

PRECESSION OF THE POLARIZATION OF PARTICLES MOVING IN A HOMOGENEOUS  
ELECTROMAGNETIC FIELD\*

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The problem of the precession of the "spin" of a particle moving in a homogeneous electromagnetic field—a problem which has recently acquired considerable experimental interest—has already been investigated for spin  $\frac{1}{2}$  particles in some particular cases.<sup>1</sup> In the literature the results were derived by explicit use of the Dirac equation, with the occasional inclusion of a Pauli term to account for an anomalous magnetic moment. On the other hand, following a remark of Bloch<sup>2</sup> in connection with the nonrelativistic case, the expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion. To solve the problem for arbitrary spin in the relativistic case, it will thus suffice to produce a consistent set of covariant classical equations of motion. Such equations have been indicated a long time ago by Frenkel<sup>3</sup> and are discussed by Kramers.<sup>4</sup> These authors use an antisymmetric tensor  $M$  as the relativistic generalization of the intrinsic angular momentum observed in the rest-frame of the particle. A formulation in terms of the (axial) four-vector  $s$  which describes the polarization in a covariant fashion<sup>5</sup>—though basically equivalent—is however much more convenient for our problem. We shall therefore derive first the equations of motion directly in terms of this four-vector  $s$ .

Let the spin of the particle be represented<sup>6</sup> in the rest-frame ( $R$ ) by  $\vec{s}$ . We assume (a) that there exists a four-vector  $s$  such that in ( $R$ ) it coincides with  $\vec{s}$ :

$$s = (s^0, \vec{s}); \quad \text{in } (R), \quad s = (0, \vec{s}). \quad (1)$$

Denoting the four-velocity of the particle by  $u = (u^0, \vec{u}) = \gamma(1, \vec{v})$  [where  $\vec{v}$  is the ordinary velocity, and  $\gamma(v) = (1 - v^2)^{-1/2}$ ], one has in every frame

$$s \cdot u = 0, \quad \text{i.e.,} \quad s^0 = \vec{s} \cdot \vec{v}. \quad (2)$$

We further assume (b) that  $\vec{s}$  obeys in ( $R$ ) the

customary equation of motion

$$d\vec{s}/d\tau = (ge/2m)(\vec{s} \times \vec{H}), \quad (R) \quad (3)$$

where  $\vec{H}$ ,  $e$ , and  $m$  have their standard meanings, while the gyromagnetic ratio  $g$  is defined by this very equation. While  $s^0$  vanishes by hypothesis in any instantaneous rest-frame,  $ds^0/d\tau$  need not. In fact, (2) implies

$$ds^0/d\tau = \vec{s} \cdot (d\vec{v}/d\tau), \quad (R) \quad (4)$$

for such frames. In general,  $du/d\tau = f/m$  (where  $f$  = four-force), while in a homogenous external electromagnetic field specified by  $F = -(\vec{E}, \vec{H})$

$$du/d\tau = (e/m)F \cdot u. \quad (5)$$

The immediate generalization of Eqs. (3) and (4) to arbitrary frames is

$$ds/d\tau = (ge/2m)[F \cdot s + (s \cdot F \cdot u)u] - [(du/d\tau) \cdot s]u, \quad (6)$$

as can be checked by reducing to the rest-frame. With (5), one has for homogeneous fields

$$ds/d\tau = (e/m)[(g/2)F \cdot s + (g/2 - 1)(s \cdot F \cdot u)u]. \quad (7)$$

(5) and (7) constitute, for any value of  $g$  and arbitrary spin  $s$ , a consistent set of equations of motion; they imply that  $s \cdot s$  and  $s \cdot u$  are constant, so that condition (2) is maintained.<sup>7</sup> For experiments of current interest, the main use of (7) is in the computation of the rate  $\Omega$  at which longitudinal polarization is transformed into a transverse one (and vice versa). For this, we express  $s$  in the laboratory frame ( $L$ ) in terms of two unit polarization four-vectors,  $e_l$  and  $e_t$ :

$$s/s = e_l \cos \phi + e_t \sin \phi,$$

where

$$s = (-s \cdot s)^{1/2},$$

$$e_l = \gamma(v, \vec{v}/v) \equiv \gamma(v, \hat{v}), \quad e_t = (0, \hat{n}),$$

$$\hat{n} \cdot \hat{n} = 1, \quad \hat{n} \cdot \hat{v} = 0. \quad (L) \quad (8)$$

