

PREMIUM CALCULATION:  
WHY STANDARD DEVIATION SHOULD BE REPLACED  
BY ABSOLUTE DEVIATION<sup>1</sup>

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ABSTRACT

Average absolute (instead of quadratic) deviation from median (instead of expectation) is better suited to determine the safety loading for insurance premiums than standard deviation: The corresponding premium functionals behave additive under the practically relevant risk sharing schemes between first insurer and reinsurer.

KEYWORDS

Premium principles; average absolute deviation; comonotonic additivity; distorted probabilities.

0. INTRODUCTION

If one looks into the extensive literature on premium principles one gets the impression that actuaries are more or less incontent with the premium principles known till now. For example there was not known a nontrivial functional on nonnegative random variables, in actuarial terms a premium principle for insurance contracts, with the following elementary and plausible requirements: **P1**. The safety loading (premium minus expected value) is nonnegative, **P2**. no ripoff, i.e. the premium does not exceed the maximal claim, **P3**. consistency, i.e. the safety loading does not change if claims are augmented by a non-random constant and **P4**. proportionality, i.e. insuring a certain percentage of total damage costs that percentage of full insurance. It should be mentioned that the proportionality property **P4** despite its practical importance is not regarded desirable by all authors (e.g. GERBER). We shall discuss that point at the end of section three.

The present article intends to make actuaries familiar with a broad class of functionals with properties **P1** through **P4**. These functionals had been developed (by SCHMEIDLER, YAARI and others) during the last decade in the context of economic decision theory with the intention to overcome the

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controversely discussed shortcomings of expected utility theory. Expected utility had been used, too, to construct premium functionals as the exponential principle, favoured e.g. by GERBER.

To make things as easy and accessible as possible we confine ourselves to an elementary one parameter class of premium functionals of the YAARI type (DENNEBERG 1985, 1988a and b). This functional resembles the standard deviation principle, where the safety loading is proportional to standard deviation. But the volatility measure standard deviation is replaced by average absolute deviation from the median and, surprisingly, all works.

In the first section we compile the properties of average absolute deviation from median, a volatility measure which nowadays is nearly forgotten, whereas in the first part of our century it enjoyed equal rights with standard deviation (e.g. in CZUBER, cf. the discussion in DENNEBERG 1988b). The median being a quantile, it is appropriate here and in the sequel to employ the quantile function instead of its inverse function, the usual distribution function.

The premium functional with safety loading proportional to absolute deviation is introduced in the second section and properties P1 through P4 and some others—here we stress only subadditivity—are verified.

In section three the basic issue of comonotonicity of several random variables is introduced which, in some sense, is opposite to independence. Comonotonicity means that the risks involved are not able to compensate each other and this property implies additivity of our premiums. If risks are shared, e.g. between first insurer and reinsurer, the partial risks are comonotonic for most risk sharing schemes, among them all practically relevant ones. Hence our premium functional is compatible with the practice of reinsurance. We discuss comonotonic additivity, a property not shared by the standard deviation principle, versus independence additivity, a property shared by the variance and exponential principles.

The final section gives an outlook on the more general class of premium functionals mentioned above. There is a further well known volatility measure, which, like absolute deviation, is associated to that class: the Gini coefficient. It might be interesting for pricing reinsurance.

## 1. QUANTILE FUNCTION AND ABSOLUTE DEVIATION

Let  $X$  be a random variable to be interpreted as claims from an insurance contract or from a portfolio of such claims. We assume the increasing distribution function  $F = F_X$  of  $X$  to be known.  $F(x)$ ,  $x \in \mathbb{R}$ , denotes the probability of the event  $X \leq x$ . For our purposes the inverse function  $\tilde{F}$  of  $F$  is better suited to represent the distribution of  $X$  than  $F$ . Since  $F$ , in general, is not one to one (e.g. for discrete distributions), we have to be cautious in defining  $\tilde{F}$ . First, for  $q$  in the unit interval  $[0, 1]$  we define the  $q$ -quantile of  $X$  to be the interval  $[\inf_{F(x) \geq q} x, \sup_{F(x) \leq q} x]$ . The  $\frac{1}{2}$ -quantile is the **median** of  $X$ . For

all  $q \in [0, 1]$  outside possibly a countable set the  $q$ -quantile of  $X$  reduces to a single point. Now we define  $\tilde{F}(q)$  to be some fixed point of the  $q$ -quantile of  $X$ .

Then  $MX := \tilde{F}(\frac{1}{2})$  is a median of  $X$ . There is possibly an arbitrariness in the definition of  $\tilde{F}$  and  $MX$  but this does not affect the values of the subsequent integrals. For short the function  $\tilde{F}$  will be called the **quantile function** of  $X$ .

The expectation of  $X$  is

$$EX = \int_{-\infty}^{\infty} x dF(x) = \int_0^1 \tilde{F}(q) dq$$

and we will make use of the absolute and quadratic norms

$$\|X\|_1 := E|X|, \quad \|X\|_2 := (E(X^2))^{1/2}.$$

The corresponding volatility parameters are **average absolute deviation from median**  $\tau = \tau(X)$  and **standard deviation**  $\sigma = \sigma(X)$ :

$$\tau := \|X - MX\|_1, \quad \sigma := \|X - EX\|_2.$$

It is natural to take the real numbers  $MX$  and  $EX$  as points of reference in defining the respective volatility parameter since these numbers minimise the respective distance from  $X$ :

$$\tau = \min_{a \in \mathbb{R}} \|X - a\|_1, \quad \sigma = \min_{a \in \mathbb{R}} \|X - a\|_2.$$

If one looks for a parameter to indicate asymmetries of distributions one encounters two main methods. Either one uses higher odd moments, e.g.  $E(x^3)$ , or semivariances. The analogous to the latter in the case of absolute deviation are

$$\tau_- := \int_0^{1/2} |\tilde{F}(q) - MX| dq, \quad \tau_+ := \int_{1/2}^1 |\tilde{F}(q) - MX| dq$$

and one has

$$\tau = \tau_+ + \tau_-$$

$$EX - MX = \tau_+ - \tau_-.$$

From these equations we derive, that the triple  $(MX, \tau_-, \tau_+)$  of parameters contains the same information about the distribution of  $X$  as the triple  $(EX, MX, \tau)$ .

