Precoloring Extension. III. Classes of Perfect Graphs

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Abstract. We continue the study of the following general problem on vertex colorings of graphs. Suppose that some vertices of a graph G are assigned to some colors. Can this "precoloring" be extended to a proper coloring of G with at most k colors (for some given k)? Here we investigate the complexity status of precoloring extendibility on some classes of perfect graphs, giving good characterizations (necessary and sufficient conditions) that lead to algorithms with linear or polynomial running time. It is also shown how a larger subclass of perfect graphs can be derived from graphs containing no induced path on four vertices.

1. Introduction

We consider finite undirected graphs $G = (V, E)$ with vertex set V and edge set E. The *clique number* or *maximum clique size* and the *chromatic number* of G are denoted by $\omega(G)$ and $\chi(G)$, respectively. The *independence number* (i.e., the clique number of the complementary graph) is denoted by $\alpha(G)$. For any vertex set $W \subseteq V$, G_W denotes the subgraph induced by W. By definition, for a given integer $k \geq 1$, a (proper) *k-coloring* is a function $f : V \to \{1, 2, \ldots, k\}$ such that $uv \in E$ implies $f(u) \neq f(v)$.

The problem we investigate in this paper was initiated in [1, 2] and is called the *PRECOLORING EXTENSION* problem, or *PrExt* in short. It can be formulated as follows.

Instance. An integer $k \geq 1$, a graph $G = (V, E)$ with $|V| \geq k$, a vertex subset $W \subseteq V$, and a proper k-coloring φ of G_W.

Question. Can φ be extended to a proper k-coloring of the entire graph G?

For an instance of PrExt we say that k is the *color bound*, and G is a *precolored* or a *partially k-colored* graph. The vertices of W and of $V - W$ are called *precolored* and *precolorless*, respectively. The *precolored classes* are the sets $W_i = \{x \in W : f(x) = i\}$, $i = 1, 2, \ldots, k$. The number of nonempty precolored classes, $|f(W)|$, is called the *precolor number*. We note that the extendibility of a precoloring with color bound $k = 1$ simply means that the graph has no edge and the precolor number is at most one. One can also see that PrExt is slightly less general than *LIST-COLORING* [30]; namely, in PrExt, the list of admissible colors of a precolorless vertex is $\{1, 2, \ldots, k\}$, while that of a precolored one has length just one.

Restricted instances. The *MONOCOLORING PROBLEM* is the special version of PrExt where the precolor number is one. Moreover, given a nonnegative integer d, the subproblem *d-PrExt* is defined as the problem in which the instances of PrExt are restricted to those partially k-colored graphs where the size of each precolored class is at most d. Note that 0-PrExt is equivalent to the usual *CHROMATIC NUMBER* problem, i.e., "Is $\chi(G) \leq k$?".

1.1. Background and summary

Many important problems in graph theory and combinatorial optimization, and some practical problems as well, can be formulated as certain instances of PrExt. We refer the reader to the papers [2, 3, 15, 20, 28] where, beside the original motivation coming from scheduling and potential applications in VLSI theory, several connections with a number of interesting and extensively studied concepts (such as e.g. *bipartite matchings*, *partial Latin squares*, *integer-valued multicommodity flows*, *coloring games*) are discussed.

In this paper we concentrate on the solvability of PrExt on various types of *perfect* graphs, including trees, bipartite graphs and their complements, split graphs, cographs, interval graphs, and complements of Meyniel graphs. Some particular instances of PrExt on the entire class of perfect graphs also are considered.

One reason justifying the study of PrExt on those particular classes is that the general problem is surprisingly hard to solve. Namely, even 1-PrExt is NP-complete on bipartite graphs, as well as 2-PrExt on interval graphs (see [20] and [2], respectively). Our general aim can be summarized as follows.

Problem 1.1. Let $\mathcal{G}$ be a given class of graphs.

(1) Decide whether PrExt (or d-PrExt, for some given $d \geq 1$) is NP-complete, or polynomially solvable, or is solvable even by a linear algorithm on the precolorings of the graphs $G \in \mathcal{G}$.

(2) Find easy-to-check necessary and sufficient conditions for a precoloring of $G \in \mathcal{G}$ to be extendible.

The paper is organized in a way strongly motivated by Problem 1.1. In Section 2 we present several conditions that are necessary (in every graph or in some particular classes) for the extendibility of precolorings. In Section 3, classes of graphs are considered for which the conditions in question are not only necessary but also sufficient. One of them, namely the *"core condition"* (cf. Section 2.2) leads to a further observation (Theorem 4.1), interesting on its own as well, that enables us to derive a wider subclass of perfect graphs from the class of cographs. The algorithmic consequences of the characterization theorems of Section 3 are discussed in Section 8.

The proofs of the characterizations (Sections 5 through 7) are grouped according to the conditions involved. On the other hand, the algorithmic results are ordered by time complexity: Section 8.1 presents linear algorithms, Section 8.2 deals with polynomial-time algorithms of a purely combinatorial nature, while Section 8.3 gives some applications of the ellipsoid method.

Some open problems are mentioned in the concluding section.

1.2. Notation and terminology

We apply the usual notation K_n, C_n, P_n for the *complete* graph, the *cycle*, and the *path* on n vertices, respectively. Here n is the *length* of C_n and of P_{n+1}, and if n is an odd number, we say that a graph isomorphic to C_n (P_{n+1}) is an *odd cycle* (*odd path*). We also use the notions *line* graph, *bipartite* graph, *cograph*, *interval* graph, *chordal* graph, *split* graph, *Meyniel* graph, and *perfect* graph in the usual sense. Let us recall that the class of cographs is the smallest class that contains the one-vertex graph and is closed under vertex-disjoint union and complementation. (Thus the complement of any connected cograph is disconnected.) Equivalently, this is the class of graphs containing no induced 4-path. A split graph is a chordal graph whose complement is also chordal. Equivalently, a graph is a split graph if and only if a 4-cycle or a 5-cycle is induced neither in the graph nor in its complement. In a split graph one clique and one independent set together cover all vertices (cf. [8, 14]). A Meyniel graph is a graph in which every odd cycle of length at least five contains at least two chords (i.e., edges joining nonconsecutive vertices on the cycle). Deleting all edges of an induced subgraph of a Meyniel graph, a *slim* graph is obtained (which is also known to be perfect [16]).

Given a graph $G = (V, E)$ and a subset $U \subseteq V$, $N(U)$ denotes the set of those vertices of $V - U$ which have at least one neighbor in U.

2. Obstacles for non-extendibility of a precoloring

Given a partially k-colored graph $G = (V, E)$ with precolored vertex set W, it is obvious that $\chi(G) \leq k$ is a necessary condition for the extendibility of the precoloring. However, even if this condition is fulfilled, there can be other obstacles, too.

2.1. Integer combinations of cliques

We shall consider sets of characteristic vectors over V for a graph $G = (V, E)$. For any $U \subseteq V$, by the *characteristic vector* of U over V we mean the $|V|$-dimensional column vector $\mathbf{u}$ with 0 or 1 in each component such that, considering the elements of V in a fixed order, the ith component of $\mathbf{u}$ is 1 (0) if the ith element of V is (is not) in U, $i = 1, 2, \ldots, |V|$. Also, for any set $\mathcal{K}$ of characteristic vectors over V, let $\Sigma\mathcal{K}$ denote the sum of the vectors of $\mathcal{K}$.

Let W be any subset of V with characteristic vector $\mathbf{w}$ over V, and let $\mathcal{L}$ ($\mathcal{M}$) denote the set of the characteristic vectors of all *maximal (maximum) cliques* of G. A subset $\mathcal{K} \subseteq \mathcal{M}$ is called a W*-knot* if there exist strictly less than $|W| + |\mathcal{K}|$ (not necessarily distinct) vectors in $\mathcal{L}$ such that their sum (as a column vector) is at least $\mathbf{w} + \Sigma\mathcal{K}$. Observe that the existence of any W-knot implies $|W| \geq 2$ and that $\mathcal{M}$ is also a W-knot. We can formulate the following condition.

Knot condition. No W-knot exists for any precolored class W.

Given two vertices $v_i \in V$ with characteristic vectors $\mathbf{v}_i$ of $\{v_i\}$ over V, $i = 1, 2$, we say that some (not necessarily distinct) vectors $\mathbf{x}_{ij} \in \mathcal{M}$, $i = 1, 2$, $j = 1, 2, \ldots, m$, *span* v_1 *and* v_2 if $\mathbf{v}_1 + \mathbf{x}_{11} + \ldots + \mathbf{x}_{1m} = \mathbf{v}_2 + \mathbf{x}_{21} + \ldots + \mathbf{x}_{2m}$.

