Prediction for future failures in Weibull Distribution under hybrid censoring

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Abstract

In this paper, we consider the prediction of a future observation based on a Type-I hybrid censored sample when the lifetime distribution of experimental units are assumed to be a Weibull random variable. Different classical and Bayesian point predictors are obtained. Bayesian predictors are obtained using squared error and LINEX loss functions. We also provide simulation consistent method for computing Bayesian prediction intervals. Monte Carlo simulations are performed to compare the performances of the different methods, and one data analysis has been presented for illustrative purposes.

Keywords: Hybrid censoring, Weibull distribution, Predictor, Prediction interval, Monte Carlo simulation.

1 Introduction

The Weibull distribution is one of the most widely used distributions in reliability and survival analysis. The probability density function (PDF) and hazard function can take variety of shapes. In this manuscript it is assumed that the two-parameter Weibull distribution has the following PDF

$$f(x;\alpha,\lambda) = \begin{cases} \alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}} & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

and the distribution function

$$F(x;\alpha,\lambda) = \begin{cases} 1 - e^{-\lambda x^{\alpha}} & \text{if} \quad x > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

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Here $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. From now on, a two-parameter Weibull distribution with the PDF (1.1) will be denoted by WE(α, λ).

Censoring is very common in most of the life testing experiments. The two most common censoring schemes are namely Type-I and Type-II censoring. Consider a sample of n units placed on a life-test at time 0. In Type-I censoring scheme, a time T, independent of the failure times, is pre-fixed so that beyond this time no failures will be observed, that is, the experiment terminates at time T. In Type-II censoring scheme, the number of observed failures is fixed, say r ($r \leq n$), and the experiment stops when the r-th failure takes place. Epstein (1954) introduced a new censoring scheme which is a mixture of Type-I and Type-II censoring schemes, and it is known as the hybrid censoring scheme. Let $X_{1:n} < \ldots < X_{n:n}$ denote the ordered lifetime of the experimental units. In hybrid censoring scheme, the experiment stops at $\min\{X_{r:n}, T\}$, where r and T are pre-fixed. From now on, we term this one as the Type-I hybrid censoring scheme. Therefore, in Type-I hybrid censoring scheme, one observes $X_{1:n} \ldots, X_{r:n}$, if $X_{r:n} < T$ (Case I), or $X_{1:n}, \ldots, X_{R:n}$ when $R < r$, and $X_{R:n} < T < X_{R+1:n}$ (Case II). Here R is a random variable and $R = 0, 1, \ldots, r-1$.

In the recent years, the hybrid censoring scheme has received a considerable attention in the reliability and life-testing experiments. Epstein (1954) first introduced Type-I hybrid censoring scheme, since then several others hybrid censoring schemes have been introduced in the literature. It has been discussed quite extensively by many others. See, for example, Fairbanks *et al.* (1982), Draper and Guttman (1987), Chen and Bhattacharya (1988), Ebrahimi (1986 , 1992), Jeong *et al.* (1996), Kundu and Gupta (1988), Childs *et al.* (2003), Kundu (2007), Kundu and Banerjee (2008) and the recent review article by Balakrishnan and Kundu (2012) on this topic.

Prediction of future observation comes up quite naturally in many life testing experiments. Extensive work on prediction problem based on frequentist and Bayesian framework can be found in the literature. Smith (1997, 1999) investigated the properties of the different predictors based on Bayes and frequentist procedures for a class of parametric family under smooth loss functions. Al-Hussaini (1999) also considered the Bayesian prediction problem for a large class of lifetime distributions. Dellaportas and Wright (1991) considered a numerical approach to Bayesian prediction for the two-parameter Weibull distribution. They assumed that the shape parameter has a uniform prior over a finite interval and the error is squared error. Recently, Kundu and Raqab (2012) considered the prediction of future observation from a Type-II censored data for the two-parameter Weibull distribution under fairly flexible priors on the shape and scale parameters.

The main aim of this paper is to consider the prediction of future observation based on

Type-I hybrid censored observation for a two-parameter Weibull distribution. We consider both the frequentist and Bayesian approaches. We obtain the maximum likelihood predictor, best unbiased predictor and the conditional median predictor. We consider the Bayesian predictor based on the assumption that the shape and scale parameters have gamma priors. It is observed that the Bayesian predictor cannot be obtained in closed form and we propose to use Gibbs sampling technique to compute the Bayes predictor. We also obtain prediction intervals based on the frequentist and Bayesian approaches. Bayesian prediction intervals are also obtained using Gibbs sampling technique. We perform some Monte Carlo simulation to compare the performances of the different methods and one data analysis has been performed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2, we provide the preliminaries and notations. Different predictors are presented in Section 3. In Section 4, we discuss different prediction intervals. Monte Carlo simulation results and the data analysis are presented in Section 5. Finally we conclude the paper in Section 6.

2 Notation and Preliminaries

Let $\mathbf{X} = (X_{1:n}, X_{2:n}, \cdots, X_{d:n})$ denote a Type-I hybrid censored sample from model (1.1). For notation simplicity, we will write $(X_1, X_2, \cdots, X_d)$ for $(X_{1:n}, X_{2:n}, \cdots, X_{d:n})$. Based on the observed data, the likelihood function for α and λ without the normalizing constant is

$$L(\alpha, \lambda) = \alpha^d \lambda^d \left\{ \prod_{i=1}^{d} x_i^{\alpha-1} \right\} e^{-\lambda\left[\sum_{i=1}^{d} x_i^{\alpha} + (n-d)T_0^{\alpha}\right]}, \tag{2.1}$$

where d denotes the number of failures and $T_0 = \min\{X_{r:n}, T\}$. So, we have

$$T_0 = \begin{cases} X_{r:n} & \text{for Case I} \\ T & \text{for Case II} \end{cases} \quad \text{and} \quad d = \begin{cases} r & \text{for Case I} \\ R & \text{for Case II.} \end{cases}$$

From (2.1), the maximum likelihood estimator (MLE) of λ can be shown to be

$$\widehat{\lambda} = \frac{d}{\sum_{i=1}^{d} X_i^{\widehat{\alpha}} + (n-d)T_0^{\widehat{\alpha}}}, \tag{2.2}$$

where $\widehat{\alpha}$ can be obtained as a solution of the following equation

$$\sum_{i=1}^{d} \ln(x_i) + d\left[\frac{1}{\alpha} - \frac{\sum_{i=1}^{d} x_i^{\alpha} \ln(x_i) + (n-d)T_0^{\alpha}\ln(T_0)}{\sum_{i=1}^{d} x_i^{\alpha} + (n-d)T_0^{\alpha}}\right] = 0. \tag{2.3}$$

In the next section we discuss different methods of the prediction of $Y = X_{s+d:n} (s = 1, 2, \ldots, n-d)$ of all the $n-d$ censored units based on observed data $\mathbf{X} = (X_1, ..., X_d)$.

