World Applied Sciences Journal 7 (Special Issue for Applied Math): 162-167, 2009
ISSN 1818-4952

Preconditioned Technique for Solving Fredholm Integral Equations of the First Kind with Orthogonal Triangular Functions

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Abstract: In this paper, a numerical approach based on an m-set of general, orthogonal triangular functions (TF) is proposed to approximate the solution of Fredholm integral equations of the first kind. By using the orthogonal triangular functions as a basis in Galerkin method, the solution of linear integral equations reduces to a system of algebric equations. If the recent system become ill-conditioned then we will use the preconditioned technique to convert above problem to well-conditioned. The convergence of the proposed method is established. Some numerical examples illustrate the proposed approach.

Key words: Integral equations . orthogonal triangular functions . preconditioner

INTRODUCTION

The problem of solving the linear system

$$Ax = b \tag{1}$$

where $x,b \in \Re^n$ and A is a matrix of dimensions $n\times n$ is an important problem in numerical linear algebra. In general, when the dimensions and the condition number of A matrix are very large, problems such as (1) are ill-posed and small changes to the problem can make very large changes to the answers obtained. So, preconditioned technique is a key fundamental for solving very large linear system and it means the transform of the original linear system into one which has the same solution. Suppose the recent system by using different basis obtained from the Fredholm integral equation of the first kind. There are different iterative methods as preconditioned techniques [1]. Krylov subspace methods are one of the most important iterative methods for this work. The rate of their convergence is slow and sometimes we encounter breakdown [1]. The iterative method of conjugate gradient is effective method for solving Eq. (1) and the speed of convergence depends on the condition number of the coefficient matrix [1]. Also conjugate gradient squared (CGS), generalized minimal residual (GMRES) and bi-conjugate gradient (BICG) methods are introduced in [1,2] for solving nonsymmetric systems. Gauss type preconditioning methods for nonnegative matrices and M-matrices linear systems are applied by Zhang in [3]. These methods are partly derived from the LU factorization method. A sparse approximate inverse preconditionear is developed in [4] for the conjugate gradient method and these recent preconditioners are considerable interest for using on parralel computers. This paper is orgonized as follows. We first review a Gauss type preconditioning method [3] and in section 3, we use orthogonal triangular basis to find numerical solution of Fredholm integral equations of the first kind. The convergence of the Gauss type preconditioning method for preconditioned linear systems is considered in section 4. Finally, section 5 contains some numerical examples which are obtained by MATLAB software to show the simplicity and accuracy method.

PRECONDITIONER

Consider the linear system (1), we use precondition P as a matrix which approximates A. We want transform system (1) into system

$$PAx = Pb \tag{2}$$

In this paper, we focus our attention on the solution of linear systems obtained from Fredholm integral equations of the first kind. We first illustrate a preconditioned iterative method based on Gauss transformation matrices [3].

Gauss type preconditioning methods: We consider preconditioers chosen to eliminate the off-diagonal elements of the coefficient matrix of a linear system. The left Gauss type preconditioners applied for nonnegative matrices and M-matrices linear systems which eliminate the strictly lower triangular elements and right Gauss type preconditioners are derived from the LU factorization method which eliminate strictly upper triangular elements [3].

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Definition 1: A matrix A is a Z-matrix if $a_{ij} \le 0$, i, j = 1,2,…,n, i≠j [3].

Definition 2: A matrix A is a M-matrix if A has the form A = sI-B, s>0, B≥0 and s>ρ(B) (the spectral radius of B) [3].

Definition 3: A nonsingular Z-matrix is called an M-matrix if $A^{-1} \ge 0$ [5].

Suppose A is a M-matrix, without loss of generality we have

$$A = I - L - U \quad (3)$$

where I, -L and -U are the identity matrix, strictly lower triangular and strictly upper triangular parts of A, respectively. If we choose M = I, N = L+U and M = I-L, N = U, the classical Jacobi and Gauss-Seidel iterative methods obtain, respectively.
The preconditioned form of the linear system (1) is

$$PAx = Pb$$

where P is called the left preconditioner and the right preconditioner, respectively. Then a kind of iterative scheme can be written.

$$x^{(k+1)} = M_p^{-1} N_p x^{(k)} + M_p^{-1} Pb \quad (4)$$

where $PA = M_p - N_p$, k = 0,1,… and M_p is nonsingular.

