external variables must come from singularities in the kernel, and that the amplitudes for the γ subsystem cannot contain singularities corresponding to bound states in the α subsystem, if $\alpha \neq \gamma$. However, this argument is plainly non-rigorous unless we can prove various boundedness and continuity properties of the amplitudes in the external variables, which we cannot do relativistically.

This question ought to be related to the work on one-particle singularities in axiomatic field theory by Zimmermann and Hepp [36], but the precise connection is not apparent to the present authors.

To obtain the S matrix formula for three-particles $\rightarrow$ three-particles explicitly, it is convenient to use the five variables $\underset{\sim}{\Omega}_\alpha$, defined by (1.24). The complete three-particle S matrix in the c.m. frame, with all its kinematic factors, is then given by

$$\begin{aligned}\langle \underset{\sim}{\Omega}_\alpha | S_{00}(\overline{W}) | \underset{\sim}{\Omega}'_\alpha \rangle &= \langle \underset{\sim}{\Omega}_\alpha | \underset{\sim}{\Omega}'_\alpha \rangle + \\ &+ 2i \sum_{\gamma=1}^{3} (p_\gamma / 4\overline{\omega}_\gamma) \langle \underset{\sim}{p}_\gamma \bar{p}^0_\gamma | T_\gamma(\overline{\omega}_\gamma + i\varepsilon) | \underset{\sim}{p}'_\gamma \, p'^0_\gamma \rangle \times \\ &\times \delta(\overline{\omega}_\gamma - \overline{\omega}'_\gamma)\, \delta(\cos\Theta_\gamma - \cos\Theta'_\gamma)\, \delta(\Phi_\gamma - \Phi'_\gamma) + 2i [p_\alpha q_\alpha p'_\alpha q'_\alpha]^{1/2} \times \\ &\times (8\overline{W})^{-1} \langle \underset{\sim}{p}_\alpha \bar{p}^0_\alpha, \underset{\sim}{q}_\alpha \bar{q}^0_\alpha | U^{(\pm)}_{00}(\overline{W} + i\varepsilon) | \underset{\sim}{p}'_\alpha \bar{p}'^0_\alpha, \underset{\sim}{q}'_\alpha \bar{q}'^0_\alpha \rangle ,\end{aligned} \tag{4.23}$$

where α is arbitrary. The disconnected terms of $U^{(\pm)}_{00}$ given by (4.14) or (4.15), will vanish on the mass shell, but rather similar terms are reinstated by (2.30). By substituting (4.20) into (4.9) or (4.10), and comparing with (2.28), we get the S matrices for bound-state scattering and disintegration. To write these, we need to define mass-shell variables for bound-state scattering

$$\bar{q}^0_{\alpha n} \equiv \frac{m_\alpha \sqrt{M^2_{\alpha n} + q^2_\alpha} - (m_1 + m_2 + m_3 - m_\alpha)\sqrt{m^2_\alpha + q^2_\alpha}}{m_1 + m_2 + m_3} . \tag{4.24}$$

The S matrix for the scattering of the nth bound state of the α subsystem plus third particle, into the mth bound state of the β subsystem plus third particle, can then be calculated to be (using polar coordinates for $\underset{\sim}{q}_\alpha$ and $\underset{\sim}{q}'_\beta$)

$$\langle \Theta_\alpha \Phi_\alpha | S_{\alpha n, \beta m}(\bar{W}) | \Theta'_\beta \Phi'_\beta \rangle = \delta_{nm}\delta_{\alpha\beta} \langle \Theta_\alpha \Phi_\alpha | \Theta'_\alpha \Phi'_\alpha \rangle +$$
$$+ 2i\pi [q_\alpha q'_\beta]^{1/2} (4\bar{W})^{-1} \int d^4 p_\alpha \, d^4 p'_\beta \, \psi^*_{\alpha n}(\hat{p}_\alpha) \times$$
$$\times \langle \hat{p}_\alpha, \underset{\sim}{q}_\alpha \bar{q}^0_{\alpha n} | U^{(\pm)}_{\alpha\beta}(\bar{W}+i\varepsilon) | \hat{p}'_\beta, \underset{\sim}{q}'_\beta \bar{q}'^0_{\beta m} \rangle \psi_{\beta m}(\hat{p}'_\beta). \qquad (4.25)$$

The S matrix for the disintegration of a bound state by the third particle, giving a three-particle final state, is

$$\langle \Theta_\alpha \Phi_\alpha | S_{\alpha n, 0}(\bar{W}) | \underset{\sim}{\Omega}'_\beta \rangle =$$
$$= 2i [2\pi q_\alpha p'_\beta q'_\beta]^{1/2} (8\bar{W})^{-1} \int d^4 p_\alpha \, \psi^*_{\alpha n}(\hat{p}_\alpha) \times$$
$$\times \langle \hat{p}_\alpha, \underset{\sim}{q}_\alpha \bar{q}^0_{\alpha n} | U^{(\pm)}_{\alpha 0}(\bar{W}+i\varepsilon) | \underset{\sim}{p}'_\beta \bar{p}'^0_\beta, \underset{\sim}{q}'_\beta \bar{q}'^0_\beta \rangle, \qquad (4.26)$$

where β is arbitrary. By substituting into the discontinuity formulae (4.19) and (4.22), using the mass-shell delta-functions to eliminate the energy integrals and transforming to the variables $\underset{\sim}{\Omega}_\alpha$ or $\cos\Theta_\alpha, \Phi_\alpha$, using the Jacobian (1.25), these S matrices can be shown to satisfy the correct unitarity formulae. The phase space in the three-particle states is weighted according to (1.24). It should be stressed that all these S matrix formulae assume that the components of $\underset{\sim}{p}_\alpha$ are defined in the <u>two</u>-particle c.m. frame, and those of $\underset{\sim}{q}_\alpha$ in the <u>three</u>-particle c.m. frame, and that they would not be correct otherwise.

Readers of the non-relativistic paper [1] will remember how, by considering all particles as composite, these formulae can be made to give the S matrices for such processes as $\pi + N \rightarrow \pi + \pi + N$.

5. THE SEPARABLE APPROXIMATION

5a SCATTERING EQUATIONS FOR COMPOSITE PARTICLES

Suppose that the two particle subsystems are each dominated by certain bound states and resonances. Then, as discussed in I and in Section 3, the two-particle off-shell amplitudes factorize in the initial and final momenta. The result is the same as if the original Bethe-Salpeter kernel I_α had been separable, provided the factors are chosen to give the correct bound-state wave function. As in the non-relativistic case [1] this leads to a great simplification of the three-particle equations.

In the symbolic notation of (3.16) we have

$$V_\alpha = 2\pi i \sum_n |\alpha n\rangle f_{\alpha n} d_\alpha^{-1} \langle \alpha n| , \tag{5.1}$$

$$T_\alpha = \sum_n |\alpha n\rangle t_{\alpha n} \langle \alpha n| . \tag{5.2}$$

Multiplying by $\langle \alpha n|$ will only remove one and not both of the four-vector arguments $\hat{p}_\alpha$, $\hat{q}_\alpha$. Thus $\langle \alpha n|A|\beta m\rangle$ is still an operator not a scalar (though in a smaller Hilbert space). For algebraic simplicity, we shall first derive the composite-particle scattering equations under the assumption that there are no three-body forces, $I_0 = 0$. The extension to the case with three-body forces will be given in Subsection 5c.

As in the non-relativistic case we define "potentials" for the scattering of composite particles by

$$Z_{\alpha n, \beta m} = (1 - \delta_{\alpha\beta}) \langle \alpha n| d_\gamma |\beta m\rangle \tag{5.3}$$

with $\gamma \neq \alpha, \beta$. In momentum space

$$\langle \hat{q}_\alpha | Z_{\alpha n, \beta m}(W) | \hat{q}'_\beta \rangle = (1-\delta_{\alpha\beta}) \int d^4 p_\alpha \, d^4 p'_\beta \times$$
$$\times g^*_{\alpha n}(\hat{p}_\alpha) \, \delta_4(\hat{q}_\gamma - \hat{q}'_\gamma) \, \delta_4(\hat{p}_\gamma - \hat{p}'_\gamma) \times \tag{5.4}$$
$$\times \left[m_\gamma^2 - i\varepsilon - \hat{k}_\gamma^2 \right]^{-1} g_{\beta n}(\hat{p}'_\beta).$$

The delta-functions enable the integrations to be done. Using (1.9) we get, for example,

$$\langle \hat{q}_1 | Z_{1n,2m}(W) | \hat{q}'_2 \rangle =$$
$$= \frac{g^*_{1n}\left(-\frac{m_2}{m_2+m_3}\hat{q}_1 - \hat{q}'_2\right) g_{2m}\left(\hat{q}_1 + \frac{m_1}{m_1+m_3}\hat{q}'_2\right)}{m_3^2 - i\varepsilon + (\underset{\sim}{q}_1 + \underset{\sim}{q}'_2)^2 - \left(\frac{m_3 W}{m_1+m_2+m_3} + q_1^0 + q_2'^0\right)^2} \tag{5.5}$$

Next we define scattering amplitudes for composite particles

$$X_{\alpha n, \beta m} = \pi \langle \alpha n | G^0_\alpha U^{(+)}_{\alpha\beta} G^0_\beta | \beta m \rangle +$$
$$+ Z_{\alpha n, \beta m} \left[1 - f_{\beta m} \langle \beta m | G^0_\beta | \beta m \rangle \right] =$$
$$= \pi \langle \alpha n | G^0_\alpha U^{(-)}_{\alpha\beta} G^0_\beta | \beta m \rangle + \tag{5.6}$$
$$+ \left[1 - f_{\alpha n} \langle \alpha n | G^0_\alpha | \alpha n \rangle \right] Z_{\alpha n, \beta m} ,$$

the equality of the two forms being easily deduced from (4.6) and (4.7). The last term is proportional to the inverse bound-state propagator, by (3.18), and therefore vanishes on the mass shell for bound-state scattering, so that $X_{\alpha n, \beta m}$ then coincides with the bound-state scattering amplitudes of the previous subsection. Just as in the non-relativistic case, this last term has been introduced to give the resulting equations a more symmetric and physically transparent form.

By substituting (5.1) and (5.2) into the Faddeev equations, and using (3.18), (5.3) and (5.6) we get

$$\begin{aligned} \langle \alpha n | G^0_\alpha U^{(\pm)}_{\alpha\beta} &= \sum_m \pi^{-1} Z_{\alpha\beta} f_{\beta m} \langle \beta m | + \\ &+ \sum_{\delta \neq \beta} \sum_m \pi^{-1} X_{\alpha n, \delta m} t_{\delta m} \langle \delta m | , \end{aligned} \tag{5.7}$$

(the first term being absent for $\beta = 0$), and substituting this into (5.6) gives the scattering equations for composite particles

$$X_{\alpha n, \beta m} = Z_{\alpha n, \beta m} + \sum_{\gamma z} X_{\alpha n, \gamma z} \tau_{\gamma z} Z_{\gamma z, \beta m} \tag{5.8}$$

where

$$\tau_{\gamma z} = (2\pi^2 i)^{-1} t_{\gamma z} d_\gamma \tag{5.9}$$

is the combined propagator for the composite-particle and third-particle scattering. The explicit form of these equations in momentum space is

$$\langle \hat{q}_\alpha | X_{\alpha n,\beta m}(W) | \hat{q}'_\beta \rangle = \langle \hat{q}_\alpha | Z_{\alpha n,\beta m}(W) | \hat{q}'_\beta \rangle$$
$$+ (2\pi^2 i)^{-1} \sum_{\gamma \xi} \int d^4 q''_\gamma \langle \hat{q}_\alpha | X_{\alpha n,\gamma \xi}(W) | \hat{q}''_\gamma \rangle \times$$
$$\times t_{\gamma\xi}([W - q''^0_\gamma]^2 - \underset{\sim}{q}''^2_\gamma) \left[m^2_\gamma + \underset{\sim}{q}''^2_\gamma - \left(\frac{m_\gamma W}{m_1 + m_2 + m_3} - \right.\right.$$
$$\left.\left. - q''^0_\gamma \right)^2 \right]^{-1} \langle \hat{q}''_\gamma | Z_{\gamma\xi,\beta m}(W) | \hat{q}'_\beta \rangle. \tag{5.10}$$

They are linear integral equations in one momentum four-vector, and have the form of two-particle Bethe-Salpeter equations, with non-local energy-dependent interactions. By (5.5), (5.9) and (3.19), the "potential" and "propagator" depend only on the masses and form factors of the composite particles involved. These equations are completely self-contained, and all the observables in three-particle processes can be deduced from them, in this approximation. The S matrices for bound-state scattering and rearrangement are, by (3.9), (4.25) and the remark after (5.6),

$$\langle \Theta_\alpha \bar{\Phi}_\alpha | S_{\alpha n,\beta m}(\bar{W}) | \Theta'_\beta \bar{\Phi}'_\beta \rangle = \delta_{nm} \delta_{\alpha\beta} \langle \Theta_\alpha \bar{\Phi}_\alpha | \Theta'_\alpha \bar{\Phi}'_\alpha \rangle +$$
$$+ 2i [q_\alpha q'_\beta]^{1/2} (4\bar{W})^{-1} \langle \underset{\sim}{q}_\alpha \bar{q}^0_{\alpha n} | X_{\alpha n,\beta m}(\bar{W} + i\varepsilon) | \underset{\sim}{q}_\beta \bar{q}^0_{\beta m} \rangle, \tag{5.11}$$

while the S matrix for the bound-state disintegration process, with a three-particle final state is, by (4.26) and (5.7)

$$\langle \Theta_\alpha \bar{\Phi}_\alpha | S_{\alpha n,0}(\bar{W}) | \underset{\sim}{\Omega}'_\beta \rangle = \sum_{\delta m} 2i [2 q_\alpha p'_\delta q'_\delta / \pi]^{1/2} \times$$
$$\times (8\bar{W})^{-1} \langle \underset{\sim}{q}_\alpha \bar{q}^0_{\alpha n} | X_{\alpha n,\beta m}(\bar{W}) | \underset{\sim}{q}'_\delta \bar{q}'^0_\delta \rangle \times \tag{5.12}$$
$$\times t_{\delta m}(\bar{w}'_\delta) g_{\delta m}(\underset{\sim}{p}_\delta \bar{p}^0_\delta).$$

The graphical significance of the composite-particle scattering equations (5.8) is shown in Fig. 5. The "potentials" (5.3) correspond to Fig. 6a, which describes the virtual disintegration of a composite particle at one vertex, with one of the constituents propagating across to give a different composite particle at the other vertex. In the separable approximation without three-body forces, this is the only way in which composite particles can interact. It obviously corresponds to the off-shell peripheral graphs (one-particle exchange) for isobar production and scattering. The full equations contain this graph as a potential, and satisfy three-particle unitarity, as we shall prove in the next subsection. They therefore provide the answer to the problem of making the peripheral approximation unitary which has exercised many authors lately [37]. The various devices used by these authors stand in the same relation to our equations as the unitarized Born approximation (i.e., Born approximation for the K matrix) does to the exact Lippmann-Schwinger equation in potential scattering. From this analogy, we would expect their devices to be accurate when the modification to the peripheral model is not too great, but to break down quantitatively if the interaction is strong enough to produce a three-particle resonance, as it appears to be in many physically interesting cases [38]. So far as we know, the present approximation is the only one yet proposed, which has any reasonable chance of describing three-particle resonances accurately.