3 Point Prediction

It is well-known that the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$ is just the distribution of $X_{s+d:n}$ given $X_d = x_d$ (due to the Markovian property of censored-order statistics). This implies that the density of Y given $\mathbf{X} = \mathbf{x}$ is the same as the density of the sth order statistic out of $n-d$ units from the population with density $f(y)/(1-F(T_0)), y \geq T_0$ (left truncated density at T_0). Therefore, the conditional density of $Y = X_{s+d:n}$ given $\mathbf{X} = \mathbf{x}$, for $y \geq T_0$, is given by

$$f(y|\mathbf{x}) = s\binom{n-d}{s} f(y) \left[F(y) - F(T_0)\right]^{s-1} \left[1 - F(y)\right]^{n-d-s} \left[1 - F(T_0)\right]^{-(n-d)}, \tag{3.1}$$

where $s = 1, 2, \ldots, n-d$. For model (1.1), (3.1) reduces to

$$\begin{aligned} f(y|\mathbf{x}, \alpha, \lambda) &= s\binom{n-d}{s} \alpha\lambda y^{\alpha-1} \left[e^{-\lambda T_0^{\alpha}} - e^{-\lambda y^{\alpha}}\right]^{s-1} \\ &\times \exp\{-\lambda \left[(n-d-s+1)y^{\alpha} - (n-d)T_0^{\alpha}\right]\}. \end{aligned} \tag{3.2}$$

Note that for $d = r$ and $T_0 = X_{r:n}$, we obtain

$$\begin{aligned} f(y|\mathbf{x}, \alpha, \lambda) &= s\binom{n-r}{s} \alpha\lambda y^{\alpha-1} \left[e^{-\lambda x_{r:n}^{\alpha}} - e^{-\lambda y^{\alpha}}\right]^{s-1} \\ &\times \exp\{-\lambda \left[(n-r-s+1)y^{\alpha} - (n-d)x_{r:n}^{\alpha}\right]\}, \end{aligned}$$

which is the conditional density of $Y = X_{s+r:n}$ given the Type-II censored sample $\mathbf{X} = (X_{1:n}, ..., X_{r:n})$, see for example Kundu and Raqab (2012).

3.1 Maximum Likelihood Predictor

Now we want to predict Y by maximum likelihood method. The predictive likelihood function (PLF) of Y and (α, λ) is given by

$$L(y, \alpha, \lambda|\mathbf{x}) = f(y|\mathbf{x}, \alpha, \lambda) f(\mathbf{x}|\alpha, \lambda) \tag{3.3}$$

Generally, if $\widehat{Y} = u(\mathbf{X})$, $\widehat{\alpha} = v_1(\mathbf{X})$ and $\widehat{\lambda} = v_2(\mathbf{X})$ are statistics for which

$$L(u(\mathbf{x}), v_1(\mathbf{x}), v_2(\mathbf{x})|\mathbf{x}) = \sup_{(y,\alpha,\lambda)} L(y, \alpha, \lambda|\mathbf{x}),$$

then $u(\mathbf{X})$ is said to be the maximum likelihood predictor (MLP) of the Y and $v_1(\mathbf{x})$ and $v_2(\mathbf{x})$ the predictive maximum likelihood estimators (PMLEs) of α and λ, respectively.

Consequently, the predictive likelihood function (PLF) of Y, α and λ, is given by

$$\begin{aligned} L(y,\alpha,\lambda) &= s\binom{n-d}{s}(\alpha\lambda)^{d+1}y^{\alpha-1}\left\{\prod_{i=1}^{d}x_i^{\alpha-1}\right\}\left[e^{-\lambda T_0^\alpha}-e^{-\lambda y^\alpha}\right]^{s-1} \\ &\times \ \exp\left\{-\lambda\left[\sum_{i=1}^{d}x_i^\alpha+(n-d-s+1)y^\alpha\right]\right\}. \end{aligned} \tag{3.4}$$

Apart from a constant term, the predictive log-likelihood function is

$$\begin{aligned} \ln L(y,\alpha,\lambda) &\propto (d+1)[\ln(\alpha)+\ln(\lambda)]+(\alpha-1)[\ln(y)+\sum_{i=1}^{d}\ln(x_i)] \\ &+ (s-1)\ln\left[e^{-\lambda T_0^\alpha}-e^{-\lambda y^\alpha}\right]-\lambda\left[\sum_{i=1}^{d}x_i^\alpha+(n-d-s+1)y^\alpha\right]. \end{aligned} \tag{3.5}$$

By using (3.5), the predictive likelihood equations (PLEs) for y, α and λ are given, respectively, by

$$\frac{\partial \ln L(y,\alpha,\lambda)}{\partial y} = \frac{\alpha-1}{y}+(s-1)\frac{\alpha\lambda y^{\alpha-1}e^{-\lambda y^\alpha}}{e^{-\lambda T_0^\alpha}-e^{-\lambda y^\alpha}}-\alpha\lambda(n-d-s+1)y^{\alpha-1}=0 \tag{3.6}$$

$$\begin{aligned} \frac{\partial \ln L(y,\alpha,\lambda)}{\partial \alpha} &= \frac{d+1}{\alpha}+[\ln(y)+\sum_{i=1}^{d}\ln(x_i)]+(s-1)\frac{\lambda y^\alpha\ln(y)e^{-\lambda y^\alpha}-\lambda T_0^\alpha\ln(T_0)e^{-\lambda T_0^\alpha}}{e^{-\lambda T_0^\alpha}-e^{-\lambda y^\alpha}} \\ &-\lambda\left[\sum_{i=1}^{d}x_i^\alpha\ln(x_i)+(n-d-s+1)y^\alpha\ln(y)\right]=0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial \ln L(y,\alpha,\lambda)}{\partial \lambda} &= \frac{d+1}{\lambda}+(s-1)\frac{y^\alpha e^{-\lambda y^\alpha}-T_0^\alpha e^{-\lambda T_0^\alpha}}{e^{-\lambda T_0^\alpha}-e^{-\lambda y^\alpha}} \\ &-\left[\sum_{i=1}^{d}x_i^\alpha+(n-d-s+1)y^\alpha\right]=0 \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we obtain the MLP of Y as

$$\hat{Y}_{MLP}=\left(T_0^{\tilde{\alpha}}-\frac{1}{\tilde{\tilde{\lambda}}}\ln\left[1-\frac{(s-1)\widetilde{\lambda}\widetilde{\alpha}T_0^{\widetilde{\alpha}}}{\widetilde{\alpha}\left(d-\lambda\sum_{i=1}^{d}x_i^{\widetilde{\alpha}}\right)+1}\right]\right)^{1/\widetilde{\alpha}}$$

where $(\tilde{\alpha},\tilde{\lambda})$ is PMLE of (α,λ) that can be obtained numerically from (3.7) and (3.8).

