PREDICTION OF OUTSTANDING LIABILITIES IN NON-LIFE INSURANCE[1]

BY RAGNAR NORBERG

University of Copenhagen

ABSTRACT

A fully time-continuous approach is taken to the problem of predicting the total liability of a non-life insurance company. Claims are assumed to be generated by a non-homogeneous marked Poisson process, the marks representing the developments of the individual claims. A first basic result is that the total claim amount follows a generalized Poisson distribution. Fixing the time of consideration, the claims are categorized into settled, reported but not settled, incurred but not reported, and covered but not incurred. It is proved that these four categories of claims can be viewed as arising from independent marked Poisson processes. By use of this decomposition result predictors are constructed for all categories of outstanding claims. The claims process may depend on observable as well as unobservable risk characteristics, which may change in the course of time, possibly in a random manner. Special attention is given to the case where the claim intensity per risk unit is a stationary stochastic process. A theory of continuous linear prediction is instrumental.

KEYWORDS

Claims reserves; marked Poisson process; stochastic claims intensity; linear prediction.

1. INTRODUCTION

A. Objective of the study and relations to existing actuarial literature

In its spirit, the present study is a follow-up of a previous paper by the author (NORBERG, 1986), where the problem of predicting incurred but not reported claims is treated for various specifications of the model assumptions and the available data. In that paper data are invariably assumed to be discretized on an annual basis, say. The fairly wide framework model allows for fluctuation of unobservable risk conditions from one year to the next, represented by iid (independent and identically distributed) latent stochastic variates. The ideas have been developed further by HESSELAGER and WITTING (1988), who specify assumptions about stochastic variation in the development pattern from one

[1] Based on invited paper presented at the 23rd ASTIN Colloquium, Stockholm, 1991.

year to another. This issue has been pursued also in a recent paper by NEUHAUS (1992). The present paper carries on in two directions of increasing generality.

In the first place it undertakes to construct a continuous time stochastic process description of the occurrence and development of insurance claims, the purpose being to gain economy of notation as compared with the discrete time set-up, and—more important—to look for possibilities of refining the model apparatus and the statistical methods. A first draft of the ideas was presented in an earlier paper by the author (NORBERG, 1989). Notable previous works on prediction of outstanding claims in a time-continuous setting are contributed by KARLSSON (1976), who discusses the effect of correlation between the claim amount and the delay time from occurrence to notification in a renewal process, by ARJAS (1989), who clarifies ideas in a general stochastic process framework, and by JEWELL (1989), who treats the special problem of predicting the number of incurred but not reported claims in the case of a homogeneous Poisson process with random claim intensity, constant in time.

In the second place, fluctuations and trends in risk conditions are accommodated in a more flexible manner than in previous works. Fluctuating unobservable risk conditions are represented by a latent stochastic process, which may be taken to have a certain inertness, and observable risk characteristics are accounted for by covariates (explanatory, exogeneously given quantities) that may change over time. Thus, one is not compelled to lump all varying risk factors into the unknown and to postulate that risk factors in different years are independent selections from some common distribution.

B. Outline of the paper

Prediction of liabilities is a major issue in any assessment of the financial strength of an insurer, whether performed in public supervision or in company management. The present study adopts the point of view of a regulatory authority conducting solvency control based on the break-up criterion, whereby the relevant liabilities are the total outstanding payments in respect of claims that the insurer is under contract to cover at the time of consideration.

Section 2 explicates the objective of the solvency control and describes the structure of the observable data. Section 3 presents the basic model, a so-called marked Poisson process: claims occur in accordance with a non-homogeneous Poisson process, and to each individual claim is associated a random "mark" representing its development from occurrence to final settlement. A key result is that the claims process may be viewed as the outcome of a Poisson number of claims with independent and identically distributed occurrence epochs and marks. Consequently, the total claim amount follows a generalized Poisson distribution. Section 4 provides further useful results on distributional properties of the process. Fixing the time of consideration, the claims are categorized into settled, reported but not settled, incurred but not reported, and covered but not incurred. It is proved that these four categories of claims arise from independent marked Poisson processes. From this basic decomposition result the distribution of the total outstanding liability is easily found, and adequate reserves can be computed. This is the topic of Section 5. In Section 6 the model is extended by letting the claim intensity be a stochastic process representing fluctuating unobservable risk conditions. A linear predictor of the outstanding liability in respect of not reported claims is obtainable by methods taken from a recent paper by the author (NORBERG, 1992) — it is an integral with respect to the process counting the reported claims. Section 7 offers a sketch of how observable risk factors can be represented by covariates that may change in the course of time. Section 8 provides some numerical illustrations.

2. THE SOLVENCY CONTROL SYSTEM AND THE DATA

A. The break-up point of view in solvency control

Consider an insurer who has been running business in one or more lines of insurance since time 0, say, and who is subjected to solvency control at some subsequent time τ, henceforth referred to as the *present moment*. The solvency assessment is based on the break-up scenario, by which underwriting of new business is assumed to cease at time τ. Thus, only outstandings for which liability has been assumed prior to time τ are relevant, and, since the premiums for this part of the business have already been collected, the reserve provided at time τ must be adequate to meet these claims. Possible excess of premiums over claims in the future cannot be counted as available for covering currently assumed liabilities (no counting of chickens before they are hatched). In this perspective it is henceforth understood that all quantities introduced are related to coverages secured by contracts in force prior to or at time τ. They could properly be equipped with an index τ, but this is omitted to save notation.

B. Data from policy records

For the time being focus is on one single line of business, a mass branch like non-industrial fire or automobile insurance. The portfolio is made up of many small risk units so that a single claim has no significant impact on the size and the composition of the portfolio and, therefore, need not be related to the individual risk from which it stems. Thus, the macro viewpoint of the collective theory of risk is adopted, and it is assumed that the information provided by the policy records at time t is adequately summarized by:

$w(t)$, the risk exposure per time unit at time t.

The exposure rate $w(t)$ may be a thought of as a simple measure of volume or size of the business, but later on it will be allowed to depend on covariates describing the composition of the portfolio. In the break-up context the function w will typically look as in Figure 1. Future exposure after time τ pertains to contracts that are currently in force. In practice they will expire in finite time so that $w(t) = 0$ for t large enough, but it will suffice here to assume that the total exposure,

$$W = \int_0^\infty w(t)\,dt, \tag{2.1}$$

is finite.

