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Preliminary Results on Asymptotic Stabilization of Hamiltonian Systems with Nonholonomic Constraints ¹

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Abstract— This paper presents some preliminary results on asymptotic stabilization of nonholonomic mechanical systems using the Hamiltonian formulation proposed in [1]. Our work seeks to establish a general formulation for designing time-varying controllers for some mechanical system described in the generalized coordinates (position and momentum). The paper gives the change of coordinates that transforms the Hamiltonian system to the form needed to apply the center manifold theorem. We also present a worked example for which stability is analyzed.

Keywords Stabilization, Hamiltonian systems, Nonholonomic systems, time-varying control.

I. INTRODUCTION

Stabilization of nonholonomic systems has recently received a lot of interest. It is now well known that the necessary condition for feedback stabilization given by Brockett [2] is not satisfied for nonholonomic systems. Thus, there exists no continuous feedback law making the origin locally asymptotically stable. Consequently, alternative solutions such as discontinuous feedback and smooth time-varying feedback have been investigated.

Time-varying strategies have been extensively studied and constructive methods have been proposed for a quite large class of driftless nonholonomic systems [3]. General results on the existence of such controllers has been explored by Coron [4] who established that almost all controllable systems are stabilizable by continuous time-periodic feedback. Throughout suitable modifications (i.e. homogeneous norms, etc) such controllers can also provide exponential convergence rates [5], [6], [7]. Recently, works of the same authors have extended the stabilization results to systems with drift by adding an integrator to the kinematic (velocity) input.

Most of these works use as a basis for the control design, a transformed driftless systems in canonical form (chained or power forms are commonly used). Since these transformations do not yet cover the com-

plete class of mechanical systems having nonholonomic constraints, it is then interesting to investigate the possibility of design controllers that do not rely on these canonical forms.

The Hamiltonian formulation used in [1] seems to be a good basis for this analysis. They show how the Hamiltonian form of equations may be used for stabilization purposes; the controller is constructed using a suitable potential-like function. However, this controller does not yield asymptotic stability but make the general system coordinates (q, p) (position and momentum) tend to an invariant set $(p = 0, q = q_0)$, which contains the origin and depends on the used potential-like function. The idea behind the controller proposed in this paper is to smoothly switch over two controllers constructed by using two different potential functions yielding two different invariant sets whose intersection is the singleton $\{0\}$. Since the switching is performed via periodic time-varying functions, the stability analysis can be carried out by the application of the center manifold theory. This requires first to find the general change of coordinates that transforms the closed-loop system into the form needed to apply the center manifold theorem and then to study the stability on the resulting reduced-order center manifold.

In this paper preliminary results are presented. Section 2 presents the Hamiltonian control formulation. Section 3 gives the time-varying control law based on the smooth switch over two potential-like functions. Section 4 gives the general change of coordinates required to apply the center manifold theory. Section 5 presents a worked example for which stability conditions can be found. Finally Section 6 gives the conclusions.

II. HAMILTONIAN CONTROL FORMULATION

In this section, we recall some of the results concerning the Hamiltonian formulation given in [1]. Let Q be an n -dimensional configuration manifold with local coordinates $q = (q_1, \dots, q_n)$. Classical con-

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straints for a mechanical system are given in local coordinates as

$$A^T(q)\dot{q} = 0 \quad (1)$$

with A a $k \times n$ matrix $k \leq n$, with entries depending smoothly on q . Throughout, we assume that $A(q)$ has rank equal to k everywhere. The constraints (1) determine a k -dimensional distribution D on Q , given in every point $q_0 \in Q$ as

$$D(q_0) = \ker A^T(q_0) \quad (2)$$

The constraints (1) are called nonholonomic if D is not involutive. Since $\text{rank } A(q) = k$, there exists locally a smooth $n \times n - k$ matrix $S(q)$ of rank $n - k$ such that

$$A^T(q)S(q) = 0. \quad (3)$$

Defining the Hamiltonian of the system $H(q, p)$ by the Legendre transformation:

$$H(q, p) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (4)$$

with $L(q, \dot{q})$ a smooth Lagrangian function satisfying the usual regularity condition, the constrained Hamiltonian equations on T^*Q are given as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} + A(q)\lambda + B(q)u \\ A^T(q)\frac{\partial H}{\partial p} &= 0 \end{aligned} \quad (5)$$

where $\lambda \in \mathbb{R}^k$ are the constraint forces, $u \in \mathbb{R}^m$; $B(q)u$ are the external forces applied to the system with $B(q)$ an $n \times m$ full rank matrix.