3.2 Best Unbiased Predictor

A statistic $\hat{Y}$ which is used to predict $Y = X_{s+d:n}$ is called a best unbiased predictor (BUP) of Y, if the predictor error $\hat{Y} - Y$ has a mean zero and its prediction error variance $\text{Var}(\hat{Y} - Y)$ is less than or equal to that of any other unbiased predictor of Y.

Since the conditional distribution of Y given $\mathbf{X} = (X_1, \cdots, X_d)$ is just the distribution of Y given $\mathbf{X}$, therefore the BUP of Y is

$$\hat{Y}_{BUP} = E(Y|\mathbf{X}) = \int_{T_0}^{\infty} y f(y|\mathbf{x}, \alpha, \lambda) dy.$$

Using (3.2) and the binomial expansion

$$\left[e^{-\lambda T_0^\alpha} - e^{-\lambda y^\alpha}\right]^{s-1} = \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^{s-j-1}\, e^{-\lambda j T_0^\alpha}\, e^{-\lambda(s-j-1)y^\alpha}, \tag{3.9}$$

we obtain

$$\begin{aligned} \hat{Y}_{BUP} &\equiv I(T_0; \alpha, \lambda) \\ &= s\binom{n-d}{s} \lambda^{-\frac{1}{\alpha}} \\ &\times \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^{s-j-1} \frac{e^{\lambda(n-d-j)T_0^\alpha}}{[n-d-j]^{1+1/\alpha}} \Gamma(1 + \frac{1}{\alpha}\ ;\ \lambda[n-d-j]T_0^\alpha), \end{aligned} \tag{3.10}$$

where $\Gamma(a; b)$ is upper incomplete gamma function and is defined as

$$\Gamma(a; b) = \int_b^{\infty} e^{-t} t^{a-1} dt.$$

Because the parameters α and λ are unknown, they have to be estimated. Thus, one would replace them by their corresponding MLEs and obtain the BUP of Y.

3.3 Conditional Median Predictor

Conditional median predictor (CMP) is another possible predictor which is suggested by Raqab and Nagaraja (1995). A predictor $\hat{Y}$ is called the CMP of Y, if it is the median of the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$, that is

$$P_\theta(Y \leq \hat{Y}|\mathbf{X} = \mathbf{x}) = P_\theta(Y \geq \hat{Y}|\mathbf{X} = \mathbf{x}). \tag{3.11}$$

Using the relation

$$P_{\alpha,\lambda}(Y \leq \widehat{Y}|\mathbf{X} = \mathbf{x}) = P_{\alpha,\lambda}\left(1 - \frac{e^{-\lambda Y^\alpha}}{e^{-\lambda T_0^\alpha}} \geq 1 - \frac{e^{-\lambda \widehat{Y}^\alpha}}{e^{-\lambda T_0^\alpha}} |\mathbf{X} = \mathbf{x}\right), \tag{3.12}$$

and using the fact that the distribution of $1-\frac{e^{-\lambda Y^{\alpha}}}{e^{-\lambda T_0^{\alpha}}}$ given $\mathbf{X}=\mathbf{x}$ is a $Beta(s,n-d-s+1)$ distribution, we obtain the CMP of Y as

$$\hat{Y}_{CMP}=\left(-\frac{1}{\lambda}\ln\left[[1-Med(B)]e^{-\lambda T_0^{\alpha}}\right]\right)^{1/\alpha}, \tag{3.13}$$

where B has $Beta(s,n-d-s+1)$ distribution and $Med(B)$ stands for median of B. By substituting α and λ with their corresponding MLEs, we obtain the CMP of Y.

3.4 Bayesian Predictors

In this section, our interest is to predict $Y=X_{s+d}$ $(s=1,2,...,n-d)$ based on the observed hybrid censored sample $\mathbf{x}=(x_1,...,x_d)$ by a Bayesian approach. If we denote the predictive density of $Y=X_{s+d}$ as $f_s^*(y|\mathbf{x})$, then

$$f_s^*(y|\mathbf{x})=\int_0^{\infty}\int_0^{\infty}f(y|\mathbf{x},\alpha,\lambda)\pi(\alpha,\lambda|\mathbf{x})d\alpha d\lambda \tag{3.14}$$

where $f(y|\mathbf{x},\alpha,\lambda)$ is the conditional density of $Y=X_{s+d}$ given $\mathbf{x}=(x_1,...,x_d)$ and $\pi(\alpha,\lambda|\mathbf{x})$ is the joint posterior density of α and λ. For finding $\pi(\alpha,\lambda|\mathbf{x})$, it's assumed that α and λ each have independent $Gamma(a_1,b_1)$ and $Gamma(a_2,b_2)$ priors respectively. Based on the priors, the joint posterior density function of α and λ given the data is

$$\pi(\alpha,\lambda|\mathbf{x})\propto g_1(\lambda|\alpha,\mathbf{x})g_2(\alpha|\mathbf{x}). \tag{3.15}$$

Here $g_1(\lambda|\alpha,\mathbf{x})$ is a gamma density function with the shape and scale parameters as $d+a_2$ and $\sum_{i=1}^{d}x_i^{\alpha}+(n-d)T_0^{\alpha}+b_2$, respectively. Also $g_2(\alpha|\mathbf{x})$ is a proper density function given by

$$g_2(\alpha|\mathbf{x})\propto\frac{e^{-b_1\alpha}\alpha^{d+a_1-1}}{\left(\sum_{i=1}^{d}x_i^{\alpha}+(n-d)T_0^{\alpha}+b_2\right)^{d+a_2}}\prod_{i=1}^{d}x_i^{\alpha-1}. \tag{3.16}$$

Substituting (3.15) in (3.14), the predictive density function $f_s^*(y|\mathbf{x})$ can be obtained as

$$f_s^*(y|\mathbf{x})=\int_0^{\infty}\int_0^{\infty}f(y|\mathbf{x},\alpha,\lambda)g_1(\lambda|\alpha,\mathbf{x})g_2(\alpha|\mathbf{x})d\alpha d\lambda. \tag{3.17}$$

Now, the Bayesian point predictors can be obtained from the predictive density function $f_s^*(y|\mathbf{x})$ and given the loss function.