Span condition. Any two spanned precolored vertices have the same color.

A special case of the following lemma was first observed by Sebő [29] in connection with the monocoloring problem on perfect graphs.

Lemma 2.1. If G is a partially k-colored graph and $k = \omega(G)$, then both the knot and span conditions are necessary for the precoloring extendibility.

This lemma will be proved in Section 7.

2.2. Local constraints

Considering an arbitrary partially k-colored graph $G = (V, E)$ with precolored classes W_i, $i = 1, \ldots, k$, a subset $U \subseteq V$ consisting of *pairwise adjacent* precolorless vertices is called a q*-core* if $1 \leq |U| \leq q$ and the number of those precolored classes W_i for which $U \subseteq N(W_i)$ is at least $q - |U|$. If $|U| = 1$, then we call U an *elementary* q*-core.*

Core condition. In a partially k-colored graph no $(k+1)$-core exists.

Obviously, the core condition is stronger than the assumption $k \geq \omega(G)$.

Starting with a partial k-coloring of $G = (V, E)$, we repeat the following procedure until it terminates. Test whether G contains a $(k+1)$-core. If this is the case then stop. Also stop if G contains no elementary k-core. Otherwise choose an elementary k-core $\{u\}$. Then there is a unique color i for which

$u \notin N(W_i)$. Color u with i, and repeat these steps in the new partial k-coloring.

Sequence condition. In a partially k-colored graph no sequence described above terminates with a $(k+1)$-core.

Obviously, the sequence condition is stronger than the core condition.

Since the colors of the vertices involved in the sequence condition are uniquely determined within the given color bound, the following observation is valid.

Lemma 2.2. If $G = (V, E)$ is a partially k-colored graph, then both the core condition and the sequence condition are necessary for the precoloring extendibility. □

2.3. Vertex decompositions

Consider an arbitrary partially k-colored graph $G = (V, E)$ with precolored classes W_i, $i = 1, \ldots, k$. For any $U \subseteq V$, let $\alpha_i(U)$ be 0 if $U \subseteq N(W_i)$; otherwise let it denote the independence number of the subgraph induced by $U - N(W_i)$.

Independence condition. In a partially k-colored graph, $|U| \leq \alpha_1(U) + \ldots + \alpha_k(U)$ holds for every set U of precolorless vertices.

The following lemma is a natural generalization of an observation of Marcotte and Seymour [28]; its validity is an easy consequence of the definitions.

Lemma 2.3. The independence condition is necessary for the precoloring extendibility. □

2.4. Menger path systems

Let $G = (V, E)$ be a partially k-colored interval graph with precolored classes W_i, $i = 1, \ldots, k$; for simplicity assume that $V = \{1, \ldots, n\}$. Let $k' \leq k$ denote the precolor number; without loss generality we may assume that $i \in W_i$ for $i = 1, \ldots, k'$, and that $W_i = \emptyset$ for $i = k' + 1, \ldots, k$. Since G is an interval graph, it contains at most n maximal cliques, and the vertex sets of those cliques can be listed as $C^1, \ldots, C^m$ such that for each vertex x the cliques containing x are consecutive. For each $i \in V$, let a_i (b_i) denote the smallest (largest) subscript j for which $i \in C^j$.

Having fixed such an ordering of the maximal cliques, define an acyclic multidigraph on the vertex set $\{-1, 0, 1, \ldots, m\}$ containing four types of arcs:
(1) an arc from -1 to b_i for $i = 1, \ldots, k'$;
(2) $k - k'$ copies of the arc from -1 to 0;

(3) an arc from $a_i - 1$ to b_i for $i = k' + 1, \ldots, k$;
(4) $k - |C^j|$ copies of the arc from $j - 1$ to j for any $j \in \{1, 2, \ldots, m\}$ with $|C^j| < k$.

Let M_j, $j = 1, 2, 3, 4$, denote the multiset of the arcs of type (j). Any proper k-coloring of G defines the partition of each C^j into its elements. Considering the heads b of the arcs of our multidigraph as elements of C^b, we gain a natural partition of $M_1 \cup M_3$ into k classes. Therefore, using the additional arcs of $M_2 \cup M_4$, one can easily find k pairwise arc-disjoint oriented paths from -1 to m. Applying the trivial half of the famous Menger theorem for our acyclic multidigraph, we obtain the following necessary condition for the precoloring extendibility.

Menger condition. For any $D \subseteq \{0, 1, \ldots, m - 1\}$, the total number of arcs from $\{-1, 0, \ldots, m - 1\} - D$ to $D \cup \{m\}$ is at least k.

Lemma 2.4. The Menger condition is necessary for the precoloring extendibility on a precolored interval graph. □

3. Good characterizations

For some graph classes, the necessary conditions studied in Section 2 provide us with good characterizations (i.e., necessary and sufficient conditions that can be tested in polynomial time) for the precoloring extension problem. In this section we just formulate these results; the proofs can be found in Sections 5 through 7.

Let $G = (V, E)$ be a partially k-colored graph with precolored classes W_i, $i = 1, \ldots, k$. We begin with the simplest nontrivial case, $k = 2$. Certainly, it is necessary to exclude odd cycles; therefore, assume for the moment that G is bipartite.

Proposition 3.1. For precolored bipartite graphs, if the color bound is 2, the following properties are equivalent.
(1) The precoloring is extendible.
(2) Both the knot and span conditions are satisfied.
(3) The sequence condition is satisfied.

From now on, the color bound can be arbitrarily large.

Proposition 3.2. In a partially k-colored forest, a precoloring is extendible if and only if the sequence condition is satisfied.

If the color bound is 2, then P_5 with endpoints precolored with distinct colors is the simplest forest for which the core condition alone is not sufficient for the precoloring extendibility. If the color bound is three, a simple example is shown in Fig. 1. (The numbers indicate the colors of the precolored vertices.)

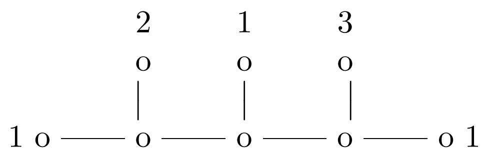

Figure 1

For any integer $m \geq 2$, by a P_m-free graph we mean a graph not containing P_m as an *induced* sugraph. The P_2-free graphs are the edgeless graphs, the P_3-free graphs are those whose connected components are cliques, and the P_4-free graphs are the cographs. The P_5-free graphs were also investigated in many papers, e.g., in [4] and [19].

Theorem 3.3. In a P_5-free bipartite graph, a precoloring is extendible if and only if the core condition is satisfied.

The partially 3-colored graphs exhibited in Fig. 1 and Fig. 2 show that P_5 can be changed neither to C_6 nor to P_6 in Theorem 3.3. What is more, in Section 8 we prove that PrExt is NP-complete on P_6-free bipartite graphs (Theorem 8.5).

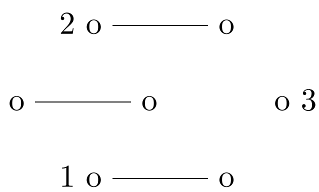

Figure 2

The graph C_5 (with no precolored vertices) shows that, without assuming bipartiteness, the exclusion of induced P_5 is not strong enough to ensure the sufficiency of the core condition. It will suffice, however, if we exclude P_4.

Theorem 3.4. In a cograph, a precoloring is extendible if and only if the core condition is satisfied.

In order to have a convenient formulation of the following four results, without loss of generality *we assume that* $|W_1| \geq |W_2| \geq \ldots \geq |W_k|$ holds for the precolored classes W_i. Note that if $|W_1| = 1$, then PrExt coincides with 1-PrExt, and the assumption $|W_2| = 0$ restricts PrExt to the monocoloring problem.

Theorem 3.5. If G is the complement of a Meyniel graph and $|W_1| \leq 1$, then a precoloring is extendible if and only if the core condition is satisfied.

The following two theorems (although in forms different from ours) were first noticed by Sebő [29].

Theorem 3.6. If G is a perfect graph, $3 \le k = \omega(G)$, and $|W_2| = 0$, then a precoloring is extendible if and only if the knot condition is satisfied.

Theorem 3.7. If G is a perfect graph, $3 \le k = \omega(G)$, $|W_1| = |W_2| = 1$, and $|W_3| = 0$, then a precoloring is extendible if and only if the span condition is satisfied.

