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Prepotential Approach to Solvable Rational Potentials and Exceptional Orthogonal Polynomials

Choon-Lin Ho

Department of Physics, Tamkang University, Tamsui 25137, Taiwan, R.O.C.

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We show how all the quantal systems related to the exceptional Laguerre and Jacobi polynomials can be constructed in a direct and systematic way, without the need of shape invariance and Darboux-Crum transformation. Furthermore, the prepotential need not be assumed a priori. The prepotential, the deforming function, the potential, the eigenfunctions and eigenvalues are all derived within the same framework. The exceptional polynomials are expressible as a bilinear combination of a deformation function and its derivative.

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§1. Introduction

In the last three years or so one has witnessed some interesting developments in the area of exactly solvable models in quantum mechanics: the number of exactly solvable shape-invariant models has been greatly increased owing to the discovery of new types of orthogonal polynomials, called the exceptional X_ℓ polynomials.[1)–18)] Unlike the classical orthogonal polynomials, these new polynomials have the remarkable properties that they still form complete sets with respect to some positive-definite measure, although they start with degree ℓ polynomials instead of a constant.

Two families of such polynomials, namely, the Laguerre- and Jacobi-type X_1 polynomials, corresponding to $\ell = 1$, were first proposed by Gómez-Ullate et al. in Ref. 1), within the Sturm-Lioville theory, as solutions of second-order eigenvalue equations with rational coefficients. The results in Ref. 1) were reformulated in the framework of quantum mechanics and shape-invariant potentials by Quesne et al.[2),3)] These quantal systems turn out to be rationally extended systems of the traditional ones which are related to the classical orthogonal polynomials. The most general X_ℓ exceptional polynomials, valid for all integral $\ell = 1, 2, \ldots$, were discovered by Odake and Sasaki[4)] (the case of $\ell = 2$ was also discussed in Ref. 3)). Later, in Ref. 5) equivalent but much simpler looking forms of the Laguerre- and Jacobi-type X_ℓ polynomials were presented. Such forms facilitate an in-depth study of some important properties of the X_ℓ polynomials, such as the actions of the forward and backward shift operators on the X_ℓ polynomials, Gram-Schmidt orthonormalization for the algebraic construction of the X_ℓ polynomials, Rodrigues formulas, and the generating functions of these new polynomials. Structure of the zeros of the exception polynomials was studied in Ref. 6).

More recently, these exceptional polynomials have been studied in many ways. For instance, possible applications of these new polynomials were considered in

Ref. 7) for position-dependent mass systems, and in Ref. 8) for the Dirac and Fokker-Planck equations. The new polynomials were also considered as solutions associated with some conditionally exactly solvable potentials.[9] These polynomials were recently constructed by means of the Darboux-Crum transformation.[3),10),11)] Rational extensions of certain shape-invariant potentials related to the exceptional orthogonal polynomials were generated by means of Darboux-Bäcklund transformation in Ref. 12). Generalizations of exceptional orthogonal polynomials to discrete quantum mechanical systems were done in Ref. 13). Structure of the X_ℓ Laguerre polynomials was considered within the quantum Hamilton-Jacobi formalism in Ref. 14). Generalizations of these new orthogonal polynomials to multi-indexed cases were discussed in Refs. 16) and 17). Recently radial oscillator systems related to the exceptional Laguerre polynomials have been considered based on higher order supersymmetric quantum mechanics.[18]

So far most of the methods employed to generate the exceptional polynomials have invoked in one way or another the idea of shape invariance and/or the related Darboux-Crum transformation. Furthermore, the so-called superpotentials (which we shall call the prepotential), which determine the potentials, have to be assumed a priori (often with good educated guesses).

The aim of this paper is to demonstrate that it is possible to generate all the quantal systems related to the exceptional Laguerre and Jacobi polynomials by a simple constructive approach without the need of shape invariance and the Darboux-Crum transformation. The prepotential (hence the potential), eigenfunctions and eigenvalues are all derived within the same framework. We call this the prepotential approach, which is an extension of the approach we employed to construct all the well-known one-dimensional exactly solvable quantum potentials in Ref. 19).