In the Bayesian inference, the most commonly used loss function is the squared error loss (SEL), $L(\phi,\phi^*)=[\phi^*-\phi]^2$, where ϕ^* is an estimate of ϕ. This loss is symmetric and its use is very popular. In life testing and reliability problems, the nature of losses are not always symmetric and hence the use of SEL is unacceptable in some situations. A useful asymmetric loss function is the linear-exponential (LINEX) loss, $L(\phi, \phi^*) \propto \exp[a(\phi^* - \phi)] - a(\phi^* - \phi) - 1, \;\; a \neq 0$. This loss function was introduced by Varian (1975) and was extensively discussed by Zellner (1986). The sign and magnitude of the shape parameter a represents the direction and degree of symmetry, respectively. When a closes to 0, the LINEX loss is approximately SEL and therefore almost symmetric.

If $\widehat{Y} = \delta(\mathbf{x})$ is a predictor of $Y = X_{s+d}$ $(s = 1, 2, ..., n-d)$, then the Bayesian point predictors of Y under a squared error loss, $\widehat{Y}_{SEP}$ and under a LINEX loss function, $\widehat{Y}_{LEP}$ are

$$\widehat{Y}_{SEP} = \int_{T_0}^{\infty} y f_s^*(y|\mathbf{x}) dy \tag{3.18}$$

and

$$\widehat{Y}_{LEP} = -\frac{1}{a} \ln \left[\int_{T_0}^{\infty} e^{-ay} f_s^*(y|\mathbf{x}) dy \right], \tag{3.19}$$

respectively. Since (3.18) and (3.19) can't computed explicitly, we adopt here Gibbs sampling procedure to obtain Bayesian point predictors. For later use, we need the following result.

Theorem 1: The conditional distribution of α given the data, $g_2(\alpha|x_d)$ is log-concave.

Proof: See the Appendix.

Devroye (1984) proposed a general method to generate samples from a general log-concave density function. Therefore, using Theorem 1, and adopting the method of Devroye (1984), it is possible to generate samples from (3.16). We use the idea of Geman and Geman (1984) to generate samples from the conditional posterior density function using the following scheme.

1. Generate α_1 from $g_2(.|\mathbf{x})$ using the method developed by Devroye (1984).
2. Generate λ_1 from $g_1(.|\alpha_1, \mathbf{x})$.
3. Repeat step 1 and step 2, M times and obtain $(\alpha_1, \lambda_1), \cdots, (\alpha_M, \lambda_M)$.

Note that in step 1, we use the Devroye algorithm as follows:

i) Compute $c = g_2(m|\mathbf{x})$. (m is the mode of $g_2(.|\mathbf{x})$)

ii) Generate U uniform on [0,2], and V uniform on [0,1].

iii) If $U \leq 1$ then $\alpha = U$ and $T = V$, else $\alpha = 1 - \ln(U-1)$ and $T = V(U-1)$.

iv) Let $\alpha = m + \frac{\alpha}{c}$.

v) If $T \leq \frac{g_2(\alpha|\mathbf{x})}{c}$, then α is a sample from $g_2(.|\mathbf{x})$ else go to Step (ii).

Using the Gibbs sampling technique, the simulation consistent estimators of $f_s^*(y|\mathbf{x})$ can be obtained as

$$\widehat{f_s^*}(y|\mathbf{x}) = \sum_{i=1}^{M} f(y|\mathbf{x}, \alpha_i, \lambda_i). \tag{3.20}$$

By using (3.2), (3.18), (3.19) and (3.20), $\widehat{Y}_{SEP}$ and $\widehat{Y}_{LEP}$ under squared error and LINEX loss function can be obtained as

$$\hat{Y}_{SEP} = \frac{1}{M}\sum_{i=1}^{M} I(\mathbf{x}, \alpha_i, \lambda_i), \qquad \hat{Y}_{LEP} = -\frac{1}{a}\ln\left[\frac{1}{M}\sum_{i=1}^{M} J(\mathbf{x}, \alpha_i, \lambda_i)\right], \tag{3.21}$$

where $I(\mathbf{x}, \alpha, \lambda)$ is defined in (3.10). Also $J(\mathbf{x}, \alpha, \lambda)$ is defined as

$$J(\mathbf{x}, \alpha, \lambda) = E(e^{-aY}|\mathbf{X}) = \int_{\mathbf{x}}^{\infty} e^{-ay} f(y|\mathbf{x}, \alpha, \lambda)dy,$$

that by using (3.9), it can be written as

$$\begin{aligned} J(\mathbf{x}, \alpha, \lambda) &= s\binom{n-d}{s}\alpha\lambda\sum_{j=0}^{s-1}\binom{s-1}{j}(-1)^{s-j-1}e^{\lambda(n-d-j)T_0^\alpha} \\ &\times \int_{T_0}^{\infty} y^{\alpha-1}\exp\{-\lambda[n-d-j]y^\alpha - ay\}dy. \end{aligned} \tag{3.22}$$

4 Prediction Intervals

In this section, we consider several methods for obtain prediction intervals (PI's) for the value $Y = X_s$ based on the Type I hybrid censored sample $\mathbf{X} = (X_1, X_2, \cdots, X_d)$.

4.1 Pivotal Method

Let us take the random variable Z as

$$Z = 1 - \frac{e^{-\lambda Y^\alpha}}{e^{-\lambda T_0^\alpha}}.$$

As mentioned before the distribution of Z given $\mathbf{X}=\mathbf{x}$ is a $Beta(s,n-d-s+1)$ distribution. So, we can consider Z as a pivotal quantity to obtain the prediction interval for Y. Now, a $(1-\gamma)100\%$ PI for Y is $(L_1(\mathbf{X}),U_1(\mathbf{X}))$ where

$$L_1(\mathbf{X})=\left[-\frac{1}{\lambda}\ln\left([1-B_{\frac{\gamma}{2}}]e^{-\lambda T_0^\alpha}\right)\right]^{\frac{1}{\alpha}},\qquad U_1(\mathbf{X})=\left[-\frac{1}{\lambda}\ln\left([1-B_{1-\frac{\gamma}{2}}]e^{-\lambda T_0^\alpha}\right)\right]^{\frac{1}{\alpha}} \tag{4.1}$$

where B_γ stands for 100γth percentile of $Beta(s,n-d-s+1)$ distribution. When α and λ are unknown, the parameters in (4.1), have to be estimated. For example, by replacing α and λ with their corresponding MLEs, the prediction limits for Y can be obtained.