We can give a common generalization of the above two good characterizations as follows. Assume that $|W_2| \le 1$. Let $r \in \{1, 2, \dots, k\}$ be the largest integer for which $W_r \ne \emptyset$ and let $W_i = \{w_i\}$, $i = 2, \dots, r$. Define a new graph G^0 from G by adding a new vertex w_0, the new edges $w_0 w_i$, $i = 2, \dots, r$, and for any $2 \le i < j \le r$, the new edge $w_i w_j$ if it was not an edge in G.

Theorem 3.8. If G^0 is perfect, $3 \le k = \omega(G)$ and $|W_2| \le 1$, then the given partial k-coloring of G is extendible if and only if no $(\{w_0\} \cup W_1)$-knot exists in G^0.

We note that if G is perfect and $r \le 2$, then G^0 is also perfect. This is not the case, however, if $r \ge 3$, even in the rather simple case $G = P_5$, where $k = 3$, the two endpoints are precolored with colors 2 and 3, and a third vertex is precolored with color 1. On the other hand, if G is the complement of a Meyniel graph, then G^0 is always perfect. This follows from the following facts: for $W = \{w_0, w_2, \dots, w_r\}$, G_W is a maximal clique of G^0, and the complement of G^0_V is a *slim* graph which is perfect (cf. [16]).

The following theorems show the sufficiency of the core condition for some further classes of perfect graphs.

Theorem 3.9. In the complements of bipartite graphs, a precoloring is extendible if and only if the core condition is satisfied.

Theorem 3.10. In split graphs, a precoloring is extendible if and only if the core condition is satisfied.

The *line graph of a multiforest* is a (chordal) graph whose vertices are represented by not necessarily distinct edges of a forest, and two such edges are adjacent if they share at least one vertex in the forest. The following theorem was proved by Marcotte and Seymour.

Theorem 3.11. [28] In line graphs of multiforests, a precoloring is extendible if and only if the independence condition is satisfied. □

Finally, we recall a good characterization for a problem studied by Biró and the authors.

Theorem 3.12. [2] An instance of 1-PrExt with color bound k is extendible on an interval graph G if and only if $\omega(G) \le k$ and the Menger condition is satisfied. □

4. Generating perfect graphs

Let $\mathcal{G}$ be a class of graphs such that any induced subgraph of any member of $\mathcal{G}$ is also in $\mathcal{G}$. Now we produce a new class $\mathcal{G}^*$ from $\mathcal{G}$ in the following way. Let $G \in \mathcal{G}$, consider any (proper) partial k-coloring of G, unite the vertices in each precolored class, and make the "united" vertices pairwise adjacent. We denote by G^* the graph obtained, and define $\mathcal{G}^*$ as the set of all graphs G^* obtained from all precolorings of all $G \in \mathcal{G}$. Obviously, G^* coincides with G if no vertex is precolored (and also if the precolored vertices induce a complete subgraph in G), thus $\mathcal{G}^*$ contains $\mathcal{G}$. It is also straightforward that any induced subgraph of any member of $\mathcal{G}^*$ is also in $\mathcal{G}^*$.

Theorem. 4.1. If every $G \in \mathcal{G}$ is perfect, and for every precoloring of every $G \in \mathcal{G}$ the core condition is sufficient for the precoloring extendibility, then every $G^* \in \mathcal{G}^*$ is a perfect graph.

A class of perfect graphs which satisfies the assumptions of the above theorem will be called *PrExt-perfect.* From the results of the previous section we obtain that the cographs, the P_5-free bipartite graphs, the complements of bipartite graphs and the split graphs are all PrExt-perfect. The most interesting class is extracted in the following result which is an easy corollary of Theorems 3.4 and 4.1.

Theorem. 4.2. If G is a cograph with an arbitrary partial k-coloring, then G^* is perfect. □

5. The sequence condition

In this section we prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Observe the following facts for a partial 2-coloring of a bipartite graph $G = (V, E)$ with at least one edge. The set of all maximum cliques is the set of all edges. Two vertices are spanned if and only if there is an even path connecting them. For any W-knot there is an odd path with endpoints in W.

(1) ⇒ (3): By Lemma 2.2.

(3) ⇒ (2): Any even path with differently precolored vertices or any odd path with endpoints precolored with the same color shows that the sequence condition is not satisfied.

(2) ⇒ (1): Assuming (2) and that no proper 2-coloring extension exists, consider a bipartition $V = V_1 \cup V_2$, and consider a connected component for which the bipartition defines no proper 2-coloring extension. Then there exist distinct vertices v_1 and v_2 precolored with colors $c, d \in \{1, 2\}$, respectively. Without loss of generality we may assume that $v_1 \in V_1$ and $c = 1$. If for any possible

choice of vertex v_2 in the connected component $v_2 \in V_d$ holds, then the bipartition defines a proper 2-coloring extension; thus there is a $v_2 \notin V_d$. Since we are in a connected component of a bipartite graph, between the endpoints v_1 and v_2 we gain either an even path with differently precolored vertices (if $d = 2$) or an odd path with precolored vertices of the same color (if $d = 1$). This contradiction completes the proof. □

Proof of Proposition 3.2. The necessity of the condition holds by Lemma 2.2. To prove the sufficiency, let $G = (V, E)$ be a partially k-colored forest, and suppose that the assertion is valid for every forest on fewer than $n = |V|$ vertices, and for every forest on n vertices in which the number of precolorless vertices is smaller than that in G. (If $n < k$ or all vertices are precolored, we have nothing to prove.)

Let $v_1, v_2, \ldots, v_q$ be a longest path of precolorless vertices in G. Then v_2 is the unique precolorless neighbor of v_1. If $\{v_1\}$ is an elementary k-core, there is precisely one admissible color, say i, at v_1. Assigning color i to v_1, we obtain a precolored graph still satisfying the sequence condition but with fewer precolorless vertices, implying that the precoloring is extendible. On the other hand, if $\{v_1\}$ is not an elementary k-core, at most $k - 2$ colors occur in the neighborhood of v_1. The graph $G - \{v_1\}$ also satisfies the sequence condition, and has fewer than n vertices, therefore the precoloring can be extended. In the extension, v_2 is the unique vertex that can exclude a further color from v_1; hence at least one admissible color remains for v_1, and the precoloring is extendible on G as well. □

6. The core condition

In this section we prove Theorems 3.3, 3.4, 3.5., 3.9, 3.10 and 4.1. For the theorems of Section 3 the necessity of the core condition is proved by Lemma 2.2; thus we have to show its sufficiency for the precoloring extendibility. This is also obvious if the precolorless vertices induce no edge. In general, on the contrary to the sufficiency we may consider a minimal graph $G = (V, E)$ with precolored vertex set W for which the precoloring is not extendible, the core condition is satisfied, and for which $|V| + |E| - |W|$ is as small as possible. Among other things this implies that G is connected, for any proper induced subgraph of G the precoloring is extendible, and the subgraph induced by the precolorless vertices is connected.

Proof of Theorem 3.3. Consider a minimal counterexample connected bipartite P_5-free graph $G = (V, E)$. Let $V = X \cup Y$ be a bipartition of G with $|X - W| \geq |Y - W|$. Since the precolorless vertices induce at least one edge, $|Y - W| > 0$. We shall prove $|X - W| = |Y - W| = 1$. Assume that $|X - W| \geq 2$; let $x', x'' \in X - W$ be two distinct vertices such that $|N(x')| \leq |N(x'')|$. Since G

is a connected P_5-free bipartite graph, we obtain that $N(x') \subseteq N(x'')$. Observe that the nonextendibility of the precoloring is preserved if we delete x' (since in case of extendibility x' could get the same color as x''); this is impossible because of the minimality of G. Therefore $|X - W| = 1$, and also $|Y - W| = 1$.

Consider the unique $x \in X - W$ and the unique $y \in Y - W$. Since neither $\{x\}$, nor $\{y\}$, nor $\{x, y\}$ is a $(k+1)$-core, there is an admissible color on each of x and y, and there are at least two admissible colors on $\{x, y\}$; hence the precoloring is extendible. This contradiction completes the proof. □

Before proving the sufficiency of the core condition for cographs, let us extend the definition of q-core for the empty set, too, as follows. We say that $\emptyset$ is a q-core if the precoloring contains at least q distinct colors. (Since we have assumed that the precoloring uses at most k colors, $\emptyset$ cannot be a $(k+1)$-core; therefore the validity of the core condition ramins unchanged.)