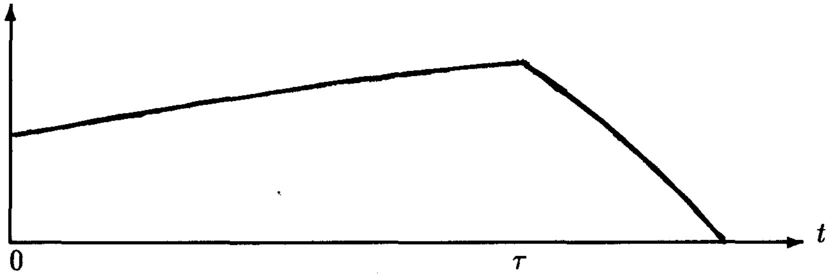

FIGURE 1. Exposure of business written up to time τ.

C. Data from claims records

The claims statistics is a file of records, one for each individual claim. Associate with a claim the following quantities, which can be read from the completed claims record after the claim has been settled:

T, the time of occurrence,
U, the waiting time from occurrence until notification to the insurer,
V, the waiting time from notification until final settlement,
$Y(v')$, the indemnity paid in respect of the claim during the time interval $[T+U, T+U+v']$, i.e. within v' time units after its notification,
$Y = Y(V)$, the final claim amount.

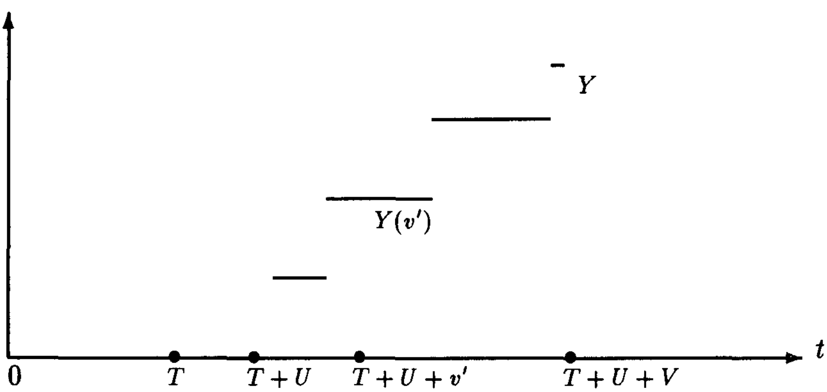

FIGURE 2. Occurrence and development of a claim.

https://doi.org/10.2143/AST.23.1.2005103 Published online by Cambridge University Press

A typical claims history, as described by these quantities, is depicted in Figure 2.

It is convenient to assemble the quantities describing the development of the claim after its occurrence in

$$Z = (U, V, Y, \{Y(v'); 0 \leq v' < V\}),$$

and to represent the claim as the pair (T, Z). The space of all possible realizations of a claim is $(0, \infty) \times \mathscr{Z}$, where $\mathscr{Z}$ is the space of all possible developments.

The claims for which liability has been assumed by time τ is a collection of points (T_i, Z_i) in the claims space. Let the claims (if any) be numbered in accordance with the chronology of their occurrence, $0 < T_1 < T_2 < \ldots$ (implying that no two claims occur simultaneously). Introduce

$N(t) = \sum_i 1_{\{T_i \leq t\}}$, the number of claims occurred by time t,

and abbreviate

$N = \lim_{t \to \infty} N(t)$, the total number of claims.

The occurrence epochs T_i determine the counts $\{N(t)\}_{t \geq 0}$, and vice versa;

$$T_i = \inf\{t; N(t) \geq i\}. \tag{2.2}$$

The claims data that will eventually be observable is

$$\{(T_i, Z_i)\}_{1 \leq i \leq N}. \tag{2.3}$$

3. THE BASIC MODEL

A. Review of some properties of Poisson distributions

Proofs of the standard results to be listed here can be looked up in e.g. BEARD et al. (1984).

A non-negative integer-valued random variable N is *Poisson distributed* with parameter $W (\geq 0)$, written $N \sim Po(W)$, if

$$P\{N = n\} = \frac{W^n}{n!} e^{-W}, \qquad n = 0, 1, \ldots \tag{3.1}$$

Let Y_i, $i = 1, 2, \ldots$, be iid selections from a distribution P_Y, all independent of $N \sim Po(W)$. Then the random variable

$$X = \sum_{i=1}^{N} Y_i \tag{3.2}$$

(interpreted as 0 if $N = 0$) is said to have a *generalized Poisson distribution* with frequency parameter W and jump size distribution P_Y, abbreviated $X \sim Po(W, P_Y)$. This distribution is given by

$$P_X = \sum_{n=0}^{\infty} \frac{W^n}{n!} e^{-W} P_Y^{n*}, \tag{3.3}$$

where the topscript $n*$ signifies n-th convolution. Its first three central moments are

$$m_X^{(k)} = W \int y^k P_Y(dy), \qquad k = 1, 2, 3, \tag{3.4}$$

provided that the first three moments of P_Y exist.

If $X^{(j)}$, $j = 1, \ldots, k$, are independent and $X^{(j)} \sim Po(W^{(j)}, P_{Y^{(j)}})$ for each j, then

$$X = \sum_{j=1}^{k} X^{(j)} \sim Po(W, P_Y), \tag{3.5}$$

with

$$W = \sum_{j=1}^{k} W^{(j)}, \tag{3.6}$$

$$P_Y = \frac{\sum_{j=1}^{k} W^{(j)} P_{Y^{(j)}}}{\sum_{j=1}^{k} W^{(j)}}. \tag{3.7}$$

Putting all jump sizes equal to 1, (3.4)-(3.7) specialize to results for Poisson distributions.

There are at least two reasons why the generalized Poisson distribution plays an important role in risk theory. First, it is widely held to be a reasonable description of claims generated by a large and fairly homogeneous portfolio of risks and, second, it is computationally feasible. In particular, the moments are given by simple explicit formulas, confer (3.4), and there exist a number of techniques for computing tail probabilities and fractiles in such distributions, see e.g. BEARD et al. (1984).