By using the left preconditioners, the iterative methods such as Jacobi and Gauss-Seidel type methods converge faster than the original ones [3]. Many researchers have proposed the preconditioning techniques. For example, Kohno *et al.* considered a parameterized preconditioner [3] and recently Zhang *et al.* proposed the left Gauss type preconditioning techniques based on Hadjidmos *et al.* and LU factorization method [3]. The following left preconditioners are all lower tringular matrices with unity diagonal entries and they are considered by Zhang in [3] by using Gauss transformation matrix

$$P_m = M_m M_{m-1} \dots M_2 M_1 \quad (5)$$

where m = 1,2,…,n-1 and the construction of Gauss transformation matrices is as follows [6]:

$$M_k = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 1 & \dots & 0 \\ 0 & \dots & -\frac{a_{k+1k}}{a_{kk}} & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -\frac{a_{nk}}{a_{kk}} & 0 & \dots & 1 \end{bmatrix} \quad (6)$$

where k = 1,2,…,n-1 and we have

$$B = (b_{ij}) = M_k A \;, \quad j = 1,2,\dots,n$$

where

$$b_{ij} = \begin{cases} a_{ij} & , \quad i = 1,2,\dots,k \\ a_{ij} - \tau_i a_{kj} & , \quad i = k+1,\dots,n \end{cases}$$

Now, we consider the elements of the preconditioned matrix $A_{Lm} = P_m A$ as

$$A_{LM} = \begin{bmatrix} a_{11}^{(l,m)} & a_{12}^{(l,m)} & \dots & a_{1m}^{(l,m)} & a_{1m+1}^{(l,m)} & \dots & \dots & a_{1n}^{(l,m)} \\ 0 & a_{22}^{(l,m)} & \dots & a_{2m}^{(l,m)} & a_{2m+1}^{(l,m)} & \dots & \dots & a_{2n}^{(l,m)} \\ \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \dots & 0 & a_{mm}^{(l,m)} & a_{mm+1}^{(l,m)} & \dots & \dots & a_{mn}^{(l,m)} \\ 0 & \dots & \dots & 0 & a_{m+1m+1}^{(l,m)} & \dots & \dots & a_{m+1n}^{(l,m)} \\ \vdots & \ddots & & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & a_{nm+1}^{(l,m)} & \dots & \dots & a_{nn}^{(l,m)} \end{bmatrix} = \begin{pmatrix} L_{m_1} & L_{m_2} \\ L_{m_3} & L_{m_4} \end{pmatrix}$$

We can write the usual splitting of A_{Lm}, namely,

$$A_{Lm} = D_{Lm} - E_{Lm} - F_{Lm}$$

where D_{Lm}, $-E_{Lm}$ and $-F_{Lm}$ are diagonal, strictly lower triangular and strictly upper triangular parts of A_{Lm} respectively. If we define $M_p = D_{Lm}$, $N_p = E_{Lm}+F_{Lm}$ and $M_p = D_{Lm} - E_{Lm}$, $N_p = F_{Lm}$, the Jacobi and Gauss-Seidel types iterations obtain, respectively. Similarly, we can obtain the preconditioned matrix AQ_m and apply its properties to obtain iteration methods. The algorithms of the right and left Gauss type preconditioners based on the elimination of elements are presented in [3].
Now, we consider the numerical solution of the integral equation based on orthogonal triangular functions.

PROBLEM STATEMENT

Definition 4: A set of BPF, $\boldsymbol{\Psi}_{\mathbf{m}}(\mathbf{t})$ containing m component functions in the semi-open interval [0,T) is given by

$$\boldsymbol{\Psi}_{\mathbf{m}}(\mathbf{t}) = \left[\psi_0(\mathbf{t}) \quad \psi_1(\mathbf{t}) \quad \dots \quad \psi_{\mathbf{i}}(\mathbf{t}) \quad \dots \quad \psi_{\mathbf{m}-1}(\mathbf{t}) \right]^{\mathbf{T}} \quad (7)$$

where $[\dots]^T$ denotes transpose.