Clearly,  $\Omega = d\phi/dt = d\phi/\gamma d\tau$ . Introducing (8) into (7), and expressing all quantities as ordinary vectors, we find

$$\Omega = (e/m)\{(\vec{E}\cdot\hat{n}/v)[(g/2 - 1) - g/2\gamma^2] + (\hat{v}\cdot\vec{H}\times\hat{n})(g/2 - 1)\}. \quad (9)$$

The relevant "anomaly" of spin- $\frac{1}{2}$  particles,  $(g/2 - 1)$ , is clearly exhibited in (9) although our derivation was classical throughout.

We now specialize (9) to some cases of practical interest (the references are to experiments):

(A)  $\vec{E}\times\vec{v} = \vec{H}\times\vec{v} = 0$ ;  $\Omega = 0$ . The character of the polarization does not change, but the transverse polarization precesses around  $\vec{v}$  in longitudinal fields with an angular frequency  $\omega = (ge/2m\gamma)H = (g/2)\omega_L$ , as follows readily from (8).

(B)<sup>8</sup>  $\vec{H}\cdot\hat{n}\times\hat{v} = H$ ,  $\vec{E} = 0$ ;  $\Omega = \omega_L(g/2 - 1)\gamma$ , where  $\omega_L$  is the Larmor frequency defined in (A) above.

(C)<sup>9</sup>  $\vec{E}\cdot\hat{n} = E$ ,  $\vec{H} = 0$ ;  $\Omega = \omega_p[-g/2\gamma + (g/2 - 1)\gamma]$ , where  $\omega_p = eE/m\gamma v$  is the angular frequency of the particle's motion in the laboratory.

(D)<sup>10</sup>  $\vec{E}\cdot\vec{H} = 0$ , rectilinear motion:  $\vec{E} = -\vec{v}\times\vec{H}$ ;  $\vec{H}\cdot\hat{n}\times\hat{v} = H$ ,  $\Omega = \omega_L(g/2\gamma)$ .

(E)<sup>11</sup>  $\vec{E}\cdot\vec{H} = 0$ ,  $\vec{H}\cdot\hat{n}\times\hat{v} = H$ ,  $\vec{E}\cdot\hat{n} = -Ev_x/v$ , trochoidal motion:  $E/H < 1$ ;  $\Omega = (e/m)[(g/2 - 1)(H - Ev_x) + Ev_x/(\gamma^2 - 1)]$ ,

$$(\Delta\phi/2\pi) \text{ per loop} = \gamma(E/H)\gamma(v)[1 - (E/H)v_x](g/2 - 1) = \gamma(v')(g/2 - 1),$$

where  $v'$  is velocity in a frame where  $E' = 0$ .

The generalization of (6) to cover particles having an intrinsic electric dipole moment  $\vec{e} = (g'e/2m)\vec{s}$  may be of interest. In the ( $R$ ) frame, the effect of  $\vec{e}$  is taken into account by adding  $\vec{e}\times\vec{E}$  to the right-hand side of (3), while leaving (4) unchanged. Thus the required change in the right-hand side of (6) is the addition of a term  $-(g'e/2m)[(F^*\cdot s) + (s\cdot F^*\cdot u)u]$ , denoting by  $F^*$  the dual of  $F$ , i.e.,  $F^* = (-\vec{H}, -\vec{E})$ . For the experiment (B) above, one obtains then  $|\Omega| = \omega_L\gamma \times [(g/2 - 1)^2 + (g'v/2)^2]^{1/2}$ .