The plan of this paper is as follows. Section 2 presents the ideas of prepotential approach to systems which are rational extensions of the traditional systems related to the classical orthogonal polynomials. In §3 the prepotential approach is employed to generate the L2 Laguerre system. Construction of the J1 and J2 Jacobi cases is then outlined in §4. Section 5 summarizes the paper. Appendix A collects some useful results on the classical Laguerre and Jacobi polynomials. The L1 Laguerre system is then summarized in Appendix B.

§2. Prepotential approach

2.1. *Main ideas*

We shall adopt the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar = 2m = 1$. Consider a wave function $\phi(x)$ which is written in terms of a function $W(x)$ as

$$\phi(x) \equiv e^{W(x)}. \tag{2·1}$$

Operating on ϕ_N by the operator $-d^2/dx^2$ results in a Schrödinger equation $\mathcal{H}\phi = 0$, where

$$\mathcal{H} = -\frac{d^2}{dx^2} + \bar{V}, \tag{2·2}$$

$$\bar{V} \equiv \dot{W}^2 + \ddot{W}. \tag{2·3}$$

The dot denotes derivative with respect to x. Since $W(x)$ determines the potential $\bar{V}$, it is therefore called the prepotential.[20] For clarity of presentation, we shall often leave out the independent variable of a function if no confusion arises.

In this work we consider the following form of the prepotential:

$$W(x,\eta) = W_0(x) - \ln \xi(\eta) + \ln p(\eta). \tag{2·4}$$

Here $W_0(x)$ is the zero-th order prepotential, and $\eta(x)$ is a function of x which we shall choose to be one of the sinusoidal coordinates, i.e., coordinates such that $\dot{\eta}(x)^2$ is at most quadratic in η, since most exactly solvable one-dimensional quantal systems involve such coordinates. The choice of $\eta(x)$ and the final form of the potential dictate the domain of the variable x. $\xi(\eta)$ and $p(\eta)$ are functions of η to be determined later. We shall assume $\xi(\eta)$ to be a polynomial in η. The function $p(\eta)$ consists of the eigenpolynomial, but itself need not be a polynomial (see §3).

With the prepotential (2·4), the wave function is

$$\phi(x) = \frac{e^{W_0(x)}}{\xi(\eta)}\, p(\eta), \tag{2·5}$$

and the potential $\bar{V} = \dot{W}^2 + \ddot{W}$ takes the form

$$\bar{V} = \dot{W}_0^2 + \ddot{W}_0 + \left[\dot{\eta}^2\left(2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi}\right) - \frac{\xi'}{\xi}\left(2\dot{W}_0\dot{\eta} + \ddot{\eta}\right)\right] + \frac{1}{p}\left[\dot{\eta}^2 p'' + \left(2\dot{W}_0\dot{\eta} + \ddot{\eta} - 2\dot{\eta}^2\frac{\xi'}{\xi}\right)p'\right]. \tag{2·6}$$

Here the prime denotes derivative with respect to η.

For $\xi(\eta) = 1$, the prepotential approach can generate exactly and quasi-exactly solvable systems associated with the classical orthogonal polynomials.[19] The presence of ξ in the denominators of $\phi(x)$ and $V(x)$ thus gives a rational extension, or deformation, of the traditional system. We therefore call $\xi(\eta)$ the deforming function.

To make $\bar{V}$ exactly solvable, we demand that: (1) W_0 is a regular function of x, (2) the deforming function $\xi(\eta)$ has no zeros in the the ordinary (or physical) domain of $\eta(x)$, and (3) the function $p(\eta)$ does not appear in V.

The requirement (3) can be easily met by setting the last term involving $p(\eta)$ in Eq. (2·6) to a constant, say "$-\mathcal{E}$", i.e.,

$$\dot{\eta}^2 p'' + \left(2\dot{W}_0\dot{\eta} + \ddot{\eta} - 2\dot{\eta}^2\frac{\xi'}{\xi}\right)p' + \mathcal{E}p = 0. \tag{2·7}$$

If W_0, ξ can be determined, and Eq. (2·7) can be solved, then we would have constructed an exactly solvable quantal system $H\psi = \mathcal{E}\psi$ defined by $H = -d^2/dx^2 + V(x)$, with the wave function (2·5) and the potential

$$V(x) \equiv \dot{W}_0^2 + \ddot{W}_0 + \left[\dot{\eta}^2\left(2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi}\right) - \frac{\xi'}{\xi}\left(2\dot{W}_0\dot{\eta} + \ddot{\eta}\right)\right]. \tag{2·8}$$