On the constrained state space:

$$\mathcal{X}_r = \{(q, p) \in T^*Q \mid A^T(q)\frac{\partial H}{\partial p}(q, p) = 0\} \quad (6)$$

the dynamic equations of motion in the local coordinates (q, \tilde{p}^1) as described in [1] are:

$$\begin{pmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{pmatrix} = J_r(q, \tilde{p}^1) \begin{pmatrix} \frac{\partial H_r}{\partial q}(q, \tilde{p}^1) \\ \frac{\partial H_r}{\partial \tilde{p}^1}(q, \tilde{p}^1) \end{pmatrix} + \begin{pmatrix} 0 \\ B_r(q) \end{pmatrix} u \quad (7)$$

with:

- $\tilde{p}^1 := S^T(q)p, \quad \tilde{p}^1 \in \mathbb{R}^{n-k}, \quad (8)$

- $J_r(q, p) = \begin{pmatrix} 0_n & S(q) \\ -S^T(q) & (-p^T[S_i, S_j](q))_{i,j=1..n-k} \end{pmatrix} \quad (9)$

a $(2n - k) \times (2n - k)$ skew-symmetric matrix; $S_i(q)$ denotes the i -th column of $S(q)$,

- H_r : the reduced Hamiltonian, which is taken of the form

$$H_r(q, \tilde{p}^1) = V(q) + \frac{1}{2}(\tilde{p}^1)^T G(q)\tilde{p}^1 \quad G(q) > 0 \quad (10)$$

$V(q)$ is the potential energy

- $B_r(q)$ has a full rank $m = n - k$

Consider the following feedback law:

$$B_r(q)u(q, \tilde{p}^1) = -S^T(q)\frac{\partial \bar{V}}{\partial q} - \frac{\partial H_r^T}{\partial \tilde{p}^1}(q, \tilde{p}^1). \quad (11)$$

together with the Lyapunov function

$$\bar{H}_r(q, \tilde{p}^1) = H_r(q, \tilde{p}^1) + \bar{V}(q) \quad (12)$$

where \bar{V} is used to shape the potential energy in the function above. Then we have that every point $(q_0, 0) \in \mathcal{X}_r$ is Lyapunov stable and the solutions of the closed-loop system (7)-(11), tend to the largest invariant set given as:

$$\Omega = \{(q, 0) \in \mathcal{X}_r \mid S^T(q)\frac{\partial(V + \bar{V})}{\partial q}(q) = 0\} \quad (13)$$

III. CONTROL LAW DESIGN

As noted in the previous section, the potential-like energy functions can be arbitrarily chosen by defining \bar{V}_i . In general, each choice of \bar{V}_i yields a different invariant set Ω_i .

In the sequel, consider control laws combining only two potential-like energy functions although more combinations may be possible. Therefore the functions \bar{V}_1 , and \bar{V}_2 should be defined so that the corresponding invariant sets Ω_1 and Ω_2 satisfy:

$$\Omega_1 \cap \Omega_2 = \{0\} \quad (14)$$

with $\Omega_i = \{(q, 0) \in \mathcal{X}_r \mid S^T(q)\frac{\partial(V + \bar{V}_i)}{\partial q}(q) = 0\}, i = 1, 2$.

The above condition seems to be necessary to ensure that the origin is the unique equilibria. Further conditions on the functions \bar{V}_i will be needed in connection with the stability requirements.