4.2 Highest Conditional Density Method

The distribution of Z given $\mathbf{X}=\mathbf{x}$ is a $Beta(s,n-d-s+1)$ distribution with pdf

$$g(z)=\frac{z^{s-1}(1-z)^{n-d-s}}{Beta(s,n-d-s+1)},\qquad 0<z<1,$$

which is a unimodal function of z, for $1<s<n-d$. Now, the $(1-\gamma)100\%$ highest conditional density (HCD) prediction limits for Y are given by

$$L_2(\mathbf{X})=\left[-\frac{1}{\lambda}\ln(1-w_1)e^{-\lambda T_0^\alpha}\right]^{\frac{1}{\alpha}},\qquad U_2(\mathbf{X})=\left[-\frac{1}{\lambda}\ln(1-w_2)e^{-\lambda T_0^\alpha}\right]^{\frac{1}{\alpha}} \tag{4.2}$$

where w_1 and w_2 are the simultaneous solutions of the following equations:

$$\int_{w_1}^{w_2} g(z)dz = 1-\alpha \tag{4.3}$$

and

$$g(w_1)=g(w_2) \tag{4.4}$$

We simplify Equations (4.3) and (4.4) as

$$B_{w_2}(s,n-d-s+1)-B_{w_1}(s,n-d-s+1)=1-\alpha, \tag{4.5}$$

and

$$\left(\frac{1-w_2}{1-w_1}\right)^{n-d-s}=\left(\frac{w_1}{w_2}\right)^{s-1}, \tag{4.6}$$

where

$$B_t(a,b)=\frac{1}{B(a,b)}\int_0^t x^{a-1}(1-x)^{b-1}\,dx,$$

is the incomplete beta function.

4.3 Bayesian Prediction Interval

Bayesian prediction intervals are obtained from the Bayes predictive density $f^*(y|\mathbf{x})$. Bayesian prediction bounds are obtained be evaluating

$$P(Y > \lambda|\mathbf{x}) = \int_{\lambda}^{\infty} f^*(y|\mathbf{x})dy$$

for some positive λ. Now, the $100(1-\gamma)\%$ Bayesian prediction interval for Y is given by $(L_3(\mathbf{X}), U_3(\mathbf{X}))$, where $L_3(\mathbf{X})$ and $U_3(\mathbf{X})$ can be obtained by solving the follow nonlinear equations simultaneously

$$P(Y > L_3(\mathbf{x})|\mathbf{x}) = \int_{L_3(\mathbf{x})}^{\infty} f^*(y|\mathbf{x})dy = 1 - \frac{\gamma}{2} \tag{4.7}$$

and

$$P(Y > U_3(\mathbf{x})|\mathbf{x}) = \int_{U_3(\mathbf{x})}^{\infty} f^*(y|\mathbf{x})dy = \frac{\gamma}{2}. \tag{4.8}$$

5 Numerical Computations

In this section, a numerical example and a Monte Carlo simulation are presented to illustrate all the prediction methods described in the preceding sections.

5.1 Real Data Analysis

Here we present a data analysis of the strength data reported by Badar and Priest (1982). This data, represent the strength measured in GPA for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10mm. These data have been used previously by Raqab and Kundu (2005), Kundu and Gupta (2006) and Kundu and Raqab (2009). The data are presented in Table 1.

Table 1. Data Set (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

We analyzed the above data set using two-parameter Weibull distribution after subtracting 1.75. After subtracting 1.75 from all the points of this data set, we observed that the Weibull based on the parameters $\alpha = 2.256$ and $\lambda = 0.413$ fit to above data set. We checked the validity of the Weibull model based on the Kolmogorov-Smirnov (K-S) test. It is observed that the K-S distance and the corresponding p-value are respectively

$$K - S = 0.072, \qquad \text{and} \qquad p - value = 0.885.$$

We consider the following two sampling schemes:

$$Scheme\ 1: \quad r = 40 \quad and \quad T = 1$$

$$Scheme\ 2: \quad r = 15 \quad and \quad T = 2.$$

For scheme 1, it is observed that it is from Case I and for Scheme 2, it is from Case II. The data for scheme 1 are

```
0.151  0.382  0.453  0.478  0.507  0.600  0.611  0.646  0.647  0.695
0.704  0.724  0.768  0.772  0.775  0.782  0.825  0.864  0.866  0.868
0.874  0.909  0.925  0.988  0.990
```

and for scheme 2 are

```
0.151  0.382  0.453  0.478  0.507  0.600  0.611  0.646  0.647  0.695
0.704  0.724  0.768  0.772  0.775.
```

For scheme 1, the MLE and PMLE of (α, λ) are (2.920,0.520), (3.040,0.543) and for scheme 2 are (3.222,0.609), (3.445,0.687), respectively.

In order to analyze these data, we took $s = 1, 3, 5, 10, 15, 25$ and using different methods discussed in Sections 3 and 4, different point Bayesian and non-Bayesian predictions and also the 95% PI's are displayed in Table 2. Note that for computing Bayesian predictions, since we do not have any prior information, we assumed that the priors on α and λ are improper, *i.e.* $a_1 = b_1 = a_2 = b_2 = 0$. The Bayesian point predictors are computed from the Gibbs samples under the squared error and LINEX loss functions and with respect to the above improper priors.

Table 2. The values of point predictions and 95% PIs for $Y = X_s$.

Scheme 1											
	Exact val.	MLP	BUP	CMP	SEP	LEP			Pivot. meth.	HCD meth.	Bay. meth.
						a=0.1	a=1	a=10			
s=1	1.106	1.006	1.007	1.002	1.009	1.009	1.009	1.007	(0.990,1.051)	—	(0.982,1.076)
s=3	1.178	1.023	1.041	1.036	1.029	1.029	1.029	1.027	(1.001,1.107)	(0.996,1.096)	(0.990,1.124)
s=5	1.187	1.056	1.075	1.071	1.045	1.045	1.045	1.042	(1.019,1.154)	(1.013,1.144)	(1.010,1.162)
s=10	1.389	1.138	1.160	1.157	1.118	1.118	1.117	1.112	(1.079,1.259)	(1.091,1.202)	(1.058,1.276)
s=15	1.493	1.221	1.248	1.245	1.256	1.256	1.254	1.241	(1.149,1.362)	(1.180,1.335)	(1.152,1.406)
s=25	1.751	1.410	1.451	1.449	1.408	1.408	1.406	1.390	(1.320,1.593)	(1.363,1.552)	(1.285,1.601)
Scheme 1											
	Exact val.	MLP	BUP	CMP	SEP	LEP			Pivot. meth.	HCD meth.	Bay. meth.
						a=0.1	a=1	a=10			
s=1	0.782	0.786	0.793	0.787	0.792	0.792	0.792	0.791	(0.775,0.838)	—	(0.761,0.850)
s=3	0.864	0.808	0.827	0.823	0.840	0.840	0.839	0.835	(0.787,0.892)	(0.780,0.865)	(0.795,0.939)
s=5	0.868	0.839	0.860	0.856	0.889	0.888	0.888	0.880	(0.805,0.935)	(0.818,0.927)	(0.848,0.990)
s=10	0.990	0.913	0.938	0.935	0.956	0.956	0.955	0.952	(0.863,1.027)	(0.876,1.016)	(0.872,1.067)
s=15	1.187	0.980	1.012	1.010	1.045	1.045	1.044	1.033	(0.925,1.108)	(0.988,1.093)	(0.965,1.169)
s=25	1.493	1.112	1.161	1.160	1.145	1.145	1.144	1.137	(1.059,1.269)	(1.074,1.235)	(1.032,1.285)