B. Basic model assumptions for the claims process

The starting point is the process of occurrences of claims. It is assumed that claims occur in accordance with a non-homogeneous Poisson process with intensity $w(t)$ at time $t \geq 0$. In short,

$$\{T_i\}_{1 \leq i \leq N} \sim Po(w(t); t \geq 0). \tag{3.8}$$

This means precisely that the process $\{N(t)\}_{t \geq 0}$ has independent increments and $N(t) - N(s) \sim Po\left(\int_s^t w(t')\,dt'\right)$ for each interval $(s, t]$. In particular, since the total exposure in (2.1) is finite, the total number of claims is finite with probability 1 and is a Poisson variate; $N \sim Po(W)$. The claims epochs are constructed from the counting process by (2.2). Let $\{Z(t)\}_{t \geq 0}$ be a family of random elements in $\mathscr{Z}$ which are mutually independent and independent of $\{N(t)\}_{t \geq 0}$, and denote the distribution of $Z(t)$ by $P_{Z|t}$. The individual claim developments are constructed as $Z_i = Z(T_i)$, $i = 1, \ldots, N$. Phrased less precisely, the claim developments Z_i are assumed to be mutually independent, and the conditional distribution of Z_i, given $\{N(t)\}_{t \geq 0}$, depends only on T_i. Adopting standard terminology, the random element Z_i is called the *mark* associated with the occurrence epoch T_i, and the process is called *the marked Poisson process* with intensity $\{w(t)\}_{t \geq 0}$ and position-dependent marking by $\{P_{Z|t}\}_{t \geq 0}$. In short,

$$\{(T_i, Z_i)\}_{1 \leq i \leq N} \sim Po(w(t), P_{Z|t};\, t \geq 0). \tag{3.9}$$

C. An alternative construction of the process

Let the statement $T \in dt$ signify that T takes its value in a neighbourhood of extension dt around t. The meaning is clear if T is real- or vector-valued, and the device can be adopted also in more general spaces. The joint probability distribution of the observables is given by

$$\begin{aligned}
(3.10)\qquad & P\{N = n,\, (T_i, Z_i) \in (dt_i, dz_i),\, i = 1, \ldots, n\} \\
&= e^{-\int_0^{t_1} w(t)\,dt}\, w(t_1)\,dt_1 \ldots e^{-\int_{t_{n-1}}^{t_n} w(t)\,dt}\, w(t_n)\,dt_n \\
&\quad e^{-\int_{t_n}^{\infty} w(t)\,dt} \prod_{i=1}^{n} P_{Z|t_i}(dz_i) \\
(3.11)\qquad &= e^{-\int_0^{\infty} w(t)\,dt} \prod_{i=1}^{n} w(t_i)\,dt_i\, P_{Z|t_i}(dz_i) \\
(3.12)\qquad &= \frac{W^n}{n!}\, e^{-W}\, n! \prod_{i=1}^{n} P_{T,Z}(dt_i, dz_i),
\end{aligned}$$

where W is the total exposure defined by (2.1) and

$$P_{T,Z}(dt, dz) = \frac{w(t)\,dt}{W}\, P_{Z|t}(dz). \tag{3.13}$$

It is seen that $P_{T,Z}$ is the joint distribution of a random pair (T, Z), where T has a marginal distribution P_T with density $w(t)/W$, $t > 0$, and Z has conditional distribution $P_{Z|t}$ for fixed $T = t$. The expression (3.12) serves to prove the following result.

Theorem 1: The marked Poisson process in (3.9) can be viewed as the outcome of a two-stage procedure: first generate $N \sim Po(W)$, then select a random sample of N pairs from the distribution $P_{T,Z}$ given by (3.13), and order them by the size of the first entry to obtain $\{(T_i, Z_i)\}_{1 \leq i \leq N}$.

D. The distribution of the total liability

Let X denote the total liability assumed by the insurer up to time τ. It is of the form (3.2), with N and the Y_i distributed as specified in Paragraph **B** above. Being a symmetric function of the Y_i, the sum on the right of (3.2) does not depend on the chronological order of the claims. Thus, by virtue of Theorem 1, the Y_i can be replaced by iid selections from the marginal distribution P_Y obtained from the joint distribution in (3.13) upon integrating out the other variates:

$$P_Y(dy) = \frac{\int_{t>0} w(t)\, P_{Y|t}(dy)\, dt}{W}. \tag{3.14}$$

(The notational device employed in P_Y and $P_{Y|t}$ is selfexplaining and will be used throughout: the marginal distribution of a variate is denoted by the symbol of the joint distribution equipped with the appropriate subscript.) This together with (3.3) and (3.4) proves the following result:

Corollary to Theorem 1: The total claim amount is a generalized Poisson variate, $X \sim Po(W, P_Y)$, with W and P_Y given by (2.1) and (3.14). Its distribution is given by (3.3), and the three first moments are

$$m_X^{(k)} = \int_{t>0} w(t) \int_{y>0} y^k\, P_{Y|t}(dy)\, dt, \qquad k = 1, 2, 3. \tag{3.15}$$

Remark: A heuristic motivation of the result is obtained from (3.5)-(3.7) upon writing $X = \int_0^\infty X(dt)$, the "sum" of claim amounts in small intervals dt, $t \geq 0$. Each $X(dt)$ is a sum of individual claim amounts, the number of claims $N(dt)$ being $Po(w(t)\,dt)$ and the individual claim amounts being iid selections from $P_{Y|t}$. □

4. FOUR CATEGORIES OF CLAIMS

A. Paid and outstanding claims

As seen at time τ, the claims may be categorized as *settled* (s), *reported-not-settled* (rns), *incurred-not-reported* (inr), or *covered-not-incurred* (cni). For each $t \geq 0$ and $g = s, rns, inr, cni$, let $\mathscr{Z}_t^g$ denote the set of all possible developments that make a claim occurred at time t belong to category g. These sets are defined precisely as

$$\mathscr{Z}_t^s = \{z;\, t+u+v \leq \tau\}, \tag{4.1}$$

$$\mathscr{Z}_t^{rns} = \{z;\, t+u \leq \tau < t+u+v\}, \tag{4.2}$$

$$\mathscr{Z}_t^{inr} = \{z;\, t \leq \tau < t+u\}, \tag{4.3}$$

$$\mathscr{Z}_t^{cni} = \{z;\, t > \tau\}. \tag{4.4}$$

By definition, $\mathscr{Z}_t^g = \emptyset$ for $t > \tau$ and $g = s, rns, inr$, $\mathscr{Z}_t^{cni} = \mathscr{Z}$ for $t > \tau$, and $\mathscr{Z}_t^{cni} = \emptyset$ for $t \leq \tau$. For each t the sets $\mathscr{Z}_t^g$ form a partition of the space $\mathscr{Z}$, that is,

$$\bigcup_g \mathscr{Z}_t^g = \mathscr{Z},\ \mathscr{Z}_t^g \cap \mathscr{Z}_t^{g'} = \emptyset,\ g \neq g'. \tag{4.5}$$

The acronyms *inr* and *rns* are shorthand for the commonly used *ibnr* and *rbns*, the redundant "but" being dropped in *incurred but not reported* and *reported but not settled.* The term *cni* (NORBERG, 1989) represents claims related to what is usually called the unearned premium reserve.