Another approximation which is included in our equations is the Peierls' mechanism [39]. This corresponds to Fig. 6a when the composite particles are unstable. The triangle graphs, which it has been suggested may seriously modify the Peierls' mechanism [40], are included in the second approximation to our equations, Fig. 6b. By solving the complete integral equations, we should therefore be able to answer the question of whether or not the Peierls' mechanism is cancelled by other graphs.

Yet another approximation included in our equations is the isobar model [41]. This corresponds to the disintegration equation (5.12), in which the production is assumed to take place in particular angular momentum states and the off-shell radial dependence of $\langle \hat{q}_\alpha | X_{\alpha n', \delta m}(W) | \hat{q}'_\delta \rangle$ and $g_{\delta m}(\hat{p}'_\delta)$ on q'_δ and p'_δ are neglected, except for the centrifugal barrier factors $(q'_\delta)^L$ and $(p'_\delta)^\ell$. Our equations therefore show the conditions under which the isobar model is valid, and enable it to be generalized. This question will be studied in detail in a forthcoming paper by Namyslowski, Razmi and Roberts [42].

5b PROOF OF THREE-PARTICLE UNITARITY

Like the two-particle Bethe-Salpeter equation, (5.8) will be integral equations in two variables, even after partial wave decomposition. This makes them intractable numerically and obscure physically, and we would like to eliminate the relative energy variable, just as we did in the two-particle case. Now despite the fact that they appear to be two-particle equations, these scattering equations for composite particles satisfy three-particle unitarity exactly. This is a rather deep result, and in the present subsection we want to analyze just how it comes about. It seems to us that by doing so, we cast considerable light on the nature of isobar and optical models in general. Furthermore, this analysis tells us, almost unambiguously, how the relative energy should be eliminated if we want three-particle unitarity to be preserved. We shall give two different derivations of three-particle unitarity, which complement each other.

In the centre-of-mass frame we can replace the argument $\hat{P}$ by W, (5.8) can then be written in the matrix form

$$X(W) = Z(W) + X(W)\tau(W)Z(W). \tag{5.13}$$

By starting from (4.12) and the second line of (5.6) we can also prove

$$X(W) = Z(W) + Z(W)\tau(W)X(W). \tag{5.14}$$

The potential $Z(W)$ will be energy-dependent and become complex above the three-particle threshold, by (5.3). In the sense of operator analyticity [12] it will possess the three-particle cut. (5.14) gives

$$\begin{aligned} Z(W+) = {} & Z(W+) - Z(W-) + X(W-) - \\ & - Z(W-)\tau(W-)X(W-), \end{aligned} \tag{5.15}$$

where $W_\pm$ is an abbreviation for $W \pm i\varepsilon$, and substituting into (5.13)

$$\begin{aligned} X(W+) = {} & Z(W+) + X(W+)\tau(W+)\left[Z(W+) - Z(W-)\right] + \\ & + X(W+)\tau(W+)X(W-) - X(W+)\tau(W+)Z(W-)\tau(W-)X(W-), \end{aligned} \tag{5.16}$$

Similarly

$$\begin{aligned} X(W-) = {} & Z(W-) + \left[Z(W-) - Z(W+)\right]\tau(W-)X(W-) + \\ & + X(W+)\tau(W-)X(W-) - X(W+)\tau(W+)Z(W+) \times \\ & \times \tau(W-)X(W-), \end{aligned} \tag{5.17}$$

and subtracting gives

$$X(W+) - X(W-) = X(W+)\left[\tau(W+) - \tau(W-)\right]X(W-) + \\ + \left\{I + X(W+)\,\tau(W+)\right\}\left[Z(W+) - Z(W-)\right] \times \\ \times \left\{I + \tau(W-)\,X(W-)\right\}, \tag{5.18}$$

This is an off-shell unitarity condition for the composite-particle scattering amplitudes. Its relation to three-particle unitarity will become apparent, when we substitute expressions for the discontinuities of the potentials $Z(W)$ and propagators $\tau(W)$. However, before doing so, we want to give the other derivation.

Here we expand the scattering equation (5.13) graphically in a perturbation series (see Fig. 6) and take the discontinuity across the three-particle cut, term by term. According to the Cutkosky rules [13], this is done by cutting each graph in as many ways as possible, so as to intersect just three internal lines, and so as to leave the initial external lines in one half of the graph, and the final ones in the other. The dotted lines in Fig. 6 show all the possible cuts. We must then replace the lines cut by mass-shell delta-functions, and sum over all graphs and all ways of cutting each. Examining Fig. 6, we see that we can sum to get the general rule

$$X(W+) - X(W-) = X(W+)\left[\tau(W+) - \tau(W-)\right]X(W-) + \\ + X(W+)\,\tau(W+)\left[Z(W+) - Z(W-)\right]\tau(W-)\,Z(W-), \tag{5.19}$$

which is illustrated graphically in Fig. 7a and b. Comparing with (5.18) we see that we have lost the following terms

$$\left[Z(W+) - Z(W-)\right] + \left[Z(W+) - Z(W-)\right]\tau(W-)\,X(W-) \\ + X(W+)\,\tau(W+)\left[Z(W+) - Z(W-)\right]. \tag{5.20}$$

This is because these terms correspond to the cutting of external lines, so that they vanish on the mass-shell for bound-state scattering. In fact it is not hard to verify directly that (5.3) can never be singular if either the initial or the final lines are on a bound-state mass shell. (This would not be true for unstable particles.)

Comparing these two proofs, we see how the discontinuities of the potentials and propagators must be interpreted. To get the discontinuity of the potential, we must rearrange it, using (2.14),

$$\begin{aligned}
&Z_{\alpha n,\beta m}(W+) - Z_{\alpha n,\beta m}(W-) = \\
&= (1-\delta_{\alpha\beta})\left[\langle \alpha n | d_\gamma(W+)|\beta m\rangle - \langle \alpha n | d_\gamma(W-)|\beta m\rangle\right] = \\
&= (1-\delta_{\alpha\beta})\,\pi(2\pi i)^2\, d_\alpha^{-1}\langle \alpha n|\left[G^0(W+) - G^0(W-)\right]\times \\
&\quad \times |\beta m\rangle\, d_\beta^{-1} = \qquad\qquad (5.21) \\
&= (1-\delta_{\alpha\beta})(2\pi i)^3\, d_\alpha^{-1}\langle \alpha n|\Delta_0|\beta m\rangle\, d_\beta^{-1},
\end{aligned}$$

where $\gamma \neq \alpha, \beta, 0$. This manipulation of singular terms is justified by the graphical argument. The factors d_α^{-1} and d_β^{-1} are Klein-Gordon operators vanishing on the mass shell. They are insulated from the mass-shell delta-functions in Δ_0 by the Hilbert space vectors $\langle \alpha\, n|$ and $|\beta m\rangle$. They will make (5.20) vanish when the external lines are on the mass shell. The term in which the discontinuity of the potential occurs internally will be non-vanishing, however. In this term the Klein-Gordon operators in (5.21) will cancel with the propagators of the third particle in (5.9), giving

$$\tau_{\alpha n}(W+)\left[Z_{\alpha n,\beta m}(W+) - Z_{\alpha n,\beta m}(W-)\right]\tau_{\beta m}(W-) =$$
$$= 2i(\pi)^{-1}\, t_{\alpha n}(\omega_\alpha[W+])\langle\alpha n|\Delta_0|\beta m\rangle \times$$
$$\times\, t_{\beta m}(\omega_\beta[W-]) \tag{5.22}$$

It is very important for the elimination of the relative energies that we can obtain an expression for the discontinuity of the potential which contains mass-shell delta-functions for all three particles.

Similarly, the discontinuity of the propagator is given by, comparing the first graph of Fig. 7 with (5.19),

$$\tau_{\alpha n}(W+) - \tau_{\alpha n}(W_-) = (2\pi^2 i)^{-1}\left[t_{\alpha n}(\omega_\alpha+) -\right.$$
$$\left. - t_{\alpha n}(\omega_\alpha -)\right] 2\pi i\Delta_\alpha =$$
$$= \frac{2i}{\pi} t_{\alpha n}(\omega_\alpha+)\langle\alpha n|\Delta_0|\alpha n\rangle t_{\alpha n}(\omega_\alpha -). \tag{5.23}$$

Here we have used the formula for the two-particle discontinuity of the composite particle propagator (3.21), and the Cutkosky rule that the complete discontinuity is obtained by taking the product of the discontinuities of the lines cut, and have considered only the three-particle singularities, not the bound-state scattering ones.

Substituting (5.22) and (5.23) into (5.19) gives

$$X_{\alpha n,\beta m}(W+) - X_{\alpha n,\beta m}(W-) = \sum_{\gamma z}\sum_{\delta s}\frac{2i}{\pi}\times$$
$$\times X_{\alpha n,\gamma z}(W+)\, t_{\gamma z}(W+)\langle\gamma z|\Delta_0|\delta s\rangle t_{\delta s}(W-)\times \tag{5.24}$$
$$\times X_{\delta s,\beta m}(W-) =$$
$$= 2\pi i\langle\alpha n|G^0_\alpha U^{(+)}_{\alpha 0}(W+)\,\Delta_0\, U^{(-)}_{0\beta} G^0_\beta|\beta m\rangle,$$

by the isobar equation (5.7). Comparing with (4.19) shows that we have indeed proved three-particle unitarity. It is important to note that, apart from the discontinuity formulae (5.22) and (5.23), we have never used the specific forms of the potentials and propagators. Therefore we have the result: <u>in order that any set of scattering equations for composite particles, of the general form (5.13) and (5.14), should satisfy three-particle unitarity, it is sufficient that the imaginary part of the potential should be given by (5.22) and the imaginary part of the propagator by (5.23).</u> The important point about this result is not merely that it is independent of the real part of the potential, but that it does not even matter how many variables are present in the integral equation. We shall see in Section 6 that this enables us to eliminate the relative energy. The fact that the composite-particle propagator has the two-particle cut, and is not just given by the pole term, is crucial for three-particle unitarity.

Figure 7 and Eq. (5.19) have a rather pretty physical interpretation. Consider the $N\pi\pi$ system, where two of the particles are identical, and there is only one stable two-particle bound state, the nucleon, which however occurs in two subsystems, so that there is a non-vanishing Z_{NN} describing its scattering. Below the $N\pi\pi$ threshold, this Z_{NN} will be real, and so therefore will be the phase shifts calculated from it. Above the inelastic threshold both Z_{NN} and phase shift become complex. As is well known, the imaginary part of the phase shift determines the total inelastic cross-section in that partial wave. This means that, given X_{NN}, there is a very simple way of calculating the total inelastic cross section. We just substitute X_{NN} into the S matrix formula (5.11) and use the imaginary part of the phase shift. There is also a much more complicated formula for the <u>differential</u> inelastic cross section, which we get by squaring (5.12). If we were to integrate this over all its five variables $\underset{\sim}{\Omega}_\alpha$, we would have to get the same result, according to three-particle unitarity. Now (5.12) contains one term for each composite particle of each subsystem. When we square it, we get two sorts of terms - direct terms and interference terms. After integration, the direct terms give the total cross section for the product of that particular isobar. They do not depend on the angular momentum state in which it is produced except in the normal way for total cross-sections. The interference terms have very peculiar effects on the Dalitz plot of the final state [43], and depend crucially on the angular momenta. For example, for N^* production in the static limit, they vanish if the extra pion comes out in an S wave [41], but not if it emerges in a P wave. This has been described as the effect of the Bose statistics of the pion, but actually it is rather more general and occurs whenever "overlapping" resonances are present. Comparing (5.24) with (5.19), we see that the direct terms of the isobar model arise from the imaginary part of the propagator τ, while the interference terms arise from the imaginary part of the potential Z. This is why the former would be expected on the two-particle analogy, whereas the latter is a characteristic three-particle effect. We shall see in Section 7 that, for definite angular momentum states, the imaginary part of the potential is proportional to the Wick recoupling coefficient [8] between the two cluster schemes involved. This brings out still further its interpretation as a measure of the overlap of the resonances.