Finally we prove subadditivity of  $\tau$  and  $\sigma$ ,

$$\tau(X+Y) \leq \tau(X) + \tau(Y), \quad \sigma(X+Y) \leq \sigma(X) + \sigma(Y).$$

For standard deviation this is simply the triangle inequality for the norm  $\|\cdot\|_2$ . In case of absolute deviation, apart from the triangle inequality for the norm  $\|\cdot\|_1$ , one needs the above minimal property of the median to cope with the fact that the median is not additive:

$$\begin{aligned} \tau(X+Y) &= \|X+Y - M(X+Y)\|_1 = \min_{a \in \mathbb{R}} \|X+Y - a\|_1 \leq \|X+Y - (MX+MY)\|_1 \\ &\leq \|X - MX\|_1 + \|Y - MY\|_1 = \tau(X) + \tau(Y). \end{aligned}$$

In section 3 there will be given a sufficient condition for additivity of  $\tau$  analogous to additivity of variance  $\sigma^2$  in case of independence.

## 2. THE ABSOLUTE DEVIATION PRINCIPLE AND ELEMENTARY PROPERTIES

Let  $\mathfrak{X}$  be an appropriate set of random variables, e.g. the linear space  $L^1$  or  $L^2$  of random variables  $X$  on a fixed probability space with finite norm  $\|X\|_1$  and  $\|X\|_2$ , respectively. In our context, a functional

$$H: \mathfrak{X} \rightarrow \mathbb{R}, \quad X \mapsto HX$$

is called a **premium functional** or **premium principle**. The properties P1 through P4 from the introduction read in formal terms

**P1.**  $HX \geq EX$

**P2.**  $HX \leq \sup X$

**P3.**  $H(X+c) = HX+c, \quad c \in \mathbb{R}$

**P4.**  $H(cX) = cHX, \quad c \geq 0.$

Under the premium principles, studied in actuarial literature till now, only the trivial functionals  $H = E$  (net premium principle) and  $H = \sup$  (maximal loss principle) have all four properties. The common **standard deviation principle**

$$HX = EX + a\sigma(X), \quad X \in L^2, \quad \text{with parameter } a > 0$$

for example, violates P2. Our new premium functional

$$HX := EX + \rho\tau(X), \quad X \in L^1, \quad \text{with parameter } 0 \leq \rho \leq 1$$

is constructed in the same way and will be called **absolute deviation principle**.

It is worth mentioning that this functional coincides with the expected value principle for special distributions: namely if  $MX = 0$  and  $\tau_-(X) = 0$ , i.e.  $X \geq 0$  and the probability of no claim is  $\geq 1/2$ . Then  $HX = (1+\rho)EX$ .

The absolute deviation principle can be expressed, too, by the three parameters median  $MX$ , average negative and positive deviation  $\tau_-(X)$  and  $\tau_+(X)$  from the median (see section 1):

$$HX = MX - (1-\rho)\tau_-(X) + (1+\rho)\tau_+(X).$$

In this form the functional can be made plausible, too. The median serves as a reference point. Positive deviations, i.e. larger claims, are weighted more than negative deviations, i.e. smaller claims, and total weight is one.

We get an integral representation for the absolute deviation principle if, in the last formula, we replace  $\tau_-$  and  $\tau_+$  by their defining integrals:

$$\begin{aligned}
 HX &= MX - (1-\rho) \int_0^{1/2} (MX - \check{F}_X(q)) dq + (1+\rho) \int_{1/2}^1 (\check{F}_X(q) - MX) dq \\
 &= \int_0^{1/2} \check{F}_X(q) (1-\rho) dq + \int_{1/2}^1 \check{F}_X(q) (1+\rho) dq.
 \end{aligned}$$

Let  $\gamma$  denote the distribution function on the unit interval with density  $1-\rho$  on  $[0, 1/2[$  and  $1+\rho$  on  $[1/2, 1]$ , then

$$HX = \int_0^1 \check{F}_X(q) d\gamma(q).$$

Now we can prove the

**Theorem.** The absolute deviation principle has properties P1 through P4 and

**P5.**  $\check{F}_X \leq \check{F}_Y$  implies  $HX \leq HY$

**P6.**  $H(X+Y) \leq HX+HY$

**P7.**  $H$  is (Lipschitz-) continuous on  $L^1$ :

$$|HX - HY| \leq (1+\rho) \|X - Y\|_1.$$

In P5 the condition  $\check{F}_X \leq \check{F}_Y$  (to be formally correct here, take e.g. right continuous quantile functions) is equivalent to  $F_X \geq F_Y$  and this condition is often called first order stochastic dominance of  $Y$  over  $X$ . Hence P5 states compatibility of  $H$  with that stochastic order. P2 is the special case  $Y \equiv \sup X$ .

Property P6 states subadditivity of the functional  $H$ . In the next section we will give conditions under which additivity holds. In the general case a formula for the deviation  $H(X+Y) - (HX+HY)$  from additivity can be found in DENNEBERG 1985.

**Proof of the theorem.**

P1 is plain from the fact that  $\rho \geq 0$ ,  $\tau(X) \geq 0$ .

P2 is, as we noted already, a special case of P5.

P3.  $\check{F}_{X+c} = \check{F}_X + c$  and the assertion follows from the integral representation of  $H$ .

P4. For  $c \geq 0$  one has  $\check{F}_{cX} = c\check{F}_X$  (for negative  $c$  the right hand side would no longer be an increasing function).

P5 is an immediate consequence of integral calculus.