5.2 Simulation Study

In this subsection, we present some results based on Monte Carlo simulations to compare the performance of the different methods for different r, T, and for different parameter values. We compare the performances of the MLP, BUP, CMP, and the Bayes prediction (with respect to the squared error and LINEX loss functions) in terms of biases, and mean square prediction errors (MSPEs). We also compare different confidence intervals in terms of the average confidence lengths, and coverage percentages.

For computing Bayesian point and interval predictors, we assume 2 priors as follows:

$$\text{Non-Informative prior:} \quad a_j = 0, \quad b_j = 0, \quad j = 1, 2,$$
$$\text{Informative prior:} \quad a_j = 2, \quad b_j = 3, \quad j = 1, 2.$$

We considered $n = 30$ and $r = 15$ and used two parameter values $(\alpha, \lambda) = (1, 2)$ and $(\alpha, \lambda) = (2, 1)$. Note that in our simulation experiments, we have not kept d to be a particular value. We have kept r fixed and T fixed, now depending on the sample, d changes for one sample to the other. For different s, T and two priors considered, Table 3 presents the average biases, and MSPEs of the Bayesian and non-Bayesian point predictors discussed in our paper over 1000 replications. All the computations are performed using Visual Maple (V12) package.

From Table 3, we observe that the BUP is the best point predictor. It provides the smallest biases and MSPEs. The CMP is the second best predictor. We also observe that the MSPEs and biases of the BUP, and CMP are very close. The MLP does not work well. Comparing the two Bayesian predictors based on two priors, as anticipated, the Bayesian predictors based on informative prior perform better than the Bayesian predictors based on non-informative prior in terms of both biases and MSPEs. We also observe that, for small a, the Bayesian predictors under the LINEX loss function are very close to the predictors under SEL function.

We also computed 95% non-Bayesian and Bayesian PIs for $Y = X_s$ by using the results given in Section 4. In Table 4, the means and coverage probabilities of the lengths of 95% classical and Bayesian PIs are reported. From Table 4, we observe that HCD prediction intervals are shorter than the other PIs. We also note that Bayesian PIs are wider than the classical PIs. From Table 4, it is evident that Bayesian PIs provide the most coverage probabilities. Also, in most of the cases considered, the coverage probabilities are quite close to the nominal level 95%.

Now one natural question is how sensitive is the choice of prior. We want to explore the sensitivity of the predictors with respect to different informative priors, having same means but different variances. We have considered the following six priors:

$$\begin{array}{llll} \text{Prior 1:} \quad a_j = b_j = 0.1, & j = 1, 2, & \text{Prior 2:} \quad a_j = b_j = 0.2, & j = 1, 2 \\ \text{Prior 3:} \quad a_j = b_j = 0.5, & j = 1, 2, & \text{Prior 4:} \quad a_j = b_j = 1, & j = 1, 2 \\ \text{Prior 5:} \quad a_j = b_j = 2, & j = 1, 2, & \text{Prior 6:} \quad a_j = b_j = 5, & j = 1, 2. \end{array}$$

The above priors have the same mean 1 but the variances as

$$\begin{array}{lll} \text{Var(Prior 1)} = 10, & \text{Var(Prior 2)} = 5, & \text{Var(Prior 3)} = 2, \\ \text{Var(Prior 4)} = 1, & \text{Var(Prior 5)} = 0.5, & \text{Var(Prior 6)} = 0.2. \end{array}$$

We then computed the MSPEs of the Bayesian point predictors based on above priors over 1000 replications. We also computed the average lengths of 95% Bayesian PIs. The results are reported in Table 5. From Table 5, by comparing the six Bayesian point predictors and Bayesian PIs based on six gamma priors , we observe that the Bayesian point predictors and Bayesian PIs based on prior 6 provide the best results. Note that prior 6 is the best informative prior, because its variance is smaller than that of other priors. From Table 5, as expected, when the variance prior is decreasing, the MSPEs and the average lengths of 95% Bayesian PIs are decreasing.

Finally, it should be mentioned here that all of the results obtained in this study can be specialized to both the Type-II censored sample by taking ($d = r, T_0 = X_{r:n}$), and the Type-I censored sample for ($d = R, T_0 = T$).

Table 3. Biases and MSPEs of point predictions for $n = 30$ and $r = 15$.

