Commun. Korean Math. Soc. **18** (2003), No. 3, pp. 521–532

PREOPEN SETS IN SMOOTH BITOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduce the notions of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets and fuzzy pairwise (r, s)-precontinuous mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [13] in his classical paper. Using the concept of fuzzy sets Chang [2] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Chattopadhyay et al. [4] and Ramadan [9] introduced new definition of smooth topological spaces as a generalization of fuzzy topological spaces. Kandil [6] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Lee et al. [7] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil's fuzzy bitopological spaces.

In this paper, we introduce the concepts of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets and fuzzy pairwise (r, s)-precontinuous mappings in smooth bitopological spaces and then we investigate some of their characteristic properties.

2. Preliminaries

In this paper, I will denote the unit interval $[0, 1]$ of the real line and

Received February 18, 2002.

2000 Mathematics Subject Classification: 54A40.

Key words and phrases: smooth bitopological spaces, preopen sets, pairwise precontinuous mappings.

$I_0 = (0, 1]$. For a set X, I^X denotes the collection of all mappings from X to I. A member μ of I^X is called a fuzzy set of X. By $\tilde{0}$ and $\tilde{1}$ we denote constant mappings on X with value 0 and 1, respectively. For any $\mu \in I^X$, μ^c denotes the complement $\tilde{1} - \mu$. All other notations are the standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X [2] is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.
(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.
(3) If $\mu_k \in T$ for each k, then $\bigvee \mu_k \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.* Members of T are called T-fuzzy open sets of X and their complements T-fuzzy closed sets of X.

A system (X, T_1, T_2) consisting of a set X with two Chang's fuzzy topologies T_1 and T_2 on X is called a *Kandil's fuzzy bitopological space.*

A *smooth topology* on X [4, 9] is a mapping $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.
(3) $\mathcal{T}(\bigvee \mu_k) \geq \bigwedge \mathcal{T}(\mu_k)$.

The pair $(X, \mathcal{T})$ is called a *smooth topological space.* For $r \in I_0$, we call μ a $\mathcal{T}$-fuzzy r-open set of X if $\mathcal{T}(\mu) \geq r$ and μ a $\mathcal{T}$-fuzzy r-closed set of X if $\mathcal{T}(\mu^c) \geq r$.

A system $(X, \mathcal{T}_1, \mathcal{T}_2)$ consisting of a set X with two smooth topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is called a *smooth bitopological space.* Throughout this paper the indices i, j take values in $\{1, 2\}$ and $i \neq j$.

Let $(X, \mathcal{T})$ be a smooth topological space. For each $r \in I_0$, an r-cut

$$\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$$

is a Chang's fuzzy topology on X.

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Then a smooth topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we obtain that if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopological space and $r, s \in I_0$, then $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$ is a Kandil's fuzzy bitopological space. Also, if (X, T_1, T_2) is a Kandil's fuzzy bitopological space and $r, s \in I_0$, then $(X, (T_1)^r, (T_2)^s)$ is a smooth bitopological space.

DEFINITION 2.1 ([8]). Let $(X, \mathcal{T})$ be a smooth topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathcal{T}\text{-Cl}(\mu, r) = \bigwedge\{\rho \mid \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}$$

and the *fuzzy r-interior*

$$\mathcal{T}\text{-Int}(\mu, r) = \bigvee\{\rho \mid \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

THEOREM 2.2 ([8]). *For a fuzzy set μ of a smooth topological space $(X, \mathcal{T})$ and $r \in I_0$, we have:*

(1) $\mathcal{T}\text{-Int}(\mu, r)^c = \mathcal{T}\text{-Cl}(\mu^c, r)$.
(2) $\mathcal{T}\text{-Cl}(\mu, r)^c = \mathcal{T}\text{-Int}(\mu^c, r)$.

DEFINITION 2.3 ([7]). Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then μ is said to be

(1) a $(\mathcal{T}_i, \mathcal{T}_j)$*-fuzzy* (r, s)*-semiopen* set if there is a $\mathcal{T}_i$-fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathcal{T}_j\text{-Cl}(\rho, s)$,
(2) a $(\mathcal{T}_i, \mathcal{T}_j)$*-fuzzy* (r, s)*-semiclosed* set if there is a $\mathcal{T}_i$-fuzzy r-closed set ρ in X such that $\mathcal{T}_j\text{-Int}(\rho, s) \leq \mu \leq \rho$.