It appears to us that it would be extremely interesting to generalize this analysis to many-particle states, and apply it to the optical model.

The discontinuity across bound-state-scattering cut can also be obtained from (5.18). The potential does not have this cut, so we need only consider the first term. By (3.28), the bound-state pole contribution to $t_{\gamma r}(\hat{K}_\gamma)$ is $(M_{\gamma r}{}^2 - \hat{K}_\gamma^2)^{-1}$, so we can apply the Cutkosky rules to get the discontinuity of (5.9)

$$\tau_{\gamma\xi}(W+) - \tau_{\gamma\xi}(W-) = 2i\,\Delta_{\gamma\xi}\,\Delta_\gamma\,, \tag{5.25}$$

where $\Delta_{\gamma r}$ is the bound-state mass-shell delta-function of (2.17), giving

γr-bound-state-scattering discontinuity of $X_{\alpha n', \beta m}(W) =$

$$= 2i X_{\alpha n, \gamma \xi}(W+) \Delta_{\gamma \xi} \Delta_{\gamma} X_{\gamma \xi, \beta m}(W-), \tag{5.26}$$

which is obviously the correct unitarity condition for bound-state scattering [cf. (4.22)]. Thus we can add to our theorem on the three-particle discontinuity: <u>in order that equations of the form of (5.13) and (5.14) satisfy the correct unitarity equations for bound-state scattering, it is sufficient that the propagator have the discontinuity (5.25), and the potential none, across the bound-state-scattering cut</u>.

5c <u>SEPARABLE APPROXIMATION WITH THREE-BODY FORCES</u>

We now consider how the equations of Section 5a must be changed when the three-body force $I_0 \neq 0$. Now the most obvious case when three-body forces must be included, is when there is an elementary particle coupled to our three-particle system. For example, in the 3π system, unless we try to explain the pion as a 3π bound state, Fig. 8a will be three-particle irreducible and must certainly be included in the equations. Similarly, in the $N\pi\pi$ system, we must include the direct nucleon pole, Figure 8b. In these cases, the three-body force will be separable in the initial and final variables

$$\langle \hat{p} \hat{q} | I_0(W) | \hat{p}' \hat{q}' \rangle =$$

$$= \langle \hat{p} \hat{q} | 0 \rangle f_0(W) \langle 0 | \hat{p}' \hat{q}' \rangle. \tag{5.27}$$

Furthermore, the compactness properties of the Faddeev equations ought to allow us to approximate the three-body force by a separable one, even if it is not separable, just as in the two-body case. We shall therefore consider only three-body forces of the form (5.27).

We now replace (5.3) by

$$Z_{\alpha n, \beta m} = 2\pi^2 i \, \varepsilon_{\alpha\beta} \langle \alpha n | G_\alpha^0 \, d_\beta^{-1} | \beta m \rangle \tag{5.28}$$

where $d_0 = 2\pi i$, and G_α^0 is given by (4.5). This is easily seen to be equivalent to the previous definition if α and β are neither $= 0$, and gives otherwise

$$\begin{aligned} Z_{\alpha n, 0} &= \pi \langle \alpha n | G_\alpha^0 | 0 \rangle , \\ Z_{0, \beta m} &= \pi \langle 0 | G_\beta^0 | \beta m \rangle , \\ Z_{00} &= \pi \langle 0 | G^0 | 0 \rangle . \end{aligned} \tag{5.29}$$

Note that, whereas multiplication by $\langle \alpha \, n |$ eliminates one of the four-vectors, $\hat{p}_\alpha$, multiplication by $\langle 0 |$ eliminates both $\hat{p}_\alpha$ and $\hat{q}_\alpha$. Thus, whereas $Z_{\alpha\beta}$ for $\alpha, \beta \neq 0$ is an operator in the Hilbert space $L_2(\hat{q})$, $Z_{\alpha 0}$ and $Z_{0\beta}$ are <u>vectors</u> in this Hilbert space, and Z_{00} is a c-number. As a consequence, we shall be able to solve completely for the three-body force (5.27).

Instead of (5.6), we now define our composite-particle-scattering amplitudes to be

$$X_{\alpha n,\beta m} = \pi \langle \alpha n | G^{o}_{\alpha} U^{(+)}_{\alpha\beta} G^{o}_{\beta} | \beta m \rangle +$$
$$+ Z_{\alpha n,\beta m} \left[1 - f_{\beta m} \langle \beta m | G^{o}_{\beta} | \beta m \rangle \times \right.$$
$$\left. \times (1 - \delta_{\beta o}) + \delta_{\beta o} Z_{\alpha n, o} \right. \tag{5.30}$$

where α, β now go from 0 to 3. Substituting this into the Faddeev equations gives

$$\langle \alpha n | G^{o}_{\alpha} U^{(+)}_{\alpha\beta} = \sum_{m} (1 - \delta_{\beta o}) \frac{1}{\pi} Z_{\alpha n,\beta m} f_{\beta m} \langle \beta m | +$$
$$+ \frac{1}{\pi} \sum_{\delta=0}^{3} \sum_{r} \varepsilon_{\beta\delta} X_{\alpha n,\delta r} t_{\delta r} \langle \delta r | \tag{5.31}$$

and thence

$$X_{\alpha n,\beta m} = Z_{\alpha n,\beta m} + (2\pi^2 i)^{-1} \times$$
$$\times \sum_{\delta=0}^{3} \sum_{r} X_{\alpha n,\delta r} t_{\delta r} d_{\delta} Z_{\delta r,\beta m} \tag{5.32}$$

where the indices go from 0 to 3.

These equations are integral for channels 1, 2, 3, but merely algebraic for channel 0. We can therefore solve algebraically for $X_{\alpha n,o}$, using (5.29), to get

$$X_{\alpha n,0} = \pi \langle \alpha n | G^0_\alpha | 0 \rangle \left[1 - f_0(W) \langle 0 | G^0 | 0 \rangle \right]^{-1} +$$

$$+ (2\pi i)^{-1} \sum_{\delta \neq 0} \sum_{z} X_{\alpha n, \delta z} \, t_{\delta z} \, d_\delta \, \times$$

$$\times \langle \delta z | G^0_\delta | 0 \rangle \left[1 - f_0(W) \langle 0 | G^0 | 0 \rangle \right]^{-1}. \tag{5.33}$$

Putting this back into (5.32) gives us a set of integral equations with indices going from 1 to 3 only, and with a new potential,

$$X_{\alpha n, \beta m} = Z'_{\alpha n, \beta m} + (2\pi^2 i)^{-1} \times$$

$$\times \sum_{\delta=1}^{3} \sum_{z} X_{\alpha n, \delta z} \, t_{\delta z} \, d_\delta \, Z'_{\delta z, \beta m} \tag{5.34}$$

where

$$Z'_{\alpha n, \beta m} = Z_{\alpha n, \beta m} +$$

$$+ \pi \langle \alpha n | G^0_\alpha | 0 \rangle \, t_0(W) \langle 0 | G^0_\beta | \beta m \rangle, \tag{5.35}$$

and

$$t_0(W) = \left[(f_0(W))^{-1} - \langle 0 | G^0 | 0 \rangle \right]^{-1} \tag{5.36}$$

is the renormalized propagator obtained by iterating Figs. 8a or 8b.

Thus the only effect of this sort of three-body force is to add a new term to the potential for composite-particle scattering. The latter now consists not only of the rearrangement graph, Fig. 9a, but also of Fig. 9b corresponding to an intermediate three-particle bound-state with renormalized propagator. The isobar equation, describing the three-particle disintegration, also acquires some extra terms. It is, by (5.31),

$$\langle \alpha n | G^0_\alpha U^{(+)}_{\alpha 0} = \langle \alpha n | G^0_\alpha | 0 \rangle t_0(W) \langle 0 | + \\ + \frac{1}{\pi} \sum_{\delta \neq 0} \sum_m X_{\alpha n, \delta m} \, t_{\delta m} \Big[\langle \delta m | + \\ + \langle \delta m | G^0 | 0 \rangle t_0(W) \langle 0 | \Big] . \tag{5.37}$$

Fig. 9c illustrates it graphically.

It is interesting to consider what happens if there is only one two-particle bound state (for the three elementary particles all different). The original $Z_{\alpha n, \beta m}$ of (5.4) will then all vanish (no rearrangement is possible), and the only interaction comes from the three-body force. In this case we can solve the equations completely, and obtain

$$X_{11} = \pi \langle 1 | G^0_1 | 0 \rangle \, x_{11}(W) \, \langle 0 | G^0_1 | 1 \rangle \tag{5.38}$$

where

$$x_{11}(W) = \Big\{ [f_0(W)]^{-1} - \\ - (2\pi i)^{-1} \langle 0 | G^0_1 | 1 \rangle t_1 d_1 \langle 1 | G^0_1 | 0 \rangle - \\ - \langle 0 | G^0 | 0 \rangle \Big\}^{-1} . \tag{5.39}$$

This is analogous to the two-particle case with separable potentials. (5.39) is the propagator for a composite particle one of whose constituents is itself composite. Its form factor is $\sqrt{\pi}\langle 1|G_1^0|0\rangle$, illustrated in Fig. 10a. This shows that a vertex function, two of whose external lines are composite, is obtained from a certain integrated product of the two wave functions. The propagator (5.38) is an iteration of two sorts of bubbles, shown in Figs. 10b,c. Of these, Fig. 10b is what we would expect from analogy with Section 3b, but Fig. 10c is a new three-particle effect. Equations such as (5.38) and (5.39) represent the next step towards a theory in which all particles are composite.

The proof of three-particle unitarity can also be extended to include the three-body force. (5.18) is still valid, but the discontinuity of the potential $Z'_{\alpha n,\beta m}(W)$ must be modified to include that of the second term in (5.35). By similar arguments to (5.21), the extra terms to be added to that equation are

$$
\begin{aligned}
&(2\pi i)^2 d_\alpha^{-1}\langle\alpha n|\Delta_0|0\rangle t_0(W-)\langle 0|G_\beta^0(W-)|\beta m\rangle + \\
&+ 2\pi i\langle\alpha n|G_\alpha^0(W+)|0\rangle t_0(W+)\langle 0|\Delta_0|0\rangle t_0(W-)\times \\
&\times\langle 0|G_\beta^0(W-)|\beta m\rangle + \\
&+ (2\pi i)^2\langle\alpha n|G_\alpha^0(W+)|0\rangle t_0(W+)\langle 0|\Delta_0|\beta m\rangle d_\beta^{-1}
\end{aligned}
\tag{5.40}
$$

where $t_o(W)$ is given by (5.36). If we now substitute into (5.18) and put the external lines on-shell, we find that the extra terms arising from (5.40) correspond exactly to the extra terms in the isobar equation (5.37), and hence we are again able to prove three-particle unitarity

$$X_{\alpha n, \beta m}(W+) - X_{\alpha n, \beta m}(W-) = \\ = 2\pi i \langle \alpha n | G^0_\alpha U^{(+)}_{\alpha 0}(W+) \Delta_0 U^{(-)}_{0\beta}(W-) G^0_\beta | \beta m \rangle . \tag{5.41}$$

6. ELIMINATION OF THE RELATIVE ENERGIES

6a WITHOUT THREE-BODY FORCES

In this section we shall eliminate the relative energy variables from the separable equations of Section 5a. First we take the equations to pieces, to see how three-particle unitarity is satisfied. This was done in Section 5b. Then we put them together again, with one fewer variable. This will be done in the present section. The requirements on these reconstituted equations are (1) that they shall still satisfy three-particle and bound-state-scattering unitarity exactly, (2) that the Born terms shall be the same on-shell, (3) that they shall possess as much time-reversal invariance as possible and (4) that they shall only involve the reduced two-particle form factors instead of the original ones. These requirements determine them uniquely.

The three-particle unitarity relation (5.24) involves three mass-shell delta-functions Δ_α. By integrating over these, we can give all three energy variables their mass-shell values

$$W = \overline{W}, \qquad p^0_\alpha = \overline{p^0_\alpha}, \qquad q^0_\alpha = \overline{q^0_\alpha}. \tag{6.1}$$

The original equations, whose discontinuity was given by three-particle unitarity, had all three of these variables off-shell. The reconstituted equations will be obtained from unitarity by performing a dispersion integral in W, as in Section 3c. They will therefore have W off-shell, but the other relative energies on-shell

$$
\begin{aligned}
W &\neq \overline{W}\,, \\
p_\alpha^\circ &= \overline{p}_\alpha^\circ\,, \\
q_\alpha^\circ &= \overline{q}_\alpha^\circ\,.
\end{aligned}
\tag{6.2}
$$

It follows from the linear relations discussed in Section 1b that all six variables p_α^o, q_α^o will be on-shell if any two of them are. This is because the equations expressing $\hat{p}_\alpha$, $\hat{q}_\alpha$ in terms of $\hat{p}_\beta$, $\hat{q}_\beta$ do not involve $\hat{P}$. For this reason, unlike the equations of Alessandrini and Omnès [9], there is no difference between our relative energy reductions in the three channels $\alpha = 1,2,3$, and we do not have to double the number of amplitudes as they do.