$\alpha = 1$ and $\lambda = 2$													
						Non-Informative Prior				Informative Prior			
			MLP	BUP	CMP	SEP	LEP			SEP	LEP		
							a=0.1	a=1	a=10		a=0.1	a=1	a=10
$T = 0.25$	$s = 1$	Bias	-0.043	-0.021	-0.027	-0.032	-0.042	-0.043	-0.048	-0.030	-0.038	-0.049	-0.044
		MSP	0.007	0.003	0.004	0.004	0.005	0.007	0.007	0.003	0.004	0.004	0.006
	$s = 3$	Bias	-0.069	-0.051	-0.054	-0.057	-0.064	-0.068	-0.072	-0.052	-0.062	-0.064	-0.077
		MSP	0.038	0.023	0.025	0.025	0.031	0.033	0.033	0.020	0.024	0.027	0.028
	$s = 5$	Bias	-0.084	-0.064	-0.072	-0.076	-0.098	-0.105	-0.109	-0.070	-0.089	-0.100	-0.102
		MSP	0.079	0.067	0.071	0.072	0.075	0.075	0.079	0.067	0.068	0.072	0.075
	$s = 10$	Bias	-0.168	-0.139	-0.143	-0.154	-0.166	-0.173	-0.180	-0.151	-0.142	-0.157	-0.176
		MSP	0.147	0.122	0.124	0.135	0.140	0.142	0.157	0.118	0.126	0.132	0.141
	$s = 15$	Bias	-0.309	-0.232	-0.249	-0.256	-0.287	-0.295	-0.313	-0.245	-0.275	-0.278	-0.301
		MSP	0.706	0.651	0.675	0.685	0.695	0.698	0.712	0.678	0.685	0.690	0.703
$T = 0.5$	$s = 1$	Bias	-0.019	-0.007	-0.011	-0.013	-0.016	-0.018	-0.022	-0.009	-0.010	-0.013	-0.017
		MSP	0.007	0.001	0.003	0.003	0.005	0.006	0.007	0.003	0.004	0.004	0.005
	$s = 3$	Bias	-0.026	-0.015	-0.018	-0.019	-0.019	-0.025	-0.029	0.014	0.015	0.022	-0.027
		MSP	0.016	0.003	0.007	0.011	0.011	0.016	0.018	0.010	0.010	0.013	0.015
	$s = 5$	Bias	-0.060	-0.029	-0.034	-0.037	-0.044	-0.051	-0.060	-0.033	-0.041	-0.048	-0.057
		MSP	0.027	0.017	0.019	0.020	0.022	0.022	0.028	0.019	0.019	0.020	0.025
	$s = 10$	Bias	-0.087	-0.042	-0.051	-0.056	-0.066	-0.077	-0.092	-0.055	-0.061	-0.069	-0.083
		MSP	0.044	0.023	0.026	0.034	0.037	0.040	0.045	0.031	0.034	0.036	0.042
	$s = 15$	Bias	-0.193	-0.152	-0.160	-0.174	-0.179	-0.183	-0.197	-0.169	0.175	-0.177	-0.194
		MSP	0.492	0.389	0.403	0.424	0.430	0.474	0.494	0.417	0.421	0.460	0.479
$\alpha = 2$ and $\lambda = 1$													
						Non-Informative Prior				Informative Prior			
			MLP	BUP	CMP	SEP	LEP			SEP	LEP		
							a=0.1	a=1	a=10		a=0.1	a=1	a=10
$T = 0.75$	$s = 1$	Bias	-0.037	-0.022	-0.024	-0.027	-0.030	-0.034	-0.038	-0.024	-0.026	-0.030	-0.031
		MSP	0.019	0.006	0.008	0.008	0.011	0.013	0.019	0.008	0.008	0.008	0.014
	$s = 3$	Bias	-0.054	-0.031	-0.034	-0.039	-0.040	-0.047	-0.056	-0.038	-0.039	-0.044	-0.049
		MSP	0.026	0.009	0.013	0.014	0.019	0.024	0.033	0.014	0.016	0.021	0.027
	$s = 5$	Bias	-0.087	-0.050	-0.058	-0.067	-0.074	-0.081	-0.090	-0.062	-0.071	-0.077	-0.085
		MSP	0.032	0.013	0.017	0.020	0.023	0.027	0.033	0.018	0.019	0.025	0.029
	$s = 10$	Bias	-0.127	-0.088	-0.095	-0.103	-0.116	-0.124	-0.140	-0.100	-0.111	-0.119	-0.131
		MSP	0.051	0.039	0.040	0.042	0.042	0.048	0.058	0.041	0.041	0.043	0.052
	$s = 15$	Bias	-0.268	-0.207	-0.221	-0.226	-0.240	-0.254	-0.273	-0.222	-0.230	-0.347	-0.561
		MSP	0.334	0.253	0.264	0.280	0.293	0.329	0.359	0.272	0.284	0.317	0.541
$T = 1$	$s = 1$	Bias	-0.019	-0.014	-0.015	-0.015	-0.017	-0.017	-0.020	-0.015	-0.015	-0.016	-0.018
		MSP	0.010	0.004	0.005	0.005	0.007	0.007	0.010	0.005	0.006	0.007	0.008
	$s = 3$	Bias	-0.045	-0.026	-0.028	-0.035	-0.038	-0.053	-0.054	-0.032	-0.034	-0.037	-0.042
		MSP	0.012	0.007	0.008	0.009	0.009	0.010	0.013	0.008	0.008	0.009	0.011
	$s = 5$	Bias	-0.062	-0.048	-0.053	-0.058	-0.058	-0.060	-0.068	-0.049	-0.049	-0.054	-0.058
		MSP	0.020	0.012	0.013	0.016	0.020	0.023	0.028	0.014	0.016	0.019	0.022
	$s = 10$	Bias	-0.113	-0.072	-0.080	-0.084	-0.085	-0.097	-0.113	-0.082	-0.084	-0.091	-0.106
		MSP	0.041	0.030	0.033	0.035	0.035	0.038	0.042	0.033	0.033	0.036	0.039
	$s = 15$	Bias	-0.220	-0.168	-0.181	-0.192	-0.199	-0.207	-0.223	-0.186	-0.194	-0.203	-0.216
		MSP	0.263	0.188	0.195	0.201	0.217	0.234	0.258	0.199	0.211	0.229	0.246

Table 4. Average confidence/credible length and coverage percentage for $n = 30$ and $r = 15$.