P6 derives from additivity of expectation and subadditivity of  $\tau$ .

$$\begin{aligned}
 \text{P7. } |HX - HY| &= \left| \int_0^1 (\check{F}_X(q) - \check{F}_Y(q)) d\gamma(q) \right| \leq \int_0^1 |\check{F}_X(q) - \check{F}_Y(q)| d\gamma(q) \\
 &\leq (1+\rho) \int_0^1 |\check{F}_X(q) - \check{F}_Y(q)| dq \leq (1+\rho) \|X - Y\|_1.
 \end{aligned}$$

The last inequality is stated and proved as a separate lemma.

**Lemma.** For  $X, Y \in L^1$  one has

$$\|\check{F}_X - \check{F}_Y\|_1 \leq \|X - Y\|_1,$$

where, on the left hand side, the norm refers to Lebesgue measure on  $[0, 1]$ .

**Proof.** Denote by  $X \vee Y$  the maximum and by  $X \wedge Y$  the minimum of the random variables  $X, Y$ . The inequalities

$$X \wedge Y \leq X, Y \leq X \vee Y$$

imply

$$\begin{aligned} \check{F}_{X \wedge Y} &\leq \check{F}_X, \check{F}_Y \leq \check{F}_{X \vee Y}, \\ |\check{F}_X - \check{F}_Y| &\leq \check{F}_{X \vee Y} - \check{F}_{X \wedge Y}. \end{aligned}$$

By integration we get

$$\|\check{F}_X - \check{F}_Y\|_1 \leq \int (\check{F}_{X \vee Y}(q) - \check{F}_{X \wedge Y}(q)) dq = E(X \vee Y - X \wedge Y) = E|X - Y| = \|X - Y\|_1.$$

### 3. COMONOTONICITY AND REINSURANCE

Here we tackle the question under what conditions on  $X$  and  $Y$  one has equality in P6, i.e. additivity of  $H$ . The condition is that  $X, Y$  are **comonotonic** random variables (a term introduced by SCHMEIDLER and YAARI), i.e. per definitionem that one of the following equivalent conditions hold:

- (i) (No risk compensation) For each  $\omega_0$  as point of reference the functions  $f := X - X(\omega_0)$  and  $g := Y - Y(\omega_0)$  don't have opposite signs, i.e.  $|f+g| = |f| + |g|$ .
- (ii)  $X = u(Z)$  and  $Y = v(Z)$  for some  $Z$  and (weakly) increasing functions  $u, v$ .
- (iii)  $X = u(X+Y)$  and  $Y = v(X+Y)$  with continuous, increasing functions  $u, v$  such that  $u(z) + v(z) = z, z \in \mathbb{R}$ .

These conditions and the proof of their equivalence (Satz 7 in DENNEBERG 1989) is valid for real functions  $X, Y$ , the distributions don't play any role. But distributions are essential in the following theorem (Satz 1 in DENNEBERG 1989):

**Theorem.** For comonotonic random variables  $X, Y$  the quantile functions behave additive,

$$\check{F}_{X+Y} = \check{F}_X + \check{F}_Y.$$

Applied to the absolute deviation principle  $H$  we get

$$\mathbf{P8.} \quad H(X+Y) = HX + HY \quad \text{for comonotonic} \quad X, Y \in L^1.$$

The proof of the theorem uses the fact that for increasing  $u$  one has  $\tilde{F}_{u(X)} = u \circ \tilde{F}_X$ . The proof is easy if all distribution functions and the functions  $u, v$  in (iii) are one to one.

Returning for a moment to the first section we, too, have

$$\tau(X+Y) = \tau(X) + \tau(Y) \quad \text{for comonotonic} \quad X, Y \in L^1.$$

Hence comonotonicity plays the same role for average absolute deviation  $\tau$  as independence plays for variance  $\sigma^2$ . But notice that independence and comonotonicity are opposite, mutual exclusive properties (except the case where  $X$  or  $Y$  is constant).

We give typical examples for comonotonic random variables.

**Example.**  $u(x) = x^+ := \max\{0, x\}$  and  $v(x) = -x^-$ , where  $x^- := (-x)^+$ , are continuous increasing functions and  $u(x) + v(x) = x$ . Hence, for a random variable  $X$ , the random variables  $X^+ = u(X)$  and  $-X^- = v(X)$  are comonotonic. If  $X$  has median  $MX = 0$  (this can be achieved by a translation) comonotonicity implies  $\tau(X) = \tau(X^+) + \tau(X^-)$ . This equation is known from Section 1 since  $\tau(X^+) = \tau_+(X)$  and  $\tau(-X^-) = \tau(X^-) = \tau_-(X)$  in case  $MX = 0$ .

**Example** (excess of loss or stop loss reinsurance). Let  $Z$  be total claims and  $a$  the priority or stop loss point. Define  $v(z) := (z-a)^+$ ,  $u(z) := z - v(z)$  and  $X := u(Z)$ ,  $Y := v(Z)$ . Then  $X$  is the part of total claims  $Z = X + Y$  to be covered by the primary insurer and  $Y$  the part to be covered by the reinsurer.  $X, Y$  being comonotonic  $H$  is compatible with this type of reinsurance,  $H[Z] = H[X] + H[Y]$ .

We know already from P4 that  $H$  is compatible, too, with proportional reinsurance. But we can state more. Condition (ii) or (iii) for comonotonicity in connection with P8 says that  $H$  is compatible with very general risk sharing schemes. One has only the restriction that both risk sharing partners have to bear (weakly) more if total claims are higher. There are forms of reinsurance of minor or lacking practical importance which injure this condition and which are not compatible with  $H$ . An example is largest claims reinsurance.

The essential properties of our new premium functional have been derived now, and before looking on possible generalisations, we will discuss the crucial properties: proportionality P4, subadditivity P6 and comonotonic additivity P8. First notice that P4 can be derived from P8 using P5 or norm continuity. We will compare P6 and P8 with independence additivity. For the discussion it is essential to specify the situation in which a premium functional is to be applied. We distinguish two situations.