Let $u(q, \tilde{p}^1)$ be the control law given by:

$$u(q, \tilde{p}^1) = u_1(q, \tilde{p}^1)\alpha(t) + u_2(q, \tilde{p}^1)(1 - \alpha(t)) \quad (15)$$

with, for $i = 1, 2$

$$B_r(q)u_i(q, \tilde{p}^1) = -S^T(q)\frac{\partial \bar{V}_i}{\partial q}(q) - \frac{\partial H_r^T}{\partial \tilde{p}^1}(q, \tilde{p}^1) \quad (16)$$

and $\alpha(t) = \frac{1 - \cos t}{2}$.

Selecting $G(q) = I, V(q) = 0$ and assume that

$$\left(-\tilde{p}^{1T}[S_i, S_j](q)\right)_{i,j=1..n-k} = 0 \quad (17)$$

(as all degree of freedom are controlled ($m = n - k$), there exists a feedback law such that (17) is satisfied) then system (7) simplifies to:

$$\begin{aligned} \dot{q} &= S(q)\tilde{p}_1 \\ \dot{\tilde{p}}_1 &= B_r(q)u. \end{aligned} \quad (18)$$

The components u_i of the control u take then the form,

$$B_r(q)u_i(q, \tilde{p}^1) = - \left(S^T(q) \frac{\partial \bar{V}_i}{\partial q}(q) + \tilde{p}_1 \right) \text{ for } i = 1, 2 \quad (19)$$

which gives through calculations

$$B_r(q)u(q, p) = -p - \frac{1}{2}S^T(q) \left(\frac{\partial \bar{V}_1}{\partial q}(q) + \frac{\partial \bar{V}_2}{\partial q}(q) \right) + \frac{\omega_2}{2}S^T(q) \left(\frac{\partial \bar{V}_2}{\partial q}(q) - \frac{\partial \bar{V}_1}{\partial q}(q) \right) \quad (20)$$

where to simplify the notation, we have redefined $p := \tilde{p}_1 \in \mathbb{R}^m$ and defined $\omega_1 = \sin(t)$ and $\omega_2 = \cos(t)$. ω_1 and ω_2 are here seen as additional variables generated by the oscillator:

$$\begin{aligned} \dot{\omega}_1 &= \omega_2 \\ \dot{\omega}_2 &= -\omega_1. \end{aligned} \quad (21)$$

Let \bar{V}_i now be of the form

$$\bar{V}_i = q^T Q_i q; \quad Q_i \geq 0 \text{ for } i = 1, 2. \quad (22)$$

Therefore, $\frac{\partial \bar{V}_i}{\partial q}(q) = Q_i q$ for $i = 1, 2$, yielding

$$B_r(q)u(q, p) = -p - S^T(q) \frac{Q_1 + Q_2}{2} q + \omega_2 S^T(q) \frac{Q_2 - Q_1}{2} q.$$

The closed-loop system is thus given by (21) and

$$\begin{cases} \dot{q} = S(q)p \\ \dot{p} = -p - S^T(q)M_1 q + \omega_2 S^T(q)M_2 q \end{cases} \quad (23)$$

where $M_1 := \frac{Q_1 + Q_2}{2}$ and $M_2 := \frac{Q_2 - Q_1}{2}$.

The matrices $Q_i \geq 0$ are not free; they have to fulfill the requirement (14), where the Ω_i 's are given by:

$$\Omega_i = \{(q, 0) \in \mathcal{X}_r \mid S^T(q)Q_i q = 0\}, i = 1, 2.$$

Assuming that $S(q)$ satisfies

$$S(0) = \begin{pmatrix} S_1(0) \\ 0 \end{pmatrix} \quad (24)$$

with $S_1(0)$ a square non-singular matrix $m \times m$, and letting Q_i be partitioned as:

$$Q_i = \begin{pmatrix} Q_{i1} & Q_{i2} \\ Q_{i3} & Q_{i4} \end{pmatrix} \begin{array}{l} Q_{i1} \in \mathbb{R}^m \times \mathbb{R}^m, \\ Q_{i2} \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \\ Q_{i3} \in \mathbb{R}^{n-m} \times \mathbb{R}^m, \\ Q_{i4} \in \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \end{array}$$

