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PRECISE VARIATIONAL FORMULAS FOR ABELIAN DIFFERENTIALS

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In the present paper, we shall study two basic types of degenerations of compact Riemann surfaces considered by Schiffer-Spencer [10] and Fay [3]. According to the simple formalism of the degeneration considered here, the precice variational formulas without error terms will be obtained for $\omega(x, y)$ the fundamental normalized Abelian differentials of the second kind (Theorems 4 and 6), from which one may deduce similar formulas for any Abelian differentials and period matrices in the usual way. It turns out, however, that all the variational formulas found in the book by Fay [3] disagree with ours and it seems to us that they are incorrect, which is, to some extent, seen from the examples in the last section of this paper. In our formulas the coefficients β_{jk} of an expansion of $\omega(x, y)$ plays an important role. In this connection a variant of Golusin's inequality will be obtained for β_{jk}'s (Theorem 5) which can be viewed as the generalized Faber coefficients. Our method is completely elementary (c.f. Fay [3]) and yields some extension of the results in [3] and [6].

1. Pinching along a cycle homologous to zero and preliminary estimates.

On any Riemann surface, it is well-known that the following orthogonal decomposition holds [1]:

$$\Gamma=\Gamma_h \oplus \Gamma_{eo} \oplus \Gamma_{eo}^* \tag{1}$$

where Γ is the Hilbert space of square integrable differential forms, Γ_h its subspace of harmonic differentials, Γ_{eo} the closure of the subspace of smooth differentials with compact supports, Γ_{eo}^* the *-conjugate of Γ_{eo}. The above decomposition easily gives a lemma concerning the "distance" between the functions each defined on one of the boundary components of an annulus.

LEMMA. 1. *Let D be an annulus $r<|z|<R$ and assume that $\phi(z)$ (resp. $\psi(z)$) is holomorphic on $|z|=r$ (resp. $|z|=R$), where they have the Laurent expansions*

$$\phi(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \qquad \psi(z)=\sum_{n=-\infty}^{\infty} b_n z^n .$$

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Then the Diriclet norm $\|df-i^*df\|_D$ *attains its minimum among the (non-void) family*

$$\mathfrak{F}=\{f\in C(\bar{D})\cap C^1(D)\,;\ \|df\|_D<\infty,\, f=\phi \text{ on } |z|=r,\, f=\psi \text{ on } |z|=R\}$$

if and only if f *is harmonic. Moreover, the minimum is given by:*

$$\min_{f\in\mathfrak{F}}\frac{1}{2\pi}\|df\text{-}i^*df\|^2=\sum_{n=-\infty}^{\infty}{}'\frac{n|b_{-n}-a_{-n}|^2}{R^{2n}-r^{2n}}+\frac{|b_0-a_0|^2}{2\ln R/r} \tag{2}$$

(The notation $\sum'$ *indicates that in the summation* $n\neq 0$.)

Proof. First note that (c. f. Weyl [11] p. 105)

$$du-dv\in\Gamma_{eo}\qquad\text{for any } u,\, v\in\mathfrak{F}\,. \tag{3}$$

The sketch of the proof of (3) goes as follows:

Choose a $\xi\in C^\infty(\boldsymbol{R})$ such that

$$\xi(x)=\begin{cases}1, & x\geqq 2\\ 0 & x\leqq 1\end{cases}\qquad 0\leqq\xi(x)\leqq 1$$

and set up the following function for $\varepsilon>0$:

$$\xi_\varepsilon(z)=\xi\left(\frac{(R-|z|)(|z|-r)}{\varepsilon}\right)\in C_0^\infty(D)\,.$$

In order to conclude that

$$\Gamma_{eo}\ni d[\xi_\varepsilon\cdot(u-v)]\to d(u-v)\qquad(\varepsilon\to 0)\,,$$

it is only necessary to use the inequality

$$\int_0^{2\pi}|w(\rho e^{i\theta})|^2d\theta\leqq\begin{cases}\ln\rho/r\cdot\|dw\|^2_{r<|z|<\rho}\\ \ln R/\rho\cdot\|dw\|^2_{\rho<|z|<R}\end{cases}\qquad(r<\rho<R)$$

with $w=u-v$ $(u,\, v\in\mathfrak{F})$, evaluating the norm of $wd\xi_\varepsilon$.

In view of (3) and the decomposition (1), the first assertion stated in Lemma 1 holds at once. It remains to compute the minimum. An easy calculation shows that the extremal harmonic function $h(z)\in\mathfrak{F}$ is given explicitly by:

$$h(z)=\sum_{n=-\infty}^{\infty}{}'\frac{R^{2n}b_n-r^{2n}a_n}{R^{2n}-r^{2n}}z^n+\sum_{n=-\infty}^{\infty}{}'\frac{b_{-n}-a_{-n}}{R^{2n}-r^{2n}}\bar{z}^n$$

$$+\frac{b_0-a_0}{\ln R/r}\ln|z|+\frac{a_0\ln R-b_0\ln r}{\ln R/r}\,.$$

By the identity $dh-i^*dh=2h_{\bar{z}}\,d\bar{z}$, (2) is immediately obtained and the proof is completed.

Let S_1 and S_2 be two compact Riemann surfaces of genus g_1, g_2 each with a point p_1, p_2 fixed and let z_1: $U_1\to\Delta=\{z\in\boldsymbol{C}\,;\ |z|<1\}$ and z_2: $U_2\to\Delta$ be coordinates in neighborhoods U_1, U_2 of these points with $z_j(p_j)=0$ $(j=1, 2)$. Set

(4) $$\rho U_j=\{p\in U_j;\ |z_j(p)|<\rho\}, \qquad \rho C_j=\{p\in U_j;\ |z_j(p)|=\rho\} \quad (j=1, 2)$$

with $0<\rho<1$. A family of compact Riemann surfaces $\{S_\varepsilon;\ \varepsilon\in \boldsymbol{C}, 0<|\varepsilon|<1\}$ formed from S_1 and S_2 is constructed by defining

$$S_\varepsilon=(S_1\backslash|\varepsilon|U_1)\cup(S_2\backslash|\varepsilon|U_2)$$

where $x\in U_1\backslash|\varepsilon|U_1$ is identified with $y\in U_2\backslash|\varepsilon|U_2$ by the equation

(5) $$z_1(x)z_2(x)=\varepsilon\,.$$

The coordinates z_1 and z_2 are called the *pinching coordinates* for S_1 and S_2 at p_1 and p_2 respectively. Clearly, S_ε is a compact Riemann surface of genus $g=g_1+g_2$. Both the pinching coordinates map conformally the "pinched region" $S_\varepsilon\backslash((S_1\backslash U_1)\cup(S_2\backslash U_2))$, denoted by P_ε, onto the annulus $|\varepsilon|<|z|<1$, so that S_ε may be regarded as the union of $S_1\backslash U_1$, $S_2\backslash U_2$ and $|\varepsilon|<|z|<1$ under appropriate identification.

From Lemma 1, we obtain the following theorem which is the basis for the derivation of the variational formulas in this paper.

THEOREM 1. *Let Ω_j be a meromorphic differential on S_j which is holomorphic on U_j except for a possible simple pole at p_j with residue $(-1)^j\alpha$ $(j=1, 2)$. Let $\phi_j(x)=\int_{p_j}^x\left(\Omega_j-\frac{(-1)^j\alpha}{z_j}dz_j\right)$ in U_j and have a Taylor expansion in terms of the coordinates z_j given by*

$$\phi_j(z)=\sum_{n=1}^{\infty}\alpha_n^{(j)}z^n\,, \qquad |z|<1 \quad (j=1, 2)\,,$$

Then there exists a meromorphic differential Ω_ε on S_ε which is holomorphic on P_ε with the same singularities as Ω_j on $S_j\backslash U_j$ $(j=1, 2)$, satisfying, for any $\rho\in(|\varepsilon|^{1/2}, 1)$,

(6) $$\sum_{j=1}^{2}\|\Omega_\varepsilon-\Omega_j\|^2_{S_j\backslash\rho U_j}\leqq\pi\sum_{n=1}^{\infty}n(|\alpha_n^{(1)}|^2+|\alpha_n^{(2)}|^2)\cdot\frac{|\rho\varepsilon|^{2n}}{\rho^{4n}-|\varepsilon|^{2n}}\,.$$

Proof. Let h_ε be the harmonic function on an annulus $|\varepsilon|/\rho\leqq|z|\leqq\rho$ such that

$$h_\varepsilon(z)=\begin{cases}\phi_1(z)=\sum_{n=1}^{\infty}\alpha_n^{(1)}z^n & \text{on} \quad |z|=\rho\,,\\[2ex] \phi_2(\varepsilon/z)=\sum_{n=1}^{\infty}\alpha_n^{(2)}\varepsilon^n z^{-n} & \text{on} \quad |z|=|\varepsilon|/\rho\,.\end{cases}$$

Then, by Lemma 1, it is seen that

$$(7)\qquad \frac{1}{2\pi}\|dh_\varepsilon - i^*dh_\varepsilon\|^2_{|\varepsilon|/\rho<|z|<\rho} = \sum_{n=1}^{\infty}\frac{n|\alpha_n^{(2)}\varepsilon^n|^2}{\rho^{2n}-|\varepsilon|^{2n}/\rho^{2n}} + \sum_{n=-1}^{-\infty}\frac{n|\alpha_{-n}^{(1)}|^2}{\rho^{2n}-|\varepsilon|^{2n}/\rho^{2n}}$$
$$= \sum_{n=1}^{\infty} n(|\alpha_n^{(1)}|^2+|\alpha_n^{(2)}|^2)\frac{|\rho\varepsilon|^{2n}}{\rho^{4n}-|\varepsilon|^{2n}}\,.$$

By passing to the usual smoothing process, it is easy to find an $h_\varepsilon^{(n)}$ ($n=1, 2, \cdots$) with the same boundary value as h_ε satisfying

(i) $\|dh_\varepsilon^{(n)} - i^*dh_\varepsilon^{(n)}\|^2_{|\varepsilon|/\rho<|z|<\rho} < \|dh_\varepsilon - i^*dh_\varepsilon\|^2_{|\varepsilon|/\rho<|z|<\rho} + \dfrac{1}{n}$,

(ii) If we define $\Phi_\varepsilon^{(n)}$ on S_ε by

$$(*)\qquad \Phi_\varepsilon^{(n)}(z)=\begin{cases}\Omega_1(z) & , \quad z\in S_1\backslash\rho U_1\\[2mm] d(h_\varepsilon^{(n)}(z))-\dfrac{dz}{z}, & \quad z\in\{z\,;\ |\varepsilon|/\rho\leqq|z|\leqq\rho\}\\[2mm] \Omega_2(z) & , \quad z\in S_2\backslash\rho U_2\,,\end{cases}$$

then $\Phi_\varepsilon^{(n)}\in\Gamma_c^1(S_\varepsilon)$, the space of closed C^1-differentials $\in\Gamma(S_\varepsilon)$.

Here the coordinate z_1 is used to identify the pinched region with $|\varepsilon|/\rho<|z|<\rho$. Note that (*) is well-defined, because of (5) and the restriction imposed on the residues of Ω_1 and Ω_2 at p_1 and p_2. Clearly,

$$(8)\qquad \Phi_\varepsilon^{(n)}-i^*\Phi_\varepsilon^{(n)}=\begin{cases}0 & , \quad z\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2)\\ dh_\varepsilon^{(n)}-i^*dh_\varepsilon^{(n)}, & \quad z\in\{|\varepsilon|/\rho<|z|<\rho\}\,.\end{cases}$$

From the decomposition (1), it follows that

$$\Phi_\varepsilon^{(n)}-i^*\Phi_\varepsilon^{(n)}=\omega_h^{(n)}+\omega_{eo}^{(n)}+\omega_{eo}^{*(n)}$$

where $\omega_h^{(n)}\in\Gamma_h$, $\omega_{eo}^{(n)}\in\Gamma_{eo}$, and $\omega_{eo}^{*(n)}\in\Gamma_{eo}^*$. Let $\tau_\varepsilon^{(n)}=\Phi_\varepsilon^{(n)}-\omega_{eo}^{(n)}$, then $\tau_\varepsilon^{(n)}$ is closed and co-closed. It is also square integrable off the poles of Ω_1 and Ω_2, so that it is harmonic there by Weyl's lemma. Noting the fact that any harmonic differential with isolated singularities is never square integrable, we see that $\tau_\varepsilon^{(n)}$ is harmonic on S_ε except for the same singularities as Ω_1 and Ω_2 off the pinched region. Now let us define $\Omega_\varepsilon^{(n)}$ by

$$\Omega_\varepsilon^{(n)}=\frac{1}{2}(\tau_\varepsilon^{(n)}+i^*\tau_\varepsilon^{(n)})$$
$$=\frac{1}{2}(\Phi_\varepsilon^{(n)}+i^*\Phi_\varepsilon^{(n)}-\omega_{eo}^{(n)}-i^*\omega_{eo}^{(n)})\,.$$

Then $\Omega_\varepsilon^{(n)}$ is meromorphic on S_ε with the same singularities as $\tau_\varepsilon^{(n)}$ and the following estimate holds:

$$\sum_{j=1}^{2}\|\Omega_\varepsilon^{(n)}-\Omega_j\|^2_{S_j\backslash\rho U_j}=\frac{1}{4}\sum_{j=1}^{2}\|\omega_{eo}^{(n)}+i^*\omega_{eo}^{(n)}\|^2_{S_j\backslash\rho U_j}$$

$$\leqq \frac{1}{4}\|\omega_{eo}^{(n)}+i^*\omega_{eo}^{(n)}\|_{S_\varepsilon}^2=\frac{1}{2}\|\omega_{eo}^{(n)}\|_{S_\varepsilon}^2$$

$$\leqq \frac{1}{2}\|\Phi_\varepsilon^{(n)}-i^*\Phi_\varepsilon^{(n)}\|_{S_\varepsilon}^2 .$$

Here we used the orthogonal decomposition (1). Thus, from (8),

$$\sum_{j=1}^{2}\|\Omega_\varepsilon^{(n)}-\Omega_j\|_{S_j\setminus\rho U_j}^2\leqq\frac{1}{2}\|dh_\varepsilon-i^*dh_\varepsilon\|_{|\varepsilon|/\rho<|z|<\rho}^2+\frac{1}{2n}$$

for $n=1, 2, \cdots$. By a normal family argument, a properly chosen subsequence of $\{\Omega_\varepsilon^{(n)}\}_{n=1}^\infty$ converges to a meromorphic differential Ω_ε on S_ε uniformly off the poles of Ω_1 and Ω_2. Letting $n\to\infty$, we conclude that

$$\sum_{j=1}^{2}\|\Omega_\varepsilon-\Omega_j\|_{S_j\setminus\rho U_j}^2\leqq\frac{1}{2}\|dh_\varepsilon-i^*dh_\varepsilon\|_{|\varepsilon|/\rho<|z|<\rho}^2 . \tag{9}$$

By combining (7) and (9), the proof is completed.