If the market for insurance is in equilibrium in the sense that it offers no arbitrage opportunity, prices are additive at least for independent risks. Thus

premium functionals which are additive on independent risks, e.g. the variance

principle  $EX + a\sigma^2(X)$  and the exponential principle  $\frac{1}{a} \ln E e^{aX}$ , are candidates

for modeling market prices.

On the other hand, subadditive but not additive premium functionals as our absolute deviation principle or the standard deviation principle are apt to depict the law of large numbers. Hence they are applicable in portfolio decisions. Here reinsurance is an important mean, may it be to reduce the ratio of the portfolios volatility to the companies equity below a desired limit, or may it be to reduce volatility of the various companies portfolios through risk exchanges such that, eventually, the companies portfolios become proportional to the market portfolio. In such decisions comonotonic additivity P8 which—as pointed out above—applies to most risk sharing schemes, is very useful and can simplify decisions. Notice that the standard deviation principle is not comonotonic additive.

#### 4. GENERALISATIONS AND THE GINI PRINCIPLE

As the reader may have guessed already, the representation

$$HX = \int_0^1 \tilde{F}_X(q) d\gamma(q)$$

of the absolute deviation principle is capable of generalisation. One can replace the piecewise linear function  $\gamma$  by any distribution function on the unit interval  $[0, 1]$ . Such a function  $\gamma$  is called a **distortion** of probabilities. Condition P1 means that the graph of  $\gamma$  lies below the diagonal,  $\gamma(q) \leq q$ . P6 is valid if  $\gamma$  is convex and has bounded density. For P7 bounded density is needed, too, and the Lipschitz constant is the supremum  $\|\gamma'\|_\infty$  of the density  $\gamma'$  ( $\|\gamma'\|_\infty = 1 + \rho$  in case of the absolute deviation principle). All the other properties remain valid without further restrictions. In DENNEBERG 1989 (see also DENNEBERG 1990) these assertions are proved and the converse, too: any functional  $H$  on  $L^1$  with properties P1 through P8 can be represented by the above integral with a convex distribution function  $\gamma$  having bounded density.

Sometimes the absolute deviation principle may not be appropriate owing to the piecewise linearity of  $\gamma$ . For excess of loss or stop loss reinsurance the latter implies that the safety loading factor remains constant with rising priority or stop loss point, respectively. In practice one rather observes rising safety loading factors, too. Already the next simple distortion allows to model this phenomenon. For the absolute deviation principle the density can be written as

$$\gamma'(q) = 1 + \rho \operatorname{sgn} \left( q - \frac{1}{2} \right).$$



Replace the signum by the next elementary odd function, the identity:

$$\gamma'(q) = 1 + \rho \left( q - \frac{1}{2} \right).$$

The corresponding distortion is the quadratic polynomial  $\gamma(q) = q + \frac{1}{2}\rho(q^2 - q)$ , which is convex on  $[0, 1]$  for  $0 \leq \rho \leq 2$ , and the premium functional is

$$HX = EX + \rho \frac{1}{2} \text{Gini } X,$$

where

$$\text{Gini } X := \int_0^1 \tilde{F}(q) dq^2 - EX = EX \text{ gini } X$$

and *gini*  $X$  is the (normed) **Gini coefficient**, which is used in economic welfare theory as an inequality measure for wealth distributions in populations. The usual definition for the Gini coefficient is twice the area between the diagonal and the **Lorenz function**

$$l(q) := \frac{1}{EX} \int_0^q \tilde{F}(p) dp,$$

$$\text{gini } X = 2 \int_0^1 (q - l(q)) dq.$$

The equivalence to the above formula is calculated easily with Fubini's theorem. Another representation of the Gini coefficient is

$$\text{Gini } X = \frac{1}{2} \|X - Y\|_1$$

where  $Y$  is a random variable such that  $X, Y$  are independent and identically distributed (see ZAGIER). This new premium functional could be called the **Gini principle**.

The above general premium functional can further be generalised. First the basic probability measure  $P$  or the distorted  $\gamma \circ P$  can be replaced by more general set functions. Second—as in expected utility—the claims in money terms can be valued by a non linear utility function. Thus the proportionality property P4 could be weakened. Functionals of this type and their axiomatic representations are investigated in economic decision theory (see e.g. WAKKER, where the literature is discussed, too).

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## WORKSHOP

### ON CHANGING THE PARAMETER OF EXPONENTIAL SMOOTHING IN EXPERIENCE RATING

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#### ABSTRACT

We consider exponential smoothing  $Y_n = \alpha X_n + (1 - \alpha)Y_{n-1}$ ,  $0 < \alpha < 1$ , in experience rating. Here the premium  $Y_n$  is determined by the policy's own claims history ( $X_n$ ). In order to uniformize the fluctuation of premiums, it is appropriate to use a bigger  $\alpha$  for the big policies than for the small ones. When the size of the policy changes with time, a need arises to change  $\alpha$  correspondingly. It has recently been shown that changing based on the size of the premiums  $Y_n$  may lead to too low a tariff level. This result is presented here and illustrated by means of simulation. Further, some general results are given how the changing can be made without a decline in the tariff level. The results are applied to a tariff system in which the linking of the smoothing parameter to the size of the policy is particularly motivated.

#### KEYWORDS

Exponential smoothing; adaptive smoothing; experience rating; Markov chains; big claims; workers' compensation insurance.

#### 1. INTRODUCTION

In this paper we consider certain questions related to applications of exponential smoothing to premium rating. Especially, we deal with problems concerning changing the smoothing parameter.