Proof of Theorem 3.4. Consider a minimal counterexample connected cograph $G = (V, E)$. Note that $|V| \geq 2$. Since G is a cograph, there exists some proper subset U of V such that for any $u \in U$ and $v \in V - U$, $uv \in E$. Let k' (k'') denote the smallest integer for which the precoloring of G_U (G_{V-U}) is extendible by using at most k' (k'') colors. By the minimality of G there is a k'-core in G_U and a k''-core in G_{V-U}. It is clear that their union is a $(k' + k'')$-core in G. Since G contains no $(k+1)$-core, $k' + k'' \leq k$. Therefore, the precoloring of G is extendible. □

Proof of Theorem 3.5. Consider a minimal counterexample connected co-Meyniel graph $G = (V, E)$. Add an extra edge between any two nonadjacent precolored vertices. Since we gain the complement of a slim graph (cf. [16]), the new graph is perfect. The core condition for the original graph means that the clique number of the new graph is not larger than the color bound for the original precolored graph. Therefore we are home by the perfectness of the new graph. □

In the next two proofs we apply the famous König theorem [23, 24].

Proof of Theorem 3.9. Consider a minimal counterexample connected graph $G = (V, E)$ which is the complement of a bipartite graph. Then there exists a proper subset U of V for which both G_U and G_{V-U} are complete subgraphs. Define a bipartite graph with bipartition $U \cup (V - U)$ by connecting a $u \in U$ and a $v \in V - U$ if they are nonadjacent in G and either at least one of them is precolorless or both are precolored with the same color. The core condition for G implies that this bipartite graph contains no more than k independent vertices. By the König theorem for the new bipartite graph we gain $|V| - k$ independent edges. For the original graph they give us a proper precoloring extension. □

Proof of Theorem 3.10. Consider a minimal counterexample connected split graph $G = (V, E)$. Then there exists a proper subset U of V for which G_U is a complete subgraph and $V - U$ is an independent set. Let $Z \subseteq V$ denote the set of the precolorless vertices; then G_Z contains at least one edge. Since the core condition is satisfied, we are home by induction if $V - U$ contains some precolorless vertex. So we may assume that $Z \subseteq U$.

Define a bipartite graph with bipartition $Z \cup \{1, 2, ..., k\}$ as follows. For any $z \in Z$ and $i \in \{1, \dots, k\}$ let zi be an edge in the bipartite graph if z has no precolored neighbor of color i in G. Observe that the core condition implies that for any $Z' \subseteq Z$, if $G_{Z'}$ is a complete subgraph then in the bipartite graph $|N(Z')| \geq |Z'|$ holds. Therefore, if $Z \subseteq U$, by the König theorem we obtain that there is a matching of size $|Z|$ in the bipartite graph, i.e. the partial k-coloring of G is extendible. □

Proof of Theorem 4.1. Observe that if H is an induced subgraph of G, then H^* is also an induced subgraph (more precisely, is isomorphic to an induced subgraph) of G^*. Conversely, any induced subgraph F of G^* can also be obtained as $F = H^*$ for some induced subgraph H of G. Therefore it suffices to show that if the conditions of Theorem 4.1 hold then $\chi(G^*) = \omega(G^*)$. To avoid the trivial cases we may assume that $\chi(G^*) > \omega(G^*) \geq 2$. Let $k = \chi(G^*) - 1$, and let w_i, $i = 1, 2, \dots$, denote those vertices of G^* which were obtained as "united" vertices. Since the united vertices are pairwise adjacent in G^*, the number of vertices w_i is at most $\omega(G^*) \leq k$. Now in $G = (V, E)$, let us create a partial k-coloring for which the ith precolored class W_i consists of exactly those vertices $w \in V$ which were united to obtain w_i, $i = 1, 2, \dots$. This partial k-coloring of G is not extendible since $\chi(G^*) > k$.

Since the core condition is sufficient for the precoloring extendibility in G, there exists a $(k+1)$-core. However, from this $(k+1)$-core we gain a complete subgraph on $k + 1$ vertices in G^*. Therefore $\omega(G^*) \geq k + 1 = \chi(G^*)$. This contradiction completes the proof. □

7. Knots and spans

In this section we first prove Lemma 2.1, then Theorems 3.6 and 3.8, and finally Theorem 3.7. Actually, Theorem 3.8 will be deduced from Theorem 3.6, and Theorem 3.7 will be deduced from Theorem 3.8.

Proof of Lemma 2.1. Consider a fixed proper k-coloring extension of the given partial k-coloring of $G = (V, E)$. Since $k = \omega(G)$, each color occurs in each maximum clique.

Knot condition: For an arbitrary precolored class W_i of color i, let $U \supseteq W_i$ denote the color class with color i in the proper k-coloring extension. Let $\mathbf{u}$ ($\mathbf{w}_i$) denote the characteristic vector of U (W_i) over V; thus $\mathbf{u} \geq \mathbf{w}_i$. Assume that for distinct $\mathbf{y}_1, \ldots, \mathbf{y}_q \in \mathcal{M}$ and for fewer than $|W_i| + q$ (not necessarily distinct) $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathcal{L}$, the relation $\mathbf{x}_1 + \ldots + \mathbf{x}_m \geq \mathbf{w}_i + \mathbf{y}_1 + \ldots + \mathbf{y}_q$ holds. Multiplying both sides from the left by $\mathbf{u}^{\mathrm{T}}$, the transpose of $\mathbf{u}$, we obtain that $\mathbf{u}^{\mathrm{T}}\mathbf{x}_1 + \ldots + \mathbf{u}^{\mathrm{T}}\mathbf{x}_m \geq \mathbf{u}^{\mathrm{T}}\mathbf{w}_i + \mathbf{u}^{\mathrm{T}}\mathbf{y}_1 + \ldots + \mathbf{u}^{\mathrm{T}}\mathbf{y}_q$. Since $\mathbf{u}^{\mathrm{T}}\mathbf{w}_i = |W_i|$ and $\mathbf{u}^{\mathrm{T}}\mathbf{x} \leq 1$ for any $\mathbf{x} \in \mathcal{L}$, we have $m \geq |W_i| + \mathbf{u}^{\mathrm{T}}\mathbf{y}_1 + \ldots + \mathbf{u}^{\mathrm{T}}\mathbf{y}_q$. Since $m < |W_i| + q$, by the *pigeonhole principle* $\mathbf{u}^{\mathrm{T}}\mathbf{y}_p = 0$ holds for some $\mathbf{y}_p$, i.e., some maximum clique of G contains no vertex colored with i in the assumed proper k-coloring extension. This contradiction shows the necessity of the knot condition.

Span condition: For differently precolored vertices $v_i \in V$ with characteristic vectors $\mathbf{v}_i$ over V, $i = 1, 2$, suppose that there are $\mathbf{x}_{ij} \in \mathcal{M}$, $i = 1, 2$, $j = 1, 2, \ldots, m$, such that $\mathbf{v}_1 + \mathbf{x}_{11} + \ldots + \mathbf{x}_{1m} = \mathbf{v}_2 + \mathbf{x}_{21} + \ldots + \mathbf{x}_{2m}$. If U is the color class with the color of v_1 in a proper k-coloring of G, then $v_2 \notin U$, and for the characteristic vector $\mathbf{u}$ of U over V we gain $\mathbf{u}^{\mathrm{T}}\mathbf{v}_1 = 1$ and $\mathbf{u}^{\mathrm{T}}\mathbf{v}_2 = 0$. On the other hand, for any $\mathbf{x} \in \mathcal{M}$, $\mathbf{u}^{\mathrm{T}}\mathbf{x} = 1$. Multiplying the equality $\mathbf{v}_1 + \mathbf{x}_{11} + \ldots + \mathbf{x}_{1m} = \mathbf{v}_2 + \mathbf{x}_{21} + \ldots + \mathbf{x}_{2m}$ by $\mathbf{u}^{\mathrm{T}}$ from the left we obtain that $1 + m = 0 + m$. This contradiction completes the proof. □

Proof of Theorem 3.6. By Lemma 2.1 it suffices to prove that if $\mathcal{M}$ is not a W_1-knot, then the given partial k-coloring of $G = (V, E)$ is extendible. Let $\mathbf{A}$ denote the $|V|$ by $|\mathcal{M}| + 1$ matrix whose first $|\mathcal{M}|$ columns are the elements of $\mathcal{M}$, and whose last column is the characteristic vector $\mathbf{w}_1$ of W_1 over V. Let H denote the graph whose vertices are the 1-entries of $\mathbf{A}$, and let two 1-entries be adjacent in H if the corresponding rows of $\mathbf{A}$ are distinct and, as the vertices of G, are adjacent in G. Thus in H the 1-entries of the jth row of $\mathbf{A}$ form an independent set, denoted I_j, $j = 1, 2, \ldots, |V|$. Since H is obtained from G by multiplication of vertices, by the observation of Fulkerson [9] and Lovász [27], H is perfect (cf. Corollary 3.5 in [10]).