We can generate two sets of orthogonal TFs, namely $\mathbf{T1_m(t)}$ and $\mathbf{T2_m(t)}$ such that

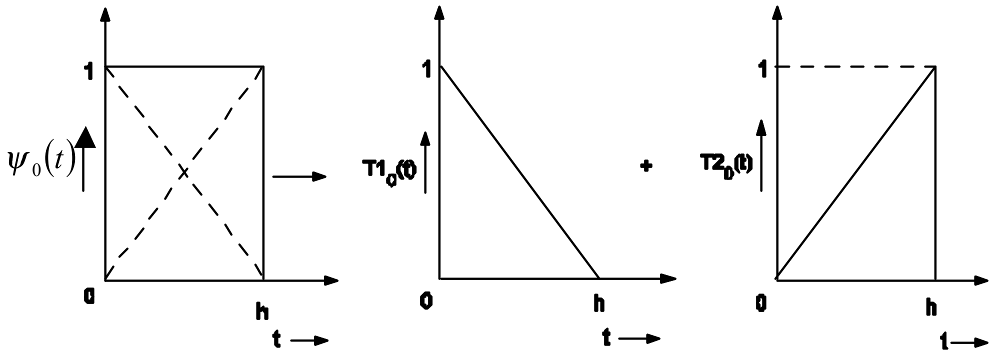

Fig. 1: Dissection of BPF into two triangular functions

$$\mathbf{\Psi_m(t) = T1_m(t) + T2_m(t)} \quad (8)$$

Let $\psi_0(t)$ be the first member of an m-set BPF, we introduce

$$\psi_0(t) = T1_0(t) + T2_0(t)$$

where $T1_0(t)$ and $T2_0(t)$ functions are shown in Fig. 1.
Consider the integral equation of the first kind as in [7]

$$\int_a^b k(s,t)f(t)dt = g(s) \quad (9)$$

where $a \le s \le b$, f (t) is the unknown function while g (s) and k (s,t) are known in $L^2[a,b]$. The Eq. (9) can be written as

$$Kf = g \quad (10)$$

where $K: X \to X$ is defined by

$$(Kf)(s) = \int_a^b k(s,t)f(t)dt \ , \quad a \le s \le b$$

We suppose in this paper $X = L^2[a,b]$.

In recent years, different methods are introduced for solving numerically the first kind of integral equations which are based on orthogonal basis such as wavelet basis [8]. In this paper, we introduce a complementary pair of orthogonal triangular function sets and we use them for solving the first kind of integral equations.

Now we define the m-set triangular function vectors as

$$\mathbf{T1_m(t)} = [T1_0(t) \ \ T1_1(t) \ \ T1_2(t) \ \ldots \ T1_i(t) \ \ldots \ T1_{m-1}(t)]^T \quad (11)$$

The ith component of the vector $\mathbf{T1_m(t)}$ is defined as

$$T1_i(t) = \begin{cases} 1 - \frac{(t-ih)}{h} & , \ ih \le t < (i+1)h \\ 0 & , \quad \text{otherwise} \end{cases} \quad (12)$$

and the ith component of the vector $\mathbf{T2_m(t)}$ is defined as

$$T2_i(t) = \begin{cases} \frac{(t-ih)}{h} & , \ ih \le t < (i+1)h \\ 0 & , \quad \text{otherwise} \end{cases} \quad (13)$$

where i = 0,1,2,…,(m-1) [9].

The condition of orthogonality for TFs is given in [9] by

$$\int_0^1 T1_i(t)T1_j(t)dt = \begin{cases} \frac{h}{3} & i = j \\ 0 & i \ne j \end{cases} ,$$

$$\int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \frac{h}{3} & i = j \\ 0 & i \ne j \end{cases}$$

Also we know that

$$\int_0^1 T1(t)T1^T(t)dt = \int_0^1 T2(t)T2^T(t)dt = \frac{h}{3} I$$

$$\int_0^1 T1(t)T2^T(t)dt = \int_0^1 T2(t)T1^T(t)dt = \frac{h}{6} I$$

where I is $m \times m$ identity matrix.

In general, a time function f (t) of Lebesgue measure may be expanded into an m-term TF series in $t \in [0,T)$ as

$$\begin{aligned} f(t) &\approx [c_0 \ \ c_1 \ \ c_2 \ \ldots \ c_i \ \ldots \ c_{m-1}]\mathbf{T1_m} \\ &+ [c_1 \ \ c_2 \ \ldots \ c_i \ \ldots \ c_{m-1} \ \ c_m]\mathbf{T2_m} \\ &= C1^T \mathbf{T1_m} + C2^T \mathbf{T2_m} \end{aligned} \quad (14)$$

where, the constant coefficients are the samples of function such that [3]

$$c_i = f(ih) \quad (15)$$

where $i = 0,1,\ldots m-1, m$.