Let  $(X_n, n \geq 1)$  be a sequence of random variables. Exponential smoothing ( $Y_n$ ) of the sequence  $(X_n)$  is defined by

$$(1.1) \quad Y_n = \alpha X_n + (1 - \alpha)Y_{n-1},$$

where  $\alpha$  is a constant,  $0 < \alpha < 1$ .

Exponential smoothing is often used for defining the premiums  $Y_n$  on the basis of the policy's own claims history ( $X_n$ ) (cf. e.g. [5] and [6]). If the random variables  $X_n$  have a common expectation, then the premiums  $Y_n$  are (asymptotically) unbiased. With suitable additional assumptions exponential smoothing leads to a strong correspondence between cumulative premiums and claims of an individual policy. E.g. if the random variables  $X_n$  are uncorrelated and have a common variance, then the variance of the cumulated difference of claims and premiums remains bounded (cf. [2], p. 288).

See e.g. [3], [4] and [7] for the connections of exponential smoothing with credibility theory.

Throughout this paper,  $X_n$  is interpreted as the total claim amount at year  $n$ , or some modification of it, of a given policy.

We consider policies consisting of distinct similar risks the number of which essentially varies from one policy to another. An example is served by the workers' compensation insurance, where the total risk consists of the risks related to the workers of the policyholder.

The bigger the smoothing parameter  $\alpha$  is, the more closely the premiums follow the fluctuation of the claims history. For policies like the above, the fluctuation of the claims history is relatively bigger for small policies than for big ones, where the size of the policy is measured by the (risk) premium. As a consequence, if the same smoothing constant  $\alpha$  were used for all policies, also the premiums of the small policies would fluctuate more than those of the big ones. Therefore, it is natural to use a smaller  $\alpha$  for the small policies than for the big ones, i.e. to define  $\alpha$  as a monotonically increasing function of the premium.

When the size of the policy changes with time, a need arises to vary  $\alpha$ , as above, monotonically increasingly with respect to the premium. In a recent paper [1], it is shown, in case  $X_n$  i.i.d., that this kind of varying procedure leads to too low a premium level, cf. (1.3) below. This result is based on the theory of Markov chains on a general state space.

The above paper considers the smoothing procedure

$$(1.2) \quad Z_n = \beta(Z_{n-1})X_n + (1 - \beta(Z_{n-1}))Z_{n-1},$$

where  $\beta$  is a monotonically increasing function  $[0, \infty) \rightarrow [c, d]$ ,  $0 < c < d < 1$ . It follows from Theorem 4 of [1] that  $EY_n$ ,  $EZ_n$  converge with a geometric convergence rate to their limits  $E_Y$ ,  $E_Z$  and that

$$(1.3) \quad E_Z < E_Y = EX_n.$$

This result is based on Theorem 3 of [1] which states that  $(Y_n)$  and  $(Z_n)$  are geometrically ergodic Markov chains (see [1] for definition). The limits  $E_Y$ ,  $E_Z$  are the means of the invariant probability distributions of the Markov chains concerned. These results are presented in Section 2 in greater detail.

We are concerned in this paper with a reduced model whose stability properties can be studied. Our purpose is to call attention to the phenomenon where the asymptotic value is below the value expected. We believe that this kind of phenomenon may occur also in more general related models.

In Section 3 the result (1.3) is illustrated by means of simulation.

Sections 4 and 5 deal with how the changing of the smoothing parameter, in certain cases, can be controlled without the above-mentioned descending of the tariff level. In Section 4 we present some general results and in Section 5 these are applied to a tariff system, in which linking the smoothing parameter to the size of the policy is motivated, not only because of the uniformizing of the fluctuation of premiums as such, but also because of the equitability of the policies.

2. VARYING THE SMOOTHING PARAMETER ON THE BASIS OF THE PREMIUMS

In the following we present results related to varying the smoothing parameter as a monotonically increasing function of the premium. These results are from [1]. For unexplained concepts the reader is referred to that paper.

Let  $X_n, n \geq 1$  be i.i.d. random variables taking non-negative values. Assume  $EX_n < \infty$  and that  $X_n$  has a continuous density function  $g > 0$  on  $(0, \infty)$  (we allow  $P(X_n = 0) > 0$ ).

Let  $Y_o, Z_o$  be random variables taking non-negative values and let iteratively for  $n \geq 1$

$$Y_n = \alpha X_n + (1 - \alpha) Y_{n-1},$$

$$Z_n = \beta(Z_{n-1}) X_n + (1 - \beta(Z_{n-1})) Z_{n-1},$$

where  $\alpha \in (0, 1)$  and  $\beta: [0, \infty) \rightarrow [c, d]$  is monotonically increasing,  $0 < c < d < 1$ .

Here  $(Y_n)$  and  $(Z_n)$  are interpreted as (alternative) sequences of premiums determined by the annual claims amounts  $(X_n)$  of a given policy.

With the above assumptions we have the following results.

**Theorem 2.1.** The sequences  $(Y_n)$  and  $(Z_n)$  are geometrically ergodic Markov chains (with state space  $S = [0, \infty)$ ).

It follows from Theorem 2.1 that the chains  $(Y_n), (Z_n)$  possess unique invariant probability measures  $\pi_Y, \pi_Z$  with corresponding means  $E_Y, E_Z$ , respectively.

**Theorem 2.2.**

(i) There exist functions  $C_Y, C_Z < \infty$  and constants  $\rho_Y, \rho_Z \in (0, 1)$  such that

(2.1)  $|EY_n - E_Y| \leq C_Y(y_o) \rho_Y^n,$

(2.2)  $|EZ_n - E_Z| \leq C_Z(z_o) \rho_Z^n$

for all  $n$ , for all initial states  $y_o$  and for almost all initial states  $z_o$  (with respect to the Lebesgue measure).

(ii)  $E_Z < E_Y = EX_n$ .