for $i = 1, 2$ and q as,

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad q_1 \in \mathbb{R}^m, \quad q_2 \in \mathbb{R}^{n-m},$$

then it can be shown - via the implicit function theorem - that condition (14) is satisfied if the following $2m \times n$ matrix is full rank:

$$\text{Rank} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = n$$

This implies that, with the controller constructed on the basis of only two quadratic potential-like functions (22), the considered class of mechanical systems is restricted to those satisfying

$$2m \geq n$$

that is systems having a number of inputs larger than one half of the number of generalized position coordinates. This class can be enlarged by increasing the number of potential-like functions and finding more involved forms.

IV. COORDINATE TRANSFORMATION

In this section we give, under the hypothesis and control structure introduced in the previous section, the required coordinate transformation that brings the closed-loop system (23) to the form required for the application of the center manifold theorem.

The matrix $S(q)$ can be expanded using a Taylor series about $q = 0$, as:

$$S(q) = S(0) + C(q) + O(\|q\|^2)$$

where $C_{ij}(q) := \left(\frac{\partial S_{ij}(q)}{\partial q} \right)_{q=0}$ and $O(\|q\|^r) \leq k \|q\|^r$ for sufficiently small $\|q\|$. Introducing this expression on (23) gives,

$$\begin{aligned} \dot{q} &= S(0)p + C(q)p + O(\|q\|^2)p \\ \dot{p} &= -p - S^T(0)(M_1 - \omega_2 M_2)q - C^T(q)(M_1 - \omega_2 M_2)q + O(\|q\|^3) \end{aligned} \quad (25)$$

Let the matrix M_i be partitioned as:

$$M_i = \begin{pmatrix} M_{i1} & M_{i2} \\ M_{i3} & M_{i4} \end{pmatrix} \begin{array}{l} M_{i1} \in \mathbb{R}^m \times \mathbb{R}^m, \\ M_{i2} \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \\ M_{i3} \in \mathbb{R}^{n-m} \times \mathbb{R}^m, \\ M_{i4} \in \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}. \end{array}$$

for $i = 1, 2$ and

$$C(q) = \begin{pmatrix} C_1(q) \\ C_2(q) \end{pmatrix} \begin{array}{l} C_1(q) \in \mathbb{R}^m \times \mathbb{R}^m, \\ C_2(q) \in \mathbb{R}^{n-m} \times \mathbb{R}^m. \end{array}$$

Let $M_{21} = 0$, yields

$$\begin{aligned} S^T(0)M_1 q &= \begin{pmatrix} S_1^T & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{13} & M_{14} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= S_1^T(0)M_{11}q_1 + S_1^T(0)M_{12}q_2 \end{aligned}$$

then system (25) can be written as

$$\begin{aligned} \dot{p} &= -p - S_1^T(0)M_{11}q_1 - S_1^T(0)M_{12}q_2 - \omega_2 S_1^T(0)M_{22}q_2 - C^T(q)(M_1 - \omega_2 M_2)q + O(\|q\|^3) \\ \dot{q}_1 &= S_1(0)p + C_1(q)p + O(\|q\|^2)p \\ \dot{q}_2 &= C_2(q)p + O(\|q\|^2)p \end{aligned} \quad (26)$$

Note that $M_{21} = 0$ implies that $Q_{11} = Q_{21}$.

Now, in order to apply the "Time-varying" Center Manifold lemma [8] the following change of coordinate is necessary.