$\alpha = 1$ and $\lambda = 2$						
			Pivot.	HCD	Bay. Meth.	
					Non-Informative prior	Informative prior
$T = 0.25$	$s = 1$	Length	0.109	—	0.125	0.117
		Cov. Prob.	0.934	—	0.935	0.936
	$s = 3$	Length	0.190	0.175	0.204	0.198
		Cov. Prob.	0.936	0.935	0.936	0.938
	$s = 5$	Length	0.291	0.281	0.317	0.309
		Cov. Prob.	0.940	0.937	0.942	0.945
	$s = 10$	Length	0.624	0.611	0.647	0.630
		Cov. Prob.	0.944	0.942	0.947	0.948
	$s = 15$	Length	2.912	—	3.091	3.036
		Cov. Prob.	0.946	—	0.948	0.949
$T = 0.5$	$s = 1$	Length	0.032	—	0.036	0.033
		Cov. Prob.	0.934	—	0.934	0.935
	$s = 3$	Length	0.178	0.165	0.199	0.184
		Cov. Prob.	0.936	0.935	0.940	0.943
	$s = 5$	Length	0.240	0.231	0.257	0.251
		Cov. Prob.	0.941	0.939	0.942	0.944
	$s = 10$	Length	0.527	0.520	0.548	0.539
		Cov. Prob.	0.945	0.944	0.948	0.948
	$s = 15$	Length	1.026	—	1.087	1.068
		Cov. Prob.	0.946	—	0.945	0.949
$\alpha = 1$ and $\lambda = 2$						
			Pivot.	HCD	Bay. Meth.	
					Non-Informative prior	Informative prior
$T = 0.75$	$s = 1$	Length	0.114	—	0.129	0.121
		Cov. Prob.	0.935	—	0.935	0.935
	$s = 3$	Length	0.235	0.221	0.247	0.241
		Cov. Prob.	0.939	0.937	0.941	0.942
	$s = 5$	Length	0.285	0.278	0.309	0.299
		Cov. Prob.	0.942	0.942	0.945	0.946
	$s = 10$	Length	0.451	0.440	0.479	0.464
		Cov. Prob.	0.945	0.942	0.946	0.947
	$s = 15$	Length	1.278	—	1.331	1.281
		Cov. Prob.	0.946	—	0.948	0.948
$T = 1$	$s = 1$	Length	0.096	—	0.113	0.105
		Cov. Prob.	0.934	—	0.936	0.936
	$s = 3$	Length	0.198	0.187	0.218	0.211
		Cov. Prob.	0.936	0.935	0.939	0.942
	$s = 5$	Length	0.352	0.341	0.365	0.355
		Cov. Prob.	0.939	0.938	0.942	0.945
	$s = 10$	Length	0.400	0.382	0.428	0.414
		Cov. Prob.	0.943	0.940	0.946	0.947
	$s = 15$	Length	0.670	—	0.692	0.681
		Cov. Prob.	0.945	—	0.946	0.949

Table 5. MSPEs of Bayesian point predictions and the average lengths of 95% Bayesian PIs based on different gamma priors, for $n = 30$ and $r = 15$.

		MSPE						Lenght					
$\alpha = 1$ and $\lambda = 2$													
		Prior 1	Prior 2	Prior 3	Prior 4	Prior 5	Prior 6	Prior 1	Prior 2	Prior 3	Prior 4	Prior 5	Prior 6
$T = 0.25$	$s = 1$	0.004	0.004	0.003	0.003	0.003	0.001	0.183	0.162	0.152	0.137	0.128	0.110
	$s = 3$	0.012	0.012	0.011	0.010	0.010	0.009	0.261	0.243	0.218	0.199	0.184	0.167
	$s = 5$	0.100	0.097	0.095	0.085	0.079	0.072	0.392	0.363	0.336	0.319	0.301	0.287
	$s = 10$	0.191	0.166	0.163	0.148	0.119	0.107	0.717	0.693	0.670	0.658	0.639	0.615
	$s = 15$	0.783	0.748	0.733	0.712	0.698	0.652	3.520	3.318	3.219	3.110	3.032	2.892
$T = 0.5$	$s = 1$	0.003	0.003	0.003	0.002	0.002	0.001	0.077	0.069	0.050	0.042	0.037	0.029
	$s = 3$	0.005	0.005	0.004	0.003	0.002	0.002	0.240	0.238	0.218	0.203	0.188	0.179
	$s = 5$	0.022	0.020	0.019	0.018	0.018	0.017	0.329	0.309	0.291	0.273	0.252	0.236
	$s = 10$	0.064	0.055	0.048	0.039	0.036	0.030	0.582	0.571	0.562	0.539	0.520	0.493
	$s = 15$	0.765	0.668	0.524	0.464	0.425	0.376	1.162	1.119	1.093	1.072	1.020	0.984
$\alpha = 2$ and $\lambda = 1$													
		MSPE						Lenght					
		Prior 1	Prior 2	Prior 3	Prior 4	Prior 5	Prior 6	Prior 1	Prior 2	Prior 3	Prior 4	Prior 5	Prior 6
$T = 0.75$	$s = 1$	0.009	0.008	0.008	0.007	0.007	0.006	0.	0.163	0.150	0.139	0.131	0.116
	$s = 3$	0.015	0.014	0.013	0.012	0.012	0.010	0.329	0.302	0.284	0.261	0.247	0.230
	$s = 5$	0.041	0.036	0.031	0.030	0.022	0.014	0.377	0.352	0.332	0.311	0.295	0.287
	$s = 10$	0.083	0.079	0.055	0.049	0.045	0.026	0.534	0.512	0.488	0.473	0.452	0.438
	$s = 15$	0.343	0.328	0.314	0.294	0.286	0.270	1.389	1.352	1.314	1.293	1.272	1.251
$T = 1$	$s = 1$	0.007	0.006	0.005	0.005	0.005	0.004	0.148	0.130	0.124	0.116	0.103	0.097
	$s = 3$	0.010	0.010	0.009	0.009	0.008	0.006	0.261	0.245	0.230	0.218	0.205	0.194
	$s = 5$	0.039	0.033	0.029	0.028	0.023	0.019	0.452	0.428	0.400	0.372	0.348	0.330
	$s = 10$	0.059	0.058	0.055	0.047	0.035	0.023	0.483	0.462	0.442	0.419	0.403	0.387
	$s = 15$	0.297	0.248	0.232	0.227	0.205	0.187	0.731	0.719	0.697	0.673	0.662	0.645

6 Conclusions

In this paper we considered the prediction of future observation of a two-parameter Weibull distribution based on a Type-I hybrid censored data. We provide both the frequentist and Bayes predictors, the corresponding prediction intervals compared their performances using Monte Carlo simulation experiments. Although, in this paper we have mainly restricted our attention for Type-I hybrid censored data, the method can be extended for other hybrid censoring schemes also. More work is needed along that direction.

7 Acknowledgement

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8 Appendix

The conditional density of α given the data is

$$g_2(\alpha|x_d) \propto \frac{e^{-b_1\alpha}\alpha^{d+a_1-1}}{\left(\sum_{i=1}^{d} x_i^{\alpha} + (n-d)T_0^{\alpha} + b_2\right)^{d+a_2}} \prod_{i=1}^{d} x_i^{\alpha-1}$$

The log-likelihood of $g_2(\alpha|x_d)$ is

$$\ln g_2(\alpha|x_d) \propto -b_1\alpha+(d+a_1-1)\ln(\alpha)-(d+a_2)\ln\left(\sum_{i=1}^{d} x_i^{\alpha} + (n-d)T_0^{\alpha} + b_2\right)+(\alpha-1)\sum_{i=1}^{d}\ln(x_i)$$

Using Lemma 1 of Kundu (2007), it follows that

$$\frac{d}{d\alpha^2}\ln\left(\sum_{i=1}^{d} x_i^{\alpha} + (n-d)T_0^{\alpha} + b_2\right) \geq 0$$

Therefore the result follows.

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