If the random variables  $X_n$  and the initial values  $Y_o, Z_o$  take their values in a finite interval  $[0, M]$ , the above result can be sharpened. In this case, under the same assumptions as above (in particular, assuming that  $g$  is continuous and positive on  $(0, M)$  and allowing  $P(X_n = 0) > 0, P(X_n = M) > 0$ ) the result (i) of Theorem 2.2 can be replaced by the following one:

(i') There exist constants  $C_Y, C_Z > 0$  and  $\rho_Y, \rho_Z \in (0, 1)$  such that

(2.3)  $|EY_n - E_Y| \leq C_Y \rho_Y^n,$

$$(2.4) \quad |EZ_n - E_Z| \leq C_Z \rho_Z^n$$

for all  $n$  and for all initial states  $y_0, z_0 \in [0, M]$ .

In this case the chains  $(Y_n)$  and  $(Z_n)$  are uniformly ergodic on the state space  $S = [0, M]$ .

### 3. A SIMULATION EXAMPLE

The purpose of the simulation is, in a simple case, to give a picture of how much smaller  $E_Z$  is than  $E_Y$ , cf. (1.3). One can broaden the picture by examining other cases and using also numerical methods in addition to simulation.

The computing was carried out by an IBM Personal Computer XT. The simulation was arranged as follows. The variables  $X_n, n \geq 1$  were taken independent and uniformly distributed over  $[0, 1]$ . We fixed

$$Y_0 = Z_0 = 0.5, \\ \alpha = 0.5.$$

As  $\beta$  we used

$$\beta(z) = 0.2 + 0.6z, \quad 0 \leq z \leq 1 \quad (\text{thus } c = 0.2 \text{ and } d = 0.8).$$

Using the constants and the function  $\beta$  above, we calculated  $Y_n, Z_n, n = 10, 50$ .

We ran the simulation 2000 times and calculated the sample means and 99%-confidence intervals (using normal approximation) for the variables  $Y_n, Z_n$  and for completeness also for  $X_n, n = 10, 50$ .

These quantities are presented in the table below. One can see that  $Z$  is considerably smaller than  $Y$  already at time  $n = 10$ .

TABLE 3.1

	Sample mean	Confidence interval	
$n = 10$			
$X$	0.5067	0.4902	0.5232
$Y$	0.5035	0.4939	0.5131
$Z$	0.4696	0.4601	0.4791
$n = 50$			
$X$	0.5077	0.4910	0.5243
$Y$	0.5042	0.4947	0.5138
$Z$	0.4689	0.4594	0.4783

The sample mean and its confidence interval were calculated for  $Z_{10}$  as follows, and similarly for the other variables:

$$\text{sample mean } \bar{Z}_{10} = \frac{1}{2000} \sum_{k=1}^{2000} Z_{10,k},$$

where  $Z_{10,k}$  is the value of  $Z_{10}$  at the  $k$ 'th run,

$$\text{confidence interval} = \bar{Z}_{10} \pm 2.58 \frac{S_{Z,10}}{\sqrt{2000}},$$

$$\text{where } S_{Z,10} = \left[ \frac{1}{2000} \sum_{k=1}^{2000} (Z_{10,k} - \bar{Z}_{10})^2 \right]^{1/2}.$$

4. VARYING THE SMOOTHING PARAMETER WITHOUT CORRELATION WITH THE CLAIMS HISTORY

In the following, we consider certain conditions under which the varying of the smoothing parameter can be done without a decline in the premium level (cf. Theorem 2.2). First, we show that this works if the varying procedure is such that the smoothing parameters do not correlate with the claims history, see Theorem 4.1 below. We then apply Theorem 4.1 to a case in which the premium  $P_n$  is presented in the form

$$(4.1) \quad P_n = V_n p_n,$$

where  $V$  is a volume measure of the policy and  $p$  a measure of risk per volume unit derived from the claims history of the policy. The volume measure  $V$  can be e.g. the number of similar subrisks. It turns out that, under certain assumptions, the varying procedure of the smoothing parameter can be based on  $V$  so that the premiums are unbiased, see Corollary 4.2 below.

**Theorem 4.1.** Let  $(X_n)$  be a sequence of random variables taking non-negative values, with  $EX_n = a < \infty$  for all  $n$ . Let  $Z_o > 0$  and let

$$Z_n = \alpha_n X_n + (1 - \alpha_n) Z_{n-1}, \quad \text{for } n \geq 1,$$

where the  $\alpha_n$ 's are random variables satisfying the conditions

$$(4.2) \quad 0 < c \leq \alpha_n \leq d < 1$$

and

$$(4.3) \quad \begin{aligned} E((1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_{k+1}) \alpha_k X_k) = \\ E((1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_{k+1}) \alpha_k) EX_k \end{aligned}$$

for all  $k, n; k \leq n$ .

Then

$$(4.4) \quad |EZ_n - a| \leq |Z_o - a|(1 - c)^n \quad \text{for all } n.$$

**Proof.** We have

$$\begin{aligned} Z_n &= \alpha_n X_n + (1 - \alpha_n) \alpha_{n-1} X_{n-1} + (1 - \alpha_n) (1 - \alpha_{n-1}) \alpha_{n-2} X_{n-2} \\ &\quad + \cdots + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_2) \alpha_1 X_1 \\ &\quad + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_1) [a + (Z_o - a)]. \end{aligned}$$

Using (4.3) we get

$$\begin{aligned} EZ_n &= aE(\alpha_n + (1 - \alpha_n) \alpha_{n-1} + \cdots + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_2) \alpha_1 \\ &\quad + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_1)) \\ &\quad + (Z_o - a) E((1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_1)). \end{aligned}$$

It is easy to see that

$$\begin{aligned} \alpha_n + (1 - \alpha_n) \alpha_{n-1} + \cdots + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_2) \alpha_1 \\ + (1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_1) = 1 \quad \text{for all } n. \end{aligned}$$

Thus

$$(4.5) \quad EZ_n - a = (Z_o - a) E((1 - \alpha_n) (1 - \alpha_{n-1}) \cdots (1 - \alpha_1)).$$

Since  $(1 - \alpha_k) \leq 1 - c$  for all  $k$ , the assertion follows from (4.5).