DEFINITION 2.4 ([7]). Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to another smooth bitopological space Y and $r, s \in I_0$. Then f is called a fuzzy pairwise (r, s)-continuous ((r, s)-open and (r, s)-closed, respectively) mapping if the induced mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is a fuzzy r-continuous (r-open and r-closed, respectively) mapping and the induced mapping $f : (X, \mathcal{T}_2) \to (Y, \mathcal{U}_2)$ is a fuzzy s-continuous (s-open and s-closed, respectively) mapping.

3. $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets

DEFINITION 3.1. Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then μ is said to be

(1) a $(\mathcal{T}_i, \mathcal{T}_j)$*-fuzzy* (r, s)*-preopen* set if $\mu \leq \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu, s), r)$,
(2) a $(\mathcal{T}_i, \mathcal{T}_j)$*-fuzzy* (r, s)*-preclosed* set if $\mathcal{T}_i\text{-Cl}(\mathcal{T}_j\text{-Int}(\mu, s), r) \leq \mu$.

THEOREM 3.2. *Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$. Then the following statements are equivalent:*

(1) *μ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen set.*
(2) *μ^c is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preclosed set.*

PROOF. It follows from Theorem 2.2. □

THEOREM 3.3. (1) *Any union of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen set.*

(2) *Any intersection of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preclosed sets is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preclosed set.*

PROOF. (1) Let $\{\mu_k\}$ be a collection of $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets. Then for each k, $\mu_k \leq \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu_k, s), r)$. So

$$\bigvee \mu_k \leq \bigvee \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\mu_k, s), r) \leq \mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(\bigvee \mu_k, s), r).$$

Thus $\bigvee \mu_k$ is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen set.

(2) It follows from (1) using Theorem 3.2. □

THEOREM 3.4. *Let μ be a fuzzy set of a smooth bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $r, s \in I_0$.*

(1) *If μ is a $\mathcal{T}_1$-fuzzy r-open set of $(X, \mathcal{T}_1)$, then μ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set of $(X, \mathcal{T}_1, \mathcal{T}_2)$.*
(2) *If μ is a $\mathcal{T}_2$-fuzzy s-open set of $(X, \mathcal{T}_2)$, then μ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preopen set of $(X, \mathcal{T}_1, \mathcal{T}_2)$.*

PROOF. (1) Let μ be a $\mathcal{T}_1$-fuzzy r-open set of $(X, \mathcal{T}_1)$. Then $\mu = \mathcal{T}_1\text{-Int}(\mu, r)$. Clearly, we have $\mu \leq \mathcal{T}_2\text{-Cl}(\mu, s)$ and hence

$$\mu = \mathcal{T}_1\text{-Int}(\mu, r) \leq \mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mu, s), r).$$

Thus μ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set of $(X, \mathcal{T}_1, \mathcal{T}_2)$.

(2) Similar to (1). □

But the converses in the above theorem need not be true which is shown by the following example.

EXAMPLE 3.5. Let $X = \{x, y\}$ and μ_1, μ_2, μ_3 and μ_4 be fuzzy sets of X defined as

$$\mu_1(x) = 0.2, \quad \mu_1(y) = 0.4;$$

$$\mu_2(x) = 0.6, \quad \mu_2(y) = 0.3;$$

$$\mu_3(x) = 0.2, \quad \mu_3(y) = 0.3;$$

and

$$\mu_4(x) = 0.5, \quad \mu_4(y) = 0.2.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $(\mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopology on X. Note that

$$\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(\mu_3, \tfrac{1}{3}), \tfrac{1}{2}) = \mathcal{T}_1\text{-Int}(\mu_2^c, \tfrac{1}{2}) = \mu_1 \geq \mu_3.$$

Thus μ_3 is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-preopen set which is not a $\mathcal{T}_1$-fuzzy $\frac{1}{2}$-open set. Also we have

$$\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(\mu_4, \tfrac{1}{2}), \tfrac{1}{3}) = \mathcal{T}_2\text{-Int}(\mu_1^c, \tfrac{1}{3}) = \mu_2 \geq \mu_4.$$

Hence μ_4 is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-preopen set which is not a $\mathcal{T}_2$-fuzzy $\frac{1}{3}$-open set.

