# Predicting zero reductions in Gröbner Basis computations

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We can reduce further using  $z^2g_2$ :

$$-y^2z^2+z^4+y^2z^2-z^4=0.$$

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Standard representations of spol $(g_2, g_1)$  and spol $(g_3, g_1)$  $\implies$  Standard representation of spol $(g_3, g_2)$ .

#### Chain criterion [3]

- Let  $f, g, h \in \mathcal{R}$ ,  $G \subset \mathcal{R}$  finite. If
  - **1.** lt(h) | lcm(lt(f), lt(g)), and
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Combined implementation of Product and Chain criterion: Gebauer-Möller Installation [10]

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▶ An element  $\alpha \in \mathscr{R}^m$  with  $\overline{\alpha} = 0$  is called a syzygy.

$$g_1 = xy - z^2, \, \mathfrak{s}(g_1) = e_1,$$
  
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Note that  $\mathfrak{s}(\operatorname{spol}(g_3, g_1)) = xye_2$  and  $\operatorname{Im}(g_1) = xy$ .

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#### Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from *P* by increasing signature.

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 Compute 1, remove 1.

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#### Sketch of proof

**1.**  $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta).$ 

**2.** All S-pairs are handled by increasing signature.  $\Rightarrow$  All relatons  $\prec \mathfrak{s}(\alpha)$  are known:

 $\alpha = \beta$  + elements of smaller signature

 $\square$ 

S-pairs in signature T

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#### S-pairs in signature T

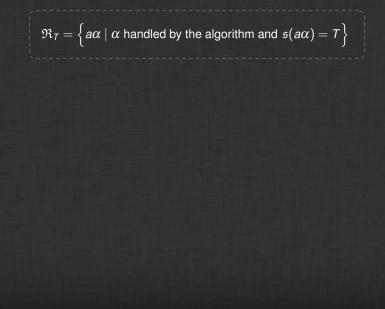
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What are all possible configurations to reach signature *T*? Define an order  $\trianglelefteq$  on  $\mathfrak{R}_T$  and choose the maximal element.



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Choose  $b\beta$  to be an element of  $\mathfrak{R}_T$  maximal w.r.t. an order  $\leq$ .

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- 2. If  $b\beta$  is not part of an S-pair  $\implies$  Go on to next signature.

## **Special cases**

 $\mathfrak{R}_{\mathcal{T}}=\Big\{ alpha \mid lpha$  handled by the algorithm and  $\mathfrak{s}(alpha)=\mathcal{T}\Big\}$ 

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Revisiting our example with ≺<sub>pot</sub>

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α added to *G* ▼ Generate all possible principal syzygies with α. (e.g. **GVW**) S-pair fulfilling Product criterion not detected by Rewrite criterion ▼ Add one corresponding syzygy. (e.g. SBA in Singular)

# **Experimental results**

## Implementation done in Singular [4]

Benchmark	STD	SBA ⊰ <sub>pot</sub>	SBA ≺ <sub>lt</sub>	
	ZR	ZR	ZR	ZR / PC
cyclic-8	4284	243	771	771 / 0
cyclic-8-h	5843	243	771	771 / 0
eco-11	3476	0	614	614/0
eco-11-h	5429	502	629	608 / 0
katsura-11	3933	0	348	304 / 0
katsura-11-h	3933	0	348	304 / 0
noon-9	25508	0	682	646 / 0
noon-9-h	25508	0	682	646 / 0
binomial-6-2	21	6	15	8 / 7
binomial-6-3	20	13	15	9 / 6
binomial-7-3	27	24	21	21 / 0
binomial-7-4	41	16	19	16/3
binomial-8-3	53	23	27	27 / 0
binomial-8-4	40	31	26	26 / 0

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#### **Ongoing work:**

- Describe in detail the connection between our conjecture and Moreno-Socías conjecture [12].
- Try to exploit even more algebraic structures for predicting zero reductions.

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