Lemma IV.1: Consider the following coordinate transformation

$$\begin{aligned}\hat{p} &= p + (\pi_{p1}\omega_1 + \pi_{p2}\omega_2)q_2 \\ \hat{q}_1 &= q_1 + (\pi_{q1} + \pi_{q2}\omega_1 + \pi_{q3}\omega_2)q_2\end{aligned}\quad (27)$$

then $\pi_{p1}, \pi_{p2}, \pi_{q1}, \pi_{q2}, \pi_{q3} \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ are uniquely defined by the set of equations

$$\begin{aligned}S_1^T(0)M_{11}\pi_{q1} - S_1^T(0)M_{12} &= 0 \\ S_1^T(0)M_{11}\pi_{q2} + \pi_{p1} - \pi_{p2} &= 0 \\ S_1^T(0)M_{11}\pi_{q3} - S_1^T(0)M_{22} + \pi_{p1} + \pi_{p2} &= 0 \\ S_1^T(0)\pi_{p1} + \pi_{q3} &= 0 \\ S_1^T(0)\pi_{p2} - \pi_{q2} &= 0\end{aligned}\quad (28)$$

provided that M_{11} is defined as a full rank matrix and hypothesis (24) is satisfied.

Proof: Computing the time derivative of \hat{p} gives,

$$\begin{aligned}\dot{\hat{p}} &= \dot{p} + (\pi_{p1}\omega_2 - \pi_{p2}\omega_1)q_2 + (\pi_{p1}\omega_1 + \pi_{p2}\omega_2)\dot{q}_2 \\ &= -\dot{p} - S_1^T(0)M_{11}\dot{q}_1 + [(S_1^T(0)M_{11}\pi_{q1} - S_1^T(0)M_{12}) \\ &\quad + \omega_1(S_1^T(0)M_{11}\pi_{q2} + \pi_{p1} - \pi_{p2}) + \omega_2(S_1^T(0)M_{11}\pi_{q3} \\ &\quad + S_1^T(0)M_{22} + \pi_{p1} + \pi_{p2})]q_2 + O(\|q\|)p + O(\|q\|^2)\end{aligned}$$

Setting the terms within the square brackets to zero gives the first three equations in (28). Proceeding similarly with \hat{q}_1 yields,

$$\begin{aligned}\dot{\hat{q}}_1 &= \dot{q}_1 + (\pi_{q2}\omega_2 - \pi_{q3}\omega_1)q_2 + (\pi_{q1} + \pi_{q2}\omega_1 + \pi_{q3}\omega_2)\dot{q}_2 \\ &= S_1(0)\dot{p} + [\pi_{q2}\omega_2 - \pi_{q3}\omega_1 - S_1(0)(\pi_{p1}\omega_1 + \pi_{p2}\omega_2)]q_2 \\ &\quad + O(\|q\|)p + O(\|q\|^2)\end{aligned}$$

the last two equations of system (28) are obtained by setting the terms within the square brackets to zero.

After the coordinate transformation, system (26) writes as

$$\begin{aligned}\begin{pmatrix} \dot{\hat{p}} \\ \dot{\hat{q}}_1 \end{pmatrix} &= \begin{pmatrix} -I_m & -S_1^T(0)M_{11} \\ S_1(0) & 0 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q}_1 \end{pmatrix} \\ &\quad + \begin{pmatrix} f_1(\hat{p}, \hat{q}_1, q_2, \omega) \\ f_2(\hat{p}, \hat{q}_1, q_2, \omega) \end{pmatrix} \\ \dot{q}_2 &= g(\hat{p}, \hat{q}_1, q_2, \omega) \\ \dot{\omega} &= H\omega\end{aligned}\quad (29)$$

where $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\begin{aligned}f_1(\hat{p}, \hat{q}_1, q_2, \omega) &= -C^T(q)(M_1 - \omega_2 M_2)q \\ &\quad - (\pi_{p1}\omega_1 + \pi_{p2}\omega_2)C_2(q)p + O(\|q\|^3) \\ f_2(\hat{p}, \hat{q}_1, q_2, \omega) &= C_1(q)p + (\pi_{q1} + \pi_{q2}\omega_1 + \pi_{q3}\omega_2)C_2(q)p \\ &\quad + O(\|q\|^2)p \\ g(\hat{p}, \hat{q}_1, q_2, \omega) &= C_2(q)p + O(\|q\|^2)p\end{aligned}$$

with q_1 and p given by (27) and $f = (f_1, f_2)^T$ and g verifying $f(0, 0, 0, \omega) = 0$, $f'(0, 0, 0, \omega) = 0$, $g(0, 0, 0, \omega) = 0$ and $g'(0, 0, 0, \omega) = 0$.