2.2. *Determining* $W_0(x), \xi(\eta)$ *and* $p(\eta)$

As mentioned before, we require that $\xi(\eta)$ has no zeros in the ordinary domain of the variable η, but $\xi(\eta)$ may have zeros in the other region in the complex η-plane. Suppose $\xi(\eta)$ satisfies the equation

$$c_2(\eta)\xi'' + c_1(\eta)\xi' + \widetilde{\mathcal{E}}\xi = 0, \quad \widetilde{\mathcal{E}} = \text{real constant}. \tag{2·9}$$

Here $c_2(\eta)$ and $c_1(\eta)$ are functions of η to be determined. We want Eq. (2·9) to be exactly solvable. This is most easily achieved by matching (2·9) with the (confluent) hypergeometric equation, and this we shall adopt in this paper. Thus $c_2(\eta)$ and $c_1(\eta)$ are at most quadratic and linear in η, respectively.

If the factor $\exp(W_0(x))/\xi(\eta)$ in Eq. (2·5) is normalizable, then $p(\eta) =$ constant (in this case we shall take $p(\eta) = 1$ for simplicity), which solves Eq. (2·7) with $\mathcal{E} = 0$, is admissible. This gives the ground state

$$\phi_0(x) = \frac{e^{W_0(x)}}{\xi(\eta)}. \tag{2·10}$$

However, if $\exp(W_0(x))/\xi(\eta)$ is non-normalizable, then $\phi_0(x)$ cannot be the ground state. In this case, the ground state, like all the excited states, must involve non-trivial $p(\eta) \neq 1$. Typically it is in such situation that the exceptional orthogonal polynomials arise. To determine states involving non-trivial $p(\eta)$, we proceed as follows. Since both ξ and ξ' appear in Eq. (2·7), we shall take the ansatz that $p(\eta)$ be a linear combination of ξ and ξ':

$$p(\eta) = \xi'(\eta)F(\eta) + \xi(\eta)G(\eta), \tag{2·11}$$

where $F(\eta)$ and $G(\eta)$ are some functions of η. Then using Eq. (2·9) we have

$$p'(\eta) = \xi'\left[-\frac{c_1}{c_2}F + F' + G\right] + \xi\left[-\frac{\widetilde{\mathcal{E}}}{c_2}F + G'\right]. \tag{2·12}$$

We demand that Eq. (2·7) be regular at the zeros of ξ. This is achieved if $p' \propto \xi$, which requires that the coefficient of ξ' in Eq. (2·12) be zero, thus giving a relation that connects F and G,

$$G = \frac{c_1}{c_2}F - F'. \tag{2·13}$$

Putting Eq. (2·13) into (2·7), we get

$$\xi'\left[-\dot{\eta}^2\left(-\frac{\widetilde{\mathcal{E}}}{c_2}F + G'\right) + \mathcal{E}F\right]$$
$$+\xi\left[\dot{\eta}^2\frac{d}{d\eta}\left(-\frac{\widetilde{\mathcal{E}}}{c_2}F + G'\right) + \left(2\dot{W}_0\dot{\eta} + \ddot{\eta}\right)\left(-\frac{\widetilde{\mathcal{E}}}{c_2}F + G'\right) + \mathcal{E}G\right] = 0. \tag{2·14}$$

Since ξ and ξ' are independent for any polynomial ξ, Eq. (2·14) implies the coefficients of ξ and ξ' are zero. Setting the terms in the two square-brackets to zero, and using

Eq. (2·13) to eliminate G, we arrive at the following equations satisfied by $F(\eta)$ and $c_1(\eta)$, respectively:

$$-\dot\eta^2 F'' + \frac{\dot\eta^2}{c_2} c_1 F' + \frac{\dot\eta^2}{c_2}\left[c_2 \frac{d}{d\eta}\left(\frac{c_1}{c_2}\right) - \widetilde{\mathcal{E}}\right] F = \mathcal{E} F, \tag{2·15}$$

and

$$\begin{aligned} c_1(\eta) &= \frac{c_2}{\dot\eta^2}\left[\frac{d}{d\eta}\left(\dot\eta^2\right) - \left(2\dot W_0 \dot\eta + \ddot\eta\right)\right] \\ &= \frac{c_2}{\dot\eta^2}\left[\frac{1}{2}\frac{d}{d\eta}\left(\dot\eta^2\right) - 2Q(\eta)\right], \end{aligned} \tag{2·16}$$

where $Q(\eta) \equiv \dot W_0 \dot\eta$, and we have used the identity $\ddot\eta = (d\dot\eta^2/d\eta)/2$ to arrive at the last line.