By putting $q_\alpha^o = \overline{q}_\alpha^o$ we eliminate the relative energy from our separable approximation equations (5.10), while putting $p_\alpha^o = \overline{p}_\alpha^o$ ensures that the form factors $g_{\alpha n}(\underset{\sim}{p}_\alpha, \overline{p}_\alpha^o)$ occurring in them is the same $g_{\alpha n}(\underset{\sim}{p}_\alpha)$ that would be obtained by solving the <u>reduced</u> two-particle equations of Section 3c. This last point is non-trivial because we are in a different Lorentz frame (three-particle c.m. instead of two-particle c.m.), so let us examine it further. The original Bethe-Salpeter radial form factor depends on two Lorentz-invariant arguments $(\hat{p}_3)^2$ and $(\hat{p}_3, \hat{K}_3)$. If we perform the two-particle reduction process in the two-particle c.m. frame, we put $\hat{K}_3 = (\underset{\sim}{0}, w_3)$, and $\hat{p}_3 = (\underset{\sim}{p}_3, \overline{p}_3^o)$, where $\overline{p}_3^o$ is given by (1.21), and $\overline{w}_3$ by (1.22). There is now only one independent (radial) variable p_3, and calculation shows that the Lorentz invariants are related by

$$
(\hat{p}_3 \cdot \hat{K}_3) = \left(\frac{m_1^2 - m_2^2}{2 m_1 m_2}\right) (\hat{p}_3)^2 . \tag{6.3}
$$

Now suppose that we perform the reduction in the three-particle c.m. frame, using (6.2). The variables on the relative-energy shells are then given by, using (1.3), (1.10), (1.14) and (1.17),

$$\hat{P}_3 = \left(\frac{m_1}{m_1+m_2} \underset{\sim}{q}_2 - \frac{m_2}{m_1+m_2} \underset{\sim}{q}_1 , \frac{m_2\sqrt{m_1^2+q_1^2} - m_1\sqrt{m_2^2+q_2^2}}{m_1+m_2} \right), \tag{6.4}$$

$$\hat{K}_3 = \left(\underset{\sim}{q}_3 \ , \sqrt{m_1^2+q_1^2} + \sqrt{m_2^2+q_2^2} \right) .$$

These now depend on three independent radial variables, q_1, q_2, q_3 and we might suppose at first sight that the two-particle form factor would be needed off the relative energy shell. However, by substituting (6.4), we find that this is not so, and the Lorentz invariants still satisfy the same relation (6.3). Therefore the reduced three-particle equations will only involve the reduced two-particle form factors $\bar{g}_{\alpha n}(\underset{\sim}{p})$. This means, among other things, that the two-particle form factors will not have the anomalous dependence on the total energy suggested by Alessandrini and Omnès [9].

We must also be careful about what happens to the normalization condition (3.25) for the two-particle form factors, when passing from the two-particle to the three-particle c.m. frames, since this would multiply the form factor by an extra kinematic factor. To avoid this, we shall use the variables $\underset{\sim}{\Omega}_\alpha$ in our discussion of three-particle unitarity. As mentioned in Section 4b, these variables include the polar co-ordinates of $\underset{\sim}{p}_\alpha$ in the two-particle rest frame, so the normalization condition (3.25) is the correct one. If we wanted to work completely in the three-particle c.m. frame, then we would acquire extra kinematic factors both in the three-particle unitarity relations and S matrices of Section 4b, and in the normalization condition for the two-particle bound-state form factors. These would eventually cancel each other.

If the two-particle Bethe-Salpeter form factors $g_{\alpha n}(\hat{p}_\alpha)$ are to be replaced throughout the equations by the reduced ones $\bar{g}_{\alpha n}(\underset{\sim}{p}_\alpha) = g_{\alpha n}(\underset{\sim}{p}_\alpha, \bar{p}^o_\alpha)$, then plainly the bound-state propagators $t_{\alpha n}(w_\alpha)$ must also be replaced by the reduced ones $\bar{t}_{\alpha n}(w_\alpha)$, defined by (3.23), which depend only on the reduced form factors. This means that the disintegration S matrix (5.12) must now become

$$\langle \Theta_\alpha \Phi_\alpha | S_{\alpha n, o}(\bar{W}) | \underset{\sim}{\Omega}'_\beta \rangle =$$

$$= \sum_{\delta m} 2i \left[2 q_\alpha p'_\delta q'_\delta / \pi \right]^{1/2} (8\bar{W})^{-1} \langle \underset{\sim}{q}_\alpha | \bar{X}_{\alpha n, \delta m}(\bar{W}) | \underset{\sim}{q}'_\delta \rangle \times \tag{6.5}$$

$$\times\, \bar{t}_{\delta m}(\bar{w}'_\delta)\, \bar{g}_{\delta m}(\underset{\sim}{p}'_\delta),$$

where $\bar{X}_{\alpha n, \beta m}$ are the reduced amplitudes for composite particle scattering, which it is our aim to define. To get three-particle unitarity for these reduced amplitudes, we must therefore replace $t_{\alpha n}$ by $\bar{t}_{\alpha n}$ in (5.24). We now want to construct reduced propagators $\bar{\tau}_{\alpha n}$ and reduced potentials $\bar{Z}_{\alpha n, \beta m}$ which have the correct discontinuities (5.23) and (5.22) (with $\bar{t}_{\alpha n}$ for $t_{\alpha n}$). We know from the arguments of Section 5b that this is sufficient for three-particle unitarity.

Reintroducing the angular integrals in (3.23) gives for the discontinuity of $\bar{t}_{3n}(w_3)$

$$\bar{t}_{3n}(w_3+) - \bar{t}_{3n}(w_3-) =$$

$$\bar{t}_{3n}(w_3+) \frac{i}{4\pi} \int d^3 p_3 |\bar{g}_{3n}(\underset{\sim}{p}_3)|^2 \frac{\bar{w}_3}{\bar{E}_{13}\bar{E}_{23}} \delta(\bar{w}_3^2 - w_3^2)\, \bar{t}_{3n}(w_3-) \tag{6.6}$$

To compare this with (5.23), we use

$$\Delta_0 \equiv \Delta_1^+ \Delta_2^+ \Delta_3^+ =$$

$$= (\bar{\omega}_3 / 4\bar{E}_{13}\bar{E}_{23}k_3^0)\, \delta(\bar{p}_3^0 - p_3^0)\, \vartheta(\omega_3)\, \delta(\bar{\omega}_3^2 - \omega_3^2)\, \delta(\bar{q}_{V_3}^0 - q_{V_3}^0), \tag{6.7}$$

which can be proved by evaluating $\Delta_1 \Delta_2$ in the c.m. frame of the (1,2) subsystem, and then using the kinematic transformation formulae of Section 1b. Substituting into (5.23) gives

$$\langle \hat{q}_{V_3} | T_{3n}(W+) - T_{3n}(W-) | \hat{q}_{V_3}' \rangle = \frac{i}{8\pi^2 k_3^0}\, t_{3n}(\omega_3 +) \times$$
$$\times \int d^3 p_3 \, |g_{3n}(\underset{\sim}{p}_3)|^2 (\bar{\omega}_3 / \bar{E}_{13}\bar{E}_{23})\, \delta(\bar{\omega}_3^2 - \omega_3^2)\, t_{3n}(\omega_3 -) \times \tag{6.8}$$
$$\times\, \delta(\bar{q}_{V_3}^0 - q_{V_3}^0)\, \delta_4(\hat{q}_{V_3} - \hat{q}_{V_3}').$$

Now, by (6.2), p_3^0 and q_3^0 will be on-shell but W will not in our reduced equations. Therefore, we must express w_3 in terms of W, $\bar{p}_3$, $\bar{q}_3$ obtaining

$$\omega_3[W] = \left[\left(W - \sqrt{m_3^2 + q_{V_3}^2} \right)^2 - q_{V_3}^2 \right]^{1/2}. \tag{6.9}$$

Comparing (6.6) and (6.8) shows that

$$\langle \underset{\sim}{q}_{V_\alpha} | \bar{T}_{\alpha n}(W) | \underset{\sim}{q}_{V_\alpha}' \rangle\, \delta(\bar{q}_\alpha^0 - q_{V_\alpha}^0)\, \delta(q_{V_\alpha}^0 - q_\alpha^{0\prime}), \tag{6.10}$$

where

$$\langle \underset{\sim}{q}_{\alpha} | \bar{T}_{\alpha n}(W) | \underset{\sim}{q}'_{\gamma_\alpha} \rangle =$$

$$= \left[2\pi \sqrt{m_\alpha^2 + q_{\gamma_\alpha}^2} \right]^{-1} \bar{t}_{\alpha n}(w_\alpha^{\gamma}[W]) \, \delta_3 (\underset{\sim}{q}_{\gamma_\alpha} - \underset{\sim}{q}'_{\gamma_\alpha}) \tag{6.11}$$

is our reduced propagator, will satisfy the three-particle discontinuity equation (5.23).

For the discontinuity of the potential, we re-express

$$\Delta_0 = \left[8 \bar{k}_1^0 \bar{k}_2^0 \bar{k}_3^0 \right]^{-1} \delta(\bar{q}_{\gamma_1}^0 - q_{\gamma_1}^0) \delta(\bar{q}_{\gamma_2}^0 - q_{\gamma_2}^0) \delta(\bar{k}_3^0 - k_3^0) , \tag{6.12}$$

and substitute into (5.22) to get

$$\langle \hat{q}_1 | T_{1n}(W+) \left[Z_{1n,2m}(W+) - Z_{1n,2m}(W-) \right] T_{2m}(W-) | \hat{q}_{\gamma_2} \rangle =$$

$$= t_{1n}(W+) \, (i/4\pi \, \bar{k}_1^0 \bar{k}_2^0 \bar{k}_3^0) \, \bar{g}_{1n}(\underset{\sim}{p}_1) \, \bar{g}_{2m}(\underset{\sim}{p}_2) \times \tag{6.13}$$

$$\times \delta(\bar{q}_{\gamma_1}^0 - q_{\gamma_1}^0) \delta(\bar{q}_{\gamma_2}^0 - q_{\gamma_2}^0) \delta(\bar{k}_3^0 - k_3^0) \, t_{2m}(W-) .$$

When the relative energies are on-shell, as in (5.2), we get

$$k_3^0 [W] = W - \sqrt{m_1^2 + q_{\gamma_1}^2} - \sqrt{m_2^2 + q_{\gamma_2}^2} . \tag{6.14}$$

The simplest way of obtaining a reduced potential with the correct discontinuity would be to replace

$$\delta(\bar{k}_3^0 - k_3^0) \longrightarrow \left[2\pi i \, (\bar{k}_3^0 - k_3^0[W] - i\varepsilon) \right]^{-1} \tag{6.15}$$

in (6.13). However, to preserve time-reversal invariance, we instead substitute

$$\delta(\bar{k}_3^\circ - k_3^\circ) \rightarrow \frac{2\,\bar{k}_3^\circ}{2\pi i}\left\{(\bar{k}_3^\circ)^2 - (k_3^\circ[W])^2 - i\varepsilon\right\}^{-1}. \tag{6.16}$$

This has the same right-hand-cut discontinuity, but also possesses a reflected cut in W. (6.13) then shows that

$$\langle \underset{\sim}{q}_{\nu_1} | \bar{Z}_{1n,2m}(W) | \underset{\sim}{q}_{\nu_2} \rangle \, \delta(\bar{q}_{\nu_1}^\circ - q_{\nu_1}^\circ)\, \delta(\bar{q}_{\nu_2}^\circ - q_{\nu_2}^\circ), \tag{6.17}$$

where

$$\langle \underset{\sim}{q}_{\nu_1} | \bar{Z}_{1n,2m}(W) | \underset{\sim}{q}_{\nu_2} \rangle = \\ = (1-\delta_{12})\,\bar{g}_{1n}(\underset{\sim}{p}_1)\left\{(\bar{k}_3^\circ)^2 - (W - \bar{k}_1^\circ - \bar{k}_2^\circ)^2 - i\varepsilon\right\}^{-1} \bar{g}_{2m}(\underset{\sim}{p}_2) \tag{6.18}$$

is our reduced potential, will satisfy the correct discontinuity equation (5.22).

Note that there is an ambiguity in reconstructing the potential from its discontinuity. We substituted the off-shell form (6.14) of the variable k_3^0 into the denominator. However, the discontinuity formula also involves $\bar{k}_3^0$ implicitly in the arguments of the off-shell form factors, because of (1.29). Here we did not make the replacement $\bar{k}_3^0 \rightarrow k_3^0[W]$, but left it on-shell. If we had put the arguments of the form factors also off-shell, then the right-hand cut and its discontinuity would be the same, but we would also acquire a left-hand cut in W, arising from the singularities of the form factors [31]. The trouble is that the form factors occurring in the reduced potential, would then not be the reduced ones $\bar{g}_{\alpha n}(\underset{\sim}{p}_\alpha) \equiv g_{\alpha n}(\underset{\sim}{p}_\alpha, \bar{p}_\alpha^0)$, but would depend in an essential manner on p^0_α off-shell. Thus, in order to calculate the potential, we would have to solve not just the reduced two-particle equations [26], but the full Bethe-Salpeter equation. Also the method of reconstructing the form factors from the phase-shifts (3.29) would break down. For practical purposes, we are therefore forced to do the reduction as in (6.18).