Assume that some maximal cliques $C^1, \ldots, C^r$ of H cover all 1-entries of $\mathbf{A}$. Let $\mathbf{x}_p$, $p = 1, 2, \ldots, r$, denote the $|V|$-dimensional column vector whose jth component is the number of the vertices in the intersection of C^p and I_j, $j = 1, 2, \ldots, |V|$; thus $\mathbf{x}_p \in \mathcal{L}$. On the other hand, $\mathbf{x}_1 + \ldots + \mathbf{x}_r$ is componentwise greater than or equal to the sum of all columns of $\mathbf{A}$, that is $\mathbf{w}_1 + \Sigma\mathcal{M}$. Assuming that $\mathcal{M}$ is not a W_1-knot, $r \geq |W_1| + |\mathcal{M}|$.

Since H is perfect and the last inequality holds for all sequences of maximal cliques $C^1, \ldots, C^r$ of H covering all vertices, there is some independent vertex set I in H of size $|\mathcal{M}| + |W_1|$. Let $\mathbf{u}$ be the characteristic vector of some $U \subseteq V$ over V defined as the set of those vertices of G whose corresponding rows in $\mathbf{A}$ meet I. Thus U is an independent set in G. For the first $|\mathcal{M}|$ columns $\mathbf{y}$ of $\mathbf{A}$, by the independence of U we have $\mathbf{y}^{\mathrm{T}}\mathbf{u} \leq 1$. Moreover, concerning the last column, $\mathbf{w}_1$, the relation ${\mathbf{w}_1}^{\mathrm{T}}\mathbf{u} \leq |W_1|$ is obvioius. Taking the sum of these $|\mathcal{M}| + 1$ inequalities, we gain $|I| \leq |\mathcal{M}| + |W_1|$. However, $|I| = |\mathcal{M}| + |W_1|$; therefore $\mathbf{y}^{\mathrm{T}}\mathbf{u} = 1$ holds for the first $|\mathcal{M}|$ columns $\mathbf{y} \in \mathcal{M}$ of $\mathbf{A}$, and we also have $\mathbf{w}_1{}^{\mathrm{T}}\mathbf{u} = |W_1|$, i.e., $W_1 \subseteq U$.

Let all vertices of U be colored with color 1. Since G is perfect and $\mathbf{y}^{\mathrm{T}}\mathbf{u} = 1$ for all $\mathbf{y} \in \mathcal{M}$, G_{V-U} is also perfect with $\omega(G_{V-U}) = k-1$. Thus G_{V-U} is properly $(k-1)$-colorable with the colors $2, \dots, k$ resulting in a proper k-coloring extension in G. □

Proof of Theorem 3.8. The given partial k-coloring of G is extendible if and only if G^0 admits such a proper k-coloring for which each vertex in $\{w_0\} \cup W_1$ gets the same color. Thus Theorem 3.6 for G^0 completes the proof. □

Proof of Theorem 3.7. By Lemma 2.1 we may assume that $W_i = \{w_i\}$, $i = 0, 1, 2$, and that the given partial k-coloring of $G = (V, E)$ is not extendible. It suffices to show that w_1 and w_2 are spanned in G. Let $\mathbf{w}_i^0$ denote the characteristic vector of W_i over $V^0 = \{w_0\} \cup V$, $i = 0, 1, 2$, and let $\mathcal{L}^0$ ($\mathcal{M}^0$) denote the set of the charateristic vectors over V^0 of all maximal (maximum) cliques of G^0. For any characteristic vector $\mathbf{x}^0$ over V^0, let $\mathbf{x}$ denote the vector obtained from $\mathbf{x}^0$ by deleting the component which corresponds to w_0, and let $|\mathbf{x}^0|$ denote the sum of the components of $\mathbf{x}^0$. Note that G^0 is a perfect graph whenever so is G.

By Theorem 3.8 for G^0 there are $\mathbf{x}_1^0, \dots, \mathbf{x}_m^0 \in \mathcal{L}^0$ and $\mathbf{y}_1^0, \dots, \mathbf{y}_q^0 \in \mathcal{M}^0$ such that $m \le q+1$ and $\mathbf{x}_1^0 + \ldots + \mathbf{x}_m^0 \ge \mathbf{w}_0^0 + \mathbf{w}_1^0 + \mathbf{y}_1^0 + \ldots + \mathbf{y}_q^0$. Since $|\mathbf{x}_p^0| \le k$, $p = 1, \dots, m$, $|\mathbf{w}_i^0| = 1$, $i = 0, 1$, and $|\mathbf{y}_r^0| = k$, $r = 1, \dots, q$, we have $m = q+1$. Since the component of $\mathbf{w}_0^0$ corresponding to w_0 is 1, at least one $\mathbf{x}_p^0$, say $\mathbf{x}_{q+1}^0$, has the same property. The only maximal clique which contains w_0 in G^0 is the edge w_0w_2, therefore $|\mathbf{x}_{q+1}^0| = 2$. By $|\mathbf{x}_1^0| + \ldots + |\mathbf{x}_m^0| \ge |\mathbf{w}_0^0| + |\mathbf{w}_1^0| + |\mathbf{y}_1^0| + \ldots + |\mathbf{y}_q^0|$ we gain that each $|\mathbf{x}_p^0| = k$, $p = 1, \dots, q$, and that $\mathbf{x}_1^0 + \ldots + \mathbf{x}_{q+1}^0 = \mathbf{w}_0^0 + \mathbf{w}_1^0 + \mathbf{y}_1^0 + \ldots + \mathbf{y}_q^0$. The latter equation can be written in the form $\mathbf{x}_1^0 + \ldots + \mathbf{x}_q^0 + \mathbf{w}_0^0 + \mathbf{w}_2^0 = \mathbf{w}_0^0 + \mathbf{w}_1^0 + \mathbf{y}_1^0 + \ldots + \mathbf{y}_q^0$, thus $\mathbf{x}_1 + \ldots + \mathbf{x}_q + \mathbf{w}_2 = \mathbf{w}_1 + \mathbf{y}_1 + \ldots + \mathbf{y}_q$. Since $\mathbf{x}_p, \mathbf{y}_p \in \mathcal{M}$, $p = 1, \dots, q$, w_1 and w_2 are spanned in G. □

8. Efficient algorithms

To avoid the trivial cases and obvious reductions, in this section we restrict ourselves to partially k-colored *connected* graphs $G = (V, E)$ where E is nonempty and $k \ge 2$. We give efficient algorithms which find either a proper k-coloring extension, or a "reason" why the precoloring is not extendible. By "reason" we mean that one of the necessary conditions listed in Section 2 fails, and this fact is proved by the algorithm.

The first variants of the algorithms presented in Sections 8.1 and 8.2 appeared in the technical report [1]. Later, Jansen and Scheffler [22] independently proved Theorems 8.1 and 8.3.

8.1. Linear-time algorithms

Trees

Theorem 8.1. PrExt is solvable on trees in linear time.

Proof. We suggest the following linear algorithm. During the algorithm we shall manage an initially empty *stack* S. We shall put triples onto S; each triple will consist of a precolorless vertex and two distinct colors. Starting from a fixed vertex w_1, apply *breadth-first search* for a partially k-colored tree $G = (V, E)$; this can be performed in linear time (cf. [10]). Let v denote the last vertex with respect to the breadth-first search. This must be a *leaf* (i.e., degree-1 vertex) of G; let v' denote its unique neighbor. If v' is precolored, then we can assign a color to v simply by choosing any color distinct from the color of v'. In this case we can cut down v from G, and for the remaining tree the breadth-first search structure remains unchanged. Therefore for the remaining graph our method can be applied recursively without affecting the claimed linearity status.