Equation (2·13) suggests that we set

$$F(\eta) = c_2(\eta)\mathcal{V}(\eta), \tag{2·17}$$

with some function $\mathcal{V}(\eta)$ in order to avoid any possible singularity from c_2. Equations (2·15) and (2·13) then reduce to

$$c_2 \mathcal{V}'' + \left(2c_2' - c_1\right)\mathcal{V}' + \left[c_2'' - c_1' + \widetilde{\mathcal{E}} + \frac{c_2}{\dot\eta^2}\mathcal{E}\right]\mathcal{V} = 0, \tag{2·18}$$

and

$$G(\eta) = \left(c_1 - c_2'\right)\mathcal{V} - c_2\mathcal{V}'. \tag{2·19}$$

As mentioned before, in this paper we shall take $c_2(\eta)$ and $c_1(\eta)$ to be at most quadratic and linear in η, respectively. This means the coefficients of the first and second terms in Eq. (2·18) are also at most quadratic and linear in η, respectively. So Eq. (2·18) can be matched with the (confluent) hypergeometric equation, provided that the coefficient of the last term in (2·18) is a constant. This then requires

$$c_2(\eta) = \pm\dot\eta^2. \tag{2·20}$$

(Note: in general one has $c_2(\eta) = \pm$ constant$\times\dot\eta^2$. But it is evident that the constant can be factored out of Eq. (2·18), together with a rescaling of $\widetilde{\mathcal{E}}$. Thus without loss of generality we set the constant to unity.) From Eq. (2·16) this leads to

$$c_1(\eta) = \pm\left[\frac{1}{2}\frac{d}{d\eta}\left(\dot\eta^2\right) - 2Q(\eta)\right]. \tag{2·21}$$

Now we summarize the procedure or algorithm for constructing an exactly solvable quantal system, whose potential as well as its eigenfunctions and eigenvalues are all determined within the some approach:

- choose $\dot\eta^2$ from a sinusoidal coordinate; this then fixes the form of c_2;

- by matching Eq. (2·9) with the (confluent) hypergeometric equation, one determines $\tilde{\mathcal{E}}$, $Q(\eta)$, c_1 and $\xi(\eta)$. Integrating $Q(x) = \dot{W}_0\dot{\eta}$ then gives the prepotential $W_0(x)$:

$$W_0(x) = \int^x dx\, \frac{Q(\eta(x))}{\dot{\eta}(x)} = \int^{\eta(x)} d\eta\, \frac{Q(\eta)}{\dot{\eta}^2(\eta)}; \tag{2·22}$$

- by matching Eq. (2·18) with the (confluent) hypergeometric equation, one determines $\mathcal{V}$, and thus $F(\eta)$, $G(\eta)$, $p(\eta)$ and $\mathcal{E}$;
- the exactly solvable system is defined by the wave function (2·5) and the potential (2·8), which, by Eqs. (2·9) and (2·20), can be recast in the form

$$V(x) \equiv \dot{W}_0^2 + \ddot{W}_0 + \frac{\xi'}{\xi}\left[2\dot{\eta}^2\left(\frac{\xi'}{\xi}\right) - \left(2\dot{W}_0\dot{\eta} + \ddot{\eta}\right) \pm c_1\right] \pm \tilde{\mathcal{E}}. \tag{2·23}$$

2.3. *Orthogonality of* $p(\eta)$

Using the relations

$$\frac{d\xi}{d\eta} = \frac{\dot{\xi}}{\dot{\eta}}, \quad \frac{dp}{dx} = \dot{\eta}p', \quad \frac{d^2p}{dx^2} = \dot{\eta}^2 p'' + \ddot{\eta}p', \tag{2·24}$$

one can recast Eq. (2·7) into a differential equation in variable x,

$$\frac{d^2}{dx^2}p(\eta(x)) + 2\left(\dot{W}_0 - \frac{\dot{\xi}}{\xi}\right)\frac{d}{dx}p(\eta(x)) + \mathcal{E}p(\eta(x)) = 0. \tag{2·25}$$