When we substitute (6.10) and (6.17) into the scattering equations (5.8), the $\delta(\bar{q}^0_\alpha - q^0_\alpha)$ factors will have the effect of removing the relative-energy integration, and putting the relative energy on-shell. Because only internal lines contribute to three-particle unitarity, the potentials directly connected to external lines are not affected by the reduction procedure, and do not acquire any relative-energy delta-functions. The over-all effect is therefore that all internal lines are put onto the relative-energy shell. As to the external lines, we are always free to put them onto the energy-shell, since the observable S matrices which we are trying to calculate are on-shell. We thus get the equation

$$\langle \underset{\sim}{q}_\alpha | \overline{X}_{\alpha n,\beta m}(W) | \underset{\sim}{q}'_\beta \rangle = \langle \underset{\sim}{q}_\alpha | Z_{\alpha n,\beta m}(W) | \underset{\sim}{q}'_\beta \rangle + \tag{6.19}$$
$$+ \sum_{\gamma r} \int d^3 q''_\gamma \langle \underset{\sim}{q}_\alpha | \overline{X}_{\alpha n,\beta m}(W) | \underset{\sim}{q}''_\gamma \rangle \overline{T}_{\gamma r}(W; q''_\gamma) \langle \underset{\sim}{q}''_\gamma | \overline{Z}_{\gamma r,\beta m}(W) | \underset{\sim}{q}'_\beta \rangle$$

where

$$\langle \underset{\sim}{q}_\alpha | \overline{X}_{\alpha n,\beta m}(W) | \underset{\sim}{q}_\beta \rangle = \langle \underset{\sim}{q}_\alpha \bar{q}^0_\alpha | X_{\alpha n,\beta m}(W) | \underset{\sim}{q}_\beta \bar{q}^0_\beta \rangle, \tag{6.20}$$

Here we have used the fact that the reduced potential (6.18) is identical with the Born term (5.5) of the original equation on-shell.

Equation (6.19) only involves integration over a three-vector $\int d^3 q''_\gamma$ instead of a four-vector $\int d^4 q''_\gamma$, and is therefore the equation we want. It employs the reduced potential (6.18) and the reduced propagator (6.11) [cf. (3.23)], which only depend on the reduced two-particle form factors $\bar{g}_{\alpha n}(\underset{\sim}{p}_\alpha)$. Apart from the relativistic kinematics, it is exactly analogous to the non-relativistic scattering equations for composite particles obtained in I. The discontinuity formulae above, together with the argument of Section 5b show that it satisfies three-particle unitarity.

The situation for bound-state scattering is more complicated. It is easily seen from (3.28) and (6.9) that bound-state-scattering unitarity is satisfied. However, the Born approximation does not coincide with that of the original equations on the bound-state-scattering mass shell. Therefore the analogue of (6.20) for bound-state scattering is not valid. This difficulty arises because the relative-energy shell for bound-state scattering (4.23) is not the same as that for three-particle scattering (1.18). The difference only affects the off-shell form factors, and not the denominator of the Born term, so it should not be too important. It is a consequence of the fact that we did not replace $\bar{k}_3^0 \to k_3^0 [W]$ in the arguments of the off-shell form factors. It is easily seen that the Born term (5.5) of the original equations, evaluated on the bound-state-scattering shell (3.28), cannot be expressed in terms of the reduced form factors $g_{\alpha n}(\underset{\sim}{p}_\alpha)$. This drawback therefore represents the inevitable price we have to pay for a one variable radial equation. The extra approximation made is of the same type as that involved in replacing the original Bethe-Salpeter form factor by the reduced one, so it should not drastically worsen the theory.

Despite appearances, the reduced scattering equation (6.19) is Lorentz invariant. The privileged position of W is due to the fact that it is the total mass, and therefore the only variable whose right-hand singularities are controlled by <u>physical</u> unitarity, not to the fact that we are working in the c.m. frame. Indeed, if we were to eliminate the relative energies in any other Lorentz frame, we would get the same result, though with more algebraic labour. This is because the mass-shell delta-functions, which define the reduced variables, are invariant under orthochronous Lorentz transformations. However, our equations are not completely time-reversal invariant under $W \to -W$, because $w_3[W]$ and $k_3^0[W]$ are not even functions. This is a pity, because the two-particle theory of Section 3c was fully time-reversal invariant, but there seems to be no way of avoiding it. This is not a serious difficulty, because the elimination of the relative energies is basically an elimination of antiparticle intermediate states, and therefore we can hardly expect full time-reversal invariance. In fact, for particles with spin the difficulty increases. There is an analogy in mass-shell S matrix theory, which may help to reassure the reader here: the MacDowell symmetry of the pion-nucleon partial wave amplitudes [44)] requires the existence of a reflected cut in W, in order that the S matrix have the general invariance properties required of strong interactions. However, this cut is so distant from the physical region that it is almost always neglected in actual calculations. The incomplete time-reversal invariance of our final equations is precisely an approximation of this sort. What fails is thus only <u>analytically continued</u> time-reversal invariance. There is no failure in the physical region, and detailed balancing is not affected.

6b WITH THREE-BODY FORCES

When three-body forces are present in the original equations, the potential for composite particle scattering acquires the additional term (5.35). We have to eliminate the relative energies in such a way as to preserve three-particle unitarity. This will be achieved if the discontinuity

$$\mathcal{T}(W+)\left[Z(W+)-Z(W-)\right]\mathcal{T}(W-) \tag{6.21}$$

as given by (5.40) and (5.9), retains the same form.

The extra term in the potential

$$\pi\langle\alpha n|G^{o}_{\alpha}|0\rangle t_{o}\langle 0|G^{o}_{\beta}|\beta m\rangle \tag{6.22}$$

is composed of three distinct factors. The outer ones will have singularities arising from the right-hand cut for two-particle scattering in G^{o}_{α}. They will therefore be non-singular on the bound-state mass shell, because of the negative energy of the bound-state. We can therefore neglect the singularities in the $\langle\alpha\, n|G^{o}_{\alpha}|0\rangle$ factors when they are connected to external lines, but not when they are internal [compare the argument after (5.20)]. The the central factor t_{o} in (6.22) always corresponds to internal lines.

We reduce the three factors of (6.22) separately. For the central factor

$$t_{o}(W) = \left[(f_{o}(W))^{-1} - \langle 0|G^{o}|0\rangle\right]^{-1}, \tag{6.23}$$

we simply rearrange

$$\Delta_0 = \frac{1}{8\bar{k}_1^0 \bar{k}_2^0 \bar{k}_3^0} \delta(\bar{p}_\alpha^0 - p_\alpha^0)\delta(\bar{q}_\alpha^0 - q_\alpha^0)\delta(\bar{W}-W), \quad (6.24)$$

[this follows from (1.7) and (2.8)], and define a reduced free Green's function

$$\bar{G}^0 = \frac{W\delta(\bar{p}_\alpha^0 - p_\alpha^0)\delta(\bar{q}_\alpha^0 - q_\alpha^0)}{4\pi \bar{k}_1^0 \bar{k}_2^0 \bar{k}_3^0 (\bar{W}^2 - W^2 - i\varepsilon)} \quad (6.25)$$

which, by (6.24), has the correct discontinuity equation (2.18) across the positive energy cut. The reduced form of (6.23) is then

$$\bar{t}_0(W) =$$
$$= \left[(f_0(W))^{-1} - \frac{1}{4\pi} \int \frac{d^3 p'_\alpha \, d^3 q'_\alpha \, \bar{W}' \, |\bar{g}_0(\underset{\sim}{p'_\alpha}, \underset{\sim}{q'_\alpha})|^2}{\bar{k}_1'^0 \bar{k}_2'^0 \bar{k}_3'^0 (\bar{W}'^2 - W^2 - i\varepsilon)} \right]^{-1}, \quad (6.26)$$

which has the correct discontinuity equation

$$\bar{t}_0(W+) - \bar{t}_0(W-) =$$
$$= 2\pi i \, \bar{t}_0(W+) \langle 0 | \Delta_0 | 0 \rangle \bar{t}_0(W-), \quad (6.27)$$

and preserves time-reversal invariance. Here

$$\bar{g}_0(\underset{\sim}{p_\alpha}, \underset{\sim}{q_\alpha}) = \langle \underset{\sim}{p_\alpha} \bar{p}_\alpha^0, \underset{\sim}{q_\alpha} \bar{q}_\alpha^0 | 0 \rangle \quad (6.28)$$

is the reduced form factor of the three-particle bound state, as defined by (5.27).

For the outer factors of (6.22), the discontinuity (5.40) contains a Klein-Gordon operator d_α^{-1} which is cancelled by the propagator of the third particle in (5.9). We therefore have to perform the reduction so as to preserve the relation

$$\tau_{\alpha n} \langle \alpha n | G_\alpha^0 | 0 \rangle = \frac{1}{\pi} t_{\alpha n} \langle \alpha n | G^0 | 0 \rangle . \qquad (6.29)$$

In order to do this, we define the reduced propagator for two free particles

$$\bar{G}_3^0 = \frac{\omega_3 \, \delta(\bar{P}_3^0 - P_3^0)}{2\pi \, \bar{E}_{13} \bar{E}_{23} \left[\bar{\omega}_3^2 - (\omega_3[W])^2 - i\varepsilon \right]} \qquad (6.30)$$

where $w_3[W]$ is given by (6.9). We know from Section 3c that $\bar{G}_3^0$ has the same discontinuity as G_3^0. The outer factors then become, e.g.,

$$\begin{aligned} \bar{\vartheta}_{3n}(\underset{\sim}{q}_3 ; W) &\equiv \langle 3n | \bar{G}_3^0 | 0 \rangle \Big|_{q_3^0 = \bar{\bar{q}}_3^0} \\ &= \frac{1}{2\pi} \int \frac{d^3 p_3' \; \bar{\omega}_3' \; \bar{g}_{3n}(\underset{\sim}{p}_3) \, \bar{\varphi}_0(\underset{\sim}{p}_3, \underset{\sim}{q}_3)}{\bar{E}_{13}' \bar{E}_{23}' \left(\bar{\omega}_3'^2 - (\omega_3[W])^2 - i\varepsilon \right)} . \end{aligned} \qquad (6.31)$$

This is essentially the vertex function between the three-particle bound state, a two-particle bound state, and an elementary particle, whereas (6.28) is the vertex function between the three-particle bound state and two elementary particles. Substituting (6.11), (6.25) and (6.31) into (6.29), we find that the equality is still satisfied.

Therefore, to include the effects of three-body forces, we must add to the reduced potential (6.18) another term

$$\langle q_\alpha | \bar{Z}'_{\alpha n,\beta m}(W) | q'_\beta \rangle = \langle q_\alpha | \bar{Z}_{\alpha n,\beta m}(W) | q'_\beta \rangle + \\ + \pi\, \bar{v}_{\alpha n}(q_\alpha; W)\, \bar{t}_0(W)\, \bar{v}^{*}_{\beta m}(q'_\beta; W), \tag{6.32}$$

where the notations are given by (6.26), (6.28) and (6.31). Instead of (6.5), the isobar formula now becomes, by (5.37),

$$\langle \Theta_\alpha \Phi_\alpha | S_{\alpha n,0}(\bar{W}) | \Omega'_\beta \rangle = \\ = \frac{2i}{8\bar{W}} [2q_\alpha/\pi]^{1/2} \Big\{ [p'_\beta q'_\beta]^{1/2} \pi\, \bar{v}_{\alpha n}(q_\alpha; \bar{W}) \times \\ \times \bar{t}_0(\bar{W})\, \bar{g}_0(p'_\beta, q'_\beta) + \sum_{\delta=1}^{3} \sum_m [p'_\delta q'_\delta]^{1/2} \times \\ \times \langle q_\alpha | \bar{X}_{\alpha n,\beta m}(\bar{W}) | q'_\delta \rangle\, \bar{t}_{\delta m}(\bar{\omega}'_\delta) \Big[\bar{g}_{\delta m}(p_\delta) + \\ + (2\pi \bar{k}^{\circ\prime}_\delta)^{-1}\, \bar{v}_{\delta m}(q'_\delta; \bar{W})\, \bar{t}_0(\bar{W})\, \bar{g}_0(p'_\beta, q'_\beta), \tag{6.33}$$

where β is arbitrary. The way we have done the reduction ensures that three-particle unitarity is still valid. This can be explicitly checked by calculating the discontinuity of (6.32), substituting into (5.18), and comparing with (6.33).

7. PARTIAL WAVE DECOMPOSITION

In this section, we decompose into angular momentum states, thus obtaining our final one-variable equations. We shall use the helicity formalism [8),45)], transforming afterwards to orbital angular momentum states [46)]. There would be considerable difficulties in performing the angular momentum analysis before the elimination of the relative energies. Firstly, we would not be able to define a two-particle rest frame if $\hat{K}_\alpha$ is spacelike. Secondly, the Wick angles would become energy dependent and extremely complicated formulae would result. Thirdly, there are difficulties in defining the helicity of a single particle, even, off its mass shell. The sensible place to perform the partial wave decomposition is therefore after the relative-energy reduction. The only variable off-shell is then the total energy, and this does not affect the angular momentum decomposition.