The above theorem is slightly stronger than what is needed for later applications. Indeed, it is sufficient to obtain the estimate

$$\sum_{j=1}^{2}\|\Omega_\varepsilon-\Omega_j\|_{S_j\setminus\rho U_j}<A\varepsilon \qquad (\varepsilon\to 0) \tag{10}$$

with some information about the bound for the constant A. If (10) is rewritten in the form

$$\sum_{j=1}^{2}\|\Omega_\varepsilon-\Omega_j\|_{S_j\setminus\rho U_j}=O(\varepsilon)\,, \tag{10$'$}$$

the constant A will be called an "implied constant" of the estimate (10)$'$. After obtaining variational formulas, we will see that the estimate $O(\varepsilon)$ in (10)$'$ cannot be replaced by $o(\varepsilon)$ in general.

2. Derivation of variational formulas.

Let us fix, once and for all, a canonical homology basis $(A^{(j)}, B^{(j)})$ for S_j where $A^{(1)}=(A_1, \cdots, A_{g_1})$, $B^{(1)}=(B_1, \cdots, B_{g_1})$, $A^{(2)}=(A_{g_1+1}, \cdots, A_g)$ and $B^{(2)}=(B_{g_1+1}, \cdots, B_g)$, and assume that every cycle in $(A^{(j)}, B^{(j)})$ is contained in $S_j\setminus U_j$ $(j=1, 2)$ without loss of generality. To choose some canonical homology basis for S_ε, let $A_1(\varepsilon), B_1(\varepsilon), \cdots, A_g(\varepsilon), B_g(\varepsilon)$ simply be a canonical basis $A_1, B_1, \cdots, A_g, B_g$ for S_1 and S_2. Let $v_{j,\varepsilon}$ $(j=1, 2, \cdots, g)$ be the normalized differential of the first kind on S_ε such that

$$\int_{A_k(\varepsilon)}v_{j,\varepsilon}=2\pi i\delta_{jk} \qquad (j,\ k=1, \cdots, g)$$

where δ_{jk} is the Kronecker δ. This normalization is used throughout the present paper.

Let O be a relatively compact region of a Riemann surface, and assume that u is a nowhere-vanishing holomorphic differential on the closure $\bar{O}$. Then a differential v defined in O is said to be *bounded* if so is the function v/u. This definition is clearly independent of the choice of u.

The uniform boundedness of $v_{j,\varepsilon}$ ($j=1, \cdots, g$) with respect to ε will now be considered, which is crucial for the later development.

LEMMA 2. *Let $z \in (S_1 \backslash \rho U_1) \cup (S_2 \backslash \rho U_2)$ with $0<\rho<1$. Then, for $j=1, \cdots, g$,*

$$v_{j,\varepsilon}(z)=O(1) \qquad (\varepsilon \to 0)$$

uniformly. (Here and hereafter estimates like $f_\varepsilon(z)=O((\varepsilon-\varepsilon_0)^n)$ $(\varepsilon \to \varepsilon_0)$ are said to be uniform if "implied constants" can be chosen independently of the variable z.)

Proof. Choose the pairs of differentials $\Omega_1^{(j)}$ and $\Omega_2^{(j)}$ on S_1 and S_2 respectively as follows:

$$(\Omega_1^{(j)}, \Omega_2^{(j)})=\begin{cases} (v_j, 0) & \text{if } 1 \leqq j \leqq g_1, \\ (0, v_j) & \text{if } g_1<j \leqq g, \end{cases}$$

where v_j for $j \leqq g_1$ (resp. $j>g_1$) are a normalized basis for the holomorphic differentials on S_1 (resp. S_2). Then, by applying Theorem 1, there exists a differential $\Omega_{j,\varepsilon}$ holomorphic on S_ε such that, for $\varepsilon \to 0$,

$$\|\Omega_{j,\varepsilon}-v_j\|_{S_1 \backslash \rho U_1}+\|\Omega_{j,\varepsilon}\|_{S_2 \backslash \rho U_2}=O(\varepsilon), \qquad 1 \leqq j \leqq g_1,$$

$$\|\Omega_{j,\varepsilon}\|_{S_1 \backslash \rho U_1}+\|\Omega_{j,\varepsilon}-v_j\|_{S_2 \backslash \rho U_2}=O(\varepsilon), \qquad g_1<j \leqq g.$$

Since this holds for any $\rho \in (0, 1)$, it follows immediately that

$$\Omega_{j,\varepsilon}(z)=O(1) \qquad (\varepsilon \to 0)$$

uniformly for $z \in (S_1 \backslash \rho U_1) \cup (S_2 \backslash \rho U_2)$. Let M_ε be the period matrix of $\Omega_{1,\varepsilon}, \cdots, \Omega_{g,\varepsilon}$ with respect to the cycles $A_1(\varepsilon), \cdots, A_g(\varepsilon)$. Then one verifies that

$$M_\varepsilon=\left(\frac{1}{2\pi i}\int_{A_j(\varepsilon)} \Omega_{k,\varepsilon}\right)_{j,k=1}^{g}=I_g+O(\varepsilon), \qquad (\varepsilon \to 0)$$

where I_g is the $g \times g$ identity matrix, since the period along a fixed cycle is a bounded linear functional on Γ_c, the space of closed square integrable differentials (c. f. Ahlfors-Sario [1], p. 284). Therefore the inverse matrix M_ε^{-1} exists for ε sufficiently small and is of the form

$$M_\varepsilon^{-1}=I_g+O(\varepsilon).$$

Consequently,

$$(v_{j,\varepsilon})_{j=1}^{g}=M_\varepsilon^{-1}(\Omega_{j,\varepsilon})_{j=1}^{g}=(\Omega_{j,\varepsilon})_{j=1}^{g}+O(\varepsilon)(\Omega_{j,\varepsilon})_{j=1}^{g}=O(1) \qquad (\varepsilon \to 0).$$

This completes the proof.

Let $\omega_1(x, y)$ (resp. $\omega_2(x, y)$, $\omega_\varepsilon(x, y)$) be the fundamental normalized differential of the second kind on S_1 (resp. S_2, S_ε), that is, the bilinear meromorphic differential with vanishing A_j-periods which is holomorphic everywhere except for a double pole along $x=y$, where, in terms of a coordinate, it has an expansion given by

$$\frac{dxdy}{(x-y)^2}+\text{regular terms.}$$

For $x\in S_1\backslash U_1$ (resp. $x\in S_2\backslash U_2$, $x\in S_\varepsilon\backslash P_\varepsilon$), let the following expansions, in terms of the pinching coordinates, hold in U_1 (resp. U_2, P_ε):

$$\begin{aligned} &\int_0^{z_j}\omega_j(\cdot, x)=\sum_{n=1}^{\infty} a_n^{(j)}(x)z_j^n, && |z_j|<1 \quad (j=1, 2) \\ &\int^{z_1}\omega_\varepsilon(\cdot, x)=\sum_{n=-\infty}^{\infty} a_{n,\varepsilon}(x)z_1^n, && |\varepsilon|<|z_1|<1. \end{aligned} \tag{11}$$

Here the constant term $a_{0,\varepsilon}(x)$ needs not to be determined. The coefficients $a_n^{(j)}(x)$ are easily seen to be extended so that these become normalized differentials of the second kind on S_j holomorphic everywhere except for a pole of order $n+1$ at p_j where, in terms of the pinching coordinates,

$$\begin{aligned} &a_1^{(j)}(z)=\omega_j(z, p_j) \\ &a_n^{(j)}(z_j)=dz_j/z_j^{n+1}+\text{regular terms} \qquad (j=1, 2;\ n=1, 2, \cdots). \end{aligned} \tag{12}$$

LEMMA 3. *The following uniform estimates hold with $0<\rho<1$:*

$$\omega_\varepsilon(x, y)=\begin{cases} \omega_j(x, y)+O(\varepsilon), & x, y\in S_j\backslash\rho U_j \\ O(\varepsilon) \quad, & x\in S_j\backslash\rho U_j,\ y\in S_{j'}\backslash\rho U_{j'} \end{cases} \qquad (j=1, 2)$$

Here and hereafter we use the convention that

$$j'=\begin{cases} 2, & j=1 \\ 1, & j=2. \end{cases}$$

Proof. Set $\Omega_1=\omega_1(\cdot, x)$, $\Omega_2=0$ and apply Theorem 1, assuming that $x\in S_1\backslash\rho U_1$ without loss of generality. Then there exists a differential $\Omega_\varepsilon(\cdot\,;x)$ meromorphic on S_ε satisfying, for positive $\rho'<\rho$,

$$\begin{aligned} &\|\Omega_\varepsilon(\cdot\,;x)-\omega_1(\cdot, x)\|^2_{S_1\backslash\rho' U_1}+\|\Omega_\varepsilon(\cdot, x)\|^2_{S_2\backslash\rho' U_2} \\ &\leqq\pi\sum_{n=1}^{\infty}\frac{n|a_n^{(1)}(x)|^2|\rho\cdot\varepsilon|^{2n}}{\rho'^{4n}-|\varepsilon|^{2n}}. \end{aligned}$$

By Cauchy's estimate, it follows that

$$|a_n^{(1)}(x)|\rho'^n\leqq K=\operatorname{Max}\left\{\left|\int_0^z\omega_1(\cdot, x)\right|;\ |z|<\rho',\ x\in S_1\backslash\rho U_1\right\}$$

Thus

$$\sum_{n=1}^{\infty} \frac{n|a_n^{(1)}(x)|^2|\rho'\varepsilon|^{2n}}{\rho'^{4n}-|\varepsilon|^{2n}} \leq K^2 \sum_{n=1}^{\infty} \frac{n|\varepsilon|^{2n}}{\rho'^{4n}-|\varepsilon|^{2n}} = O(\varepsilon^2)$$

uniformly in $x \in S_1 \backslash \rho U_1$. Analogous to Lemma 2, the following uniform estimates hold:

$$\Omega_\varepsilon(y\,;\,x) = \begin{cases} \omega_1(x, y) + O(\varepsilon) & x, y \in S_1 \backslash \rho U_1\,, \\ O(\varepsilon) & x \in S_1 \backslash \rho U_1,\ y \in S_2 \backslash \rho U_2\,, \end{cases}$$

$$\int_{A_j(\varepsilon)} \Omega_\varepsilon(y\,;\,x) = O(\varepsilon) \qquad (j=1, 2, \cdots g)\,.$$

In order to conclude the proof, it is sufficient to note that

$$\omega_\varepsilon(x, y) = \Omega_\varepsilon(y\,;\,x) - \sum_{j=1}^{g} \frac{1}{2\pi i}\Big(\int_{A_j(\varepsilon)} \Omega_\varepsilon(\cdot, x)\Big) v_{j,\varepsilon}(y)$$

and that $v_{j,\varepsilon}(y) = O(1)$ uniformly by Lemma 2.