In accordance with the categorization (4.1)-(4.4) the process in (2.3) decomposes into the component processes

(4.6) $$\{(T_i^g, Z_i^g)\}_{1 \le i \le N^g}, \; g = s, rns, inr, cni.$$

The construction is obvious: for each g the process counting the g-claims is defined by $N^g(t) = \sum_i 1_{\{T_i \le t, \, Z_i \in \mathscr{Z}^g_{T_i}\}}$, the epoch of occurrence of the i-th g-claim is $T_i^g = \inf\{t; N^g(t) \ge i\}$, and its development is $Z_i^g = Z(T_i^g)$.

The total liability in (3.2) decomposes accordingly into

(4.7) $$X = X^s + X^{rns} + X^{inr} + X^{cni},$$

where the components on the right are the total liabilities in respect of claims in the different categories;

(4.8) $$X^g = \sum_i 1_{\{Z_i \in \mathscr{Z}^g_{T_i}\}} Y_i = \sum_{1 \le i \le N^g} Y_i^g,$$

$g = s, rns, inr, cni.$

By time τ the s-part is paid, the *rns*-part splits into a *paid* part,

$$X^{prns} = \sum_{1 \le i \le N^{rns}} Y_i^{rns}(\tau - T_i^{rns} - U_i^{rns}),$$

and an *outstanding* part,

(4.9) $$X^{orns} = \sum_{1 \le i \le N^{rns}} (Y_i^{rns} - Y_i^{rns}(\tau - T_i^{rns} - U_i^{rns})),$$

and the *inr- and cni*-parts are outstanding. The last-mentioned two components can conveniently be lumped into

(4.10) $$X^{nr} = X^{inr} + X^{cni} = \sum_{1 \le i \le N^{nr}} Y_i^{nr},$$

the liability in respect of *not reported* (*nr*) claims defined in accordance with (4.1)-(4.4) by

(4.11) $$\mathscr{Z}_t^{nr} = \mathscr{Z}_t^{inr} \cup \mathscr{Z}_t^{cni} = \{z; t+u > \tau\}.$$

The process of *nr*-claims is well defined by (4.6) for $g = nr$.

The total outstanding liability at time τ is

(4.12) $$X^o = X^{orns} + X^{nr},$$

and a major issue in the solvency assessment is the prediction of this quantity.

B. The joint probability distribution of the component claims processes

The following result is basic in the sequel.

Theorem 2: The component processes in (4.6) are independent, and for each $g = s, rns, inr, cni$,

$$\{(T_i^g, Z_i^g)\}_{1 \le i \le N^g} \sim Po\left(w^g(t), P^g_{Z|t};\, t \ge 0\right),$$

with

$$w^g(t) = w(t)\, P_{Z|t}\{\mathscr{Z}_t^g\}, \tag{4.13}$$

$$P^g_{Z|t}(dz) = \frac{P_{Z|t}(dz)}{P_{Z|t}\{\mathscr{Z}_t^g\}}, \quad z \in \mathscr{Z}_t^g. \tag{4.14}$$

Proof: The component processes in (4.6) are determined by the aggregate process in (3.9) and vice versa, and so the event appearing in (3.10) can be equivalently stated in terms of the component processes. From (4.5) it follows that $\sum_g P_{Z|t}\{\mathscr{Z}_t^g\} = 1$ for each t. Thus, the likelihood in (3.12) is easily rewritten as

$$\begin{aligned} &P\{\textstyle\bigcap_g \{N^g = n^g, (T_i^g, Z_i^g) \in (dt_i^g, dz_i^g), i = 1, \ldots, n^g\}\} \\ &= \prod_g \left(\frac{(W^g)^{n^g}}{n^g!} e^{-W^g} n^g! \prod_{i=1}^{n^g} P^g_{T,Z}(dt_i, dz_i) \right), \end{aligned} \tag{4.15}$$

where

$$W^g = \int_0^\infty w^g(t)\, dt, \tag{4.16}$$

with $w^g(t)$ defined by (4.13), and

$$P^g_{T,Z}(dt, dz) = \frac{P_{T,Z}(dt, dz)}{W^g} = \frac{w^g(t)\, dt}{W^g} P^g_{Z|t}(dz), \quad z \in \mathscr{Z}_t^g. \tag{4.17}$$

The result now follows by the product form of (4.15) and comparison with (3.10)-(3.13) and Theorem 1. □

The result says that g-claims occur with an intensity which is the claim intensity times the probability that the claim belongs to the category g, and the development of the claim is governed by the conditional distribution of the mark, given that it is in category g. Accordingly, the quantity W^g in (4.16) may be termed the total exposure in respect of claims of category g or just the *g-exposure*. The distribution $P^g_{T,Z}$ is just the conditional distribution of (T, Z), given that it is a g-claim.

5. PREDICTION IN THE BASIC MODEL

A. The prediction problem

A reserve must be provided at time τ to meet the outstanding liability. Let $\mathcal{F}_\tau$ denote the statistical information available by time τ. It consists of the histories up to time τ of all individual claims that have been *reported* (r) by that time, that is, occurred at some time $t \leq \tau$ and with development in

$$\mathscr{X}_t^r = \mathscr{X}_t^s \cup \mathscr{X}_t^{rns} = \{z ; t+u \leq \tau\}. \tag{5.1}$$

The problem of providing an adequate reserve amounts to predicting X^o in (4.12) on the basis of its conditional distribution, given $\mathcal{F}_\tau$.

B. Predicting the *orns*-liability

As for the term X^{orns} on the right of (4.12), the *rns*-claims are conditionally independent, given $\mathcal{F}_\tau$. Thus, the predictive distribution of X^{orns} is the convolution of the conditional distributions of the individual terms in (4.9). Now, the conditional distribution of the size of an individual claim Y_i^{rns} may depend in a complex manner on its development up to time τ. Matters may be simplified greatly by discarding the detailed information $\{Y_i^{rns}(v'); 0 \leq v' < \tau - Y_i^{rns} - U_i^{rns}\}$, and conditioning only on $T_i^{rns} = t_i$, $U_i^{rns} = u_i$, and $Y_i^{rns} > Y_i^{rns}(\tau - T_i^{rns} - U_i^{rns}) = y_i$. This operation involves only the bivariate distribution of (U, Y) for fixed $T = t_i$, and the relevant predictive distribution of Y_i^{rns} is given by

$$P_{Y_i^{rns}|t_i, u_i, y_i}(dy) = \frac{P_{U, Y|t_i}(du_i, dy)}{\int_{y' > y_i} P_{U, Y|t_i}(du_i, dy')}, \quad y > y_i. \tag{5.2}$$