First we separate all the form factors $\bar{g}_{\alpha j}(\underset{\sim}{p}_\alpha)$ into their radial and angular parts, as in (3.14),

$$\langle \alpha j h | = Y_j^h(\vartheta_\alpha \varphi_\alpha)\, \overline{g}_{\alpha j}(p_\alpha) \,. \tag{7.1}$$

This decomposition is in the two-particle rest frame. h is therefore the helicity of the composite particle [8)]. The scattering equations for composite particles will then be identical, as far as the angular momentum is concerned, with a many-channel two-particle problem in which particles of spin j and helicity h are scattered by ones of zero spin. We can therefore use the formulae of Jacob and Wick [47)] to perform the partial wave decomposition in the three-particle c.m. system

$$\langle q_1 j h | \bar{X}^J_{12}(W) | q_2' j' h' \rangle =$$

$$= 2\pi \int_{-1}^{1} d\cos\chi_3 \, d^J_{hh'}(\chi_3) \langle q_1 0 0 | \bar{X}_{1jh, 2j'h'}(W) | q_2' \chi_3 0 \rangle, \tag{7.2}$$

where q_1 and q_2 have been expressed in radial co-ordinates. Its inverse is

$$\langle q_1 0 0 | \bar{X}_{1jh, 2j'h'}(W) | q_2' \Theta_2' \Phi_2' \rangle =$$

$$= \sum_{J=0}^{\infty} (2J + 1/4\pi) \, d^J_{hh'}(\Theta_2') \exp[i(h-h')\Phi_2'] \times \tag{7.3}$$

$$\times \langle q_1 j h | \bar{X}^J_{12}(W) | q_2' j' h' \rangle,$$

where $\underset{\sim}{q}_1$, $\underset{\sim}{q}_2'$ are in radial co-ordinates, with $\underset{\sim}{q}_1$ along the z axis. Substituting (6.18) and (7.1) gives us the partial wave decomposition of the potential

$$\langle q_1 j h | \bar{Z}^J_{12}(W) | q_2 j' h' \rangle =$$

$$= \tfrac{1}{2}(1-\delta_{12})[(2j+1)(2j'+1)]^{1/2} \int_{-1}^{1} d\cos\chi_3 \, d^J_{hh'}(\chi_3) \, d^j_{h0}(\theta_1) \times \tag{7.4}$$

$$\times d^{j'}_{h'0}(\theta_2) \, \bar{g}_{1j}(p_1) \, \bar{g}_{2j'}(p_2) \Big[m_3^2 + q_1^2 + q_2^2 + 2 q_1 q_2 \cos\chi_3 -$$

$$- \left(W - \sqrt{m_1^2 + q_1^2} - \sqrt{m_2^2 + q_2^2} \right)^2 - i\varepsilon \Big]^{-1}.$$

Here p_1, p_2 and the Wick angles ϑ_1, ϑ_2 have to be expressed in terms of q_1, q_2 and χ_3 by the formulae of Section 1c. Note that the angular factor of (7.4) is just the Wick recoupling coefficient [8),48)] for three spin-zero particles and the recoupling scheme $\langle(23)1/(31)2\rangle$. If we calculate the imaginary part of the potential, the delta-function will eliminate the integration, and the result will be proportional to the Wick recoupling coefficient. This is a mathematical expression of the fact that the imaginary part of the potential describes the overlap of the resonances (see Section 5b above).

The propagator will be independent of the total angular momentum, so the scattering equations (6.19) now become

$$\begin{aligned}\langle q_{V\alpha}\, j\, h|\, \overline{X}^{J}_{\alpha\beta}(W)\,|q'_{V\beta}\, j'\, h'\rangle &= \langle q_{V\alpha}\, j\, h|\, \overline{Z}^{J}_{\alpha\beta}(W)\,|q'_{V\beta}\, j'\, h'\rangle + \\ &+ \sum_{\gamma=1}^{3}\sum_{j''}\sum_{h''=-j''}^{j''}\int_0^\infty dq''_{V\gamma}\,\langle q_{V\alpha}\, j\, h|\,\overline{X}^{J}_{\alpha\gamma}(W)\,|q''_{V\gamma}\, j''\, h''\rangle \times \\ &\times (q''_{V\gamma})^2\left[2\pi\sqrt{m_\gamma^2+q''^2_{V\gamma}}\right]^{-1}\overline{t}_{\gamma j''}(\omega''_\gamma[W,q''_{V\gamma}]) \times \\ &\times \langle q''_{V\gamma}\, j''\, h''|\,\overline{Z}^{J}_{\gamma\beta}(W)\,|q'_{V\beta}\, j'\, h'\rangle ,\end{aligned} \qquad (7.5)$$

which involve only one integration. The sums over j" and h" will be finite in practice, since only a limited number of two-particle angular momentum states will be resonant. It is convenient to re-express this in terms of orbital angular momentum states, firstly because they include the reduction in the number of amplitudes arising from parity conservation, and secondly because they have definite centrifugal barrier threshold behaviour in q_α and q'_β, whereas the helicity formalism mixes terms with different centrifugal barriers, which is very awkward in effective range treatments. The transition from helicity states to states with definite orbital angular momentum (between the resonance and third particle) can easily be performed [46]. In the present case, the zero spin of the elementary particles makes it particularly simple, and we get

$$\langle q_1 j L | X^J_{12}(W) | q_2' j' L' \rangle =$$
$$= \sum_{h=-j}^{j} \sum_{h'=-j'}^{j'} \left[(2L+1)(2L'+1)\right]^{1/2} (2J+1)^{-1} C^{L\,j\,J}_{0\,h\,h} C^{L'\,j'\,J}_{0\,h'\,h'} \times \qquad (7.6)$$
$$\times \langle q_1 j h | \overline{X}^J_{12}(W) | q_2' j' h' \rangle .$$

The inverse formula, needed for (7.3), is given by the same transformation matrix with the sums over L, L' instead of h, h'. By considering the behaviour of (7.4) under $h \rightarrow -h$, $h' \rightarrow -h'$, we see that (7.6) vanishes for the potential, and therefore also for the amplitude, unless $(-1)^{L+j} = (-1)^{L'+j'}$. This expresses parity conservation.

8. CONCLUSIONS

8a FINAL FORMULAE

In this section we collect together our results. We shall try to make it readable independently of the rest of the paper.

The physical situation is that we have three particles of masses m_1, m_2, m_3, which in the present paper are supposed to have spin 0. Resonances and bound states can be formed between any two pairs of these particles. We suppose that their masses, widths and angular momenta are known. The problem is, given these properties of the two-particle systems, to predict the behaviour of the three-particle system. It should be stressed that overlapping resonances are allowed. Also, by considering for example the nucleon as a pion-nucleon bound state, we can treat coupled two-particle and three-particle systems such as $N\pi \rightleftarrows N\pi\pi$.

Label the two-particle subsystems by $\alpha = 1,2,3$ according to the particle not included in them. Suppose that the α subsystem contains a finite number of composite particles (bound states, virtual bound states, or resonances) of angular momenta j. We shall suppose there is at most one per partial wave. This assumption can easily be generalized at the expense of some algebraic complexity. To each of these composite particles, we assign an off-shell form factor $\bar{g}_{\alpha j}(p_\alpha)$, where p_α is the momentum in the two-particle c.m. system. We suggested as a plausible approximate form for these

$$\bar{g}_{\alpha j}(p_\alpha) = N_{\alpha j} \cdot (p_\alpha)^j \left[\mu_{\alpha j}^2 + p_\alpha^2\right]^{-j-1} . \qquad (8.1)$$

Here $\mu_{\alpha j}$ is an adjustable parameter, which is roughly the average mass of the particles whose exchange produces the resonance. $N_{\alpha j}$ is a normalization constant. In the case of a bound state of mass $M_{\alpha j}$, it must be chosen to make (e.g., $\alpha = 3$)

$$\frac{1}{2\pi}\int_0^\infty \frac{dp_3 \cdot p_3^2 \, \overline{\omega}_3 \, \left|\overline{g}_{3j}(p_3)\right|^2}{\sqrt{(m_1^2+p_3^2)(m_2^2+p_3^2)} \, (\overline{\omega}_3^{\,2} - M_{3j}^2)^2} = 1 , \tag{8.2}$$

where

$$\overline{\omega}_3 = \sqrt{m_1^2 + p_3^2} + \sqrt{m_2^2 + p_3^2} \tag{8.3}$$

In the case of an unstable particle, we can take $N_{\alpha j} = 1$ without loss of generality. Using this off-shell form factor, we set up a propagator for the composite particle

$$\overline{t}_{3j}(\omega_3) = \left\{ f_{3j}^{-1} - \frac{1}{2\pi}\int_0^\infty \frac{dp_3' \, p_3'^{\,2} \, \overline{\omega}_3' \, \left|\overline{g}_{3j}(p_3')\right|^2}{\sqrt{(m_1^2+p_3'^{\,2})(m_2^2+p_3'^{\,2})} \, (\overline{\omega}_3'^{\,2} - \omega_3^2 - i\varepsilon)} \right\}^{-1} , \tag{8.4}$$

where $\overline{w}_3'$ is expressed in terms of p_3' by (8.3). This propagator depends on w_3, the mass of the two-particle subsystem. $f_{\alpha j}$ is another adjustable parameter, which describes roughly the strength of the forces producing the composite particle. These two parameters $\mu_{\alpha j}$, $f_{\alpha j}$ can be determined by fitting the experimental two-porticle phase shift to the formula

$$(p_3/\overline{\omega}_3)\left|\overline{g}_{3j}(p_3)\right|^2 \cot \delta_j(p_3) = 4 \, \mathrm{Re}\left[\overline{t}_{3j}(\overline{\omega}_3)\right]^{-1} , \tag{8.5}$$

where $\bar{w}_3$ is expressed in terms of p_3 by (8.3). In particular, they can be determined if the mass and width of the resonance are known. For further details on all this, see Section 3.

Now we pass to the three-particle system. The essential assumption is that each of the two-particle subsystems is dominated by these bound states and resonances we have been discussing. It can then be shown that the study of the three-particle system reduces to the study of the scattering of these composite particles by the third particle. (By scattering here, we include rearrangement processes, in which we start out say with an N^* and a pion, and end up with a nucleon and a ρ .) The Faddeev approach is important because it enables this apparently woolly statement to be given a precise and demonstrable meaning. The three-particle system will then be reduced to a many-channel two-particle system, and will have equations which are essentially of a two-particle type. We emphasize that this does not depend on any crude approximation such as replacing unstable particles by stable ones.

There will be as many channels in our final equations as there are composite particles in all the two-particle subsystems. Each channel corresponds to the scattering of this composite particle by the third particle. We can therefore label the channels by $\alpha = 1,2,3$, according to which of the two-particle subsystems contains the composite particle, and by its angular momentum j. We express the scattering amplitudes for composite particles in the three-particle c.m. system. The variables involved will be W the total energy, which is an off-shell variable as in the Lippmann-Schwinger equation, q_α the momentum of the third particle in the initial state, q'_β the momentum of the third particle in the final state, J the total angular momentum, and L, L' the orbital angular momenta between composite particle and third particle in the initial and final states. The scattering amplitude for an initial channel containing a composite particle in the α subsystem of spin j, and a final channel containing a composite particle in the β subsystem of spin j', will be

$$\langle q_\alpha j L | \overline{X}^J_{\alpha\beta}(W) | q'_\beta j' L' \rangle . \tag{8.6}$$

These scattering amplitudes satisfy a system of coupled linear integral equations in one variable

$$\begin{aligned}
&\langle q_\alpha j L | \overline{X}^J_{\alpha\beta}(W) | q'_\beta j' L' \rangle = \\
&= \langle q_\alpha j L | \overline{Z}^J_{\alpha\beta}(W) | q'_\beta j' L' \rangle + \\
&+ \sum_{\gamma \neq \beta} \sum_{j''} \sum_{L'' = J - j''}^{J + j''} \int_0^\infty dq''_\gamma \langle q_\alpha j L | \overline{X}^J_{\alpha\gamma}(W) | q''_\gamma j'' L'' \rangle \times \\
&\times (q''_\gamma)^2 \left[2\pi \sqrt{m_\gamma^2 + q''^2_\gamma} \right]^{-1} \bar{t}_{\gamma j''}\left(w''_\gamma [W, q''_\gamma] \right) \times \\
&\times \langle q''_\gamma j'' L'' | \overline{Z}^J_{\gamma\beta}(W) | q'_\beta j' L' \rangle .
\end{aligned} \tag{8.7}$$

Here $\bar{t}_{\gamma j''}(w''_\gamma [W, q''_\gamma])$ is the propagator for the composite particle, (8.3), with the argument

$$w''_\gamma [W, q''_\gamma] = \left[\left(W - \sqrt{m_\gamma^2 + q''^2_\gamma}\right)^2 - q''^2_\gamma \right]^{1/2} . \tag{8.8}$$

The "potential" $\bar{Z}^{J}_{\alpha\beta}(W)$ can be expressed in terms of the off-shell form factors of the composite particles, (8.1), by a formula involving a single integral, which is made complicated by the relativistic angular momentum analysis. This formula is

$$\begin{aligned}
&\langle q_1 j L | \bar{Z}^{J}_{12}(W) | q_2 j' L' \rangle = \\
&= \tfrac{1}{2}(1-\delta_{12}) \sum_{h=-j}^{j} \sum_{h'=-j'}^{j'} \left[(2j+1)(2j'+1)(2L+1)(2L'+1)\right]^{1/2} \times \\
&\times (2J+1)^{-1} \, C^{L j J}_{0 h h} \, C^{L' j' J}_{0 h' h'} \int_{-1}^{1} d\cos\chi_3 \, \times \\
&\times d^{J}_{hh'}(\chi_3) \, d^{j}_{h0}(\vartheta_1) \, d^{j'}_{h'0}(\vartheta_2) \, \bar{g}_{1j}(p_1) \, \bar{g}_{2j'}(p_2) \, \times \\
&\times \Big[m_3^2 + q_1^2 + q_2^2 + 2 q_1 q_2 \cos\chi_3 - \big(W - \sqrt{m_1^2+q_1^2} - \\
&\quad - \sqrt{m_2^2+q_2^2}\,\big)^2 - i\varepsilon \Big]^{-1} .
\end{aligned} \tag{8.9}$$

Here $1-\delta_{12}$ indicates that $\bar{Z}_{\alpha\beta}$ vanishes unless $\alpha \neq \beta$. ($\bar{Z}_{\alpha\alpha}$ will be non-vanishing if there are three-body forces, however.) p_1 and ϑ_1 are to be expressed in terms of q_1, q_2, $\cos\chi_3$ by

$$\begin{aligned}
p_1^2 = &\Big\{ \Big[q_2^2 + q_3^2 - q_1^2 + 2\sqrt{(m_2^2+q_2^2)(m_3^2+q_3^2)} \Big]^2 - \\
&- 4 m_2^2 m_3^2 \Big\} \Big/ \Big\{ 4 \Big[\sqrt{m_2^2+q_2^2} + \sqrt{m_3^2+q_3^2} \Big]^2 - 4 q_1^2 \Big\} ,
\end{aligned} \tag{8.10}$$

$$\cos\vartheta_1 = \Big\{\sqrt{m_2^2+q_{v_2}^2}\,\big[q_{v_2}^2-q_{v_3}^2-q_{v_1}^2\big] + \sqrt{m_3^2+q_{v_3}^2}\times$$

$$\times\big[q_{v_2}^2-q_{v_3}^2+q_{v_1}^2\big]\Big\}(q_{v_1})^{-1}\Big\{\Big[q_{v_2}^2+q_{v_3}^2-q_{v_1}^2+2\sqrt{(m_2^2+q_{v_2}^2)(m_3^2+q_{v_3}^2)}\Big]^2 - \qquad (8.11)$$

$$-\,4\,m_2^2\,m_3^2\Big\}^{-1/2},$$

where

$$q_{v_3}^2 = q_{v_1}^2 + q_{v_2}^2 + 2\,q_{v_1}q_{v_2}\cos\chi_3\,. \qquad (8.12)$$

p_2 and ϑ_2 are given by the same formulae (8.10) and (8.11) with cyclic interchange of indices. These formulae include the effects of the Lorentz transformation between the two-particle and three-particle c.m. frames. C and d are the normal Clebsch-Gordan coefficients and rotation matrices. The potential will vanish unless $(-1)^{L+j} = (-1)^{L'+j'}$, which expresses parity conservation.