For our later development, it will be useful to derive an identity which comes from the method of contour integration. For simplicity, let us define $\omega_0(x, y)$ by:

$$\omega_0(x, y) = \begin{cases} \omega_j(x, y)\,, & x, y \in S_j \\ 0\,, & x \in S_j,\ y \in S_{j'} \end{cases} \qquad (j=1, 2)\,. \tag{13}$$

LEMMA 4. *Let $\varepsilon, \varepsilon_0 \in \boldsymbol{C}$ and $\rho \in \boldsymbol{R}$ satisfy* Max $\{|\varepsilon|^{1/2}, |\varepsilon_0|^{1/2}\} < \rho < 1$. *Then the following identity holds: for $x, y \in (S_1 \backslash \rho U_1) \cup (S_2 \backslash \rho U_2)$*

$$\omega_\varepsilon(x, y) - \omega_{\varepsilon_0}(x, y) = \frac{1}{2\pi i} \int_{\rho C_1 + \rho C_2} \Big(\int^z \omega_{\varepsilon_0}(x, \cdot)\Big) \omega_\varepsilon(y, z)\,.$$

Proof. *Case* 1. $x, y \in S_j \backslash \rho U_j$ $(j=1, 2)$: Integration along the boundary of $S_j \backslash \rho U_j$ canonically dissected yields

$$\omega_\varepsilon(x, y) - \omega_{\varepsilon_0}(x, y) = \frac{1}{2\pi i} \int_{\rho C_j} \Big(\int^z (\omega_{\varepsilon_0}(x, \cdot) - \omega_\varepsilon(x, \cdot))\Big) \omega_\varepsilon(y, z)\,.$$

Here the Riemann bilinear relation and the residue theorem were used. The term

$$\int_{\rho C_j} \Big(\int^z \omega_\varepsilon(x, \cdot)\Big) \omega_\varepsilon(y, z)$$

vanishes because the integrand is holomorphic on $S_\varepsilon \backslash (S_j \backslash \rho U_j)$ where Cauchy's integral theorem can be applied. On the other hand, the same theorem again shows

$$\frac{1}{2\pi i} \int_{\rho C_{j'}} \Big(\int^z \omega_{\varepsilon_0}(x, \cdot)\Big) \omega_\varepsilon(y, z) = 0$$

since the integrand is holomorphic on $S_{j'}\backslash\rho U_{j'}$. This completes the proof of Case 1.

Case 2. $x\in S_j\backslash\rho U_j$ and $y\in S_{j'}\backslash\rho U_{j'}$ $(j=1, 2)$: Similar reasoning as above gives

$$\frac{1}{2\pi i}\int_{\rho C_j}\left(\int^z(\omega_{\varepsilon_0}(x, \cdot)-\omega_\varepsilon(x, \cdot))\right)\omega_\varepsilon(y, z)=0\,.$$

The residue theorem implies

$$\frac{1}{2\pi i}\int_{\rho C_j}\left(\int^z\omega_\varepsilon(x, \cdot)\right)\omega_\varepsilon(y, z)=\omega_\varepsilon(x, y)\,.$$

Thus

$$\omega_\varepsilon(x, y)=\frac{1}{2\pi i}\int_{\rho C_j}\left(\int^z\omega_{\varepsilon_0}(x, \cdot)\right)\omega_\varepsilon(y, z)\,.$$

By symmetry and Stokes' theorem, it is seen that

$$\omega_{\varepsilon_0}(x, y)=\omega_{\varepsilon_0}(y, x)=\frac{1}{2\pi i}\int_{\rho C_{j'}}\left(\int^z\omega_\varepsilon(y, \cdot)\right)\omega_{\varepsilon_0}(x, z)$$

$$=-\frac{1}{2\pi i}\int_{\rho C_{j'}}\left(\int^z\omega_{\varepsilon_0}(x, \cdot)\right)\omega_\varepsilon(y, z)\,.$$

This completes the proof of Case 2, so that Lemma 4 is proved.

We are now in position to obtain the variational formulas of arbitrary order for $\omega_1(x, y)$ and $\omega_2(x, y)$. To this end, however, it is important first to recognize that $\omega_\varepsilon(x, y)$ is holomorphic in ε. Thus the first or second order variational formulas for $\omega_\varepsilon(x, y)$ are needed in advance. Let $\omega_j(x, y)$ have an expansion near $x=y=p_j$, in terms of the pinching coordinate, given by

$$\omega_j(x, y)=\frac{1}{(x-y)^2}+\sum_{k,l=0}^{\infty}\beta_{kl}^{(j)}x^ky^l \qquad (j=1, 2)\,. \tag{14}$$

THEOREM 2. *$\omega_\varepsilon(x, y)$ has an expansion*

$$\omega_\varepsilon(x, y)=\begin{cases}\omega_j(x, y)+\beta_{00}^{(j')}\varepsilon^2\omega_j(x, p_j)\omega_j(y, p_j)+O(\varepsilon^3), & x, y\in S_j\backslash\rho U_j\\ -\varepsilon\omega_j(x, p_j)\omega_{j'}(y, p_{j'})+O(\varepsilon^2), & x\in S_j\backslash\rho U_j,\ y\in S_{j'}\backslash\rho U_{j'}\end{cases} \tag{15}$$

near $\varepsilon=0$ with $0<\rho<1$. Here the estimates $O(\varepsilon^2)$ and $O(\varepsilon^3)$ are uniform and the differentials $\omega_\varepsilon(x, y)$, $\omega_j(x, p_j)$ and $\omega_j(y, p_j)$ are all evaluated in terms of the pinching coordinates.

Proof. Let us fix ρ' and ρ'' with $|\varepsilon|^{1/2}<\rho'<\rho<\rho''<1$ and assume that $x\in S_1\backslash\rho U_1$ without loss of generality. From Lemma 4 with $\varepsilon_0=0$, (11) and Cauchy's integral theorem, it is seen that, for $y\in S_2\backslash\rho U_2$,

$$\omega_\varepsilon(x, y)=\frac{1}{2\pi i}\int_{\rho' C_1}\Big(\int_{p_1}^{z}\omega_1(x, \cdot)\Big)\omega_\varepsilon(y, z)$$

$$=\sum_{n=1}^{\infty} a_n^{(1)}(x)\frac{1}{2\pi i}\int_{\rho' C_1} z_1^n\omega_\varepsilon(y, z_1)$$

$$=-\sum_{n=1}^{\infty}\varepsilon^n a_n^{(1)}(x)\frac{1}{2\pi i}\int_{\rho' C_2}\omega_\varepsilon(y, z_2)/z_2^n .$$

Thus Lemma 3 combined with the residue theorem and the equation (12) shows

$$\omega_\varepsilon(x, y)=-\varepsilon a_1^{(1)}(x)\frac{1}{2\pi i}\int_{\rho' C_2}\omega_\varepsilon(y, z_2)/z_2+O(\varepsilon^2)$$

$$=-\varepsilon\omega_1(x, p_1)\frac{1}{2\pi i}\int_{\rho' C_2}\omega_2(y, z_2)/z_2+O(\varepsilon^2)$$

$$=-\varepsilon\omega_1(x, p_1)\omega_2(y, p_2)+O(\varepsilon^2) .$$

Here, the estimates $O(\varepsilon^2)$ are all uniform for $x\in S_1\backslash\rho U_1$ and $y\in S_2\backslash\rho U_2$.

When $y\in S_1\backslash\rho U_1$, a similar reasoning shows

$$\omega_\varepsilon(x, y)=\omega_1(x, y)+\frac{1}{2\pi i}\int_{\rho' C_1}\Big(\int_{p_1}^{z_1}\omega_1(x, \cdot)\Big)\omega_\varepsilon(y, z)$$

$$=\omega_1(x, y)+\sum_{n=1}^{\infty} a_n^{(1)}(x)\frac{1}{2\pi i}\int_{\rho' C_1} z_1^n\omega_\varepsilon(y, z_1)$$

$$=\omega_1(x, y)-\sum_{n=1}^{\infty}\varepsilon^n a_n^{(1)}(x)\frac{1}{2\pi i}\int_{\rho'' C_2}\omega_\varepsilon(y, z_2)/z_2^n .$$

By definition $y\in S_1\backslash\rho U_1$ and $\rho'' C_2\subset S_2\backslash\rho U_2$, so that the result already obtained above can be applied to give

$$\omega_\varepsilon(x, y)=\omega_1(x, y)-\varepsilon a_1^{(1)}(x)\frac{1}{2\pi i}\int_{\rho'' C_2}\omega_\varepsilon(y, z_2)/z_2+O(\varepsilon^3)$$

$$=\omega_1(x, y)+\varepsilon^2 a_1^{(1)}(x)\omega_1(y, p_1)\frac{1}{2\pi i}\int_{\rho'' C_2}\omega_2(p_2, z_2)+O(\varepsilon^3)$$

$$=\omega_1(x, y)+\beta_{00}^{(2)}\varepsilon^2\omega_1(x, p_1)\omega_1(y, p_1)+O(\varepsilon^3) .$$

Again, the estimates $O(\varepsilon^3)$ are all uniform. This concludes the proof.

THEOREM 3. *$\omega_\varepsilon(x, y)$ has an expansion near $\varepsilon=\varepsilon_0$ $(\neq 0)$*

$$\omega_\varepsilon(x, y)=\omega_{\varepsilon_0}(x, y)-\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}\sum_{n=-\infty}^{\infty} n^2 a_{n,\varepsilon_0}(x)a_{-n,\varepsilon_0}(y)+O((\varepsilon-\varepsilon_0)^2) \tag{16}$$

uniformly for $x, y\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2)$ with $|\varepsilon_0|^{1/2}<\rho<1$.

Proof. Now let us fix a ρ' with $|\varepsilon_0|^{1/2}<\rho'<\rho$. From Lemma 4 it is seen that

$$\omega_\varepsilon(x, y)-\omega_{\varepsilon_0}(x, y)=\frac{1}{2\pi i}\int_{\rho' C_1}\sum_{n=-\infty}^{\infty} a_{n,\varepsilon_0}(x) z_1^n \omega_\varepsilon(y, z_1)$$

$$+\frac{1}{2\pi i}\int_{\rho' C_2}\sum_{n=-\infty}^{\infty} a_{n,\varepsilon_0}(x)\varepsilon^n/z_2^n \omega_\varepsilon(y, z_2)$$

$$=-\sum_n \frac{1}{2\pi i}\int_{\rho' C_2} a_{n,\varepsilon_0}(x)(\varepsilon^n/z_2^n)\omega_\varepsilon(y, z_2) \tag{17}$$

$$+\sum_n \frac{1}{2\pi i}\int_{\rho' C_2} a_{n,\varepsilon_0}(x)(\varepsilon_0^n/z_2^n)\omega_\varepsilon(y, z_2)$$

$$=\sum_n (\varepsilon_0^n-\varepsilon^n) a_{n,\varepsilon_0}(x)\frac{1}{2\pi i}\int_{\rho' C_2} w_\varepsilon(y, z_2)/z_2^n .$$

Lemma 3 shows that the estimate

$$\omega_\varepsilon(y, z_2)=O(1) \qquad y\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2), \qquad z_2\in\rho' C_2$$

holds uniformly. In addition, the identity

$$\varepsilon^n-\varepsilon_0^n=n\varepsilon_0^{n-1}(\varepsilon-\varepsilon_0)+(\varepsilon-\varepsilon_0)^2 R_n$$

where

$$R_n=\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{|z|=r_1}\frac{z^n dz}{(z-\varepsilon_0)^2(z-\varepsilon)}, & n\geqq 0 \\ -\dfrac{1}{2\pi i}\displaystyle\int_{|z|=r_2}\frac{z^n dz}{(z-\varepsilon_0)^2(z-\varepsilon)}, & n<0 \end{cases}$$

with $0<r_2<|\varepsilon_0|<r_1$, implies that

$$R_n=\begin{cases} O((\sqrt{\varepsilon_0})^n) & n\to+\infty , \\ O((\varepsilon_0\sqrt{\rho'})^n) & n\to-\infty . \end{cases} \tag{18}$$

Thus the estimate

$$\omega_\varepsilon(x, y)-\omega_{\varepsilon_0}(x, y)=O(\varepsilon-\varepsilon_0) \qquad (\varepsilon\to\varepsilon_0) \tag{19}$$

holds uniformly for $x, y\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2)$. Since ρ is arbitrary, (19) also holds for $x, y\in(S_1\backslash\rho'' U_1)\cup(S_2\backslash\rho'' U_2)$ with $|\varepsilon_0|^{1/2}<\rho''<\rho'$. Therefore, if (19) with ρ replaced by ρ'' is substituted in (17), it follows easily that

$$\omega_\varepsilon(x, y)-\omega_{\varepsilon_0}(x, y)$$

$$=\sum_n (\varepsilon_0^n-\varepsilon^n) a_{n,\varepsilon_0}(x)\frac{1}{2\pi i}\int_{\rho' C_2}\omega_{\varepsilon_0}(y, z_2)/z_2^n+O((\varepsilon-\varepsilon_0)^2)$$

$$=\sum_n (\varepsilon^n-\varepsilon_0^n) a_{n,\varepsilon_0}(x)\frac{\varepsilon_0^{-n}}{2\pi i}\int_{\rho' C_1} z_1^n\omega_{\varepsilon_0}(y, z_1)+O((\varepsilon-\varepsilon_0)^2)$$

$$=\sum_{n}(\varepsilon^{n}-\varepsilon_{0}^{n})\varepsilon_{0}^{-n}a_{n,\varepsilon_{0}}(x)(-n)a_{-n,\varepsilon_{0}}(y)+O((\varepsilon-\varepsilon_{0})^{2})$$

$$=-\frac{\varepsilon-\varepsilon_{0}}{\varepsilon_{0}}\sum_{n}n^{2}a_{n,\varepsilon_{0}}(x)a_{-n,\varepsilon_{0}}(y)+O((\varepsilon-\varepsilon_{0})^{2}).$$

Here we used (18), (19) and Cauchy's integral theorem.