Define F as

$$F = \begin{pmatrix} -I_m & -S_1^T(0)M_{11} \\ S_1(0) & 0 \end{pmatrix}$$

with $M_{11} = Q_{11} + Q_{21}$ be such that all the eigenvalues of F have negative real part. It is now possible to apply to system (29) the following modified version of the center manifold theorem:

Lemma IV.2: ("Time-varying" Center Manifold)[8] Consider the system

$$\begin{aligned}\dot{y} &= Fy + f(y, z, \omega) \\ \dot{z} &= Gz + g(y, z, \omega) \\ \dot{\omega} &= H\omega\end{aligned}\quad (30)$$

with $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $\omega \in \mathbb{R}^p$ and the eigenvalues of F have negative real part and the eigenvalues of G and H have zero real part. The function f, g and h are C^2 with $f(0, 0, \omega) = 0$, $f'(0, 0, \omega) = 0$, $g(0, 0, \omega) = 0$ and $g'(0, 0, \omega) = 0$. Then, given $M > 0$, there exists a center manifold for (30), $y = h(z, \omega)$ for $|\omega| < M$, $|z| < \delta(M)$, for some $\delta > 0$ and dependent on M , where h is C^2 and $h(0, \omega) = 0$, $h'(0, \omega) = 0$.

Therefore, there exists a center manifold for the system (29) together with the relations (27) which can be expressed in the form (30) by defining $y^T = [\hat{p}^T, \hat{q}_1^T]$, $z = q_2$ and $G = 0$. The center manifold $y = h(z, \omega)$ is thus of the form:

$$\begin{aligned}\hat{p} &= h_p(q_2, \omega) \\ \hat{q}_1 &= h_q(q_2, \omega).\end{aligned}\quad (31)$$

Stability of the system remains to be determined from the analysis of the reduced order system:

$$\dot{q}_2 = g(h_p(q_2, \omega), h_q(q_2, \omega), q_2, \omega)$$

where the determination of the lower power matrices of the expansion of $h(\omega, q_2)$ should be obtained from the application of the approximation theorem. Finally, the stability of the complete system can be studied by using averaging analysis. Next section shows how this procedure is applied to the knife edge example.

V. KNIFE EDGE EXAMPLE

The dynamic equations of motion of the knife edge moving in point contact on a plane surface described in the local coordinates (q, \tilde{p}^1) are:

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos \varphi \\ 0 & 0 & 0 & 0 & \sin \varphi \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos \varphi & -\sin \varphi & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ p_1 \\ p_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\end{aligned}\quad (32)$$

where (x, y) are the cartesian coordinates of the contact point, φ the heading angle of the knife edge, u_1 the control in the direction of the heading angle, and u_2 the control torque about the vertical axis. The generalized position coordinates are $q = (\varphi \ x \ y)^T \in \mathbb{R}^3$, and the reduced generalized momenta is $\tilde{p}^1 = (p_1 \ p_2)^T \in \mathbb{R}^2$ as defined in (8). The reduced Hamiltonian $H_r(p_1, p_2)$ is then given by:

$$H_r(p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$$

Note that assumptions (24) and (14) are satisfied for this system.

Taking

$$\bar{V}_1(q) = \frac{1}{2}q^T Q_1 q \quad Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} > 0$$

and computing the feedback law using (11):

$$u_1 = \begin{cases} u_{11} = -x \cos \varphi - y \sin \varphi - p_2 \\ u_{12} = -\varphi - p_1 \end{cases} \quad (33)$$

yields the associated invariant set:

$$\Omega_1 = \{(q, 0) \mid \varphi = 0, x = 0\}.$$

Let now \bar{V}_2 defined as:

$$\bar{V}_2(q) = \frac{1}{2}q^T Q_2 q \quad Q_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \geq 0$$

The resulting feedback is thus given by:

$$u_2 = \begin{cases} u_{21} = -x \cos \varphi - (\varphi + y) \sin \varphi - p_2 \\ u_{22} = -(\varphi + y) - p_1 \end{cases} \quad (34)$$

leading to the invariant set:

$$\Omega_2 = \{(q, 0) \mid \varphi + y = 0, x \cos \varphi = 0\}.$$

With this definition, we can easily check that the intersection of Ω_1 and Ω_2 is the singleton $\{0\}$. Therefore conditions $2m \geq n$ and the rank condition of Q_0 are, with the proposed choice of Q_1 and Q_2 , satisfied.