REMARK 3.6. That $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-semiopen sets and $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy (r, s)-preopen sets are independent notions is shown by the following example.

EXAMPLE 3.7. Let $X = \{x, y\}$ and μ_1, μ_2, μ_3, μ_4, μ_5 and μ_6 be fuzzy sets of X defined as

$$\mu_1(x) = 0.1, \quad \mu_1(y) = 0.7;$$

$$\mu_2(x) = 0.8, \quad \mu_2(y) = 0.2;$$

$$\mu_3(x) = 0, \quad \mu_3(y) = 0.6;$$

$$\mu_4(x) = 0.1, \quad \mu_4(y) = 0.8;$$

$$\mu_5(x) = 0.5, \quad \mu_5(y) = 0.6;$$

and

$$\mu_6(x) = 0.9, \quad \mu_6(y) = 0.2.$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{3} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $(\mathcal{T}_1, \mathcal{T}_2)$ is a smooth bitopology on X. The fuzzy set μ_3 is $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-preopen which is not $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen and μ_4 is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-semiopen set which is not a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy $(\frac{1}{2}, \frac{1}{3})$-preopen set. Also μ_5 is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-preopen set which is not a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-semiopen set and μ_6 is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-semiopen set which is not a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy $(\frac{1}{3}, \frac{1}{2})$-preopen set.

4. Fuzzy pairwise (r, s)-precontinuous mappings

DEFINITION 4.1. Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping from a smooth bitopological space X to another smooth bitopological space Y and $r, s \in I_0$. Then f is called

(1) a *fuzzy pairwise (r, s)-precontinuous* mapping if $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set of X for each $\mathcal{U}_1$-fuzzy r-open

set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s,r)-preopen set of X for each $\mathcal{U}_2$-fuzzy s-open set ν of Y,

(2) a *fuzzy pairwise* (r,s)*-preopen* mapping if $f(\rho)$ is a $(\mathcal{U}_1,\mathcal{U}_2)$-fuzzy (r,s)-preopen set of Y for each $\mathcal{T}_1$-fuzzy r-open set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2,\mathcal{U}_1)$-fuzzy (s,r)-preopen set of Y for each $\mathcal{T}_2$-fuzzy s-open set λ of X,

(3) a *fuzzy pairwise* (r,s)*-preclosed* mapping if $f(\rho)$ is a $(\mathcal{U}_1,\mathcal{U}_2)$-fuzzy (r,s)-preclosed set of Y for each $\mathcal{T}_1$-fuzzy r-closed set ρ of X and $f(\lambda)$ is a $(\mathcal{U}_2,\mathcal{U}_1)$-fuzzy (s,r)-preclosed set of Y for each $\mathcal{T}_2$-fuzzy s-closed set λ of X.

REMARK 4.2. It is obvious that every fuzzy pairwise (r,s)-continuous mapping is also a fuzzy pairwise (r,s)-precontinuous mapping. But the converse need not be true which is shown by the following example.

EXAMPLE 4.3. (1) *A fuzzy pairwise* (r,s)*-precontinuous mapping need not be a fuzzy pairwise* (r,s)*-continuous mapping.*

Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be a smooth bitopological space as described in Example 3.5. Define $\mathcal{U}_1 : I^X \to I$ and $\mathcal{U}_2 : I^X \to I$ by

$$\mathcal{U}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{U}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $(\mathcal{U}_1,\mathcal{U}_2)$ is a smooth bitopology on X. Consider the mapping $f : (X,\mathcal{T}_1,\mathcal{T}_2) \to (X,\mathcal{U}_1,\mathcal{U}_2)$ defined by $f(x) = x$ and $f(y) = y$. Then f is a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-precontinuous mapping which is not a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-continuous mapping.