Theorems 2 and 3 clearly show that $\omega_{\varepsilon}(x, y)$ is holomorphic in ε at $\varepsilon=0$. To obtain the Taylor expansions of $\omega_{\varepsilon}(x, y)$ with respect to ε, the following observations are in order: let

$$\alpha_{kl}^{(j)}=\frac{1}{2\pi i}\int_{\rho C_{j}}a_{k}^{(j)}(z_{j})/z_{j}^{l}\qquad (j=1, 2;\ k, l=1, 2, \cdots) \tag{20}$$

with $0<\rho<1$ and set, for $|\varepsilon|<\rho^{2}$,

$$\omega_{\varepsilon}(x, y)=\sum_{n=0}^{\infty}\varepsilon^{n}\Omega_{n}(x, y),\qquad x, y\in(S_{1}\backslash\rho U_{1})\cup(S_{2}\backslash\rho U_{2}) \tag{21}$$

where $\Omega_{0}(x, y)\equiv\omega_{0}(x, y)$ defined before. From (11) and (14) it follows easily that

$$\alpha_{kl}^{(j)}=\beta_{k-1,l-1}^{(j)}/k\qquad (j=1, 2;\ k, l=1, 2, \cdots). \tag{22}$$

Thus the symmetry $\beta_{kl}^{(j)}=\beta_{lk}^{(j)}$ implies

$$k\alpha_{kl}^{(j)}=l\alpha_{lk}^{(j)}\qquad (j=1, 2;\ k, l=1, 2, \cdots). \tag{23}$$

$\Omega_{n}(x, y)$ $(n=1, 2, \cdots)$ are bilinear holomorphic differentials on $(S_{1}\backslash\rho U_{1})\cup(S_{2}\backslash\rho U_{2})$, since the singularities of $\omega_{\varepsilon}(x, y)$ are cancelled out by those of $\Omega_{0}(x, y)$.

With the above preparation, the following theorem concerning the variational formulas of any order will now be demonstrated.

THEOREM 4. *The n-th order variational coefficients* $\Omega_{n}(x, y)$ $(n=1, 2, \cdots)$ *are given by: for* $j=1, 2$ *and* $0<\rho<1$,

$$\Omega_{n}(x, y)=\begin{cases}\displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,jj}^{hk}a_{h}^{(j)}(x)a_{k}^{(j)}(y), & x, y\in S_{j}\backslash\rho U_{j},\\ \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,jj'}^{hk}a_{h}^{(j)}(x)a_{k}^{(j')}(y)-na_{n}^{(j)}(x)a_{n}^{(j')}(y), & \\ \qquad x\in S_{j}\backslash\rho U_{j},\ y\in S_{j'}\backslash\rho U_{j'} & \end{cases} \tag{24}$$

where

$$\Omega_{n,jj}^{hk}=h\sum\alpha_{ht_{1}}^{(j')}\alpha_{t_{1}t_{2}}^{(j)}\alpha_{t_{2}t_{3}}^{(j')}\cdots\alpha_{t_{2s}k}^{(j')}, \tag{25}$$

$$\Omega_{n,jj}^{hk}=-h\sum\alpha_{ht_{1}}^{(j')}\alpha_{t_{1}t_{2}}^{(j)}\alpha_{t_{2}t_{3}}^{(j')}\cdots\alpha_{t_{2s+1}k}^{(j)} \tag{25$'$}$$

with summation taken over all integral vectors (t_{j}) *such that*

$$n-h-k=\sum_{j=1}^{2s}t_{j},\qquad t_{j}\geqq 1,\ s\geqq 0,\ s\in\boldsymbol{Z}$$

and

$$n-h-k=\sum_{j=1}^{2s+1} t_j, \qquad t_j\geqq 1,\ s\geqq 0,\ s\in \boldsymbol{Z}$$

respectively.

Proof. From Lemma 4, (5), (21) and Cauchy's integral theorem, it is seen that, for $x\in S_j\backslash\rho' U_j$ with $0<|\varepsilon|^{1/2}<\rho'<\rho<1$ and $j=1, 2$,

$$\begin{aligned}\sum_{n=1}^{\infty} \varepsilon^n \Omega_n(x, y) &= \frac{1}{2\pi i}\int_{\rho' C_j}\Big(\sum_{n=1}^{\infty} a_n^{(j)}(x) z_j^n\Big)\omega_\varepsilon(y, z) \\ &= -\frac{1}{2\pi i}\int_{\rho' C_{j'}}\Big(\sum_{n=1}^{\infty} \varepsilon^n a_n^{(j)}(x) z_{j'}^{-n}\Big)\Big(\sum_{m=0}^{\infty} \varepsilon^m \Omega_m(y, z_{j'})\Big) \\ &= -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \varepsilon^{n+m} a_n^{(j)}(x)\frac{1}{2\pi i}\int_{\rho' C_{j'}}\Omega_m(y, z_{j'})/z_{j'}^n .\end{aligned}$$

Comparing coefficients of like powers of ε, we obtain

$$\Omega_n(x, y)=-\sum_{h=1}^{n} a_h^{(j)}(x)\frac{1}{2\pi i}\int_{\rho' C_{j'}}\Omega_{n-h}(y, z_{j'})/z_{j'}^h \tag{26}$$

for $n=1, 2, \cdots$, $x\in S_j\backslash\rho' U_j$ $(j=1, 2)$ and $y\in(S_1\backslash\rho' U_1)\cup(S_2\backslash\rho' U_2)$. Since ρ' is arbitrary, the repeated use of (26) gives

$$\Omega_n(x, y)=\begin{cases}\displaystyle\sum_{h,k=1}^{h+k\leq n} a_h^{(j)}(x) a_k^{(j)}(y)\frac{1}{(2\pi i)^2}\int_{\rho' C_{j'}\times\rho' C_{j'}}\Omega_{n-h-k}(z, w)/z^h w^k \\ \qquad \text{for}\quad x, y\in S_j\backslash\rho U_j, \\ \displaystyle\sum_{h,k=1}^{h+k\leq n} a_h^{(j)}(x) a_k^{(j')}(y)\frac{1}{(2\pi i)^2}\int_{\rho' C_{j'}\times\rho' C_j}\Omega_{n-h-k}(z, w)/z^h w^k \\ \qquad -n a_n^{(j)}(x) a_n^{(j')}(y)\quad \text{for}\quad x\in S_j\backslash\rho U_j,\ y\in S_{j'}\backslash\rho U_{j'} .\end{cases} \tag{27}$$

For it is easily seen that $\Omega_0\equiv\omega_0$ satisfies

$$\frac{1}{2\pi i}\int_{\rho' C_{j'}}\Omega_0(y, z_{j'})/z_{j'}^n=\begin{cases}0 & y\in S_j\backslash\rho U_j \\ n a_n^{(j')}(y) & y\in S_{j'}\backslash\rho U_{j'}\end{cases}$$

by definitions (11) and (13). On setting (for $n=1, 2, \cdots$; $h, k\geqq 1$, $h+k\leqq n$; $j=1, 2$)

$$\begin{aligned}\Omega_{n,jj}^{hk} &= \frac{1}{(2\pi i)^2}\int_{\rho C_{j'}\times\rho C_{j'}}\Omega_{n-h-k}(z, w)/z^h w^k , \\ \Omega_{n,jj'}^{hk} &= \frac{1}{(2\pi i)^2}\int_{\rho C_{j'}\times\rho C_j}\Omega_{n-h-k}(z, w)/z^h w^k ,\end{aligned} \tag{28}$$

it remains only to show that the formulas (25) and (25)′ hold. But this is easy, if one notes the following recurrence formulas for $\Omega_{n,jj}^{hk}$ and $\Omega_{n,jj'}^{hk}$ obtained by substituting (24) in the integrand $\Omega_{n-h-k}(z, w)$ on the right hand side of (28): for $n=1, 2, \cdots$; $h, k\geqq 1$, $h+k\leqq n$; $j=1, 2$,

$$\Omega^{hk}_{n,jj}=\begin{cases}\displaystyle\sum_{p,q=1}^{p+q\leq n-h-k}\Omega^{pq}_{n-h-k,j'j'}\alpha^{(j')}_{ph}\alpha^{(j')}_{qk} & h+k<n\\ h\alpha^{(j')}_{hk} & h+k=n\end{cases}\tag{29}$$

$$\Omega^{hk}_{n,jj'}=\sum_{p,q=1}^{p+q\leq n-h-k}\Omega^{pq}_{n-h-k,j'j}\alpha^{(j')}_{ph}\alpha^{(j)}_{qk}-(n-h-k)\alpha^{(j')}_{n-h-k,h}\alpha^{(j)}_{n-h-k,k}.\tag{29$'$}$$

By induction on n, it turns out that (25) and (25)$'$ are the direct consequences of (23), (29) and (29)$'$. This completes the proof of Theorem 4.

Clearly, (23), (25) and (25)$'$ show that the important quantities $\Omega^{hk}_{n,ij}$ have the symmetry:

$$\Omega^{hk}_{n,jj}=\Omega^{kh}_{n,jj}, \qquad \Omega^{hk}_{n,jj'}=\Omega^{kh}_{n,j'j}\tag{30}$$

for $n=1, 2, \cdots$; $h, k\geqq 1$, $h+k\leqq n$; $j=1, 2$.

Remark. The coefficients $\Omega^{hk}_{n,jj}$ satisfy the following identity: for $n=1, 2, \cdots$; $h, k=1, 2, \cdots, h+k\leqq n$; $j=1, 2$,

$$(n-h-k)\Omega^{hk}_{n,jj}=\sum_{\nu=1}^{n-1}\sum_{l=1}^{l+h\leq\nu}\sum_{m=1}^{m+k\leq n-\nu}(\alpha^{(j)}_{lm}+\alpha^{(j)}_{ml})\Omega^{hl}_{\nu,jj}\Omega^{km}_{n-\nu,jj}.\tag{31}$$

To show (31) we calculate $a_{n,\varepsilon}(x)$ $(n=\pm 1, \pm 2, \cdots)$ explicitly by using Theorem 4. For $x\in S_1\backslash U_1$ termwise integration gives

$$\int^{z_1}\omega_\varepsilon(\cdot, x)=\text{const.}+\sum_{n=1}^{\infty}a^{(1)}_n(x)z_1^n+\sum_{n=1}^{\infty}\sum_{h,k=1}^{h+k\leq n}\varepsilon^n\Omega^{hk}_{n,11}a^{(1)}_h(x)\int^{z_1}a^{(1)}_k.$$

By (20) the integrals on the right hand side have expansions

$$\int^{z_1}a^{(1)}_k=-\frac{1}{kz_1^k}+\text{const.}+\sum_{l=1}^{\infty}\frac{\alpha^{(1)}_{kl}}{l}z_1^l, \qquad k=1, 2, \cdots.$$

Thus, from the definition (11), it is seen that $a_{n,\varepsilon}(x)$ is given by: for $x\in S_1\backslash U_1$ and $n=1, 2, \cdots$,

$$a_{-n,\varepsilon}(x)=-\frac{1}{n}\sum_{m,h=1}^{h+n\leq m}\varepsilon^m\Omega^{hn}_{m,11}a^{(1)}_h(x),$$

$$a_{n,\varepsilon}(x)=a^{(1)}_n(x)+\frac{1}{n}\sum_{m=1}^{\infty}\sum_{h,k=1}^{h+k\leq m}\varepsilon^m\alpha^{(1)}_{kn}\Omega^{hk}_{m,11}a^{(1)}_h(x).$$

Hence it follows from Theorems 3 and 4 that, for $x, y\in S_1\backslash U_1$,

$$\sum_{n=1}^{\infty}n\varepsilon^n\Omega_n(x, y)=\sum_{n=1}^{\infty}(\sum\varepsilon^m\Omega^{hn}_{m,11}a^{(1)}_h(x))(na^{(1)}_n(y)+\sum\varepsilon^m\alpha^{(1)}_{kn}\Omega^{hk}_{m,11}a^{(1)}_h(y))$$

$$+\sum_{n=1}^{\infty}(\sum\varepsilon^m\Omega^{hn}_{m,}a_{11h}^{(1)}(y))(na^{(1)}_n(x)+\sum\varepsilon^m\alpha^{(1)}_{kn}\Omega^{hk}_{m,11}a^{(1)}_h(x)).$$

Comparing coefficients of ε^n in the above expansions, we find an expression of $\Omega_n(x, y)$ different from (24), which, together with (24), easily implies (31). Similar identities will also be obtained by choosing x or $y \in S_2 \backslash U_2$ in the above calculations.

Now that Theorem 4 is obtained, it will be possible to derive variational formulas for any meromorphic differentials on S_ε which is holomorphic on P_ε. However, instead of computing complicated formulas for the general case, we restrict ourselves to the case of the normalized differentials of the first and the third kind.