Finally, the time-varying control law (15) is given as:

$$u(q, \bar{p}^1) = \alpha(t)u_1 + (1 - \alpha(t))u_2. \quad (35)$$

The change of coordinates given in Section 4 can be now performed. The following computations results from the above definitions:

$$M_1 = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$S(q) = \begin{pmatrix} 1 & 0 \\ 0 & \cos \varphi \\ 0 & \sin \varphi \end{pmatrix}, \quad S(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C(q) = \begin{pmatrix} 0 & 0 \\ 0 & \varphi \end{pmatrix}, \quad P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad q_1 = \begin{pmatrix} \varphi \\ x \end{pmatrix}, \quad q_2 = y.$$

$$s_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_1(q) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad c_2(q) = \begin{pmatrix} 0 & v \end{pmatrix},$$

$$M_{11}(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{12}(q) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

$$M_{21}(q) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{22}(q) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

With this choice, the matrix F

$$F = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is stable and $M_{21} = 0$.

Solving (28) gives the following change of coordinates:

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{\varphi} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ \varphi \\ x \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} + \frac{1}{2}\omega_1 \\ 0 \end{pmatrix} \omega_2 y \quad (36)$$

The first approximation of the center manifold is given as

$$\hat{p} = h_p(\omega, y) = h_1(\omega)y^2 + O(y^3) \quad (37)$$

$$\hat{q} = h_q(\omega, y) = h_2(\omega)y^2 + O(y^3) \quad (38)$$

Then evaluating the time derivatives of (37)-(38) along the closed-loop equations gives

$$\begin{cases} \frac{\partial h_1}{\partial \omega_1} \omega_2 - \frac{\partial h_1}{\partial \omega_2} \omega_1 + h_1(\omega) + h_2(\omega) + \nu(\omega) = 0 \\ \frac{\partial h_2}{\partial \omega_1} \omega_2 - \frac{\partial h_2}{\partial \omega_2} \omega_1 - h_1(\omega) = 0 \end{cases} \quad (39)$$

where

$$\nu(\omega) = \begin{pmatrix} 0 \\ \frac{1}{8}(1 + \omega_1)^2(1 - \omega_2) + \frac{1}{2}(1 + \omega_1) \end{pmatrix}$$

and the dynamics in the reduced order manifold is thus,

$$\begin{aligned} \dot{y} &= p_2 \varphi + O(y^4) \\ &= \hat{p}_2(\hat{\varphi} - \frac{1}{2}(1 + \omega_1)y) + O(y^4) \\ &= -\frac{1}{2}(1 + \omega_1)\hat{p}_2 y + O(y^4) \\ &= -\frac{1}{2}(1 + \omega_1)h_{12}y^3 + O(y^4) \end{aligned}$$

where only the second component of the vector $h_1 = (h_{11} \ h_{12})^T$ is important for stability analysis. By inspecting (39) it can be concluded that h_{12} should be of the form

$$h_{12}(\omega) = \sum_{i+j=3} a_{ij_2} \omega_1^i \omega_2^j$$

which gives,

$$\begin{aligned} \dot{y} &= -\frac{1}{2}(1 + \omega_1) \sum_{i+j=3} a_{ij_2} \omega_1^i \omega_2^j y^3 + O(y^4) \\ &= -\left(\frac{1}{2} \sum_{i+j=3} a_{ij_2} \omega_1^i \omega_2^j + \frac{1}{2} \sum_{i+j=3} a_{ij_2} \omega_1^{i+1} \omega_2^j\right) y^3 + O(y^4) \end{aligned}$$