This can further be put in the Sturm-Liouville form

$$\frac{d}{dx}\left[\mathcal{W}^2\frac{d}{dx}p(\eta(x))\right] + \mathcal{E}\mathcal{W}^2 p(\eta(x)) = 0, \tag{2·26}$$

where

$$\mathcal{W}(x) \equiv \exp\left(\int^x dx\left(\dot{W}_0 - \frac{\dot{\xi}}{\xi}\right)\right) = \frac{e^{W_0(x)}}{\xi(\eta(x))}. \tag{2·27}$$

According to the standard Sturm-Liouville theory, the functions $p_{\mathcal{E}}$ (here we add a subscript to distinguish p corresponding to a particular eigenvalue $\mathcal{E}$) are orthogonal, i.e.,

$$\int dx\; p_{\mathcal{E}}(\eta(x))p_{\mathcal{E}'}(\eta(x))\mathcal{W}^2(x) \propto \delta_{\mathcal{E},\mathcal{E}'} \tag{2·28}$$

in the x-space, or

$$\int d\eta\; p_{\mathcal{E}}(\eta)p_{\mathcal{E}'}(\eta)\frac{\mathcal{W}^2(x(\eta))}{\dot{\eta}} \propto \delta_{\mathcal{E},\mathcal{E}'} \tag{2·29}$$

in the η-space.

§3. L2 Laguerre case

We now employ the above algorithm to generate the deformed radial oscillator as given in Refs. 2)–4).

Let us choose $\eta(x) = x^2 \in [0, \infty)$. Then $\dot{\eta}^2 = 4\eta$. For c_2 and c_1, we take the positive signs in Eqs. (2·20) and (2·21) (the opposite situation is considered in Appendix B). Thus $c_2(\eta) = 4\eta$ and $c_1 = 2(1 - Q(\eta))$.

3.1. W_0, ξ and $\tilde{\mathcal{E}}$

Equation (2·9) becomes

$$\eta\xi'' + \frac{1}{2}(1 - Q(\eta))\,\xi' + \frac{\tilde{\mathcal{E}}}{4}\xi = 0. \tag{3·1}$$

Comparing Eq. (3·1) with the Laguerre equation

$$\eta L_\ell''^{(\alpha)} + (\alpha + 1 - \eta)\,L_\ell'^{(\alpha)} + \ell L_\ell^{(\alpha)} = 0, \quad \ell = 0, 1, 2, \ldots, \tag{3·2}$$

where $L_\ell^{(\alpha)}(\eta)$ is the Laguerre polynomial, we have

$$\xi(\eta) \equiv \xi_\ell(\eta; \alpha) = L_\ell^{(\alpha)}(\eta), \quad \tilde{\mathcal{E}} = 4\ell, \quad Q(\eta) = 2\left(\eta - \alpha - \frac{1}{2}\right). \tag{3·3}$$

For $\xi_\ell(\eta; \alpha)$ not to have zeros in the ordinary domain $[0, \infty)$, we must have $\alpha < -\ell$ (see Appendix A). By Eq. (2·22), the form of $Q(\eta)$ gives

$$W_0(x) = \frac{x^2}{2} - \left(\alpha + \frac{1}{2}\right)\ln x. \tag{3·4}$$

We shall ignore the constant of integration as it can be absorbed into the normalization constant.

3.2. $p(\eta)$, $\phi(\eta)$ and $\mathcal{E}$

The above results implythat $\exp(W_0) \propto \exp(x^2/2)x^{-(\alpha+\frac{1}{2})}$ $(\alpha < -\ell)$. The term $\exp(x^2/2)$ will make $\phi(x)$ non-normalizable if $p(\eta) = 1$, or if $\mathcal{V}(\eta)$ is a polynomial in η. To remedy this, we try $\mathcal{V} = \exp(-\eta)U(\eta)$ with some function $U(\eta)$. Equation (2·18) becomes

$$\eta U'' + (-\alpha + 1 - \eta)\,U' + \left(\frac{\mathcal{E} + \tilde{\mathcal{E}}}{4} + \alpha\right)U = 0. \tag{3·5}$$