Let $\omega_{a-b}^{(j)}$ (resp. $\omega_{a-b,\varepsilon}$) be the normalized differential of the third kind on S_j (resp. S_ε) with simple poles of residue 1 and -1 at $a, b \in S_j$ (resp. S_ε) respectively. Then the Riemann bilinear relation gives

$$v_k(x) = \begin{cases} \displaystyle\int_{B_k} \omega_1(\cdot, x) & x \in S_1,\ 1 \leqq k \leqq g_1 \\ \displaystyle\int_{B_k} \omega_2(\cdot, x) & x \in S_2,\ g_1+1 \leqq k \leqq g, \end{cases} \tag{32}$$

$$\omega_{a-b}^{(j)}(x) = \int_b^a \omega_j(\cdot, x) \qquad x \in S_j,\ j=1, 2$$

with the path of integration from b to a taken in S_j cut along its homology basis. For notational convenience, let the following expansion holds in terms of the pinching coordinates:

$$\int_{p_j}^{z} \eta = \sum_{n=1}^{\infty} \gamma_n^{(j)}[\eta] z_j^n, \qquad |z_j| < 1 \tag{33}$$

where η is any differential holomorphic on U_j $(j=1, 2)$. Thus (11) may be rewritten as

$$\gamma_n^{(j)}[\omega_j(\cdot, x)] = a_n^{(j)}(x).$$

Furthermore, let us write for short

$$\gamma_n^{(j)}[v_k] = \gamma_{nk}^{(j)}, \qquad \gamma_n^{(j)}[\omega_{a-b}^{(j)}] = \gamma_n^{(j)}(a, b), \qquad \gamma_n^{(j)}[\omega_{a-p_j}^{(j)} + dz_j/z_j] = \gamma_n^{(j)}(a). \tag{34}$$

Analogous to (32), the Riemann bilinear relation again gives

$$\int_{B_k} a_n^{(j)}(\cdot) = \gamma_{nk}^{(j)}, \qquad \int_b^a a_n^{(j)}(\cdot) = \gamma_n^{(j)}(a, b). \tag{35}$$

If (21) is integrated term by term along the cycle $B_k(\varepsilon) = B_k$, an expansion of $v_{k,\varepsilon}$ will be obtained at once in view of (32) and (35). Indeed, this is legitimate since B_k is contained in the region where Theorem 4 is valid.

COROLLARY 1. *The normalized differentials of the first kind on S_ε have expansions near $\varepsilon=0$: for $i=1, 2, \cdots, g_1$ and $0<\rho<1$,*

(36)
$$v_{i,\varepsilon}(x)=\begin{cases} v_i(x)+\varepsilon^2\beta_{00}^{(2)}v_i(p_1)\omega_1(x,\ p_1)+O(\varepsilon^3) & x\in S_1\backslash\rho U_1 \\ -\varepsilon v_i(p_1)\omega_2(x,\ p_2)+O(\varepsilon^2) & x\in S_2\backslash\rho U_2 \end{cases}$$

where $\beta_{00}^{(2)}$ *is the constant defined by* (14) *and the estimates* $O(\varepsilon^2)$ *and* $O(\varepsilon^3)$ *are uniform.*
More precisely: for $i=1,\cdots,g$ *and* $x\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2)$

(37)
$$v_{i,\varepsilon}(x)=\sum_{n=0}^{\infty}\varepsilon^n(v_i)_n(x)$$

where, for $i=1,\cdots,g_1$ *and* $n=1,2,\cdots,$

$$(v_i)_0(x)=\begin{cases} v_i(x) & x\in S_1, \\ 0 & x\in S_2, \end{cases}$$

$$(v_i)_n(x)=\begin{cases} \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,11}^{hk}\gamma_{hi}^{(1)}a_k^{(1)}(x) & x\in S_1\backslash\rho U_1, \\ \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,12}^{kh}\gamma_{hi}^{(1)}a_k^{(2)}(x)-n\gamma_{ni}^{(1)}a_n^{(2)}(x) & x\in S_2\backslash\rho U_2. \end{cases}$$

For $i=g_1+1,\cdots,g$, *similar formulas are obtained by symmetry.*

If (37) is integrated term by term along B_k once again, a variational formula for the period matrix for S_ε, denoted by τ_ε, is obtained.

COROLLARY 2. *The period matrix* τ_ε *has an expansion near* $\varepsilon=0$

(38)
$$\tau_\varepsilon=\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}-\varepsilon\begin{pmatrix} 0 & {}^tR_1R_2 \\ {}^tR_2R_1 & 0 \end{pmatrix}+O(\varepsilon^2)$$

where τ_1 *and* τ_2 *are the period matrices for* S_1 *and* S_2 *respectively, and*

$$R_1=(v_1(p_1),\cdots,v_{g_1}(p_1))\in\boldsymbol{C}^{g_1},$$

$$R_2=(v_{g_1+1}(p_2),\cdots,v_g(p_2))\in\boldsymbol{C}^{g_2}.$$

More precisely:

(39)
$$\tau_\varepsilon=\sum_{n=0}^{\infty}\varepsilon^n\tau_n$$

where $\tau_n=(\tau_{n,ij})_{i,j=1}^{g}$ *and, for* $n=1,2,\cdots,$

(40)
$$\tau_{n,ij}=\begin{cases} \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,11}^{hk}\gamma_{hi}^{(1)}\gamma_{kj}^{(1)} & 1\leq i,\ j\leq g_1, \\ \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega_{n,22}^{hk}\gamma_{hi}^{(2)}\gamma_{kj}^{(2)} & g_1+1\leq i,\ j\leq g, \end{cases}$$

$$\tau_{n,ij}=\tau_{n,ji}=\sum_{h,k=1}^{h+k\leq n}\Omega_{n,12}^{hk}\gamma_{hi}^{(1)}\gamma_{kj}^{(2)}-n\gamma_{ni}^{(1)}\gamma_{nj}^{(2)} \qquad \textit{for} \quad 1\leqq i\leqq g_1,\ g_1+1\leqq j\leqq g\,.$$

Here $\gamma_{ki}^{(j)}$ $(j=1, 2)$ *is the constant defined by* (33) *and* (34).

Similarly, variational formulas for $\omega_{a-b,\varepsilon}$ are obtained if both a and $b\in S_j\backslash\rho U_j$ with $0<\rho<1$ $(j=1, 2)$. On the other hand, if $a\in S_j\backslash\rho U_j$ and $b\in S_{j'}\backslash\rho U_{j'}$ $(j=1, 2)$, such a simple method as above fails immediately since the path of integration must across the pinched region. In this case, however, we can proceed as follows: analogous to Lemma 4, $\omega_{a-b,\varepsilon}(x)$ is given in terms of $\omega_\varepsilon(x, y)$ by

$$\omega_{a-b,\varepsilon}(x)-\omega_{a-b,0}(x)=\frac{1}{2\pi i}\int_{\rho C_j}\left(\int_{p_j}^{z}(\omega_{a-p_j}^{(j)}+dz_j/z_j)\right)\omega_\varepsilon(x, z) -\frac{1}{2\pi i}\int_{\rho C_{j'}}\left(\int_{p_{j'}}^{z}(\omega_{b-p_{j'}}^{(j')}+dz_{j'}/z_{j'})\right)\omega_\varepsilon(x, z) \tag{41}$$

where $a\in S_j\backslash\rho U_j$, $b\in S_{j'}\backslash\rho U_{j'}$, $x\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2)$ and

$$\omega_{a-b,0}(x)=\begin{cases}\omega_{a-p_1}^{(1)}(x) & x\in S_1\,,\\ \omega_{p_2-b}^{(2)}(x) & x\in S_2\,.\end{cases}$$

(41) follows from a similar reasoning as in the proof of Lemma 4, so that the proof may be omitted. If the expansion (21) is substituted in the right hand side of (41), the desired variational formula is obtained from Theorem 4 by termwise integration. The results are summarized as follows.

COROLLARY 3. *The normalized differential of the third kind* $\omega_{a-b,\varepsilon}(x)$ *has an expansion near* $\varepsilon=0$:

(i) *for* $a, b\in S_j\backslash\rho U_j$,

$$\omega_{a-b,\varepsilon}(x)=\begin{cases}\omega_{a-b}^{(j)}(x)+\varepsilon^2\beta_{00}^{(j')}\omega_{a-b}^{(j)}(p_j)\omega_j(x, p_j)+O(\varepsilon^3) & x\in S_j\backslash\rho U_j\,,\\ -\varepsilon w_{a-b}^{(j)}(p_j)\omega_{j'}(x, p_{j'})+O(\varepsilon^2) & x\in S_{j'}\backslash\rho U_{j'}\,,\end{cases} \tag{42}$$

(ii) *for* $x, a\in S_j\backslash\rho U_j$ *and* $b\in S_{j'}\backslash\rho U_{j'}$,

$$\omega_{a-b,\varepsilon}(x)=\omega_{a-p_j}^{(j)}(x)+\varepsilon d_{j'}(b)\omega_j(x, p_j)+O(\varepsilon^2) \tag{43}$$

where, in terms of the pinching coordinates,

$$d_j(a)=\lim_{z\to p_j}[\omega_{a-p_j}(z)+1/z_j(z)]\in \boldsymbol{C} \qquad (j=1, 2)$$

and all the estimates $O(\varepsilon^2)$, $O(\varepsilon^3)$ *are uniform.*

More precisely:

$$\omega_{a-b,\varepsilon}(x)=\sum_{n=0}^{\infty}\varepsilon^n(\omega_{a-b})_n(x) \qquad x\in(S_1\backslash\rho U_1)\cup(S_2\backslash\rho U_2) \tag{44}$$

where, for $n=1, 2, \cdots$,

(i)′ *for* $a, b\in S_j\backslash\rho U_j$,

$$(\omega_{a-b})_n(x)=\begin{cases}\displaystyle\sum_{h,k\leq 1}^{h+k\leq n}\Omega^{hk}_{n,jj}\gamma^{(j)}_h(a, b)a^{(j)}_k(x), & x\in S_j\backslash\rho U_j,\\ \displaystyle\sum_{h,k=1}^{h+k\leq n}\Omega^{hk}_{n,jj'}\gamma^{(j)}_h(a, b)a^{(j')}_k(x)-n\gamma^{(j)}_n(a, b)a^{(j')}_n(x), & x\in S_{j'}\backslash\rho U_{j'},\end{cases}\tag{45}$$

(ii)′ *for* $x, a\in S_j\backslash\rho U_j$ *and* $b\in S_{j'}\backslash\rho U_{j'}$,

$$(\omega_{a-b})_n(x)=\sum_{h,k=1}^{h+k\leq n}\{\Omega^{hk}_{n,jj}\gamma^{(j)}_h(a)-\Omega^{hk}_{n,j'j}\gamma^{(j')}_h(b)\}\, a^{(j)}_k(x)+n\gamma^{(j')}_n(b)a^{(j)}_n(x).\tag{46}$$

Here $\gamma^{(j)}_h(a, b)$ and $\gamma^{(j)}_h(a)$ are the constants defined by (33) and (34).

On account of the importance of the coefficients $\alpha^{(j)}_{nk}$, we make mention of the close connection between the differentials $a^{(j)}_n(x)$ $(n=1, 2, \cdots)$ and the Faber polynomials.

For convenience, let us omit the letter "j" in our notation and write

$$S=S_j,\qquad U=U_j,\qquad a_n(x)=a^{(j)}_n(x),\qquad \text{etc.}$$

The local coordinate z: $U\to\Delta$ is, on the other hand, regarded as a univalent mapping $\phi(t)=z^{-1}(1/t)$: $\{t;\ |t|>1\}\to S$. In the case where $S=\hat{C}$ (the Riemann sphere) and $p=\infty$ $(\in\hat{C})$, ϕ is a complex-valued function and the expansion (11) reduces to

$$\sum_{n=1}^{\infty}a_n(x)t^{-n}=\int_{\infty}^{t}\frac{\phi'(s)ds}{(\phi(s)-x)^2}=\frac{1}{x-\phi(t)}.\tag{47}$$

Recall that a generating function for the Faber polynomials p_n $(n=1, 2, \cdots)$ belonging to ϕ is given by

$$\frac{x\phi'(t)}{\phi(t)-x}=1+\frac{p_1(x)}{t}+\frac{p_2(x)}{t^2}+\cdots\tag{48}$$

[2]. From (47) and (48) it is easily seen that

$$p_n(x)-p_n(x_0)=-n\int_{x_0}^{x}a_n(t)dt\qquad(n=1, 2, \cdots).\tag{49}$$

In view of this identity, it is natural to call the Abelian integrals $\mathcal{F}_n(x)=-n\int^{x}a_n dt$ *the Faber integrals belonging to a local coordinate* z, which agree, up to a constant, the Faber polynomials if $S=\hat{C}$ and $p=\infty$. From (20) $\mathcal{F}_n\circ\phi$ $(n=1, 2, \cdots)$ has an expansion

$$\mathcal{F}_n\circ\phi(w)=w^n+\text{const.}-\sum_{k=1}^{\infty}\frac{n}{k}\alpha_{nk}w^{-k}, \qquad |w|>1. \tag{50}$$

With this analogy, α_{nk} $(n, k=1, 2, \cdots)$ may be called the generalized Faber coefficients and the equation (23) corresponds to the classical Grunsky law of symmetry [2]. Furthermore, a generalization of Golusin inequality is obtained from a straightforward analog of area theorems as follows.