Suppose that v' is precolorless. In $\mathcal{O}(|N(v')|)$ time we check if $\{v'\}$ is an elementary $(k+1)$-core or an elementary k-core. In more details: we scan all neighbors u of v', and if u is precolored with color i, then we assign the ordered pair (v', u) to color i. By scanning the pairs assigned to the colors, we can decide whether $\{v'\}$ is an elementary $(k+1)$-core, or an elementary k-core, or we gain two distinct colors $i, i' \in \{1, \ldots, k\}$ such that none of them occurs in $N(v')$. If $|N(v')| \geq k-1$, then this scanning procedure needs just $\mathcal{O}(|N(v')|)$ time. However, if $|N(v')| \leq k-2$, we just have to check $|N(v')| + 2$ colors; we surely obtain i and i'. If we find a $(k+1)$-core, we are done since the precoloring is not extendible (cf. Lemma 2.2). If we find that $\{v'\}$ is an elementary k-core, then we color v' with the unique admissible color, i.e., we actually perform one step to check the sequence condition. Now there is no difficulty in coloring v if it was precolorless. Then v can be cut down from G, and our algorithm can be continued keeping its linear status.

Therefore the only case left is where v' is precolorless and has precolored neigbors of at most $k-2$ distinct colors. In $\mathcal{O}(|N(v')|)$ time we gain distinct colors $i, i' \in \{1, \ldots, k\}$, such that none of them occurs in $N(v')$. Any precolorless leaf neighbor of v' can be cut down from G because no matter what color finally v' gets, at least one of i and i' remains free for all precolorless leaf neighbors of v'. We put the triple (v', i, i') onto S. Note that all administration can be performed in $\mathcal{O}(|N(v')|)$ time along with the reconstuction of the breadth-first search structure of the remaining graph.

Assume that all leaf neighbors of the precolorless vertex v' are precolored. Since v was the last with respect to the breadth-first search, there is at most one non-leaf precolorless neighbor of v'. Thus v' can be cut down from G with all its leaf neighbors because at least one of the colors i and i' remains admissible for coloring v'. (The reconstuction of the breadth-first search structure of the remaining graph can be done in $\mathcal{O}(|N(v')|)$ steps.)

Finally we either gain an extension of the precoloring (we have to go through S to decide which of i and i' should be assigned to v', and to color v' with such an admissible color; this means a scanning of the vertices in the order as they appear in the breadth-first search procedure) or a proof by the sequence condition that the precoloring is not extendible. Since the sum of the $\mathcal{O}(|N(v')|)$ complexities occuring in the algorithm is just $\mathcal{O}(|V|)$, the algorithm is linear.

We note that if the precoloring is not extendible, then a minimal non-k-colorable subtree can also be found. □

Connected bipartite graphs with color bound $k = 2$

Theorem 8.2. PrExt is solvable on connected bipartite graphs with color bound $k = 2$ in linear time.

Proof. If G is connected and properly 2-colorable, then the bipartition is uniquely determined. This bipartition can be computed in linear time (by applying, for example, breadth-first search starting from a fixed vertex), and meanwhile one can check whether or not this bipartition is compatible with the given partial 2-coloring. □

Connected cographs

Cographs (or *complement reducible graphs*) are defined as the class of graphs formed from a single vertex under the closure of the operations of vertex disjoint union and complement. Cographs were independently discovered under various names and were shown by Lerchs (see [6]) to have a unique tree representation. This tree, called *cotree*, forms the basis for fast polynomial algorithms.

In the cotree representation the leaves of the cortree represent the vertices of the graph. Internal nodes of the cotree are labelled 0 or 1 in such a way that (0) nodes and (1) nodes alternate along every path starting from the root which always is a (1) node. The root will have more than one (0)-node child if and only if the represented cograph is connected. Two vertices x and y of the cograph are adjacent if and only if the unique path from x to the root of the tree meets the unique path from y to the root at a (1) node of the tree. A linear time algorithm for recognizing cographs and constructing their cotree representations can be found in [7].

Given a partially k-colored connected cograph with its representing cotree, we shall give a linear algorithm which finds either a proper precoloring extension or a $(k + 1)$-core (cf. Theorem 3.4).

Theorem 8.3. PrExt is solvable in connected cographs in linear time.

Proof. We consider a partially k-colored connected cograph $G = (V, E)$ with its representing cotree T. To avoid the trivial cases, we may assume that $k < |V|$. Since G is connected, $|V| = \mathcal{O}(|E|)$. Note that T has fewer than $2|V|$ nodes. In linear time, we can also check whether or not G_W is properly k-colored. (In case of the negative answer we have nothing to do.)

Let N denote the set of the nodes of T. For any $\nu \in N$, let $G^\nu = (V^\nu, E^\nu)$ denote the cograph which corresponds to the subcotree with root ν. Observe that G^ν is actually an induced subgraph of G. If ν is the root of T, then $G^\nu = G$, and if ν is a leaf of T, then G^ν is just the single vertex ν. In our algorihm, in linear time, we shall find either a $(k+1)$-core, or optimal vertex partitions (compatible with the precoloring) into independent sets for all G^ν, $\nu \in N$, one by one according to a *"postorder" manner* with respect to a breadth-first search of T starting from its root.

By color we mean one of the numbers $1, 2, \ldots, k$; their set will be denoted by K. We shall manage an auxiliary bipartite graph B with bipartition $N \cup K$; initially B is edgeless. For each $\nu \in N$ we shall also manage a list W^ν that contains one precolored vertex from each color class of V^ν.

Here by precolored vertices we mean all vertices of G which were *originally* precolored, and also some *newly* precolored vertices. The latter will be precolored one by one during our algorithm.

In addition, for each $\nu \in N$ we also manage a partition of the set U^ν of the *currently* precolorless vertices of G^ν into k^ν nonempty independent sets. This partition will be represented in the form of a list L^ν consisting of the elements of U^ν (each element occurs once) and $k^\nu - 1$ copies of a separation sign "**;**" if $k^\nu > 1$; the partition classes will be formed by the sublists separated by the separation sign. We shall require that if U^ν is nonempty, then $\chi({G^\nu}_{U^\nu}) = k^\nu$. Obviously, the length of L^ν is $\mathcal{O}(|V^\nu|)$, and a sufficient condition for the precoloring extendibility of the current partial k-coloring of G^ν is that the precolor number of G^ν is at most $k - k^\nu$.

Initially, we consider the leaves of T. If ν is a leaf, we distinguish between two cases. If ν is precolored, we join ν and its color in B, and set W^ν as the singleton ν and L^ν as the empty list. If ν is precolorless, we set W^ν empty and L^ν as the singleton ν.

The key idea of our algorithm is as follows. Let ν be a (0) or a (1) node of T. We claim that W^ν and L^ν can be constructed from W^μ and L^μ of the childreen μ of ν in $\mathcal{O}(|V^\nu|)$ time, and that the sum of the $|V^\nu|$ is $\mathcal{O}(|E|)$. Meanwhile we have to manage the above declared properties. Note that $|V^\nu|$ is the sum of the numbers $|V^\mu|$ for the children μ of ν.

First we study the case where ν is a (0) node. The procedure consists of three phases. In the first phase, going through the children μ of ν, and for each μ going through W^μ, we consider the colors i of the vertices w of $W\mu$. If $i\nu$ is not an edge in B yet, then we join i and ν, and insert w at the end of the list $W\nu$. Otherwise we take the next element of W^μ.

In the second phase, we deal with the lists L^μ. For each μ, we scan at most $|W^\mu| + |L^\mu| \leq |V^\mu|$ elements of W^μ as follows. For $i \in W^\nu$ we check whether $i\mu$ is an edge in B. If so, we take the next element of W^ν. Otherwise, we assign color i to the vertices of the first partition class of L^μ, and we also delete them from L^μ together with the "**;**" sign following them (if any). We stop when W^ν or L^μ is exhausted.

In the last phase we prepare a two-way list M^ν from those children μ of ν for which L^μ remains nonempty after the second phase. Originally, L^ν is the empty list. For each $\mu \in M^\nu$ we copy the first segment of L^μ into L^ν, and delete it from L^μ. If L^μ is still nonempty, then we delete "**;**" from the beginning of L^μ and go to the next member of M^μ. On the other hand, if L^μ becomes empty, then we delete μ from M^μ, maintaining the two-way structure of $M\mu$. Reaching the end of M^μ, we check whether M^μ is the empty list. If so, we finish the procedure at ν. Otherwise we place a "**;**" at the end of L^ν and go through the elements of M^ν again.