Thus, the non-central predictive moments of $Y_i^{rns} - Y_i^{rns}(\tau - T_i^{rns} - U_i^{rns})$ are

$$a_i^{(k)} = \int_0^\infty (y - y_i)^k \, P_{Y_i^{rns}|t_i, u_i, y_i}(dy),$$

and the first three central moments are

$$m_i^{(1)} = a_i^{(1)}, \tag{5.3}$$

$$m_i^{(2)} = a_i^{(2)} - (a_i^{(1)})^2, \tag{5.4}$$

$$m_i^{(3)} = a_i^{(3)} - 3\, a_i^{(2)} a_i^{(1)} + 2\, (a_i^{(1)})^3. \tag{5.5}$$

Since the terms in (4.9) are independent, the three first predictive moments of X^{orns} are

$$m^{(k)}_{X^{orns}|\mathcal{F}_\tau} = \sum_{1 \leq i \leq N^{rns}} m_i^{(k)}, \qquad k = 1, 2, 3. \tag{5.6}$$

C. Predicting the *nr*-liability

The term X^{nr} on the right of (4.12) is independent of $\mathscr{F}_\tau$, and is to be assessed on the basis of its marginal distribution. By Theorem 2, all results in Paragraphs 3C-D carry over to the process of *nr*-claims — just insert topscript *nr* in all symbols. In particular, formula (3.15) applies and, recalling

$$w^{nr}(t) = w(t)\,(1 - P_{U|t}(\tau - t)), \tag{5.7}$$

$$P^{nr}_{Y|t}(dy) = \frac{\int_{u>\tau-t} P_{U,Y|t}(du, dy)}{1 - P_{U|t}(\tau - t)}, \tag{5.8}$$

one obtains

$$\begin{aligned} m^{(k)}_{X^{nr}} &= \int_{t>0} w^{nr}(t) \int_{y>0} y^k\, P^{nr}_{Y|t}(dy)\, dt \\ &= \int_{t>0} w(t) \int_{u>\tau-t} \int_{y>0} y^k\, P_{U,Y|t}(du, dy)\, dt, \qquad k = 1, 2, 3. \end{aligned} \tag{5.9.}$$

D. Predicting the total outstanding liability

By Theorem 2, the liability components X^{orns} and X^{nr} are conditionally independent, given $\mathscr{F}_\tau$. Thus the first three predictive moments of the outstanding claims X^o in (4.12) are just the sums of the corresponding moments in (5.6) and (5.9);

$$m^{(k)}_{X^o|\mathscr{F}_\tau} = m^{(k)}_{X^{orns}|\mathscr{F}_\tau} + m^{(k)}_{X^{nr}}, \qquad k = 1, 2, 3. \tag{5.10}$$

An appropriate reserve is the first moment given by (5.10) with $k = 1$. A fluctuation loading may be provided by adding a multiple of the standard deviation. As an alternative to this ad hoc method, one may take the upper ε-fractile (e.g. $\varepsilon = 0.01$) of the predictive distribution, or some approximation of it based on the first three moments in (5.10).

6. UNOBSERVABLE RISK CHARACTERISTICS MODELLED BY RANDOMLY FLUCTUATING CLAIM INTENSITY

A. Model assumptions

In addition to observable risk characteristics there may be risk conditions that are not observable or whose effects are difficult to model explictly by covariates. If they are judged to be of importance, they can be modelled by a stochastic process representing the unknown.

Only random fluctuations in the claim intensity will be discussed here. It is assumed that the intensity is of the multiplicative form

$$w(t)\,\Theta(t), \tag{6.1}$$

with $\{w(t)\}_{t\geq 0}$ a non-negative and nonrandom function representing an observable measure of risk exposure per time unit, and $\Theta(t)$ is a non-negative "proportional hazard" factor representing unobservable risk conditions that may vary over time. Assume that $\Theta = \{\Theta(t)\}_{t\geq 0}$ is a stationary stochastic process with mean and *covariance function* denoted by

$$E\,\Theta(t) = \beta, \tag{6.2}$$

$$\mathrm{Cov}\,(\Theta(s), \Theta(t)) = \rho(|t-s|), \tag{6.3}$$

The set-up of Section 3, with $w(t)$ replaced by (6.1), is taken as the conditional model, given Θ.

B. Prediction of the *orns*-liability

The predictive distribution of X^{orns} remains the same as in the previous section since the marks $\{Z(t)\}_{t\geq 0}$ are independent of $\{N(t), \Theta(t)\}_{t\geq 0}$. In particular, the first three predictive moments are given by (5.6).

C. Linear prediction of the *nr*-liability

Prediction of X^{nr} now becomes more intricate since $\mathscr{F}_\tau$ contains partial information on Θ, hence also on the *nr*-claims. A feasible approach is provided by the theory of linear prediction in time-continuous processes presented recently by the author, see NORBERG (1992) for a full account with proofs. The procedure goes as follows.

The liability X^{nr} is to be predicted from the observed claims process, $\{N^r(t)\}_{0<t\leq\tau}$, which is a sufficient statistic. Consider inhomogeneous linear predictors of the form

$$\check{X}^{nr} = g_0 + \int_{0<s\leq\tau} g(s)\,N^r(ds) = g_0 + \sum 1_{\{T_i+U_i\leq\tau\}}\,g(T_i). \tag{6.4}$$

The performance of a predictor $\check{X}^{nr}$ is measured by the mean squared error, $\|\check{X}^{nr}-X^{nr}\|^2 = E(\check{X}^{nr}-X^{nr})^2$, which is the squared norm of $\check{X}^{nr}-X^{nr}$ with respect to the usual inner product in the space of squared integrable random variables; $\langle\!\langle X, Y\rangle\!\rangle = E(XY)$ and $\|X\|^2 = \langle\!\langle X, X\rangle\!\rangle$. The optimal linear predictor is the projection of X^{nr} onto the linear space of functions of the form (6.4). Denote it by

$$\bar{X}^{nr} = \gamma_0 + \int_{0<s\leq\tau} \gamma(s)\,N^r(ds). \tag{6.5}$$

The constant term γ_0 and the function $\{\gamma(t)\}_{0<t\leq\tau}$ are determined by the following normal equations, which are analogous to those for the discrete case (think of the integral in (6.4) as a sum of terms $N^r(ds)$ multiplied by coefficients $g(s)$):

$$EX^{nr} = \gamma_0 + \int_{0<s\leq\tau} \gamma(s)\, EN^r(ds), \tag{6.6}$$

$$\text{Cov}\,(X^{nr}, N^r(dt)) = \int_{0<s\leq\tau} \gamma(s)\, \text{Cov}\,(N^r(ds), N^r(dt)), \tag{6.7}$$

$0 < t \leq \tau$. The moments appearing in these equations are easily obtained by heuristic reasoning. First, introducing

$$m^{nr}(t) = w(t) \int_{u>\tau-t} \int_{y>0} y\, P_{U,Y|t}(du, dy), \tag{6.8}$$

$$w^r(t) = w(t)\, P_{U|t}(\tau-t),\ 0 < t \leq \tau, \tag{6.9}$$

form the conditional moments

$$E(X^{nr}|\Theta) = \int_0^\infty m^{nr}(s)\, \Theta(s)\, ds,$$

$$E(N^r(dt)|\Theta) = w^r(t)\, \Theta(t)\, dt,$$
$$\text{Cov}\,(X^{nr}, N^r(dt)|\Theta) = 0,$$
$$\text{Cov}\,(N^r(ds), N^r(dt)|\Theta) = \delta_{s,t}\, w^r(t)\, \Theta(t)\, dt,$$

where $\delta_{s,t}$ is the Kronecker delta, 1 for $s = t$ and 0 otherwise.