Now we return to integral equation (9), If we expand the f (t) function into an m-term TF series in $t\in[0,1)$ as Eq. (14), we will have

$$f(t)\approx f_m(t)=\sum_{k=0}^{m-1}[c_k\, T1_k(t)+c_{k+1}\, T2_k(t)] \qquad (16)$$

The constant coefficients $\{c_k\}_{k=0}^{m}$ are unknown, substituting Eq. (16) into Eq. (9), yields

$$\begin{aligned} r_m(s) &= g(s)-\int_a^b k(s,t)f(t)dt \\ &= g(s)-\sum_{k=0}^{m-1}c_k\int_a^b k(s,t)T1_k(t)dt \qquad (17) \\ &\quad -\sum_{k=0}^{m-1}c_{k+1}\int_a^b k(s,t)T2_k(t)dt \end{aligned}$$

where $a\le s\le b$ and we expect the residual $r_m(s)$ tends to zero.

It means that [7]

$$(r_m, T1_k) = 0 \qquad (18)$$

where $k = 0,1,\ldots,m-1$, (.,.) denotes the inner product and one of the Eqs. (19) in the following is considered simultaneously:

$$(r_m, T2_k) = 0 \qquad (19)$$

where $k = 0,1,\ldots,m-1$.

By using Eqs. (17-19), we get the linear system of (m+1) equations and (m+1) unknown which gives the coefficient of $f_m(t)$.

CONVERGENCE THEOREMS

We consider a comparison between the Gauss preconditioner method and the Gauss-Seidel iterative method in the following theorems so we need some definitions and theorems.

Definition 5: [5] Let A be an $n\times n$ real matrix and $A = M-N$ be a splitting of A, then the splitting is called

(i) Regular splitting if M is nonsingular, $M^{-1}\ge 0$ and $N\ge 0$.
(ii) Convergent splitting if $\rho(M^{-1}N)<1$.

Lemma 1: (Zhang *et al*. [3, Lemma 1]) If A is an M-matrix, then for any $1\le m\le n-1$

(i) A_{Lm} exists.
(ii) $M_m\ge 0$ and $P_m\ge 0$.
(iii) Both A_{Lm} and L_{m4} are M-matrices.

Lemma 2: (Niki *et al*. [5,Theorem 2.4]) Let $A = M-N$ be a regular splitting of matrix A. Then, A is nonsingular with $A^{-1}\ge 0$, if and only if $\rho(M^{-1}N)<1$.

Theorem 1: Suppose $A = I-L-U$ be a nonsingular M-matrix. Then the preconditioned matrix $A_{Lm} = M_p-N_p$ is regular and convergent splitting.

Proof: If we consider the splitting matrices of A_{Lm} which is introduced in [3],

$$D_{Lm}=\begin{pmatrix} D_{L1} & 0 \\ 0 & D_{L4}\end{pmatrix}, E_{Lm}=\begin{pmatrix} 0 & 0 \\ 0 & E_{L4}\end{pmatrix}, F_{Lm}=\begin{pmatrix} F_{L1} & F_{12} \\ 0 & F_{14}\end{pmatrix} (20)$$

we will have

$$M_p^{-1}=(D_{Lm}-E_{Lm})^{-1}=\begin{pmatrix} D_{L1}^{-1} & 0 \\ 0 & (D_{L4}-E_{L4})^{-1}\end{pmatrix}, N_p = F_{Lm}$$

Since both A_{Lm} and $(D_{L4}-E_{L4})$ have nonzero diagonal elements, then D_{L1}^{-1} and $(D_{L4}-E_{L4})^{-1}$ exist. so M^{-1} is exist. For a Z-matrix A, "A is a nonsingular M-matrix if and only if there is a positive vector x such that Ax>0", [10, Theorem 6-2.3]. Therefore, for a positive vector $x\ (>0)\in\Re^n$, we have from Lemma.1 $A_{Lm}x = P_mAx>0$. In addition to, since for a Z-matrix the statement "A is a nonsingular M-matrix" is equivalent to "all the principal minors of A are positive", [10, p. 136]. Therefore, $A_{Lm} = M_p-N_p$ is a regular splitting of matrix A. By using Lemma.1 and multiplying the relation $M_px\ge N_px$ by M^{-1}, we have $M_p^{-1}N_px\le x$. Therefore [5, Theorem 2.7] is shown that $A_{Lm} = M_p\ge N_p$ is convergent splitting.