THEOREM 5. *Let $\{x_n\}$ be an arbitrary sequence of complex numbers. Then the coefficients α_{nk} satisfy*

$$\sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{n=1}^{N}x_n\alpha_{nk}\right|^2\leqq\sum_{n=1}^{N}\frac{1}{n}|x_n|^2, \qquad (N=1, 2, \cdots) \tag{51}$$

Equality holds for a non-zero sequence $\{x_n\}$ if and only if the complement of the image of ϕ in S has areal measure zero.

Proof. Let us evaluate the norm of $\sum_{n=1}^{N}x_na_n(x)$ on $S\backslash\rho U$ with $0<\rho<1$. The Riemann bilinear relation and (50) give

$$\begin{aligned}
&\frac{1}{2\pi}\left\|\sum_{n=1}^{N}x_na_n(\cdot)\right\|_{S\backslash\rho U}^2\\
&=-\frac{1}{2\pi i}\iint_{S\backslash\rho U}\sum_{n=1}^{N}x_na_n(\cdot)\wedge\overline{\sum_{n=1}^{N}x_na_n(\cdot)}\\
&=\frac{1}{2\pi i}\int_{\rho C}\overline{\left(\int^x\sum_{n=1}^{N}x_na_n(\cdot)\right)}\sum_{n=1}^{N}x_na_n(x)\\
&=\frac{1}{2\pi i}\int_{|w|=1/\rho}\overline{\sum_{n=1}^{N}\frac{x_n}{n}\mathcal{F}_n\circ\phi(w)}d\left[\sum_{n=1}^{N}\frac{x_n}{n}\mathcal{F}_n\circ\phi(w)\right]\\
&=\sum_{n=1}^{N}\frac{|x_n|^2}{n}\rho^{-2n}-\sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{n=1}^{N}x_n\alpha_{nk}\right|^2\rho^{2k}.
\end{aligned}$$

Letting $\rho\to1$, we have

$$\frac{1}{2\pi}\left\|\sum_{n=1}^{N}x_na_n(\cdot)\right\|_{S\backslash U}^2=\sum_{n=1}^{N}\frac{1}{n}|x_n|^2-\sum_{k=1}^{\infty}\frac{1}{k}\left|\sum_{n=1}^{N}x_n\alpha_{nk}\right|^2,$$

which obviously implies Theorem 5. The equality statement is a direct consequence of the linear independence of $a_n(x)$'s.

When $S=\hat{C}$ the inequality (51) has been already obtained by Jenkins [5], Milin [7] and Pommerenke [9]. Applying the Cauchy inequality to (51), we have at once a version of Grunsky inequality: let $\{x_n\}$ be an arbitrary sequence of complex numbers. Then,

$$\left|\sum_{k,n=1}^{N}\beta_{n-1,k-1}x_nx_k\right|\leqq\sum_{n=1}^{N}n|x_n|^2 \qquad (N=1, 2, \cdots) \tag{52}$$

where β_{nk} (n, $k=0, 1, \cdots$) are the coefficients of the expansion (14). (Note the identity (22).) Equality condition is the same as in Theorem 5.

In particular the important quantities $\beta_{00}^{(j)}$ appearing in Theorem 2 and Corollaries 3 and 4 satisfy

$$|\beta_{00}^{(j)}| \leqq 1 \qquad (j=1, 2). \tag{53}$$

Remark. Schiffer and Spencer have proved an inequality more general than (52) in their book [10] where they generalized, to the case of finite bordered Riemann surfaces, Grunsky's necessary and sufficient condition for the univalence of an analytic function defined on the exterior of the unit circle. Since (51) implies (52), their theorem 5.5.3. [10, p. 168] can be restated. For the sake of completeness, we record this fact as a

COROLLARY. *Let ϕ map a neighborhood of $0 \in \Delta$ conformally into a neighborhood of $p=\phi(0) \in S$. Using a local coordinate ϕ^{-1} around p, one may calculate the series expansion* (14). *Then ϕ can be extended over Δ to give an analytic imbedding of Δ into S if and only if the inequalities* (51) *hold for every sequence $\{x_n\}$ of complex numbers.*

3. Pinching along a non-zero homology cycle.

Here, the notation and the definitions in the previous sections are used unless otherwise stated.

Let S be a compact Riemann surface of genus g and choose coordinates $z_1: U_1 \to \Delta$ and $z_2: U_2 \to \Delta$ in disjoint neighborhoods U_1 and U_2 of two points $p_1, p_2 \in S$. Again, a family of compact Riemann surfaces $\{S_\varepsilon; \varepsilon \in \boldsymbol{C}, 0<|\varepsilon|<1\}$ formed from S is constructed by identifying U_1 and U_2 under the condition (5). S_ε is a compact Riemann surface of genus $g+1$ while the pinched region $P_\varepsilon = S_\varepsilon \backslash (S \backslash (U_1 \cup U_2))$ is usually identified by the pinching coordinates z_1 and z_2 with the annulus $|\varepsilon|<|z|<1$ as before. To choose some canonical homology basis for S, let $A_1(\varepsilon), B_1(\varepsilon), \cdots, A_g(\varepsilon), B_g(\varepsilon)$ simply be a canonical basis $A_1, B_1, \cdots, A_g, B_g$ for S lying in $S \backslash (U_1 \cup U_2)$. In addition let $A_{g+1}(\varepsilon) = \rho C_2$ with any ρ satisfying $|\varepsilon|<\rho<1$ and let $B_{g+1}(\varepsilon)$ be any path from $z_1^{-1}(\sqrt{\varepsilon})$ to $z_2^{-1}(\sqrt{\varepsilon})$ lying within $S \backslash (|\varepsilon| U_1 \cup |\varepsilon| U_2)$ cut along the homology basis for S.

Corresponding to Theorem 1, the following analogous theorem holds with trivial modification, so that proof will be omitted.

THEOREM 1′. *Let Ω be a meromorphic differential on S which is holomorphic on U_1 and U_2 except for possible simple poles at p_1 and p_2 with residues $-\alpha$ and α respectively, and let*

$$\alpha_n^{(j)} = \gamma_n^{(j)}[\Omega - (-1)^j \alpha dz_j / z_j]$$

for $j=1, 2$ and $n=1, 2, \cdots$. Then there exists a meromorphic differential Ω_ε on S_ε which is holomorphic on P_ε with the same singularities as Ω on $S \backslash (U_1 \cup U_2)$, satisfying

(6)′ $$\|\Omega_\varepsilon-\Omega\|^2_{S\setminus(\rho U_1\cup\rho U_2)}\leqq\pi\sum_{n=1}^{\infty}n(|\alpha_n^{(1)}|^2+|\alpha_n^{(2)}|^2)\frac{|\rho\varepsilon|^{2n}}{\rho^{4n}-|\varepsilon|^{2n}}$$

with $|\varepsilon|^{1/2}<\rho<1$.

On applying Theorem 1′ and using the identity analogous to Lemma 4

(54) $$\omega_\varepsilon(x, y)-\omega(x, y)=\frac{1}{2\pi i}\int_{\rho C_1+\rho C_2}\left(\int^z\omega(\cdot, x)\right)\omega_\varepsilon(z, y)$$

for $x, y\in S\setminus(\rho U_1\cup\rho U_2)$ with $|\varepsilon|^{1/2}<\rho<1$, it is now easy to deduce such variational formulas as in Theorems 2, 3 and 4 by a method similar to the one used in section 2. For instance, the uniform boundedness of $v_{j,\varepsilon}$ $(j=1, \cdots, g+1)$ will be shown immediately by choosing

$$\Omega=\begin{cases} v_j & \text{if } j=1, \cdots, g \\ \omega_{p_2-p_1} & \text{if } j=g+1\end{cases}$$

and applying Theorem 1′. (This is the reason why simple poles at p_1 and p_2 must be permitted for the Ω in Theorem 1′ as the singularity.)

Now the main results in this section will be summarized almost without proof in the form of a theorem and corollaries. In order to state these, let us define

(20)′ $$\alpha_{pl}^{sj}=\frac{1}{2\pi i}\int_{\rho C_j}a_p^{(s)}(z_j)/z_j^l \qquad (s, j=1, 2;\ p, l=1, 2, \cdots)$$

which, corresponding to (20), are important to express the variational coefficients. Again, it follows the symmetry:

(23)′ $$l\alpha_{lm}^{jk}=m\alpha_{ml}^{kj} \qquad (j, k=1, 2;\ l, m=1, 2, \cdots)$$

THEOREM 6. *$\omega_\varepsilon(x, y)$ has an expansion near $\varepsilon=0$: for $x, y\in S/(\rho U_1\cup\rho U_2)$ with $0<\rho<1$,*

(15)′ $$\omega_\varepsilon(x, y)=\omega(x, y)-\varepsilon[\omega(x, p_1)\omega(y, p_2)+\omega(x, p_2)\omega(y, p_1)]+O(\varepsilon^2)$$

where the estimate $O(\varepsilon^2)$ is uniform.

More precisely: for $|\varepsilon|^{1/2}<\rho<1$

(21)′ $$\omega_\varepsilon(x, y)=\sum_{n=0}^{\infty}\varepsilon^n\Omega_n(x, y) \qquad x, y\in S\setminus(\rho U_1\cup\rho U_2)$$

where $\Omega_0(x, y)=\omega(x, y)$ and, for $n=1, 2, \cdots,$

(24)′ $$\Omega_n(x, y)=\sum_{j,k=1}^{2}\sum_{l,m=1}^{l+m\leqq n}\Omega_{n,jk}^{lm}a_l^{(j)}(x)a_m^{(k)}(y)-\sum_{j=1}^{2}n\,a_n^{(j)}(x)a_n^{(j')}(y).$$

The coefficients $\Omega_{n,jk}^{lm}$ are given by: for $l, m, n=1, 2, \cdots$ $(l+m\leqq n)$ and $j, k=1, 2,$

(25)″ $$\Omega_{n,jk}^{lm}=l\sum(-1)^d\alpha_{lt_1}^{j's_1}\alpha_{t_1t_2}^{s_1's_2}\alpha_{t_2t_3}^{s_2's_3}\cdots\alpha_{t_dm}^{s_d'k'}$$

with summation taken over all vectors (s_p) *and* $(t_q)\in \boldsymbol{Z}^d$ *such that*

$$n-l-m=\sum_{j=1}^{d} t_j, \qquad t_j\geqq 1,\ s_j=1,\ 2,\ d\geqq 0\,.$$

Instead of (29), the recurrence formula for $\Omega^{lm}_{n,jk}$ is given by

$$(29)' \qquad \Omega^{lm}_{n,jk}=\begin{cases} \displaystyle\sum_{p,q=1}^{p+q\leq n-l-m} \Omega^{pq}_{n-l-m,st}\alpha^{sj'}_{pl}\alpha^{tk'}_{qm}-\sum_{s=1}^{2}(n-l-m)\alpha^{sj'}_{n-l-m,l}\alpha^{s'k'}_{n-l-m,m} & \text{if } l+m=n\,, \\ l\alpha^{j'k'}_{lm} & \text{if } l+m<n\,. \end{cases}$$

By induction on n, (25)″ is verified from (23)′ and (29)′ as before.

Integration of (21)′ along the cycle $B_j(\varepsilon)$ $(j=1,\cdots,g)$ immediately yields

COROLLARY 4. *For* $i=1,\cdots,g$, $v_{i,\varepsilon}(x)$ *has an expansion near* $\varepsilon=0$: *for* $x\in S\backslash(\rho U_1\cup\rho U_2)$ *with* $0<\rho<1$,

$$(36)' \qquad v_{i,\varepsilon}(x)=v_i(x)-\varepsilon[v_i(p_2)\omega(x,\ p_1)+v_i(p_1)\omega(x,\ p_2)]+O(\varepsilon^2)$$

where the estimate $O(\varepsilon^2)$ *is uniform.*

More precisely:

$$v_{i,\varepsilon}(x)=\sum_{n=0}^{\infty}\varepsilon^n(v_i)_n(x) \qquad x\in S\backslash(\rho U_1\cup\rho U_2)$$

where $(v_i)_0(x)=v_i(x)$ *and, for* $n=1,\ 2,\ \cdots$,

$$(v_i)_n(x)=\sum_{j,k=1}^{2}\sum_{l,m=1}^{l+m\leq n}\Omega^{lm}_{n,jk}\gamma^{(j)}_{li}a^{(k)}_m(x)-\sum_{j=1}^{2}n\gamma^{(j)}_{ni}a^{(j')}_n(x)$$

with $\gamma^{(j)}_{nk}=\gamma^{(j)}_n[v_k]$.