(2) *A fuzzy pairwise* (r,s)*-preopen mapping need not be a fuzzy pairwise* (r,s)*-open mapping.*

Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ and $(X,\mathcal{U}_1,\mathcal{U}_2)$ be smooth bitopological spaces as described in (1). Consider the mapping $f : (X,\mathcal{U}_1,\mathcal{U}_2) \to (X,\mathcal{T}_1,\mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = y$. Then f is a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-preopen mapping which is not a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-open mapping.

(3) *A fuzzy pairwise* (r,s)*-preclosed mapping need not be a fuzzy pairwise* (r,s)*-closed mapping.*

Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ and $(X,\mathcal{U}_1,\mathcal{U}_2)$ be smooth bitopological spaces as described in (1). Consider the mapping $f : (X,\mathcal{U}_1,\mathcal{U}_2) \to (X,\mathcal{T}_1,\mathcal{T}_2)$

defined by $f(x) = x$ and $f(y) = y$. Then f is a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-preclosed mapping which is not a fuzzy pairwise $(\frac{1}{2}, \frac{1}{3})$-closed mapping.

THEOREM 4.4. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ be a mapping and $r, s \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy pairwise (r, s)-precontinuous mapping.*
(2) *$f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preclosed set of X for each $\mathcal{U}_1$-fuzzy r-closed set μ of Y and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preclosed set of X for each $\mathcal{U}_2$-fuzzy s-closed set ν of Y.*
(3) *For each fuzzy set μ of Y,*

$$\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r) \leq f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r))$$

and

$$\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mu), r), s) \leq f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)).$$

(4) *For each fuzzy set ρ of X,*

$$f(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r) \leq \mathcal{U}_1\text{-Cl}(f(\rho), r)$$

and

$$f(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho, r), s) \leq \mathcal{U}_2\text{-Cl}(f(\rho), s).$$

PROOF. (1) $\Rightarrow$ (2) Let μ be any $\mathcal{U}_1$-fuzzy r-closed set and ν any $\mathcal{U}_2$-fuzzy s-closed set of Y. Then μ^c is a $\mathcal{U}_1$-fuzzy r-open set and ν^c is a $\mathcal{U}_2$-fuzzy s-open set of Y. Since f is a fuzzy pairwise (r, s)-precontinuous mapping, $f^{-1}(\mu^c)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set and $f^{-1}(\nu^c)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preopen set of X. But $f^{-1}(\mu^c) = f^{-1}(\mu)^c$ and $f^{-1}(\nu^c) = f^{-1}(\nu)^c$. By Theorem 3.2, $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preclosed set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preclosed set of X.

(2) $\Rightarrow$ (1) Let μ be any $\mathcal{U}_1$-fuzzy r-open set and ν any $\mathcal{U}_2$-fuzzy s-open set of Y. Then μ^c is a $\mathcal{U}_1$-fuzzy r-closed set and ν^c is a $\mathcal{U}_2$-fuzzy s-closed set of Y. By (2), $f^{-1}(\mu^c)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preclosed set and $f^{-1}(\nu^c)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preclosed set of X. But $f^{-1}(\mu^c) = f^{-1}(\mu)^c$ and $f^{-1}(\nu^c) = f^{-1}(\nu)^c$. By Theorem 3.2, $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preopen set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preopen set of X. Thus f is a fuzzy pairwise (r, s)-precontinuous mapping.

(2) $\Rightarrow$ (3) Let μ be any fuzzy set of Y. Then $\mathcal{U}_1\text{-Cl}(\mu, r)$ is a $\mathcal{U}_1$-fuzzy r-closed set and $\mathcal{U}_2\text{-Cl}(\mu, s)$ is a $\mathcal{U}_2$-fuzzy s-closed set of Y. By (2),

$f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r))$ is $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preclosed and $f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s))$ is $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preclosed of X. Thus

$$\begin{aligned} f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r)) &\geq \mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mathcal{U}_1\text{-Cl}(\mu, r)), s), r) \\ &\geq \mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)) &\geq \mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mathcal{U}_2\text{-Cl}(\mu, s)), r), s) \\ &\geq \mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\mu), r), s). \end{aligned}$$