that can be compactly written as:

$$\dot{y} = -\frac{1}{2}(a_1 + a_2)y^3 + O(y^4) \quad (40)$$

Now, applying the coordinate change given in Appendix 1, stability of system (40) can be concluded from studying the stability of the following transformed averaged system:

$$\dot{\zeta} = -\frac{1}{2}(\bar{a}_1 + \bar{a}_2)\zeta^3 + O(\zeta^4)$$

where \bar{a}_1 and \bar{a}_2 are respectively the mean value of a_1 and a_2 . The coefficients a_{ij_2} for which the averaging value of $\omega_1^i \omega_2^j$ and $\omega_1^{i+1} \omega_2^j$ is different from zero, are:

$$a_{20_2} = -a_{02_2}, \quad a_{10_2} = \frac{1}{4}, \quad a_{12_2} = \frac{9}{73.4}, \quad a_{30_2} = -\frac{3}{73.4}$$

We finally obtain

$$\bar{a}_1 = 0, \quad \bar{a}_2 = \frac{\pi}{4}$$

which gives a locally asymptotically stable averaged system:

$$\dot{\zeta} = -\frac{\pi}{8}\zeta^3 + O(\zeta^4) \quad (41)$$

VI. CONCLUSIONS

We have presented some preliminary results on asymptotic stabilization of nonholonomic mechanical systems using the Hamiltonian formulation proposed in [1]. Our work seeks to establish a general formulation for designing time-varying controllers for some mechanical system described in the generalized coordinates (position and momentum). We have explicitly stated the change of coordinates required to transform the Hamiltonian system to the form needed to apply the center manifold theorem. We have also presented the knife edge example for which stability is analyzed. One possible generalization of this approach lies in the consideration of more than two potential-like functions with eventually a reduced Hamiltonian in a more general form.

VII. APPENDIX

Let consider the system

$$\dot{y} = -g(t)y^3 + O(y^4) \quad (42)$$

where $g(t)$ is a T -periodic function of class C^r and the coordinate transformation

$$y = \zeta + \beta(t)\zeta^3. \quad (43)$$

Then, differentiation gives

$$\begin{aligned} \dot{y} &= \dot{\zeta} + 3\beta(t)\zeta^2\dot{\zeta} + \frac{\partial\beta}{\partial t}\zeta^3 \\ \dot{y} &= (1 + 3\beta(t)\zeta^2)\dot{\zeta} + \frac{\partial\beta}{\partial t}\zeta^3 \end{aligned} \quad (44)$$

and

$$\begin{aligned} \dot{\zeta} &= (1 + 3\beta(t)\zeta^2)^{-1}(\dot{y} - \frac{\partial\beta}{\partial t}\zeta^3) \\ &= (1 - 3\beta(t)\zeta^2 + O(\zeta^4))(-g(t)y^3 - \frac{\partial\beta}{\partial t}\zeta^3 + O(y^4)) \\ &= (1 - 3\beta(t)\zeta^2 + O(\zeta^4))(-g(t)(\zeta + \beta(t)\zeta^3)^3 \\ &\quad - \frac{\partial\beta}{\partial t}\zeta^3 + O(\zeta^4)) \\ &= (1 - 3\beta(t)\zeta^2 + O(\zeta^4))(-g(t)\zeta^3 - \frac{\partial\beta}{\partial t}\zeta^3 + O(\zeta^4)) \end{aligned} \quad (45)$$

which, with the choice

$$\tilde{g}(t) = g(t) - \bar{g},$$

where \bar{g} is the time average of $g(t)$ gives

$$\dot{\zeta} = -\bar{g}\zeta^3 - \tilde{g}(t)\zeta^3 - \frac{\partial\beta}{\partial t}\zeta^3 + O(\zeta^4).$$

Now, we choose $\beta(t)$ so that

$$\frac{\partial\beta}{\partial t} = -\tilde{g}(t)$$

and we obtain

$$\dot{\zeta} = -\bar{g}\zeta^3 + O(\zeta^4).$$

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