Comparing this equation with the Laguerre equation (3·2) (replacing ℓ by another integer $n = 0, 1, 2 \ldots$. In the rest of this paper, the index n will always take on these values), one has

$$U(\eta) = L_n^{(-\alpha)}(\eta), \quad \mathcal{E} \equiv \mathcal{E}_n = 4(n - \alpha) - \tilde{\mathcal{E}} = 4(n - \alpha - \ell). \tag{3·6}$$

From Eqs. (2·17) and (2·19), one eventually obtains

$$\begin{aligned} p(\eta) &\equiv p_{\ell,n}(\eta) = \xi' F + \xi G \\ &= 4e^{-\eta} P_{\ell,n}(\eta;\alpha), \end{aligned} \tag{3·7}$$

$$\begin{aligned} P_{\ell,n}(\eta;\alpha) &\equiv \eta L_n^{(-\alpha)} \xi'_\ell + \left(\alpha L_n^{(-\alpha)} - \eta L_n'^{(-\alpha)}\right) \xi_\ell \\ &= \eta L_n^{(-\alpha)} \xi'_\ell + (\alpha - n) L_n^{(-\alpha-1)} \xi_\ell. \end{aligned} \tag{3·8}$$

Use has been made of Eqs. (A·1)–(A·3) in obtaining the last line in Eq. (3·8). We note that $P_{\ell,n}(\eta;\alpha)$ is a polynomial of degree $\ell+n$. It is just the L2 type exceptional Laguerre polynomial. We will show in the next subsection that it is equivalent to the form presented in Ref. 5) (to be called HOS form for simplicity). By Eq. (2·29), one finds that $P_{\ell,n}(\eta;\alpha)$'s are orthogonal in the sense

$$\int_0^\infty d\eta \, \frac{e^{-\eta}\eta^{-(\alpha+1)}}{\xi_\ell^2} P_{\ell,n}(\eta;\alpha) P_{\ell,m}(\eta;\alpha) \propto \delta_{nm}. \tag{3·9}$$

The exactly solvable potential is given by Eq. (2·23) with $W_0(x)$ and $\xi_\ell(\eta;\alpha)$ given by Eqs. (3·4) and (3·3), respectively. The eigenvalues $\mathcal{E}_n$ are given in Eq. (3·6), i.e. $\mathcal{E}_n = 4(n - \alpha - \ell)$. Explicitly, the potential is

$$V(x) = x^2 + \frac{\left(\alpha+\frac{1}{2}\right)\left(\alpha+\frac{3}{2}\right)}{x^2} + 8\frac{\xi'_\ell}{\xi_\ell}\left[\eta\left(\frac{\xi'_\ell}{\xi_\ell} - 1\right) + \alpha + \frac{1}{2}\right] + 2(2\ell - \alpha). \tag{3·10}$$

It is easily shown that $V(x)$ is equivalent to the potential for L2 Laguerre case in Refs. 4), 5), 11) with $\alpha = -g - \ell - \frac{1}{2}$ $(g > 0)$. Particularly, it is exactly equal to the form given in Eq. (2.21) of Ref. 11). The complete eigenfunctions are

$$\phi_{\ell,n}(x;\alpha) \propto \frac{e^{-\frac{x^2}{2}} x^{-(\alpha+\frac{1}{2})}}{\xi_\ell} P_{\ell,n}(\eta;\alpha), \quad \alpha < -\ell. \tag{3·11}$$

For $\ell = 0$, we have $\xi_0 = 1$ and $\xi'_\ell = 0$, and the system reduces to the radial oscillator. From Eq. (3·8) one has $P_{\ell,n} \to L_n^{(-\alpha-1)}$ and $\alpha < -1/2$.

3.3. *Reducing $P_{\ell,n}(\eta;\alpha)$ to HOS form*

The polynomial $P_{\ell,n}(\eta;\alpha)$ is expressed as a bilinear combination of $\xi_\ell(\eta;\alpha)$ and its derivative $\xi'_\ell(\eta;\alpha)$. The HOS form instead expresses the exceptional polynomial as a bilinear combination of $\xi_\ell(\eta;\alpha)$ and its shifted form, i.e., $\xi_\ell(\eta;\alpha-1)$.