Theorem 2: Let A be a M-matrix and both $A = M-N$ and $A_{Lm} = M_p-N_p$ be the Gauss-seidel convergent splitting. Then the following inequality holds.

$$\rho(M_p^{-1}N_p)\le\rho(M^{-1}N)<1$$

Proof: Since

$$\begin{aligned} A_{Lm}^{-1} &= (P_mA)^{-1} \\ &= (M_mM_{m-1}\ldots M_2M_1A)^{-1} \\ &= A^{-1}M_1^{-1}M_2^{-1}\ldots M_{m-1}^{-1}M_m^{-1} \end{aligned}$$

From the construction of Gauss transformation matrices M_k, for $k = 1,2,\dots,m$ in Eq. (6) and Lemma.2, we have $A^{-1} \ge A_{Lm}^{-1} \ge 0$. Therefore,

$$0 \le A_{Lm}^{-1}x = (I - M_p^{-1}N_p)^{-1} M_p^{-1} N_p x$$

$$\le A^{-1}Nx = (I - M^{-1}N)M^{-1}Nx = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}x$$

by using [5, Theorem 2.7], we get

$$\frac{\rho(M_p^{-1}N_p)}{1-\rho(M_p^{-1}N_p)} \le \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}$$

it implies

$$\rho(M_p^{-1}N_p) \le \rho(M^{-1}N) < 1 \qquad (21)$$

Remark 1: Let us present an error analysis in the orthogonal triangular function domain. If we note to the Eq. (15), we will see the coefficients ci for $i = 0,1,\dots,m$ are not optimal and this show that the minimum integral square error (MISE) is not minimized [9]. Of course, we know obviously that the coefficients ci's and di's are samples of f (t) for deriving a piecewise linear solution and we don't require the integration formula. Furthermore, it can be shown that the optimal coefficients and error estimations for optimal approximations with triangular function can reduce e_{MISE}, [9]. Now, we denote the e_{MISE} by

$$e_{MISE} = \| f_m(t) - \bar{f}(t) \| \qquad (22)$$

where f_m (t) and $\bar{f}(t)$ show the approximate and exact solutions of the first kind Fredholm Integral Equations, respectively. By using MATLAB software, the solution of the linear resolve system is approximated.

NUMERICAL EXAMPLES

In this section, we considered some examples from the first kind of Fredholm integral equations by using the proposed preconditioned technique for $m = 8$ and the representational errors e_{MISE} are shown in Table 1. Also numerical results are shown in Fig. 2 for $m = 16$ to illustrate the efficiency of this technique.