Once these composite-particle scattering matrices have been calculated by the integral equations (8.7), all observables in the three-particle, or coupled three-particle and two-particle, systems can be expressed in terms of them. Three-particle bound states will be poles in W, from whose residues their wave functions can be calculated. The S matrix for the scattering or rearrangement collision of a bound state will be

$$\langle 00,h|\, S_{\alpha j,\beta j'}(\overline{W})\,|\Theta'_\beta \Phi'_\beta h'\rangle =$$

$$= \delta_{\alpha\beta}\,\delta_{jj'}\,\delta_{hh'}\,\delta(1-\cos\Theta'_\beta)\,\delta(\Phi'_\beta) +$$

$$+2i\,(q_\alpha q'_\beta)^{1/2}\,(16\pi\overline{W})^{-1}\sum_{J=0}^{\infty}\;\sum_{L=J-j}^{J+j}\;\sum_{L'=J-j'}^{J+j'} \times \tag{8.13}$$

$$\times\left[(2L+1)(2L'+1)\right]^{1/2} d^J_{hh'}(\Theta'_\beta)\,\exp\left[i(h-h')\Phi'_\beta\right]\times$$

$$\times\, C^{LjJ}_{0hh}\; C^{L'j'J}_{0h'h'}\;\langle q_\alpha jL|\,\overline{X}^{J}_{\alpha\beta}(\overline{W})\,|q'_\beta j'L'\rangle .$$

Here h and h' are the helicities of the initial and final bound state, and

$$\overline{W} = \sqrt{M^2_{\alpha j}+q^2_\alpha} + \sqrt{m^2_\alpha + q^2_\alpha} =$$
$$= \sqrt{M^2_{\beta j'}+q'^2_\beta} + \sqrt{m^2_\beta + q'^2_\beta} . \tag{8.14}$$

The S matrix for the disintegration of the bound state during the scattering process, giving a three-particle final state, will be

$$\langle 00,h|\, S_{\alpha j,0}(\overline{W})\,|\vartheta'_\beta \varphi'_\beta \varpi'_\beta \Theta'_\beta \Phi'_\beta\rangle =$$

$$= \sum_{\delta=1}^{3}\sum_{j'}\sum_{J=0}^{\infty}\;\sum_{h'=-j'}^{j'}\;\sum_{L=J-j}^{J+j}\;\sum_{L'=J-j'}^{J+j'} \times$$

$$\times\, 2i\left[2q_\alpha p'_\delta q'_\delta\right]^{1/2}(4\overline{W})^{-1}\left[(2L+1)(2L'+1)\right]^{1/2}(4\pi)^{-3/2}\times \tag{8.15}$$

$$\times\, C^{LjJ}_{0hh}\; C^{L'j'J}_{0h'h'}\; d^J_{hh'}(\Theta'_\delta)\,\exp\left[i(h-h')\Phi'_\beta\right]\times$$

$$\times\,\langle q_\alpha jL|\,\overline{X}^{J}_{\alpha\delta}(\overline{W})\,|q'_\delta j'L'\rangle\, \overline{t}_{\delta j'}(\overline{\varpi}'_\delta)\,\overline{g}_{\delta j'}(p'_\delta)\,Y^{h'}_{j'}(\vartheta'_\delta\varphi'_\delta).$$

Here $q'_3, \pi - \Theta'_3, -\Phi'_3$ are the polar co-ordinates of the momentum of final particle three in the three-particle c.m. frame, and $p'_3, \vartheta'_3, \varphi'_3$ are the polar co-ordinates of the momentum of final particle one in the c.m. frame of the (1,2) subsystem, the others being given by cyclic interchange of indices. $\bar{w}'_\delta$ is given by (8.3), and Y is the ordinary spherical harmonic. Also

$$\begin{aligned} \bar{W} &= \sqrt{M^2_{\alpha\gamma} + q^2_\alpha} + \sqrt{m^2_\alpha + q^2_\alpha} = \\ &= \sqrt{m_1^2 + q_1'^2} + \sqrt{m_2^2 + q_2'^2} + \sqrt{m_3^2 + q_3'^2} . \end{aligned} \tag{8.16}$$

Note that this three-particle S matrix has an isobar structure in which an unstable particle is produced by the scattering, which then decays. These S matrices can be proved to satisfy three-particle unitarity <u>exactly</u> (Sections 5b and 6). The disintegration S matrix (8.15) is the most interesting one, since it provides a model for processes like $N + \pi \rightarrow N + \pi + \pi$, if we consider the initial nucleon as a πN bound state.

These scattering equations for composite particles contain no arbitrary constants, apart from those already present in the two-particle subsystems. The form in which we have stated them is without three-body forces, however we expect the most important three-body forces to be separable, and these can be included. (8.9) and (8.15) then acquire additional terms, given in Section 6b.

The modifications for three identical particles have already been adequately discussed in the non-relativistic paper [1]. Briefly, they are that the subscripts α, β can be omitted from $\bar{Z}^J$ and $\bar{X}^J$, the sum over γ in (8.7) omitted, and $\bar{Z}^J$ multiplied throughout by two to compensate. The sum over δ in (8.15) remains, however, and the right-hand side must be multiplied by 1/3.

Isospin can easily be included, using 6j symbols, as in the non-relativistic paper [1]. The extension to unitary spin has been given by Murtaza [49].

8b DISCUSSION

If two-particle scattering can be approximated by the formation and disintegration of composite particles, then three-particle scattering ought to reduce to the interaction of a composite particle with an elementary one. This in turn ought to be governed by equations not unlike those for the interaction of two elementary particles. The purpose of this paper is to give this simple idea a concrete and verifiable form. Many previous authors have sought to formulate such isobar models, but all came to grief on the problems of "overlapping" resonances, and three-particle unitarity. These difficulties we have completely understood and solved. The overlapping of resonances, the Peierls mechanism, the fact that the one-particle exchange graph becomes complex in the physical region for unstable particles, the effects of triangle graphs, all these are included in our theory. Three-particle unitarity is also exactly satisfied, despite the isobar nature of the equations. Furthermore, this three-particle unitarity is not something tacked on afterwards, as in the unitarized peripheral model, but flows out of the theory in a particularly beautiful and physical way.

Our theory is an off-shell one - it flouts the canons which have ruled theoretical physics for the last seven years. As we have pointed out, it is S matrix theory that is unphysical when applied to unstable particles, and not ours, because an unstable particle is observed experimentally as a continuous distribution of masses, and not as a pole on an unphysical sheet.

This distinction is more than theological - it is the basic reason why S matrix isobar models have been unable to satisfy three-particle unitarity.

We share with S matrix theorists however the vision of making all particles composite, and we believe that it is in our theory that this ideal finds its true expression. We sketched in the non-relativistic paper [1] a resulting plan for calculating everything from vertex functions, N particle unitarity being included at the same time as (N-2) particle exchange. The present paper climbs the first hill we saw - the three-particle theory is made relativistic. A number of people have recently looked at the four-particle non-relativistic theory [50], and in particular Mishima and Takahashi have verified that the composite-particle scattering equations then include two-particle exchange automatically.

The relativistic equations for three-meson systems are given in Section 8a. The previous hundred or so pages contain an outline of the proof. It is an outline firstly because we left out all derivations which took us less than a week, and secondly because we do not know how to give it the complete mathematical rigour that is possible non-relativistically. The reader may be struck by a certain disproportion. It is indeed easy to speculate that such equations should exist, and even to give them a vague symbolic or graphical form. What is not easy, however, is to determine their precise analytical expression [51]. The only way to decide this, in which we feel any confidence, is to derive the equations from field theory. In doing so we met with a whole series of obstacles, ranging from off-shell relativistic kinematics to the many-time nature of the Bethe-Salpeter equation. Our aim in the present paper has been to flatten them so thoroughly that they need never bother anyone again. (We might also mention that Alessandrini and Omnès recently tried to do the job in half the space [9], and in our opinion got it wrong.)

The equations for three-meson systems given in Section 8a are complete and ready for the computer. We intend to apply them to such problems as the dynamics of the ω , A_1, A_2, and their strange-particle analogues, to τ decay and its relation to the $\pi\pi$ interaction, to the possibility that the pion itself may be a three-pion bound state [52], and to the dynamical origin of SU(6). Two questions have been shunted into separate papers, mainly because of notational complexity. These are the extension of the equations to elementary particles of any spin, and the formulae for the density of the Dalitz plot and other observables.

We hope that a large area of elementary particle physics, rich in experimental material, and hitherto inhabited only by crude and barbaric models, will be opened up by this work to precise dynamical calculation. Only then will these experiments be able to play their full rôle in elucidating the fundamental interactions.

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47) M. Jacob and G.C. Wick, Ann. Phys. 7, 404 (1959). In the case when the elementary particles have spin, the three-particle nature of the problem becomes more evident.

48) Our kinematic factors differ from Wick's because of the normalization of the two-particle states. His normalization unfortunately introduces a spurious kinematic singularity which eventually cancels. For this reason, care must be taken in applying the formulae of Ref. [8] directly.

49) G. Murtaza, "SU_3 and Three-Particle States", Imperial College (London) preprint (1965).

50) L.D. Faddeev, private communication;

J. Weyers, "Diffusion dans des Systèmes de Quatre Corps", Louvain preprint (1965);

N. Mishima and Y. Takahashi, "Composite-Composite Particles Scattering in a Four-Particle System", Tokyo preprint (1965);

L. Rosenberg, "Generalized Faddeev Integral Equations for Multi-Particle Scattering Amplitudes", New York preprint (1965).

51) There have been attempts to circumvent field theory by formulating one-time relativistic potential scattering directly. However, the potential then becomes completely uncertain. See

F. Coester, Helv. Phys. Acta 38, 7 (1965), and earlier references given there, and also

Yu. A. Gol'fand, Doklady Akad. Nauk S.S.S.R. 138, 331 (1961);

H. Enatsu, Progr. Theoret. Phys. 30, 236 (1963);

L.B. Redei, J. Math. and Phys. 6, 487 (1965).

52) A. Ahmadzadeh and J.A. Tjon, "A New Reduction of the Faddeev Equations and its Application to the Pion as a Three-Particle Bound State", San Diego preprint (1965). Encouraging calculations of the ω mass have also been made by

A.N. Mitra, Phys. Rev. 127, 1342 (1962), and

A. Ahmadzadeh, Berkeley preprint UCRL-11749 (1964); and of τ decay by

A.N. Mitra, Nuclear Phys. 6, 404 (1958); 18, 502 (1960);

A.N. Mitra and S. Ray, Ann. Phys. 21, 439 (1963); and

R. Prasad, Nuovo Cimento 35, 682 (1965).

However, all these calculations are rather crude, and cannot be taken as definitive confirmation of the theory. In particular, the Lorentz transformation between the two-particle and three-particle c.m. frame seems to have been usually ignored.

FIGURE CAPTIONS

Figure 1 :
a) A typical two-particle-reducible graph;
b) a typical two-particle-irreducible one;
c) the two-particle Bethe-Salpeter equation, which describes how the former are obtained by iterating the latter.

Figure 2 : The three-particle analogue of the Bethe-Salpeter equation. The sum is over the three pairings of final particles.

Figure 3 :
a) How to get right-hand singularities in the total energy;
b) how to get normal thresholds in external mass. The cut lines are to be put on the mass shell. Alessandrini and Omnès [9] seem to have assumed that the former decomposition will give the latter singularity.

Figure 4 : Right-hand cut contributions with a) two-particle, b) three-particle, c) four-particle intermediate states, all included in the ladder approximation. The Blankenbecler-Sugar reduction eliminates graphs like c) in which the particles go in different directions in time, and b) which is not a right-hand singularity for stable particles on the mass shell. b) will be included in our three-particle equations, however.

Figure 5 :
a) The composite-particle-scattering equation (5.8) in graphical form;
b) structure of the composite-particle propagator (3.18).

Figure 6 : Typical terms in the perturbation expansion of the composite-particle-scattering equations. The dotted lines show the various ways they have to be cut to give three-particle unitarity.