Second we study the case where ν is a (1) node of T. Considering the lists W^μ for all children, any two occuring colors must be distinct. Therefore we just have to concatenate these list to obtain W^ν. As a byproduct, we have $|W^\nu|$. Going through this list we join each occuring color to ν in B. To obtain L^ν we just have to concatenate the lists L^μ by inserting separation signs between any two consecutive lists L^μ. The total time complexity is $\mathcal{O}(|V^\nu|)$.

Finally, having completed the above process for the root ρ of the cotree, we have to finish the solution of the precoloring extension problem. We go through K and delete all vertices which are adjacent to ρ. Then we go through the list of the remaining colors in K, and in L^ρ we color the vertices with the these colors such that we change the color after each separation sign.

If we find that the number of distinct colors in K is not enough for coloring all precolorless vertices, then by Theorem 3.4 we know that there is a $(k+1)$-core. In fact, such a core C^ρ can also be found recursively: If ν is a leaf, then $C^\nu = \{\nu\}$ if ν is precolorless, and empty otherwise. If ν is a (1) node, then C^ν is the union of the cores C^μ of the children μ of ν (i.e., their concatenation when the C^μ are stored in lists). On the other hand, if ν is a (0) node, then we set $C^\nu = C^\mu$ where C^μ is a k'-core with the largest possible value of k' among all children of ν.

At the end of the proof we show that the sum of the numbers $|V^\nu|$ for all nodes ν of the cotree is $\mathcal{O}(|E|)$. Recall that two vertices x and y of the cograph are adjacent if and only if the unique path from x to the root of the tree meets the unique path from y to the root at a (1) node of the tree. For each (1) node ν, let m^ν denote the number of those edges of G which are "realized" via node

ν. Thus the sum of those numbers is exactly $|E|$. We define m^ν for a leaf node ν as 1 and for a (0) node ν as the sum of the numbers m^μ for the childreen μ of ν. Note that each such μ is either a leaf or a (1) node. Therefore the sum of the numbers m^ν for the entire cotree is no more than $\mathcal{O}(|E|)$. Hence it suffices to show that if ν is either a (0) node or a (1) node, then $|V^\nu| \leq 2m^\nu$.

We can prove this inequality by induction on the distance of ν from the leaf set V in T. The statement is obvious if the distance is 1. For larger distances, we are also done by induction for the (0) nodes. On the other hand, if ν is a (1) node, the graph on V^ν defined by the edges "realized" via ν is connected. Therefore, $|V^\nu| \leq m^\nu + 1 \leq 2m^\nu$. □

Connected P_5-free bipartite graphs

The connected P_5-free bipartite graphs are also known as connected *difference graphs* and all such graphs $G = (V, E)$ are characterized by the property that they admit a bipartition $V = V_1 \cup V_2$ and such an ordering $v_{i1}, v_{i2}, \ldots$ of the vertices in each V_i for which $N(v_{i1}) \supseteq N(v_{i2}) \supseteq \ldots$ (cf. [4, 19]). We assume that a connected P_5-free bipartite graph $G = (V, E)$ is given. In linear time, one can easily find V_i and v_{i1}, $i = 1, 2$.

Theorem 8.4. PrExt is solvable on a connected P_5-free bipartite graph in linear time.

Proof. We just have to check if $\{v_{11}\}$ or $\{v_{21}\}$ or $\{v_{11} \cup v_{21}\}$ is a $(k+1)$-core. This can be done in linear time. If some of them is a $(k+1)$-core, we are done by Lemma 2.2. Otherwise, all precolorless vertices of V_i may get the final color of v_{i1}, $i = 1, 2$, no matter if v_{i1} is precolored or prelolorless. Therefore, the precoloring is extendible in linear time. □

A *chordal bipartite graph* is a bipartite graph that contains no induced $(2s)$-cycle for $s \geq 3$; such graphs were studied e.g. in [11] and [18]. Note that the P_5-free bipartite graphs are chordal bipartite. The next theorem shows that in the previous result the assumption that the graphs are P_5-free is essential.

Theorem 8.5. PrExt is NP-complete in P_6-free chordal bipartite graphs.

Proof. We apply a reduction from 3-SAT (satisfiability of conjunctive normal forms with three literals per clause), similarly to the NP-completeness proofs of [22] and [26].

Let $\Phi = C^1 \wedge \ldots \wedge C^m$ be a formula over n variables, with clauses $C^j = y_{j1} \vee y_{j2} \vee y_{j3}$, $j = 1, 2, \ldots, m$, where each literal y_{js} is taken from the set $\{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}$. We construct a partially $(2n)$-colored chordal bipartite graph G_Φ on $n + m + (2n-2)n + (2n-3)m$ vertices as follows. The precolorless vertices of G_Φ induce a complete bipartite subgraph with bipartition $\{x_1, \ldots, x_n\} \cup \{C^1, \ldots, C^m\}$. Each "variable-vertex" x_i has $2n-2$

precolored neighbors, each of degree 1, assigned to the colors from the set $\{1,\ldots,2n\}-\{i,n+i\}$. The "clause-vertices" C^j have $2n-3$ degree-1 precolored neighbors each, on which all but 3 colors of the set $\{1,\ldots,2n\}$ appear; if x_i ($\neg x_i$) occurs in clause C^j, then color i ($n+i$) does not occur in the neighborhood of the clause-vertex C^j.

Every longest induced path in G_Φ begins and ends at a degree-1 vertex, and it can contain at most 3 precolorless vertices. Hence, G_Φ is P_6-free. Moreover, since the vertices of degree at least 2 induce a complete bipartite subgraph, G_Φ is a *chordal* bipartite graph. Representing color i at x_i by the truth assignment $x_i =$ **false** and color $n+i$ by $x_i =$ **true**, one can verify that Φ is satisfiable if and only if the precoloring of G_Φ is extendible. Thus the NP-completeness follows by the well-known theorem of Cook [5]. □

8.2. Flows and bipartite matchings

Assume that and interval graph $G=(V,E)$ is given on n vertices along with the numbers a_i, b_i, $i=1,2,\ldots,n$ and the cliques C^j, $j=1,2,\ldots,m$ exactly as it was studied at the end of Section 2. Assume that G is partially k-colored where each color occurs at most once. We point out that PrExt can be solved by applying network flow techniques.

The straightforward construction of the multidigraph associated to G can be performed in $\mathcal{O}(nk)$ time. Next we have to check whether or not the Menger condition holds. Applying standard network flow techniques, in polynomial time we either find k pairwise arc-disjoint oriented paths form -1 to m or a proof that the Menger condition does not hold. In the latter case we are home by Lemma 2.4. In the former case, starting from the k oriented paths, we can construct a proper precoloring extension. (See [2] for more details.) We can summarize our results as follows:

Theorem 8.6. 1-PrExt can be solved in polynomial time on interval graphs. □

Now we turn our attention to the complements of bipartite graphs and to the split graphs. The proofs of Theorems 3.9 and 3.10 actually show how to gain polynomial algorithms which solve PrExt on these classes of graphs. The details of such algorithms can be found in [20].

Given a bipartite graph on n vertices, a maximum matching can be found in $\mathcal{O}(n^{5/2})$ time [17]. Applying this result, the following theorem can be deduced.

Theorem 8.7. [20] If G is a partially k-colored graph on n vertices, and either G is a split graph or the complement of G is bipartite, then PrExt can be solved in $\mathcal{O}(n^{5/2})$ time. □

As a matter of fact, the time complexity of PrExt on those two classes of graphs is exactly the same as that of the bipartite matching problem (i.e., any improvement on the former would improve on the latter as well).

8.3. General perfect graph algorithms

By the famous results of Grötschel, Lovász and Schrijver [12, 13], the chromatic number problem can be solved in polynomial time on perfect graphs. Given a perfect graph G, the polynomial algorithm computes $\omega(G) = \chi(G)$ and constructs both a maximum clique and a proper coloring with $\omega(G)$ colors. From our point of view this result implies the following theorem which can easily be deduced from the definitions.

Theorem 8.8. PrExt can be solved in polynomial time on PrExt-perfect graphs. □

We gain similarly the following theorem (cf. Theorem 3.5).

Theorem 8.9. 1-PrExt can be solved in polynomial time on the complements of Meyniel graphs. □

Consider a partially k-colored prefect graph G for which the class of all maximum cliques, $\mathcal{M}$ is also given. The proofs of Theorems 3.6 through 3.8 show that both the knot and span conditions can be cheched in polynomial time. Furthermore, we can deduce the following theorem.