On the other hand, Theorem 6 and the identity

$$(55) \qquad v_{g+1,\varepsilon}(x)-\omega_{p_2-p_1}(x)=\frac{1}{2\pi i}\int_{\rho C_1}\left(\int^z(\omega_{p_2-p_1}+dz_1/z_1)\right)\omega_\varepsilon(z,\ x)$$
$$+\frac{1}{2\pi i}\int_{\rho C_2}\left(\int^z(\omega_{p_2-p_1}-dz_2/z_2)\right)\omega_\varepsilon(z,\ x)$$

for $x\in S\backslash(\rho U_1\cup\rho U_2)$ with $|\varepsilon|^{1/2}<\rho<1$ give

COROLLARY 5. $v_{g+1,\varepsilon}(x)$ *has an expansion near* $\varepsilon=0$: *for* $x\in S\backslash(\rho U_1\cup\rho U_2)$

$$(56) \qquad v_{g+1,\varepsilon}(x)=\omega_{p_2-p_1}(x)-\varepsilon[\gamma_2\omega(x,\ p_1)+\gamma_1\omega(x,\ p_2)]+O(\varepsilon^2)$$

where the estimate $O(\varepsilon^2)$ *is uniform and the constants*

$$(57) \qquad \gamma_j=\lim_{x\to p_j}[\omega_{p_2-p_1}(x)-(-1)^j dz_j(x)/z_j(x)] \qquad (j=1,\ 2)$$

are evaluated in terms of the pinching coordinates.

More precisely:

(58) $$v_{g+1,\varepsilon}(x)=\sum_{n=0}^{\infty}\varepsilon^n(v_{g+1})_n(x) \qquad x\in S\backslash(\rho U_1\cup\rho U_2)$$

where $(v_{g+1})_0(x)=\omega_{p_2-p_1}(x)$ *and, for* $n=1, 2, \cdots$,

(59) $$(v_{g+1})_n(x)=\sum_{j,k=1}^{2}\sum_{l,m=1}^{l+m\leq n}\Omega_{n,jk}^{lm}\gamma_l^{(j)}a_m^{(k)}(x)-\sum_{j=1}^{2}n\gamma_n^{(j)}a_n^{(j')}(x)$$

with $\gamma_n^{(j)}=\gamma_n^{(j)}[\omega_{p_2-p_1}-(-1)^j dz_j/z_j]$.

Let

$$\tau_\varepsilon=\left(\begin{array}{c|c}\tau_{ij,\varepsilon} & \sigma_{i,\varepsilon}\\ \hline \sigma_{j,\varepsilon} & \sigma_\varepsilon\end{array}\right)_{i,j=1}^{g}\in GL(g+1, \boldsymbol{C})$$

be the period matrix for S_ε with respect to a canonical basis $A_1(\varepsilon)$, $B_1(\varepsilon)$, $\cdots$, $A_{g+1}(\varepsilon)$, $B_{g+1}(\varepsilon)$. From Corollaries 4 and 5, it is easy to calculate the Taylor expansion of τ_ε at $\varepsilon=0$ except for the $(g+1, g+1)$-element σ_ε for which the path of integration $B_{g+1}(\varepsilon)$ must across the pinched region. The next lemma shows that σ_ε can be expressed through the line integrals whose paths of integration avoid the pinched region.

LEMMA 5. *For* $\varepsilon\in\boldsymbol{C}$ *and* $\rho\in\boldsymbol{R}$ *satisfying* $0<|\varepsilon|^{1/2}<\rho<1/2$, *the following identity holds:*

(60) $$\sigma_\varepsilon=\ln 4\varepsilon+\int_{z_1^{-1}(1/2)}^{z_2^{-1}(1/2)}\omega_{p_2-p_1}+\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}(\omega_{p_2-p_1}+dz_1/z_1)\right)v_{g+1,\varepsilon}(z)$$
$$+\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(1/2)}^{z}(\omega_{p_2-p_1}-dz_2/z_2)\right)v_{g+1,\varepsilon}(z)\,.$$

(*The proper choice of the logarithm depends on the path chosen to define the cycle* $B_{g+1}(\varepsilon)$.)

Proof. Cauchy's integral theorem and the bilinear relation give

$$\int_{\rho C_1+\rho C_2}\left(\int_{z_1^{-1}(1/2)}^{z}(v_{g+1,\varepsilon}-\omega_{p_2-p_1})\right)v_{g+1,\varepsilon}(z)=0\,.$$

Hence

$$\int_{z_1^{-1}(1/2)}^{z_2^{-1}(1/2)}(v_{g+1,\varepsilon}-\omega_{p_2-p_1})$$
$$=\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}+\frac{dz_1}{z_1}\right)\right)v_{g+1,\varepsilon}(z)$$
$$+\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}-\frac{dz_2}{z_2}\right)\right)v_{g+1,\varepsilon}(z)$$

$$-\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}\left(v_{g+1,\varepsilon}+\frac{dz_1}{z_1}\right)\right)v_{g+1,\varepsilon}(z)$$
$$-\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(1/2)}^{z}\left(v_{g+1,\varepsilon}-\frac{dz_2}{z_2}\right)\right)v_{g+1,\varepsilon}(z).$$

On the other hand, a change of parameter by using $z_1z_2=\varepsilon$ yields

$$\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}\left(v_{g+1,\varepsilon}+\frac{dz_1}{z_1}\right)\right)v_{g+1,\varepsilon}(z)$$
$$=-\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(2\varepsilon)}^{z}\left(v_{g+1,\varepsilon}-\frac{dz_2}{z_2}\right)\right)v_{g+1,\varepsilon}(z).$$

Therefore

$$\int_{z_1^{-1}(1/2)}^{z_2^{-1}(2\varepsilon)}v_{g+1,\varepsilon}-\int_{z_1^{-1}(1/2)}^{z_2^{-1}(1/2)}\omega_{p_2-p_1}-\int_{z_2^{-1}(1/2)}^{z_2^{-1}(2\varepsilon)}dz_2/z_2$$
$$=\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}+\frac{dz_1}{z_1}\right)\right)v_{g+1,\varepsilon}(z)$$
$$+\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}-\frac{dz_2}{z_2}\right)\right)v_{g+1,\varepsilon}(z).$$

Note that the path of integration from $z_1^{-1}(1/2)$ to $z_2^{-1}(2\varepsilon)$ can be identified with $B_{g+1}(\varepsilon)$, so that the proof is completed.

From the above lemma, it is seen that the constant term in the expansion (60) is given by:

$$\ln 4+\int_{z_1^{-1}(1/2)}^{z_2^{-1}(1/2)}\omega_{p_2-p_1}+\frac{1}{2\pi i}\int_{\rho C_1}\left(\int_{z_1^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}+\frac{dz_1}{z_1}\right)\right)\omega_{p_2-p_1}(z)$$
$$+\frac{1}{2\pi i}\int_{\rho C_2}\left(\int_{z_2^{-1}(1/2)}^{z}\left(\omega_{p_2-p_1}-\frac{dz_2}{z_2}\right)\right)\omega_{p_2-p_1}(z)$$
$$=\ln 4+\int_{z_1^{-1}(1/2)}^{z_2^{-1}(1/2)}\omega_{p_2-p_1}-\int_{z_1^{-1}(1/2)}^{z_1^{-1}(0)}\left(\omega_{p_2-p_1}+\frac{dz_1}{z_1}\right)+\int_{z_2^{-1}(1/2)}^{z_2^{-1}(0)}\left(\omega_{p_2-p_1}-\frac{dz_2}{z_2}\right).$$

This, in turn, is seen to be equal to the constant

$$\lim_{x\to 0}\left[\int_{z_1^{-1}(x)}^{z_2^{-1}(x)}\omega_{p_2-p_1}-2\ln x\right].$$

Corollaries 4, 5 and Lemma 5 give immediately the expansion of the period matrix.

COROLLARY 6. *Let γ_1, γ_2, $\gamma_n^{(j)}$ and $\gamma_{nk}^{(j)}$ be defined as in Corollaries 4 and 5. Then the period matrix for S_ε has an expansion*

(38)′
$$\tau_\varepsilon=\begin{pmatrix} \tau_{ij}+\varepsilon\sigma_{ij} & a_i+\varepsilon\sigma_i \\ a_j+\varepsilon\sigma_j & \ln\varepsilon+c_0+c_1\varepsilon \end{pmatrix}+O(\varepsilon^2)$$

where $(\tau_{ij})_{i,j=1}^g$ *is the period matrix for* S, $a_i=\int_{p_1}^{p_2} v_i$,

$$\sigma_{ij}=-(v_i(p_1)v_j(p_2)+v_i(p_2)v_j(p_1)), \qquad \sigma_i=-(\gamma_1 v_i(p_2)+\gamma_2 v_i(p_1)),$$

$c_0=\lim_{x\to 0}\left[\int_{z_1^{-1}(x)}^{z_2^{-1}(x)}\omega_{p_2-p_1}-2\ln x\right]$ *and* $c_1=-2\gamma_1\gamma_2$.

More precisely:

(i)
$$\tau_{ij,\varepsilon}=\sum_{n=0}^{\infty}\varepsilon^n(\tau_{ij})_n \qquad i,\ j=1,\ 2,\ \cdots,\ g$$

where $(\tau_{ij})_0=\tau_{ij}$ *and, for* $n=1,\ 2,\ \cdots$

(61)
$$(\tau_{ij})_n=\sum_{s,t=1}^{2}\sum_{l,m=1}^{l+m\leq n}\Omega_{n,st}^{lm}\gamma_{li}^{(s)}\gamma_{mj}^{(t)}-\sum_{s=1}^{2}n\gamma_{ni}^{(s)}\gamma_{nj}^{(s')}.$$

(ii)
$$\sigma_{i,\varepsilon}=\sum_{n=0}^{\infty}\varepsilon^n(\sigma_i)_n \qquad i=1,\ \cdots,\ g$$

where $(\sigma_i)_0=\int_{B_i}\omega_{p_2-p_1}=\int_{p_1}^{p_2}v_i$ *and, for* $n=1,\ 2,\ \cdots$,

(62)
$$(\sigma_i)_n=\sum_{s,t=1}^{2}\sum_{l,m=1}^{l+m\leq n}\Omega_{n,st}^{lm}\gamma_l^{(s)}\gamma_{mi}^{(t)}-\sum_{s=1}^{2}n\gamma_n^{(s)}\gamma_{ni}^{(s')}.$$

(iii)
$$\sigma_\varepsilon=\ln\varepsilon+\sum_{n=0}^{\infty}\varepsilon^n(\sigma)_n$$

where $(\sigma)_0=c_0$ *and, for* $n=1,\ 2,\ \cdots$,

(63)
$$(\sigma)_n=\sum_{s,t=1}^{2}\sum_{l,m=1}^{l+m\leq n}\Omega_{n,st}^{lm}\gamma_l^{(s)}\gamma_m^{(t)}-2n\gamma_n^{(1)}\gamma_n^{(2)}.$$

From Theorem 6, it is also possible to derive a variational formula for the prime form $E(x, y)$. For the basic properties of $E(x, y)$ the reader may consult [3, Chap. 2].

Since the multipliers of $E(x, y)$ and $E_\varepsilon(x, y)$ (the prime form for S_ε) along the cycles ρC_1 and ρC_2 are both equal to 1 (c.f. [3]), we can choose a single-valued branch of $\ln(E_\varepsilon(x, y)/E(x, y))$ over $S\backslash(\rho U_1\cup\rho U_2)$ canonically dissected so as to satisfy

(64)
$$\lim_{x,y\to q}\ln(E_\varepsilon(x, y)/E(x, y))=0$$

for any $q\in S\backslash(\rho U_1\cup\rho U_2)$ [3, Corollary 2.5]. From now on, all paths of integration are taken within a fixed canonical dissection containing ρU_1 and ρU_2. With this agreement we have

COROLLARY 7. *$\ln E_{\varepsilon}(x, y)$ has an expansion: for $x, y \in S \backslash (\rho U_1 \cup \rho U_2)$ with $|\varepsilon|^{1/2} < \rho < 1$,*

$$\ln E_{\varepsilon}(x, y) = \ln E(x, y) - \varepsilon \omega_{y-x}(p_1) \omega_{y-x}(p_2) + O(\varepsilon^2) \tag{65}$$

where the estimate $O(\varepsilon^2)$ is uniform.