(3) $\Rightarrow$ (4) Let ρ be any fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. By (3),

$$\begin{aligned} f^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho), r)) &\geq \mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}f(\rho), s), r) \\ &\geq \mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho), s)) &\geq \mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}f(\rho), r), s) \\ &\geq \mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho, r), s). \end{aligned}$$

Hence

$$\mathcal{U}_1\text{-Cl}(f(\rho), r) \geq ff^{-1}(\mathcal{U}_1\text{-Cl}(f(\rho), r) \geq f(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(\rho, s), r)$$

and

$$\mathcal{U}_2\text{-Cl}(f(\rho), s) \geq ff^{-1}(\mathcal{U}_2\text{-Cl}(f(\rho), s) \geq f(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(\rho, r), s).$$

(4) $\Rightarrow$ (2) Let μ be any $\mathcal{U}_1$-fuzzy r-closed set and ν any $\mathcal{U}_2$-fuzzy s-closed set of Y. Then $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are fuzzy sets of X. By (4),

$$\begin{aligned} f(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r)) &\leq \mathcal{U}_1\text{-Cl}(ff^{-1}(\mu), r) \\ &\leq \mathcal{U}_1\text{-Cl}(\mu, r) = \mu \end{aligned}$$

and

$$\begin{aligned} f(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu), r), s)) &\leq \mathcal{U}_2\text{-Cl}(ff^{-1}(\nu), s) \\ &\leq \mathcal{U}_2\text{-Cl}(\nu, s) = \nu. \end{aligned}$$

So

$$\begin{aligned}\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r) &\leq f^{-1}f(\mathcal{T}_1\text{-Cl}(\mathcal{T}_2\text{-Int}(f^{-1}(\mu), s), r)) \\ &\leq f^{-1}(\mu)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu), r), s) &\leq f^{-1}f(\mathcal{T}_2\text{-Cl}(\mathcal{T}_1\text{-Int}(f^{-1}(\nu), r), s) \\ &\leq f^{-1}(\nu).\end{aligned}$$

Therefore $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r, s)-preclosed set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s, r)-preclosed set of X. □

In general, it need not be true that if f and g are fuzzy pairwise (r, s)-precontinuous ((r, s)-preopen and (r, s)-preclosed, respectively) then so is $g \circ f$. But we have the following theorem.

THEOREM 4.5. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$, $(Y, \mathcal{U}_1, \mathcal{U}_2)$ and $(Z, \mathcal{V}_1, \mathcal{V}_2)$ be smooth bitopological spaces and let $f : X \to Y$ and $g : Y \to Z$ be mappings and $r, s \in I_0$. Then the following statements are true.*

(1) *If f is fuzzy pairwise (r, s)-precontinuous and g is fuzzy pairwise (r, s)-continuous then $g \circ f$ is fuzzy pairwise (r, s)-precontinuous.*
(2) *If f is fuzzy pairwise (r, s)-open and g is fuzzy pairwise (r, s)-preopen then $g \circ f$ is fuzzy pairwise (r, s)-preopen.*
(3) *If f is fuzzy pairwise (r, s)-closed and g is fuzzy pairwise (r, s)-preclosed then $g \circ f$ is fuzzy pairwise (r, s)-preclosed.*

PROOF. Straightforward. □

The next two theorems show that a fuzzy pairwise precontinuous mapping is a special case of a fuzzy pairwise (r, s)-precontinuous mapping.

THEOREM 4.6. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ and $(Y, \mathcal{U}_1, \mathcal{U}_2)$ be smooth bitopological spaces and let $r, s \in I_0$. Then $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is fuzzy pairwise (r, s)-precontinuous if and only if $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is fuzzy pairwise precontinuous.*

PROOF. Let $\mu \in (\mathcal{U}_1)_r$ and $\nu \in (\mathcal{U}_2)_s$. Then $\mathcal{U}_1(\mu) \geq r$ and $\mathcal{U}_2(\nu) \geq s$ and hence μ is a $\mathcal{U}_1$-fuzzy r-open set and ν is a $\mathcal{U}_2$-fuzzy s-open set of Y. Since $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a fuzzy pairwise (r,s)-precontinuous mapping, $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r,s)-preopen set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s,r)-preopen set of $(X, \mathcal{T}_1, \mathcal{T}_2)$. So $f^{-1}(\mu)$ is $((\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$-fuzzy preopen and $f^{-1}(\nu)$ is $((\mathcal{T}_2)_s, (\mathcal{T}_1)_r)$-fuzzy preopen of $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$. Thus $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is a fuzzy pairwise precontinuous mapping.