Theorem 8.10 Given a partially k-colored perfect graph G where $3 \leq k = \omega(G)$ and given the class of all maximum cliques, assume that either just 2 vertices are precolored or all precolored vertices have the same color. Then PrExt can be solved in polynomial time. □

As we already mentioned, 2-PrExt is NP-complete on chordal graphs [2]. On the other hand, the maximum cliques of a chordal graph can be listed in polynomial time; therefore, PrExt becomes polynomially solvable if the precolor number is 1 or all but two vertices are precolorless.

9. Concluding remarks and open problems

Since the area of precoloring extensions is quite a new branch in the theory of graph colorings, it offers lots of interesting open problems for further research. Below are collected some of those questions which are closely related to the results of the present note.

First of all, with reference to Section 4, let us say that a graph G^* is a *cograph contraction* if it is obtained from a cograph G by replacing some mutually disjoint independent sets $S_1, \ldots, S_t$ by new vertices $x_1, \ldots, x_t$ and joining x_i $(1 \leq i \leq t)$ to all x_j $(j \neq i)$ and also to all those vertices $v \in$

$V(G) - (S_1 \cup \ldots \cup S_t)$ which have at least one neighbor in S_i. We have seen in Theorem 4.2 that every cograph contraction is perfect.

Problem 9.1. Characterize the class of cograph contractions.

Problem 9.2. Characterize PrExt-perfect graphs. In particular, describe further classes $\mathcal{G}$ of perfect graphs such that every $G^* \in \mathcal{G}^*$ is perfect.

Problem 9.3. Determine the complexity of PrExt for some further classes of perfect graphs, e.g. *unit interval* graphs and *permutation* graphs.

Note that both on comparability graphs and on co-comparability graphs, the problem is NP-complete as shown by the classes of bipartite and interval graphs, respectively.

Problem 9.4. Decide whether 1-PrExt is polynomially solvable on chordal graphs or on some of their subclasses, e.g. on strongly chordal graphs.

One such particular result related to undirected path graphs will appear in [21].

We have seen in Theorem 3.6 that precoloring extendibility on perfect graphs G with color bound $\omega(G)$ and precolor number 1 has a good characterization. Thus it is natural to ask what happens if the precolor number is 2. Kratochvíl (private communication) noticed that the problem becomes NP-hard in this case. Namely, 1-PrExt with color bound 3 is NP-hard on bipartite graphs [25, 2], and it is easy to eliminate color 3 by joining the color-3 precolored vertex to two pandent vertices precolored 1 and 2. Obviously, by joining the two pandent vertices, the clique size becomes 3. Nevertheless, we believe that $k = \omega(G)$ with precolor number 2 admits a polynomial algorithm if some reasonable structural restriction is imposed on the perfect graph G.

Problem 9.5. In which results of Section 8.3 can the ellipsoid method be replaced by a purely combinatorial algorithm?

In order to guarantee the linearity of the PrExt algorithm on cographs (Theorem 8.3) we implicitly assumed that a completely clear memory space of $\mathcal{O}(nk)$ fields is available (n is the number of the vertices, k is the color bound). More precisely, we need a representation of the bipartite graph B in the proof of Theorem 8.3 for which we can check the adjacency of any pair of vertices in constant time, and this can easily be managed, e.g. if B is represented by a matrix whose rows represent the nodes of the cotree and whose colums represent the colors, and the ith element of row ν is 1 (0) if ν is adjacent (not adjacent) to i. However, the representing matrix should be empty at the beginning. The $\mathcal{O}(nk)$ space requirement can be replaced by others, for example that the sum or difference of binary vectors of lenght k can be computed in constant time. Without such assumptions we can prove a slightly worse upper bound on the time complexity, namely $\mathcal{O}(m+n\sqrt{\log k})$, where m denotes the number of edges of the connected cograph.

Problem 9.6. Decide whether the factor $\sqrt{\log k}$ can be eliminated.

Perhaps the answer is affirmative; to prove this, it would suffice to find a more concise representation of B that still admits constant-time access to any edge of B.

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References

[1] Biró, M., Hujter, M., and Tuza, Zs., Precoloring extension, Preprint (Computer and Automation Institute, Budapest, 1990).

[2] Biró, M., Hujter, M., and Tuza, Zs., Precoloring extension. I. Interval graphs, Discrete Math. **100** (1992) 267–279.

[3] Bodlaender, H. L., Jansen, K., and Woeginger, G. J., Scheduling with in compatible jobs, Preprint (Technical University, Graz, 1992).

[4] Bacsó, G., and Tuza, Zs., Dominating cliques in P_5-free graphs, Periodica Math. Hungar. bf 21 (1990) 303–308.

[5] Cook, S. A., The complexity of theorem proving procedures, Proc. 3rd Annual ACM Symposium on Theory of Computing (1971) 151–158.

[6] Corneil, D. G., Lerchs, H., and Stewart Burlingham, L., A linear recognition algorithm for cographs, SIAM J. Comput. **14** (1985) 926–934.

[7] Corneil, D. G., Perl, Y., and Stewart, L. K., Complement reducible graphs, Discrete Appl. Math. **3** (1981) 163–174.

[8] Földes, S., and Hammer, P. L., Split graphs, Congr. Numer. **19** (1977) 311–315.

[9] Fulkerson, D. R., Blocking and anti-blocking sets pairs of polyhedra, Math. Programming **1** (1971) 50-71

[10] Golumbic, M. C., Algorithmic Graph Theory and Perfect Graphs, (Academic Press, New York, 1980).

[11] Golumbic, M. C., and Goss, C. F., Perfect elimination and chordal bipartite graphs, J. Graph Theory **2** (1978) 155–163.

[12] Grötschel, M., Lovász, L., and Schrijver, A., The ellipsoid method and its consequences in combinatorial optimization, Combinatorica **1** (1981) 169–197.

[13] Grötschel, M., Lovász, L., and Schrijver, A., Polynomial algorithms for perfect graphs, Ann. Discrete Math. **21** (1984) 325–356.

[14] Gyárfás, A., and Lehel, J., A Helly-type problem in trees, in: Colloq. Math. Soc. J. Bolyai, 4., Combinatorial Theory and Its Applications, Vol. **2** (North-Holland, Amsterdam, 1970) 571–584.
[15] Harary, F., and Tuza, Zs., Two graph-coloring games, Bull. Austral. Math. Soc. **48** (1993) 151–159.
[16] Hertz, A., Slim graphs, Graphs and Combinatorics **5** (1989) 149–157.
[17] Hopcroft, J., and Karp, R. M., An $n^{5/2}$ algorithm for maximum matching in bipartite graphs, SIAM J. Computing **2** (1973) 225–231.
[18] Hammer, P. L., Maffray, F., and Preissmann, M., A characterization of chordal bipartite graphs, Rutcor Research Report #16-89 (Rutgers University, New Brunswick, New Jersey, 1989).
[19] Hammer, P. L., Peled, U. N., and Sun, X., Difference graphs, Discrete Appl. Math. **28** (1990) 35–44.
[20] Hujter, M., and Tuza, Zs., Precoloring extension. II. Graph classes related to bipartite graphs, Acta Math. Univ. Comenianae **62** (1993) 1–11.
[21] Hujter, M., and Tuza, Zs., Precoloring extension. IV. General bounds and list colorings, in preparation.
[22] Jansen, K., and Scheffler, P., Generalized coloring for tree-like graphs (1992) submitted.
[23] König, D., Gráfok és alkalmazásuk a determinánsok és halmazok elméletében (Hungarian), [Graphs and their applications in the theories of determinants and sets] Matematikai és Természettudományi Értesítő **34** (1916) 104–119. [German translation: Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. **77** (1916) 453–465.]
[24] König, D., Theorie der Endlichen und Unendlichen Graphen (German), [Theory of finite and infinite graphs] Akademische Verlagsgesselschaft, Leipzig (1936). [Reprinted Chelsea, New York (1950); Translated to English by R. McCoart (Birkhäuser, 1990).]
[25] Kratochvíl, J., Precoloring extension with fixed color bound, to appear.
[26] Kratochvíl, J., and Tuza, Zs., Algorithmic complexity of list colorings (1992) submitted.
[27] Lovász, L., A characterization of perfect graphs, J. Combin. Theory **B 13** (1972) 95–98.
[28] Marcotte, O., and Seymour, P. D., Extending an edge-coloring, J. Graph Theory **14** (1990) 565–573.
[29] Sebő, A., private communication (1992).
[30] Vizing, V. G., Critical graphs with a given chromatic class (Russian), Diskret Analiz. **5** (1965) 9–17.