Note that condition (4.3) does not hold true for  $(Z_n)$  in case of Theorems 2.1 and 2.2. We apply Theorem 4.1 to a claims process satisfying the following assumption.

**Assumption (4.6).** Let a risk at year  $n$  consist of  $N_n$  subrisks  $\eta_{n,1}, \dots, \eta_{n,N_n}$  for which

$$(1^\circ) \quad E_{\eta_{n,k}} = a < \infty \quad \text{for all } n, k.$$

The numbers of subrisks  $N_n$  are here random variables,  $EN_n < \infty$  for all  $n$ . We assume that the variables  $\eta_{n,k}$  are independent of the process  $(N_n)$ . (We allow mutual correlation of the variables  $\eta_{n,k}$  as well as that of the variables  $N_n$ .)

Denote  $\xi_n = \sum_{i=1}^{N_n} \eta_{n,i}$  and  $X_n = \xi_n / N_n$ .

Under assumption (4.6) and using the notation above, we have the following corollary of Theorem 4.1.

**Corollary 4.2.** Let  $\alpha$  be a function of the number of the subrisks  $N$  satisfying  $0 < c \leq \alpha(N) \leq d < 1$ . Then

(i) relation (4.4) holds true for the process

$$(4.7) \quad Z_n = \alpha_n X_n + (1 - \alpha_n) Z_{n-1}, \quad n \geq 1,$$

where  $\alpha_n = \alpha(N_{n-1})$ ,

(ii) if  $P_n = Z_n N_n$ , then



$$(4.8) \quad \frac{|EP_n - E\xi_n|}{EN_n} \leq |a - z_o| (1 - c)^n,$$

where  $z_o$  is the initial value of the process (4.7).

**Proof.**

(i) Clearly,  $EX_n = a$  for all  $n$ . We have to verify condition (4.3). Denote  $(1 - \alpha_n) \cdots (1 - \alpha_{k+1})\alpha_k = \alpha(k, n)$ . Since the variables  $\eta_{k,i}$  are independent of the process  $(N_n)$ , they are independent of  $\alpha(k, n)$ . Hence

$$(4.9) \quad E(\alpha(k, n)\eta_{k,i}) = E\alpha(k, n)E\eta_{k,i} = aE\alpha(k, n)$$

for all  $k, n, i; k \leq n, i \leq N_k$ . Condition (4.3) then follows from (4.9).

(ii) By the independence assumption of (4.6),  $E\xi_n = aEN_n$ . By a similar reasoning as in the proof of Theorem 4.1 and in that of item (i) above, we see that  $EP_n = aEN_n + (z_o - a)E((1 - \alpha_n) \cdots (1 - \alpha_1)N_n)$ . Accordingly, the assertion follows from condition  $c \leq \alpha_n$ .

Consequently, in this case where the varying of the smoothing parameter is based on the volume measure  $N_n$  instead of the premiums, the premiums keep the right level.

Note that the variables  $X_n$  in assumption (4.6) need not be independent of the parameters  $\alpha_n$ . If, for instance,  $\alpha$  is a monotonically increasing function of the number of the subrisks  $N$ , then in case  $\alpha$  is big (small) then also  $N$  is big (small) and  $X$  is more (less) concentrated. Note also that the variables  $\eta_{n,i}, i = 1, \dots, N_n$  are allowed to be mutually correlated. This fact has significance in the application presented in Section 5.

Condition 1° in assumption (4.6) can be weakened. For example, it is easy to see that Collorary 4.2 remains valid if condition (1°) is replaced by the following one:

(2°) The portfolio is composed of classes  $C^j, j = 1, \dots, m$  such that each class consists of  $N_n^j$  subrisks  $\eta_{n,1}^j, \dots, \eta_{n,N_n^j}^j$  for which

$$E\eta_{n,k}^j = a_j < \infty \quad \text{for all } n, k,$$

and that the ratios  $r_j = N_n^j/N_n$  are constants. (In this case  $a = \sum_j r_j a_j$ .)

In workers' compensation insurance the subclasses  $C^j$  can be interpreted as different occupational groups.

5. A TARIFF SYSTEM IN WORKERS' COMPENSATION INSURANCE

The following presentation is founded on a Finnish tariff system for workers' compensation insurance. The tariff (in the following FT) is intended for medium-sized and bigger employers and is applied since 1983. The tariff aims

at a relatively strong correspondence between claims and premiums of individual policies. The premium is mainly determined by smoothing the policy's own claims history. However, in order to decrease the effects of large claims amounts, also a collective part of the premium is charged.

For each policy, the premium  $P_n$  of year  $n$  is presented in the form (cf. 4.1)

$$(5.1) \quad P_n = V_n p_n.$$

Here  $V_n$  is the payroll of the policyholder at year  $n$ . The risk per wage monetary unit measure  $p_n$  will be defined later. Let

$$X_n = \xi_n / V_n,$$

where  $\xi_n$  is the policy's total claims amount at year  $n$ . The purpose of the tariff is to smooth the possibly strongly fluctuating time series ( $X_n$ ) to a less fluctuating sequence of coefficients ( $p_n$ ).

We restrict, in this context, the presentation of the application of the tariff to the case where there is not any significant trend in ( $X_n$ ) and the risk structure of the policy does not essentially change. (Note that the indemnifications are, for the most part, tied to the wage-index and follow this index closely. As a consequence, a trend in ( $X_n$ ), caused by inflation, can be regarded as negligible.) In case of a trend or structural change the tariff will not be straightforwardly applied. (See e.g. [2] for controlling this type of situations.)