More precisely.

$$\ln E_{\varepsilon}(x, y) = \ln E(x, y) - \sum_{n=1}^{\infty} n \varepsilon^n \int_x^y a_n^{(1)} \int_x^y a_n^{(2)} + \frac{1}{2} \sum_{n=1}^{\infty} \varepsilon^n \sum_{j,k=1}^{2} \sum_{l,m=1}^{l+m \leq n} \Omega_{n,jk}^{lm} \int_x^y a_l^{(j)} \int_x^y a_m^{(k)} . \tag{66}$$

Proof. First we note the identity [3, Corollary 2.6]: for $x, y \in S$,

$$\omega(x, y) = \frac{\partial^2}{\partial x \partial y} \ln E(x, y) dx dy . \tag{67}$$

To prove (66), set

$$F(x, y) = \ln (E_{\varepsilon}(x, y)/E(x, y)) + \sum_{n=1}^{\infty} n \varepsilon^n \int_x^y a_n^{(1)} \int_x^y a_n^{(2)} - \frac{1}{2} \sum_{n=1}^{\infty} \varepsilon^n \sum_{j,k=1}^{2} \sum_{l,m=1}^{l+m \leq n} \Omega_{n,jk}^{lm} \int_x^y a_l^{(j)} \int_x^y a_m^{(k)}$$

and consider $\frac{\partial^2}{\partial x \partial y} F(x, y)$. (67) and Theorem 6 show

$$\frac{\partial^2}{\partial x \partial y} F(x, y) = 0 .$$

On account of the symmetry $F(x, y) = F(y, x)$ (c.f. [3]), it is seen that $F(x, y)$ has the form

$$F(x, y) = h(x) + h(y)$$

where $h(x)$ is single-valued and holomorphic on $S \backslash (\rho U_1 \cup \rho U_2)$ canonically dissected, since $a_n^{(j)}(x)$ ($j=1, 2$ and $n=1, 2, \cdots$) has no residues at p_1 and p_2. (64) implies that $F(x, x)=0$ or $h(x)=0$, so that $F(x, y)=0$. This gives (66) while (65) is proved by recalling the identity (c.f. [3])

$$\omega_{a-b}(x) = \int_b^a \omega(\cdot, x) \tag{68}$$

and the proof is completed.

Remark. Let $g(x, y)$ be the Green's function on a planar regular region D. Then it can be verified that

$$g(x, y) = \ln \left| \frac{E(x, \bar{y})}{E(x, y)} \right| \qquad x, y \in D \tag{69}$$

where $E(x, y)$ is the prime form for the double of D with respect to a suitable canonical homology basis and $\bar{y}$ is the conjugate point of $y \in D$. (69) shows that Robin's constant $c(x)$ is given by

$$c(x)=\ln |E(x, \bar{x})| . \tag{70}$$

By using the representations (69) and (70), Corollary 7 will yield variational formulas for $g(x, y)$ and $c(x)$, but we do not enter into these calculations.

4. Examples.

To guarantee the validity of our formulas, we consider here two cases where $\omega_\varepsilon(x, y)$ can be calculated easily by other methods.

EXAMPLE 1. Let S_1 and S_2 be the extended complex plane $\hat{C}$. Then the fundamental normalized differential $\omega_j(x, y)$ is given by

$$\omega_j(x, y)=\frac{dxdy}{(x-y)^2} \qquad (j=1, 2). \tag{71}$$

If $|\kappa_j|<1$, the function $\phi_j(z)=\frac{1}{z}+\kappa_j z$ maps conformally the unit disk $\varDelta$ onto $U_j \subset \hat{C}$ with $\phi_j(0)=\infty$ $(j=1, 2)$. Hence it is possible to take ϕ_j^{-1} as a coordinate z_j: $U_j \rightarrow \varDelta$ on S_j centered at $p_j=\infty$ $(j=1, 2)$. Since S_ε has genus zero, it is well-known that, for any fixed $x \in S_1 \backslash U_1$, there exists a conformal mapping f_ε: $S_\varepsilon \rightarrow \hat{C}$ satisfying $f_\varepsilon(x)=\infty$. To calculate $\omega_\varepsilon(x, y)$ on S_ε, we shall first study the mapping f_ε itself. Let f_j be the restriction of f_ε to $S_j \backslash |\varepsilon| U_j$ $(j=1, 2)$, and assume without loss of generality that f_1 is holomorphic on $S_1 \backslash |\varepsilon| U_1$ except for a simple pole at x with residue 1. In view of the equation (5), f_1 and f_2 must satisfy

$$f_1(\phi_1(z))=f_2(\phi_2(\varepsilon/z)) \qquad \text{for} \quad |\varepsilon|<|z|<1 . \tag{72}$$

From the functional equations

$$\phi_j(z)=\phi_j\left(\frac{1}{\kappa_j z}\right) \qquad (j=1, 2) \tag{73}$$

(for simplicity assume $\kappa_1\kappa_2 \neq 0$ here, as in the sequel), it follows that (1) is extended meromorphically to the function $F(z)$ which is now defined on $0<|z|<\infty$. By (72) and (73) $F(z)$ satisfies

$$F(z)=F\left(\frac{1}{\kappa_1 z}\right)=F\left(\frac{\kappa_2\varepsilon^2}{z}\right),$$

so that

$$F(z)=F(\kappa_1\kappa_2\varepsilon^2 z), \qquad 0<|z|<\infty .$$

Thus $F(e^z)$ becomes a doubly periodic function with periods $2\pi i$ and

$\alpha(=\ln \kappa_1\kappa_2\varepsilon^2)$ which is holomorphic except for simple poles at $z\equiv\ln\beta$ and $z\equiv-\ln\kappa_1\beta$ (mod periods) with residues $\left(\kappa_1\beta-\frac{1}{\beta}\right)^{-1}$ and $-\left(\kappa_1\beta-\frac{1}{\beta}\right)^{-1}$ respectively; here β denotes a number satisfying $\phi_1(\beta)=x$. As is seen from the theory of elliptic functions, $F(e^z)$ has an explicit representation:

$$F(e^z)=\left(\kappa_1\beta-\frac{1}{\beta}\right)^{-1}[\zeta(z-\ln\beta)-\zeta(z+\ln\kappa_1\beta)]+\text{const.} \tag{74}$$

where $\zeta(z)=\zeta(z;\,2\pi\imath,\,\alpha)$ is the Weierstrassian zeta-function. On the other hand, it is well-known (see [4, p, 477]) that $\zeta(z)$ has a series expansion given by

$$\zeta(z)=\eta z+\frac{1}{2}\frac{e^z+1}{e^z-1}+\sum_{n=1}^{\infty}\frac{h^n e^{-z}}{1-h^n e^{-z}}-\sum_{n=1}^{\infty}\frac{h^n e^z}{1-h^n e^z} \tag{75}$$

where

$$h=e^\alpha=\kappa_1\kappa_2\varepsilon^2 \quad\text{and}\quad \eta=\frac{1}{2\pi\imath}(\zeta(z+2\pi i)-\zeta(z)).$$

Hence, if (75) is substituted in (74), we have

$$\begin{aligned}\left(\kappa_1\beta-\frac{1}{\beta}\right)F(z)&=\text{const.}+\frac{1}{2}\frac{z+\beta}{z-\beta}+\sum_{n=1}^{\infty}\frac{h^n}{z-h^n\beta}-\sum_{n=1}^{\infty}\frac{h^n z}{\beta-h^n z}\\&\quad-\frac{1}{2}\frac{\kappa_1\beta z+1}{\kappa_1\beta z-1}-\sum_{n=1}^{\infty}\frac{h^n}{\kappa_1\beta z-h^n}+\sum_{n=1}^{\infty}\frac{h^n\kappa_1\beta z}{1-h^n\kappa_1\beta z}\\&=\text{const.}+\frac{\kappa_1\beta-\beta^{-1}}{\phi_1(z)-x}+\sum_{n=1}^{\infty}\sum_{d=1}^{\infty}h^{nd}\kappa_1^{-d}(\kappa_1^d\beta^d-\beta^{-d})(z^{-d}+\kappa_1^d z^d).\end{aligned} \tag{76}$$

Observe that, for fixed $x\in S_1\backslash U_1$, $\omega_\varepsilon(x,\,\phi_1(z))$ is given by

$$\omega_\varepsilon(x,\,\phi_1(z))=-F'(z)/\phi_1'(z).$$

Thus, if (76) is differentiated, it follows that

$$\omega_\varepsilon(x,\,\phi_1(z))=\frac{1}{(\phi_1(z)-x)^2}-\sum_{n=1}^{\infty}\sum_{d=1}^{\infty}d\kappa_1^{-d}h^{nd}\frac{(\kappa_1\beta)^d-\beta^{-d}}{\kappa_1\beta-\beta^{-1}}\frac{(\kappa_1 z)^d-z^{-d}}{\kappa_1 z-z^{-1}} \tag{77}$$

for $x,\,\phi_1(z)\in S_1\backslash U_1$. To show that (77) agrees with the expansion given by Theorem 4, let us determine the differentials $a_n(x)$ and the coefficients $\Omega^{hk}_{n,11}$. By (71) and the definition (11), $a_n^{(1)}(x)$ $(n=1,\,2,\,\cdots)$ are given by the expansion

$$\sum_{n=1}^{\infty}a_n^{(j)}(x)z^n=\frac{1}{x-\phi_j(z)}\qquad(j=1,\,2),$$

and thus, after easy calculation, it is seen that

$$a_n^{(1)}(x)=\frac{(\kappa_1\beta)^d-\beta^{-d}}{\kappa_1\beta-\beta^{-1}},\qquad a_n^{(1)}(\phi_1(z))=\frac{(\kappa_1 z)^d-z^{-d}}{\kappa_1 z-z^{-1}}. \tag{78}$$

If (78) is substituted in (77), we conclude finally

$$(79)\qquad \omega_\varepsilon(x,\ y)=\frac{1}{(x-y)^2}-\sum_{n=1}^{\infty}h^n\sum_{d\mid n}d\kappa_1^{-d}a_d^{(1)}(x)a_d^{(1)}(y)$$

with $h=\kappa_1\kappa_2\varepsilon^2$. On the other hand, one verifies that

$$(80)\qquad \alpha_{nm}=-\delta_{nm}\kappa_j^n \qquad (j=1,\ 2)$$

with δ_{nm} the Kronecker δ, since the expansion (14) has the form

$$\phi_j^*\omega_j(x,\ y)=\frac{1}{(x-y)^2}-\frac{\kappa_j}{(1-\kappa_j xy)^2}=\frac{1}{(x-y)^2}-\sum_{n=0}^{\infty}(n+1)\kappa_j^{n+1}x^n y^n .$$

From (80) and (25) the variational coefficients $\Omega_{n,11}^{hk}$ are easily calculated. The result is that $\Omega_{n,11}^{hk}$'s all vanish except when n is even and $h=k$. In the exceptional case, $\Omega_{2n,11}^{dd}$ is given by

$$\Omega_{2n,11}^{dd}=\begin{cases} -d(\kappa_1\kappa_2)^n\kappa_1^{-d} & \text{if } \ d\mid n\,, \\ 0 & \text{otherwise.}\end{cases}$$

Hence (79) completely agrees with our Theorem 4.

EXAMPLE 2. With the same notation as in section 3, we set:

$$S=\hat{C},\ U_1=\{z\,;\ |z|<r\},\ U_2=\{z\,;\ |z|>R\},\ p_1=0,\ p_2=\infty,\ z_1=z/r,\ z_2=R/z$$

where r and R are numbers satisfying $0<r<R$. Thus, by (5), $z\in U_1$ and $w\in U_2$ are identified if and only if $z=\dfrac{\varepsilon r}{R}w$. Similar reasoning as in example 1 at once shows

$$(81)\qquad \omega_\varepsilon(x,\ y)=[\wp(\ln x/y)-\eta]\frac{dxdy}{xy}\,, \qquad x,\ y\in S\backslash(U_1\cup U_2)$$

where $\wp(z)=\wp(z\,;\ 2\pi i,\ \ln \varepsilon r/R)$ is the Weierstrassian pe-function with $\eta=\dfrac{1}{2\pi i}(\zeta(z+2\pi i)-\zeta(z))$. Again, it is well-known (see [4, p. 477]) that $\wp(z)$ has an expansion given by

$$(82)\qquad \wp(z)-\eta=\frac{e^z}{(e^z-1)^2}+\sum_{n=1}^{\infty}\frac{h^ne^{-z}}{(1-h^ne^{-z})^2}+\sum_{n=1}^{\infty}\frac{h^ne^z}{(1-h^ne^z)^2}$$

with $h=\varepsilon r/R$. Thus (81) and (82) give

$$(83)\qquad \omega_\varepsilon(x,\ y)=\frac{1}{(x-y)^2}+\sum_{n=1}^{\infty}\varepsilon^n\sum_{\substack{d\mid n\\ d>0}}d(r/R)^n\Big(\frac{y^{d-1}}{x^{d+1}}+\frac{x^{d-1}}{y^{d+1}}\Big).$$

On the other hand, from (71) and (11) it is seen that

$$a_n^{(1)}(x)=\frac{r^n}{x^{n+1}}\,,\qquad a_n^{(2)}(x)=-\frac{x^{n-1}}{R^n}\qquad (n=1,\ 2,\ \cdots)$$

and

$$\alpha_{jk}^{11}=\alpha_{jk}^{22}=0\,,\qquad \alpha_{jk}^{12}=\alpha_{jk}^{21}=-\delta_{jk}(r/R)^{j}\,,\qquad (j,\ k=1,\ 2,\ \cdots)\,.$$

Hence, by (25)'', $\Omega_{n,jk}^{lm}$'s all vanish except when $l=m$ and $j=k$. In the exceptional case, we have

$$\Omega_{n,12}^{dd}=\Omega_{n,21}^{dd}=\begin{cases} -d(r/R)^{n-d} & \text{if } \ d\,|\,n\ (d<n)\,,\\ 0 & \text{otherwise,}\end{cases}$$

concluding that (83) agrees with our Theorem 6.

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