The first relation is obtained upon replacing $w(t)$ in (5.9) by the product in (6.1). The remaining relations follow easily from the fact that $\{N^r(t)\}_{0<t\leq\tau}|\Theta \sim Po(w^r(t)\,\Theta(t);\ 0 < t \leq \tau)$ and the conditional independence of X^{nr} and $\{N^r(t)\}_{0<t\leq\tau}$. Now, form the unconditional moments by use of the general rules $EX = EE(X|\Theta)$ and $\text{Cov}\,(X, Y) = \text{Cov}\,(E(X|\Theta), E(Y|\Theta)) + E\,\text{Cov}\,(X, Y|\Theta)$:

$$EX^{nr} = \int_0^\infty m^{nr}(s)\, ds\, \beta,$$

$$EN^r(dt) = w^r(t)\, dt\, \beta,$$

$$\text{Cov}\,(X^{nr}, N^r(dt)) = \int_0^\infty m^{nr}(s)\, \rho(|t-s|)\, ds\, w^r(t)\, dt,$$

$$\text{Cov}\,(N^r(ds), N^r(dt)) = w^r(s)\, \rho(|t-s|)\, w^r(t)\, ds\, dt + \delta_{s,t}\, w^r(t)\, \beta\, dt.$$

Upon substituting these expressions, (6.6) becomes

$$\int_0^\infty m^{nr}(s)\,ds\,\beta = \gamma_0 + \int_0^\tau \gamma(s)\,w^r(s)\,ds\,\beta, \tag{6.10}$$

and, after dividing by $w^r(t)\,dt$ on both sides, (6.7) becomes

$$\int_0^\infty m^{nr}(s)\rho(|t-s|)\,ds = \int_0^\tau \gamma(s)w^r(s)\rho(|t-s|)\,ds + \gamma(t)\beta,\ 0 < t \le \tau. \tag{6.11}$$

Thus one is left with the problem of first solving $\gamma(\cdot)$ from (6.11), which is an integral equation of the Fredholm type. Then one solves the constant term γ_0 from (6.10). In general, a numerical procedure must be arranged for each specification of $P_{Z|t}$ and ρ.

D. The case with exponential covariance function

A reasonable specification of the covariance function in (6.3) is

$$\rho(|t-s|) = \lambda e^{-\kappa|t-s|}. \tag{6.12}$$

Inserting (6.12) in (6.11) and dividing on both sides by λ yields

$$\begin{aligned} &\int_0^t m^{nr}(s)\,e^{\kappa(s-t)}\,ds + \int_t^\infty m^{nr}(s)\,e^{\kappa(t-s)}\,ds \\ &= \int_0^t \gamma(s)\,w^r(s)\,e^{\kappa(s-t)}\,ds + \int_t^\tau \gamma(s)\,w^r(s)\,e^{\kappa(t-s)}\,ds + \gamma(t)\frac{\beta}{\lambda}. \end{aligned} \tag{6.13}$$

Assume that the functions m^{nr} and w^r are continuous. Differentiating with respect to t in (6.13) and cancelling terms that sum to zero on both sides gives

$$\begin{aligned} &\int_0^t m^{nr}(s)\,e^{\kappa(s-t)}\,ds(-\kappa) + \int_t^\infty m^{nr}(s)\,e^{\kappa(t-s)}\,ds\,\kappa \\ &= \int_0^t \gamma(s)\,w^r(s)\,e^{\kappa(s-t)}\,ds(-\kappa) + \int_t^\tau \gamma(s)\,w^r(s)\,e^{\kappa(t-s)}\,ds\,\kappa + \gamma'(t)\frac{\beta}{\lambda}. \end{aligned} \tag{6.14}$$

Differentiating once more with respect to t gives

$$\begin{aligned} &\int_0^t m^{nr}(s)\,e^{\kappa(s-t)}\,ds\,\kappa^2 + \int_t^\infty m^{nr}(s)\,e^{\kappa(t-s)}\,ds\,\kappa^2 - 2m^{nr}(t)\,\kappa \\ &= \int_0^t \gamma(s)\,w^r(s)\,e^{\kappa(s-t)}\,ds\,\kappa^2 + \int_t^\tau \gamma(s)\,w^r(s)\,e^{\kappa(t-s)}\,ds\,\kappa^2 \\ &\quad - 2\gamma(t)\,w^r(t)\kappa + \gamma''(t)\frac{\beta}{\lambda}. \end{aligned} \tag{6.15}$$

Now, multiply (6.13) by κ^2 and subtract it from (6.15) to obtain the following second order differential equation for the optimal coefficient function:

$$(6.16)\qquad \gamma''(t) - \left(\kappa^2 + 2\frac{\lambda\kappa}{\beta} w^r(t)\right)\gamma(t) + 2m^{nr}(t)\frac{\lambda\kappa}{\beta} = 0,\ 0 < t < \tau.$$

To solve this equation by standard procedures, one needs two boundary conditions, e.g. on $\gamma(0)$ and $\gamma'(0)$. Putting $t = 0$ in (6.13) and (6.14) gives

$$(6.17)\qquad \gamma'(0) = \kappa\gamma(0)$$

and

$$(6.18)\qquad \int_0^\infty m^{nr}(s)\, e^{-\kappa s}\, ds - \int_0^\tau \gamma(s)\, w^r(s)\, e^{-\kappa s}\, ds = \gamma(0)\frac{\beta}{\lambda}.$$

The latter condition involves the solution γ, and so it does not serve immediately as a boundary condition. One can, however, search by trial and error as follows. Fix a value of $\gamma(0)$, whereby (6.17) fixes $\gamma'(0)$. Find the solution of (6.16) for these boundary conditions. Compute the expression on the left of (6.18) and compare with the expression on the right. If they are equal, the problem is solved. If not, adjust the choice of $\gamma(0)$ and repeat the procedure. Continue in this manner until (approximate) equality is attained in (6.18).