Figure 7 : Illustrating the proof of three-particle unitarity. Summing Fig. 6 leads to three graphs like a) coming from the discontinuity of the propagators, and six like b) coming from the discontinuity of the potential. Substituting the two-particle discontinuity of the composite particle propagator, as given by c) /both right-hand side external lines have propagators/ gives d) which is the square of the three-particle disintegration amplitude e) according to the isobar equation (5.12).

Figure 8 : Important three-particle-irreducible graphs, for a) the 3π system, b) the $N\pi\pi$ system. They must be included unless we can correctly predict the pion and nucleon, respectively, as dynamical bound states.

Figure 9 : When separable three-body forces are included, the potential for composite-particle-scattering acquires terms like b), besides the usual recoupling graph a). The isobar equation describing the three-particle disintegration becomes c), while d) gives the effective propagator of the three-particle bound state occurring in these equations.

Figure 10 : When a composite particle is itself composed of a composite and an elementary particle, then its form factor is a), and its propagator is an iteration of the two types of bubbles, b) and c).

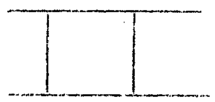


Fig. 1a

Fig. 1b

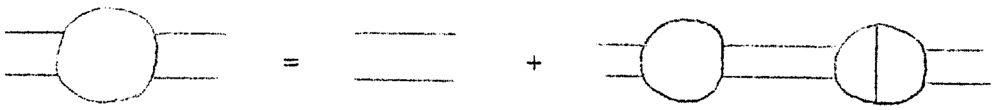


Fig. 1c

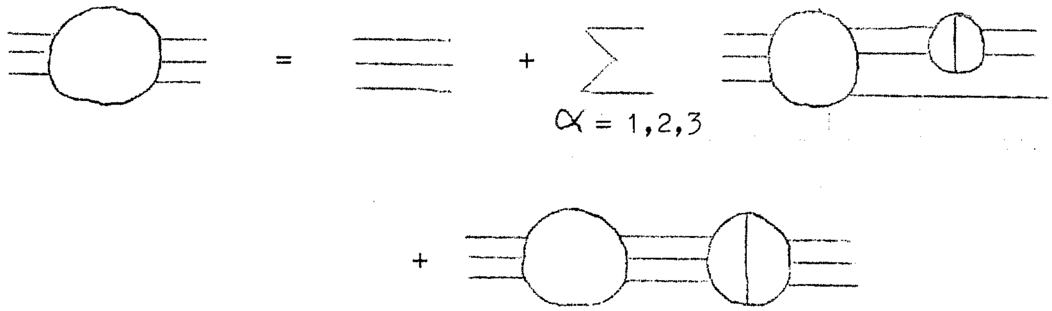


Fig. 2

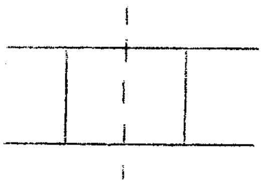


Fig. 3a

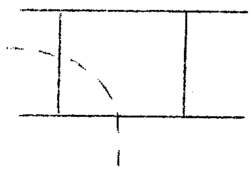


Fig. 3b

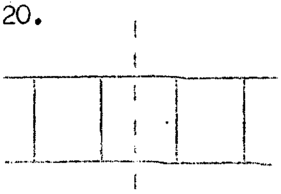
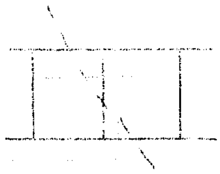


Fig. 4a



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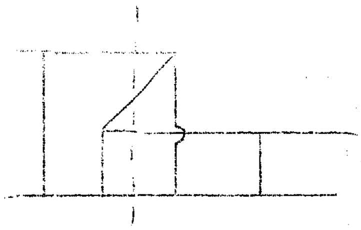

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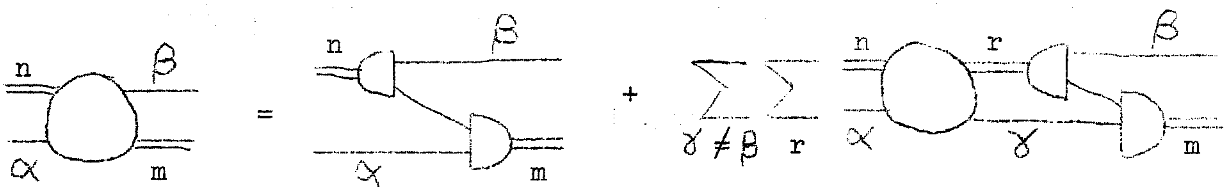

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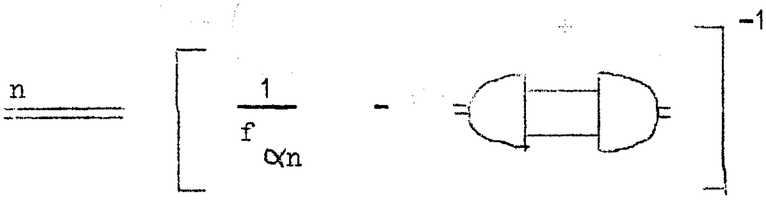


Fig. 5b

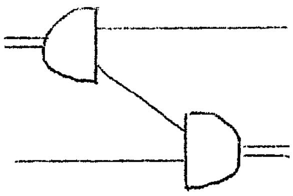


Fig. 6a

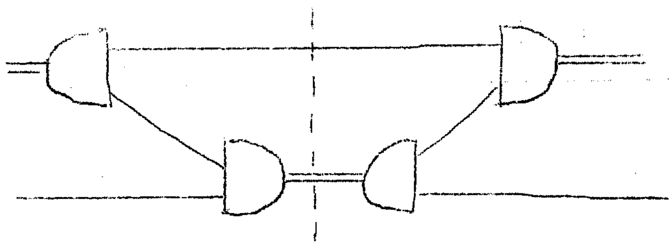


Fig. 6b

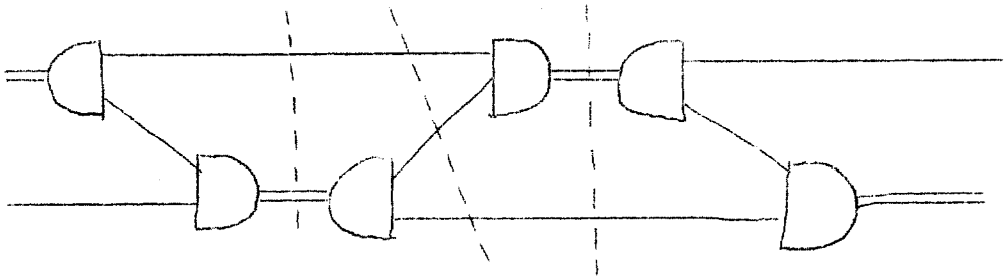


Fig. 6c

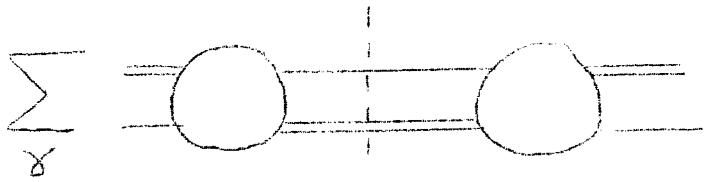


Fig. 7a

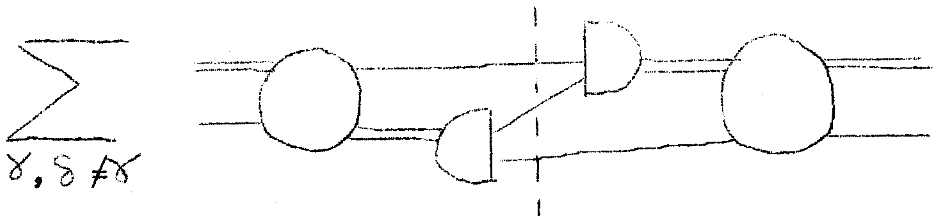
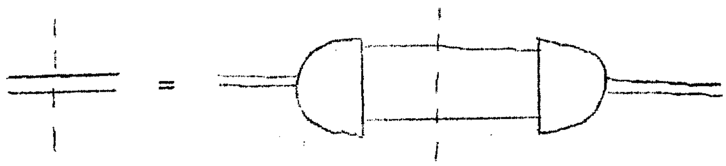


Fig. 7b



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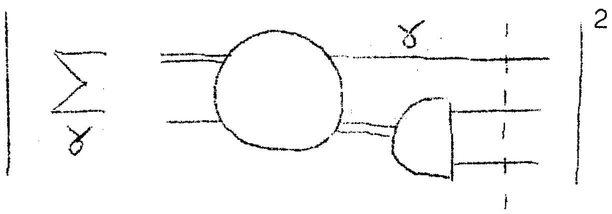


Fig. 7d

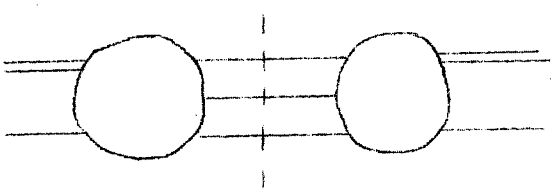


Fig. 7e

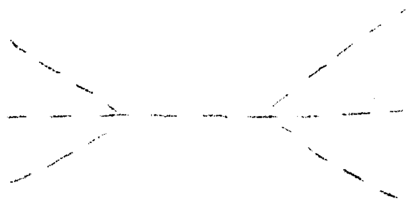
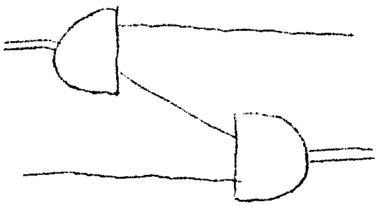


Fig. 8a

Fig. 8b



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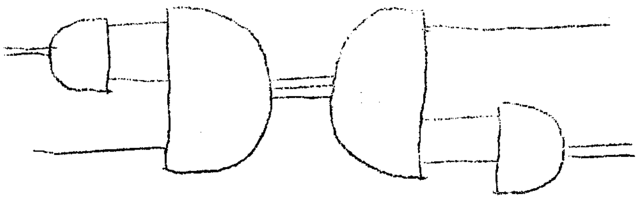
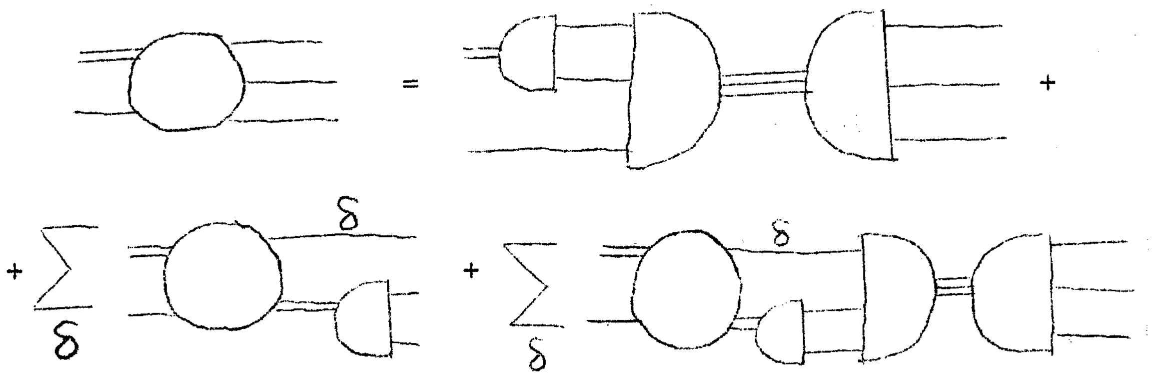


Fig. 9b



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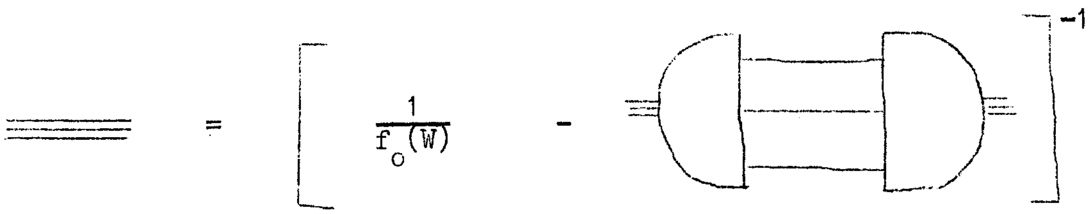

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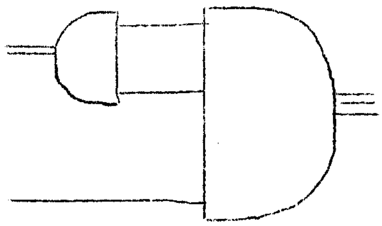

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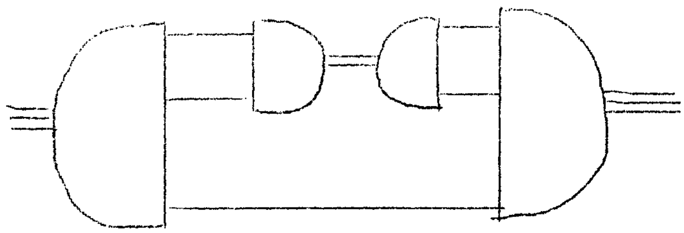

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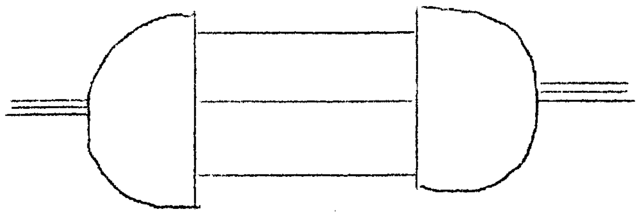

Fig. 10c