The time series ( $X_n$ ) is smoothed, first, by taking a moving average

$$(5.2) \quad X_n^* = \sum_{k=1}^m c_k X_{n-k},$$

where  $c_i \geq 0$ ,  $\sum c_i = 1$ ,  $m < \infty$ . (In FT  $c_1 = 0.5$ ,  $c_2 = 0.3$ ,  $c_3 = 0.2$ .)

Then exponential smoothing is applied to ( $X_n^*$ ) resulting in the sequence ( $Z_n$ ), defined as follows

$$(5.3) \quad \begin{aligned} Z_0 &= z_0, \\ Z_n &= \alpha_n X_n^* + (1 - \alpha_n) Z_{n-1}, \quad \text{when } n \geq 1, \end{aligned}$$

where  $z_0$  is an initial value and  $\alpha_n$ 's are the smoothing parameters. (A motivation for the double smoothing is that the change in the premiums, caused by a large claims amount, has a flatter shape than what would be the case if only exponential smoothing were applied.) We will revert to the definition of  $\alpha_n$ 's later. The varying procedure of the smoothing parameters will be such that ( $Z_n$ ) can be considered to represent a correct tariff level (cf. Section 4 and the discussion at the end of this section).

The smoothing procedure (5.3) is such that, as an effect of a big claims amount in some year, ( $Z_n$ ) can increase remarkably. Consequently, ( $Z_n$ ) is not suitable for ( $p_n$ ), since an even development of the premiums is desirable from the policyholder's point of view. For this reason, the increase of the premiums is reduced by introducing the variables  $q_n$ , defined as follows

$$(5.4) \quad \begin{aligned} q_0 &= z_0, \\ q_n &= \min(Z_n, hq_{n-1}), \quad \text{for } n \geq 1, \end{aligned}$$

where  $h > 1$  is a coefficient limiting the increase of the process  $(q_n)$ . (In FT  $h = 1.5$ .) See Figure 5.1.

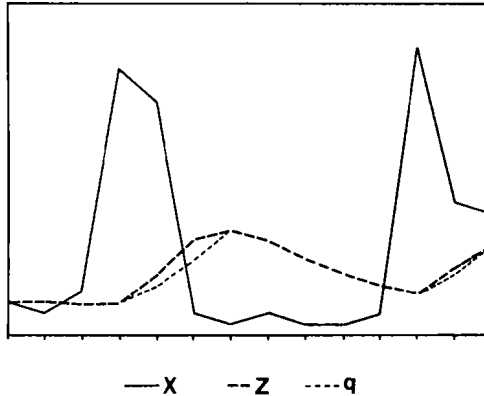


FIGURE 5.1.

Clearly,  $q_n \leq Z_n$ . As mentioned above,  $(Z_n)$  can be considered to represent a correct tariff level. Due to this, all policies are charged a collective part of the premium corresponding to the expected difference  $Z_n - q_n$  over the whole insurance portfolio. The coefficient  $p_n$  (cf. 5.1) is defined as follows

$$p_n = (1 + r_n) q_n,$$

where the coefficient  $r_n$  is common for all policies and  $r_n q_n V_n$  is the collective part of the premium of the individual policy in question. (In FT  $r_n$  has been 0.02.)

We turn to the definition of the smoothing parameter  $\alpha$ . Since the policies consist of several similar risks the number of risks varying from one policy to another, the fluctuation of the claims process is steeper for small policies than for bigger ones. This steeper fluctuation would be transmitted also to  $(Z_n)$  if the same smoothing constant were applied for all policies. In this case  $(q_n)$  of the small policies would differ more from the correct tariff level  $(Z_n)$  than  $(q_n)$  of the big policies. As a consequence, the small policies would line their pockets at expense of the big ones. For this reason, and also for harmonizing the fluctuation of the premiums, it is reasonable to use a smaller  $\alpha$ -coefficient for small policies than for big ones.

In the following, we first consider the choosing of  $\alpha$  when a policy starts in the tariff system considered. The insurance portfolio is divided, on the basis of the size of the premiums, into classes  $C_i, i = 1, \dots, l$ , where the classes are in increasing order according to the premium size. The limits of the classes are adjusted yearly on the basis of the wage-index. For the smoothing parameter of

the "medium-sized" policies has been chosen  $\alpha = 0.2$ . The purpose of this choice is that the fluctuation of the premiums would be of a suitable magnitude from the policyholders point of view. On the basis of the insurer's statistics, a parameter  $\alpha^i$  has been associated with every class  $C_i$  so that the gain caused by the truncation (5.4) would be approximately of equal size for the different classes. This has led to an increasing sequence of parameters  $\alpha^i$ ,  $0.1 \leq \alpha^i \leq 0.28$ ,  $i = 1, \dots, l$ . A new policy starts in the tariff system with the smoothing parameter  $\alpha^i$  defined by its initial premium.

The changing of  $\alpha$  is carried out on the basis of the payroll  $V_n$  of the policyholder (cf. 5.1). Every fifth year the size of the policy, measured by the payroll proportioned to the wage-index, is checked, and if the size has changed essentially, the parameter  $\alpha$  is changed correspondingly.

The changing procedure is in accordance with assumption (4.6) (with  $2^\circ$ ). Note first that it follows from assumption (4.6) that condition (4.3) holds true even if  $X_k$  in (4.3) is replaced by  $X_{k-1}$ ,  $X_{k-2}$  or  $X_{k-3}$ . As a consequence, Corollary 4.2 remains valid if  $X_n^*$  (see 5.2) is substituted for  $X_n$  in (4.7). Further, it is reasonable to assume that the risks  $\eta_{n,k}$  associated with single wage monetary units  $k$  are independent of the size of the total payrolls. In addition, the straightforward use of the tariff is restricted to the case where there is not a significant trend in  $(X_n)$  and the risk structure is stable. Note also that assumption (4.6) allows mutual correlations of the variables  $\eta_{n,k}$ .

Accordingly, by Corollary 4.2 ( $Z_n$ ), and hence  $(p_n)$ , can be considered to represent a correct tariff level.

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