7. OBSERVABLE RISK CHARACTERISTICS MODELLED BY COVARIATES

In Paragraph 2B it was mentioned that the intensity $w(t)$ may depend on covariates representing observable risk characteristics of the portfolio at time t. The same goes for the mark distribution $P_{Z|t}$. The risk characteristics may be fluctuating "external" conditions, like weather and other driving conditions in automobile insurance, or "internal" portfolio characteristics that can be read from the policy records.

As an example of the latter, imagine a portfolio that is heterogeneous with respect to observable risk characteristics, e.g. a portfolio of fire insurance policies where the fire peril and the typical claim size vary substantially from one house to another, depending on size of the building, building materials, fire preventing measures, and possibly other characteristics. As explained by NORBERG and SUNDT (1985), in such a situation one cannot rely on aggregate statistics on claims frequencies and claim sizes for the portfolio as a whole: changes in the composition of the portfolio may cause considerable changes of the claims process in the course of time.

To concretize, suppose the portfolio is composed of q risk classes with marked Poisson claims processes $\{(T_i^{(p)}, Z_i^{(p)})\}_{1 \le i \le N^{(p)}}$, $p = 1, \ldots, q$. Assume that they all have the same mark space $\mathscr{Z}$ (just a matter of definition). For the aggregate claims process of the whole portfolio the process counting the claims is

$$N(t) = \sum_{p=1}^{q} N^{(p)}(t), \tag{7.1}$$

the claims epochs are

$$T_i = \inf\{t; N(t) \geq i\}, \tag{7.2}$$

and the marks are

$$Z_i = \sum_{p=1}^{q} Z^{(p)}(T_i)\, 1_{\{N^{(p)}(T_i) - N^{(p)}(T_i -) = 1\}}. \tag{7.3}$$

Theorem 3: Assume that the claims processes of the individual classes are mutually independent and are of marked Poisson type;

$$\{(T_i^{(p)}, Z_i^{(p)})\}_{1 \leq i \leq N^{(p)}} \sim Po(w^{(p)}(t), P_{Z|t}^{(p)}; t \geq 0), \qquad p = 1, \ldots, q.$$

Then the aggregate process defined by (7.1)-(7.3) is also marked Poisson,

$$\{(T_i, Z_i)\}_{1 \leq i \leq N} \sim Po(w(t), P_{Z|t}; t \geq 0),$$

with

$$w(t) = \sum_{p=1}^{q} w^{(p)}(t), \tag{7.4}$$

$$P_{Z|t} = \frac{\sum_{p=1}^{q} w^{(p)}(t)\, P_{Z|t}^{(p)}}{\sum_{p=1}^{q} w^{(p)}(t)}. \tag{7.5}$$

Proof: The likelihood of the aggregate process is

$$P\{N = n, (T_i, Z_i) \in (dt_i, dz_i), i = 1, \ldots, n\}$$
$$= e^{-\int_0^{t_1} \sum_{p=1}^{q} w^{(p)}(t)dt} \left(\sum_{p=1}^{q} w^{(p)}(t_1)\, dt_1\, P_{Z|t_1}^{(p)}(dz_1) \right) \ldots$$
$$e^{-\int_{t_{n-1}}^{t_n} \sum_{p=1}^{q} w^{(p)}(t)dt} \left(\sum_{p=1}^{q} w^{(p)}(t_n)\, dt_n\, P_{Z|t_n}^{(p)}(dz_n) \right)$$
$$e^{-\int_{t_n}^{\infty} \sum_{p=1}^{q} w^{(p)}(t)dt},$$

which, in terms of the quantities in (7.4) – (7.5), is easily recast as (3.11) with $w(t)$ and $P_{Z|t}$ defined by (7.4) and (7.5). □

The model can be specified further as follows. Assume that each class p is homogeneous and consists of $n^{(p)}(t)$ identical risks at time t, and that each individual risk in the class catches fire with intensity $w^{(p)}$ and has claims developing in accordance with the distribution $P_Z^{(p)}$. The statistical assessment of these entities must be based on data as per risk class. The quantities in (7.4)-(7.5) can be taken to be $w^{(p)}(t) = n^{(p)}(t)\, w^{(p)}$ and $P_{Z|t}^{(p)} = P_Z^{(p)}$; $t > 0$. In this case the relevant covariate at time t is the vector $\boldsymbol{c}(t) = (n^{(1)}(t), \ldots, n^{(q)}(t))$.

The impact of observable external risk conditions may be assessed on the basis of statistics from risk cohorts that are fully developed and settled by time τ. However, an additional difficulty arises as such risk conditions typically are subject to random fluctuations and, contrary to internal risk characteristics, their future course is unknown. Thus, at time τ only the external conditions governing the *rns* and *inr* claims are known, whereas those governing the *cni* claims are unknown and can only be predicted in an extended model specifying some assumptions as to the nature of the fluctuations of external risk conditions.

8. EXAMPLES

A. A special model

Suppose the joint distribution of the waiting time from occurrence to notification and the claim amount for a claim occurred at time t is independent of t and that $Y \sim \mathrm{Ga}\,(\rho, \sigma)$, the gamma distribution with shape parameter ρ and scale parameter σ^{-1}, and $U|_{Y=y} \sim \mathrm{Ga}\,(1, \mu y)$, the exponential distribution with parameter μy. This means that large claims tend to be reported more promptly than small claims. The joint density of (U, Y) is

(8.1)
$$p_{U,Y}(u, y) = \mu y\, e^{-\mu y u} \frac{\sigma^\rho}{\Gamma(\rho)} y^{\rho-1} e^{-\sigma y}$$
$$= \frac{\mu \sigma^\rho}{\Gamma(\rho)} y^{\rho+1-1} e^{-(\mu u + \sigma) y}.$$

The predictive moments in (5.3)-(5.5) and (5.9) involve integrals of the type

$$I_{r,s}(y') = \int_{y'}^{\infty} y^{r-1} e^{-sy}\, dy,\ s > 0,\ r = \rho+1, \rho+2, \ldots,$$

which can be calculated by the recursive relation (integration by parts)

$$I_{r,s}(y') = \frac{1}{s} (y')^{r-1} e^{-sy'} + \frac{r-1}{s} I_{r-1,s}(y'),\ r > 1,$$

starting from the numerically computed value of $I_{\rho+1,s}(y')$. In particular

$$I_{r,s}(0) = \frac{\Gamma(r)}{s^r}, \qquad r, s > 0,$$

and one easily finds

$$\int_{u>\tau-t}\int_{y>0} y^k\, p_{U,Y}(u,y)\,du\,dy = \frac{\sigma^\rho(\rho+k-1)^{(k)}}{(\mu(\tau-t)+\sigma)^{\rho+k}}. \tag{8.2}$$