Conversely, let μ be any $\mathcal{U}_1$-fuzzy r-open set and ν any $\mathcal{U}_2$-fuzzy s-open set of $(Y, \mathcal{U}_1, \mathcal{U}_2)$. Then $\mathcal{U}_1(\mu) \geq r$ and $\mathcal{U}_2(\nu) \geq s$. So $\mu \in (\mathcal{U}_1)_r$ and $\nu \in (\mathcal{U}_2)_s$. Since $f : (X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s) \to (Y, (\mathcal{U}_1)_r, (\mathcal{U}_2)_s)$ is a fuzzy pairwise precontinuous mapping, $f^{-1}(\mu)$ is a $((\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$-fuzzy preopen set and $f^{-1}(\nu)$ is a $((\mathcal{T}_2)_s, (\mathcal{T}_1)_r)$-fuzzy preopen set of $(X, (\mathcal{T}_1)_r, (\mathcal{T}_2)_s)$. So $f^{-1}(\mu)$ is a $(\mathcal{T}_1, \mathcal{T}_2)$-fuzzy (r,s)-preopen set and $f^{-1}(\nu)$ is a $(\mathcal{T}_2, \mathcal{T}_1)$-fuzzy (s,r)-preopen set of $(X, \mathcal{T}_1, \mathcal{T}_2)$. Thus $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a fuzzy pairwise (r,s)-precontinuous mapping. □

THEOREM 4.7. *Let (X, T_1, T_2) and (Y, U_1, U_2) be Kandil's fuzzy bitopological spaces and let $r, s \in I_0$. Then $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise precontinuous mapping if and only if $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise (r,s)-precontinuous mapping.*

PROOF. Let μ be a $(U_1)^r$-fuzzy r-open set and ν a $(U_2)^s$-fuzzy s-open set of $(Y, (U_1)^r, (U_2)^s)$. Then $(U_1)^r(\mu) \geq r$ and $(U_2)^s(\nu) \geq s$ and hence $\mu \in U_1$ and $\nu \in U_2$. Since $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise precontinuous mapping, $f^{-1}(\mu)$ is a (T_1, T_2)-fuzzy preopen set and $f^{-1}(\nu)$ is a (T_2, T_1)-fuzzy preopen set of (X, T_1, T_2). So $f^{-1}(\mu)$ is a $((T_1)^r, (T_2)^s)$-fuzzy (r,s)-preopen set and $f^{-1}(\nu)$ is a $((T_2)^s, (T_1)^r)$-fuzzy (s,r)-preopen set of $(X, (T_1)^r, (T_2)^s)$. Thus $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise (r,s)-precontinuous mapping.

Conversely, let $\mu \in U_1$ and $\nu \in U_2$. Then $(U_1)^r(\mu) \geq r$ and $(U_2)^s(\nu) \geq s$, and hence μ is a $(U_1)^r$-fuzzy r-open set and ν is a $(U_2)^s$-fuzzy s-open set of Y. Since $f : (X, (T_1)^r, (T_2)^s) \to (Y, (U_1)^r, (U_2)^s)$ is a fuzzy pairwise (r,s)-precontinuous mapping, $f^{-1}(\mu)$ is a $((T_1)^r, (T_2)^s)$-fuzzy (r,s)-preopen set and $f^{-1}(\nu)$ is a $((T_2)^s, (T_1)^r)$-fuzzy (s,r)-preopen set of $(X, (T_1)^r, (T_2)^s)$. So $f^{-1}(\mu)$ is a (T_1, T_2)-fuzzy preopen set and $f^{-1}(\nu)$ is a (T_2, T_1)-fuzzy preopen set of (X, T_1, T_2). Thus $f : (X, T_1, T_2) \to (Y, U_1, U_2)$ is a fuzzy pairwise precontinuous mapping. □

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