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<sup>1</sup>H. A. Tolhoek and S. R. de Groot, *Physica* **17**, 17 (1951); H. Mendlowitz and K. M. Case, *Phys. Rev.* **97**, 33 (1955); L. M. Carrassi, *Nuovo cimento* **7**, 524 (1958).

<sup>2</sup>F. Bloch, *Phys. Rev.* **70**, 460 (1946).

<sup>3</sup>J. Frenkel, *Z. Physik* **37**, 243 (1926).

<sup>4</sup>H. A. Kramers, *Quantum Mechanics* (North Holland Publishing Company, Amsterdam, 1957), p. 226 *et seq.* Kramers' Eq. (4) (p. 229) does not correspond to our Eq. (6). His conclusion that already classically  $g=2$  is implied for an electron, based on the relativistic equation he uses, stems from the fact that that equation corresponds in the rest-frame to  $ds^0/d\tau = (ge/2m)\vec{s}\cdot\vec{E}$ . Comparing with our (4), with  $d\vec{v}/d\tau = (e/m)\vec{E}$ , one sees that the "derived" result is, in fact, built into the theory from the start. A more general equation is mentioned by Kramers on p. 231, in fine print, and attributed to Frenkel. The inconsistencies arising in Kramers' discussion of what he calls "spin-orbit" forces [i.e., of the form  $(\nabla\vec{H})\cdot\vec{u}$  in the rest-frame] are connected with the fact that neither of his equations of motion applies when the field is inhomogeneous. In that case,  $du/d\tau$  is not given by (5) alone, but has to include an additional term (the covariant analog of the gradient force just mentioned) before being introduced into (6).

<sup>5</sup>The covariant polarization four-vector  $s$  is essentially the expectation value of the operator  $w$  used by Bargmann and Wigner [Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948)] to characterize representations of the inhomogeneous Lorentz group:  $\langle w\cdot w \rangle = -s(s+1)m^2$ ,  $s = \text{spin}$ .  $s$  can be expressed in terms of the skew tensor  $M$  of Frenkel (which satisfies  $M\cdot u = 0$ ), and vice versa:  $s = M^* \cdot u$ ,  $M^* = s \times u$ , i.e.,  $M^{*ik} = s^i u^k - s^k u^i$ . For the quantum-mechanical applications of  $s$  see, e.g., C. Bouchiat and L. Michel, *Phys. Rev.* **106**, 170 (1955).

<sup>6</sup>Our notation is:  $c=1$ ,  $\hbar=1$  throughout; coordinate four-vector of components  $x^0=t, x^1, x^2, x^3$ :  $x = (x^0, \vec{x})$ ,  $\vec{x} = \{x^\alpha\}$  ( $\alpha=1, 2, 3$ ); metric of signature (+ ---);  $\tau = \text{proper time}$ ; a dot between symbols, contraction of neighboring indices with the metric tensor, e.g.,  $x\cdot x = (x^0)^2 - \vec{x}^2$ ; skew tensor of components  $T^{ik}$  indicated as  $T = (\vec{T}^t, \vec{T}^s)$ ,  $\vec{T}^t = \{T^{0\alpha}\}$ ,  $\vec{T}^s = \{T^{\beta\gamma}\}$ ;  $\alpha, \beta, \gamma=1, 2, 3$ ; its dual by  $T^* = (\vec{T}_s, -\vec{T}_t)$ .

<sup>7</sup>Equations (5) and (7) can be integrated explicitly by reference to four orthonormal four-vectors  $e^{(i)}$  such that each of them obeys (5), and  $e^{(0)} = u$ .

<sup>8</sup>Crane, Pidd, and Louisell, *Bull. Am. Phys. Soc. Ser. II*, **3**, 369 (1958).

<sup>9</sup>H. Frauenfelder *et al.*, *Phys. Rev.* **106**, 386 (1957).

<sup>10</sup>P. E. Cavanagh *et al.*, *Phil. Mag.* **2**, 1105 (1957).

<sup>11</sup>P. S. Farago, *Proc. Phys. Soc. (London)* **72**, 891 (1958).