For $k = 0$ in (8.2), one gets the probability of $U > \tau-t$ involved in the *nr*-intensity in (5.7) and the *r*-intensity in (6.9);

$$1-P_U(\tau-t) = \frac{\sigma^\rho}{(\mu(\tau-t)+\sigma)^\rho}. \tag{8.3}$$

For $k = 1$ one gets the integral involved in m^{nr} in (6.8);

$$\int_{u>\tau-t}\int_{y>0} y\, p_{U,Y}(u,y)\,du\,dy = \frac{\sigma^\rho \rho}{(\mu(\tau-t)+\sigma)^{\rho+1}}. \tag{8.4}$$

B. Numerical results

Consider the model Paragraph 6D, with (U, Y) distributed as described in Paragraph A above. First, let $\rho = \sigma = 2$, which implies that Y has expected value $\rho/\sigma = 1$, coefficient of variation (standard deviation divided by expected value) $\rho^{-1/2} = 0.707$, and skewness (here defined as the third root of third central moment divided by standard deviation) $2^{1/3}\rho^{-1/6} = 1.122$. Next, let $\rho = \sigma = 0.09$, which implies that Y has expected value 1, coefficient of variation 3.33, and skewness 1.88. Throughout take $\beta = 1$, $\tau = 1$, $w(t) = 100$ for $0 < t \leq 1$ and $w(t) = 0$ for $t > 1$ (no *cni*-claims).

Table 1 displays the coefficients and the constant term of the optimal linear predictor of the outstanding (here *inr*−) liability for some choices of λ, κ, and μ. The results are easy to explain, so only a few comments shall be rendered here.

TABLE 1

OPTIMAL LINEAR PREDICTOR OF THE *inr*-LIABILITY

λ	κ	ρ	σ	μ	$\gamma(0)$	$\gamma(.2)$	$\gamma(.4)$	$\gamma(.6)$	$\gamma(.8)$	$\gamma(1)$	γ_0
.01	0	2	2	10	0.053	0.053	0.053	0.053	0.053	0.053	5.31
.01	1	2	2	10	0.022	0.027	0.036	0.046	0.059	0.064	6.42
.01	1	2	2	1	0.244	0.291	0.332	0.360	0.364	0.327	44.98
.10	0	2	2	1	1.282	1.282	1.282	1.282	1.282	1.282	12.54
.10	1	2	2	1	0.740	0.940	1.230	1.568	1.834	1.771	19.29
.10	1	2	2	10	0.010	0.017	0.040	0.095	0.211	0.319	2.18
.01	1	.09	.09	1	0.092	0.110	0.127	0.143	0.152	0.144	18.30
.01	1	.09	.09	10	0.013	0.015	0.018	0.022	0.026	0.027	3.00
.10	0	.09	.09	1	0.832	0.832	0.832	0.832	0.832	0.832	7.98
.10	1	.09	.09	1	0.421	0.521	0.653	0.810	0.960	0.970	10.45

In those cases where $\kappa = 0$, the coefficient function $\gamma(\cdot)$ is constant. The situation is reduced to that of standard credibility models, with $\Theta(t) = \Theta$ random, but not changing over time. Then $N^r(\tau) = \int_{0<s\leq\tau} N^r(ds)$ is sufficient, and the optimal coefficient function should be constant.

Large values of λ give large coefficients, that is, much weight on the observed claims reports. This effect is well-known from standard credibility theory.

Large values of μ mean that claims are quickly reported, and then both the constant term and the coefficients get small since few claims are outstanding.

The roles of κ, ρ, and σ are not easy to interpret.

C. An illustration: estimation of the cumulative hazard in a model with no delays

Finally, to acquire a feeling of how the entities in the model act on the linear predictor, consider a special case simple enough to allow for comparison with well-known results from standard credibility theory. Let the model be as in Paragraph 6D, with no delays in reporting of claims, hence $w^r(t) = w(t)$. As in the previous paragraph, take $\beta = 1$, $\tau = 1$, $w(t) = w$ constant for $t \leq 1$.

Consider the problem of estimating the cumulative intensity $w \int_0^1 \Theta(s)\, ds$ from the observations $\{N(t)\}_{0\leq t\leq 1}$. The normal equations of the optimal linear estimator are as in Paragraph 6D with $w^r(t) = m^{nr}(t) = w$ for $t \leq 1$ and $m^{nr}(t) = 0$ for $t > 1$. Table 2 shows the results for some selected values of λ, κ, and w.

As in the previous example, $\kappa = 0$ yields constant coefficients. The coefficient function is invariably symmetric around the midpoint, $t = 0.5$, with the larger values in the middle, and this effect is more pronounced the larger κ is. This is so because the observations in the center shed light on Θ-values to both sides.

TABLE 2

OPTIMAL LINEAR ESTIMATOR OF THE CUMULATIVE HAZARD RATE

λ	κ	w	$\gamma(0)$	$\gamma(.2)$	$\gamma(.4)$	$\gamma(.6)$	$\gamma(.8)$	$\gamma(1)$	γ_0
.10	0.0	100	.909	.909	.909	.909	.909	.909	9.09
.10	1.0	100	.779	.879	.914	.914	.879	.779	12.18
.10	5.0	100	.553	.774	.797	.797	.774	.553	24.43
.01	0.0	100	.500	.500	.500	.500	.500	.500	50.00
.01	1.0	100	.365	.421	.448	.448	.421	.365	57.68
.01	5.0	100	.154	.244	.270	.270	.244	.154	75.84
.10	0.0	1000	.990	.990	.990	.990	.990	.990	9.90
.10	1.0	1000	.929	.991	.995	.995	.991	.929	14.24
.00	1.0	100	.000	.000	.000	.000	.000	.000	100.00

ACKNOWLEDGEMENTS

The present paper is an excerpt of a report prepared for the Norwegian Banking, Insurance and Securities Commission. My thanks are due to former

Chief Actuary Oddvar Rygge and former Managing Director Svein Aasmundstad for their willingness to stake on research of this kind as a part of insurance supervision and to Arild Kristiansen for fruitful cooperation on statistical problems during the past five years.

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RAGNAR NORBERG
Laboratory of Actuarial Mathematics
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.

https://doi.org/10.2143/AST.23.1.2005103 Published online by Cambridge University Press