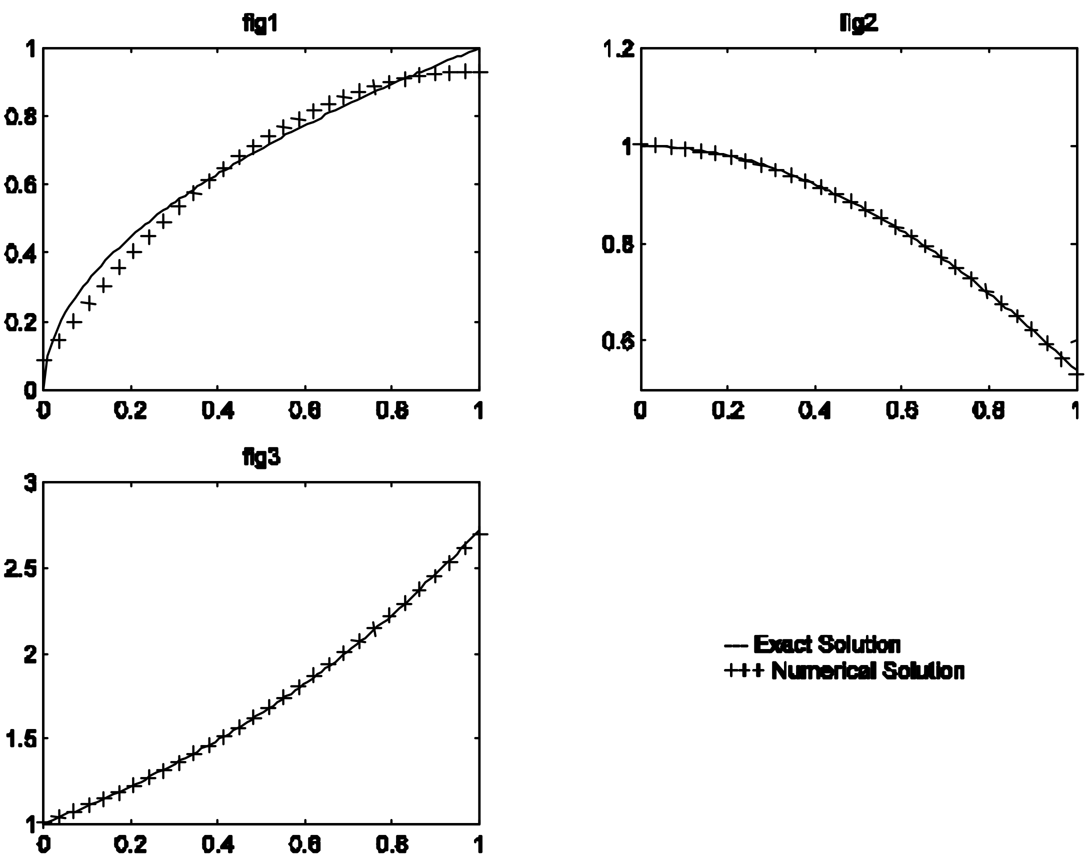

Fig. 2: Results for Examples (1-3) with m = 16

Table 1: Errors e_{MISE} for examples 1-3 with m = 8

t	Example.1	Example.2	Example.3
0.000	0	0	0
0.125	0.1E-9	0.7E-9	0.1E-8
0.250	0.5E-9	0.4E-9	0.1E-8
0.375	0.2E-9	0.1E-9	0.1E-8
0.500	0.2E-9	0.6E-7	0.1E-8
0.625	0.2E-9	0.6E-9	0.7E-8
0.750	0.1E-8	0.3E-9	0.1E-8
0.875	0.1E-9	0.2E-9	0

Example 1: Consider the following integral equation,

$$\int_0^1 \frac{(t-s)^2}{1+t^2} f(t)\,dt = -0.487495s^2 - 0.532108s + 0.179171$$

and exact solution $f(t)=\sqrt{t}$, $0 \le t \le 1$.

Example 2: Consider the following integral equation,

$$\int_0^1 (\sin(s+t) + \exp(t)\cos(s-t))f(t)\,dt$$
$$= 1.4944\cos(s) + 1.4007\sin(s)$$

and exact solution f (t) = cox (t), $0 \le t \le 1$.

Example 3: Consider the following integral equation,

$$\int_0^1 \exp(st)f(t)\,dt = \frac{\exp(s+1)-1}{s+1}$$

and exact solution f (t) = exp (t), $0 \le t \le 1$.

It is noted that by using the TF (optimal), the representational error is the least. For obtaining more information about the recent results and comparing them with the results of the other basis such as BPF and SHF, one can see [9].

CONCLUSION

Based on the structural properties of orthogonal basis, a complementary pair of orthogonal triangular function (TF) sets was extended to approximate the solution of the first kind Fredholm integral equations. Also for the solution of Ax = b which is obtained from the Galerkin method, some preconditioners based on Gauss type preconditioning techniques are applied and these preconditioners retain the efficiency and robustness of the primitive version.

REFERENCES

1. Saad, Y., 2000. Iterative methods for sparse linear systems. Second Edition with Corrections.
2. Molavi-Arabshahi, S.M. and M. Dehghan, 2007. Preconditioned techniques for solving large sparse linear systems arising from the discretization of the elliptic partial differential equations. Appl. Math. Comp., 188: 1371-1388.
3. Zhang, Y., T.-Z. Huang and X. Liu, 2007. Gauss type preconditioning techniques for linear systems. Appl. Math. Comp., 188: 612-633.
4. Benzi, M., C.D. Meyer and M. Tuma, 1996. A sparse approximate inverse preconditioner for the conjugate gradient method. SIAM J. Sci. of Comp., 17: 1135-1149.
5. Niki, H., T. Kohno and M. Morimoto, 2008. The preconditioned Gauss-Siedel method faster than the SOR method. Comp. Appl. Math., 219: 59-71.
6. Van Loan, C. and G. Golub, 1989. Matrix computations, Second Edition. The Johns Hopkins university press, Baltimore.
7. Delves, L.M. and J.L. Mohamed, 1985. Computational methods for Integral Equations. Cambridge University Press.
8. Shang, X. and D. Han, 2007. Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets. Appl. Math. Comp., 191: 440-444.
9. Deb, A., A. Dasgupta and G. Sarkar, 2006. A new set of orthogonal functions and its application to the analysis of dynamic systems. J. Franklin Institute, 343 (1): 1-26.
10. Berman, A. and R.J. Plemmons, 1994. Nonnegative matrices in the mathematical sciences, Academic Press, New York, 1979; Reprinted by SIAM, Philadelphia, PA.