To show the equivalence between $P_{\ell,n}(\eta;\alpha)$ and the HOS form, we make use of the identities Eqs. (A·1) and (A·3) to express $\eta\xi'_\ell(\eta;\alpha)$ in the first term of $P_{\ell,n}(\eta;\alpha)$ as

$$\begin{aligned} \eta\xi'_\ell(\eta;\alpha) &= -\eta L_{\ell-1}^{(\alpha+1)}(\eta) \\ &= -\alpha L_{\ell-1}^{(\alpha)}(\eta) + \ell L_\ell^{(\alpha-1)}(\eta). \end{aligned} \tag{3·12}$$

Then we have

$$\begin{aligned} P_{\ell,n}(\eta;\alpha) &= \left(-\alpha L_{\ell-1}^{(\alpha)} + \ell L_\ell^{(\alpha-1)}\right) L_n^{(-\alpha)} + \left(\alpha L_n^{(-\alpha)} - \eta L_n'^{(-\alpha)}\right) L_\ell^{(\alpha)} \\ &= \left(\alpha\left(L_\ell^{(\alpha)} - L_{\ell-1}^{(\alpha)}\right) + \ell L_\ell^{(\alpha-1)}\right) L_n^{(-\alpha)} - \eta L_n'^{(-\alpha)} L_\ell^{(\alpha)}. \end{aligned} \tag{3·13}$$

Using Eq. (A·2) we have $L_\ell^{(\alpha)}(\eta) - L_{\ell-1}^{(\alpha)}(\eta) = L_\ell^{(\alpha-1)}(\eta)$. Finally, we arrive at

$$P_{\ell,n}(\eta;\alpha) = (\alpha+\ell)\, L_n^{(-\alpha)}(\eta)\xi_\ell(\eta;\alpha-1) - \eta L_n'^{(-\alpha)}(\eta)\xi_\ell(\eta;\alpha), \tag{3·14}$$

$$\xi_\ell(\eta;\alpha-1) \equiv L_\ell^{(\alpha-1)}(\eta). \tag{3·15}$$

Setting $\alpha = -g-\ell-\frac{1}{2}$ and $\xi_\ell(\eta;g) \equiv L_\ell^{(-g-\ell-\frac{1}{2})}(\eta)$ into (3·14), we have

$$P_{\ell,n}(\eta;\alpha) = -\left[\left(g+\frac{1}{2}\right) L_n^{(g+\ell+\frac{1}{2})}(\eta)\,\xi_\ell(\eta;g+1) \right.$$
$$\left. +\eta L_n'^{(g+\ell+\frac{1}{2})}(\eta)\,\xi_\ell(\eta;g)\right]. \tag{3·16}$$

This is, up to a multiplicative constant, the HOS form of the L2 Laguerre polynomial.

The example in this section demonstrates that the prepotential approach described in §2 can indeed generate the exactly solvable quantal system which has the L2 Laguerre polynomials as the main part of its eigenfunctions. The prepotential $W_0(x)$, the potential $V(x)$, the deforming function $\xi_\ell(\eta;\alpha)$, the eigenfunction $\phi_{\ell,n}(x;\alpha)$ and eigenvalues $\mathcal{E}_n$ are all determined from first principle.

In the next section and in the Appendix, we shall generate systems associated with the exceptional Jacobi and L1 Laguerre polynomials. Our description for these cases will be concise, since the main steps are similar to those described in this section.

§4. Exceptional Jacobi cases

Let us choose $\eta(x) = \cos(2x) \in [-1,1]$. In this case it turns out, as can be easily confirmed, that both the upper and the lower signs in Eqs. (2·20) and (2·21) for c_2 and c_1 give the same equations that determine ξ and $\mathcal{V}$, i.e. Eqs. (2·9) and (2·18). So for definiteness, we shall take the upper signs, which give $c_2(\eta) = 4(1-\eta^2)$ and $c_1 = -2(2\eta + Q(\eta))$.

4.1. W_0, ξ *and* $\tilde{\mathcal{E}}$

Equation determining ξ is

$$(1-\eta^2)\xi''(\eta) + \left(-\eta - \frac{Q(\eta)}{2}\right)\xi'(\eta) + \frac{\tilde{\mathcal{E}}}{4}\xi(\eta) = 0. \tag{4·1}$$

Comparing this with the differential equation satisfied by the Jacobi polynomial $P_\ell^{(\alpha,\beta)}(\eta)$, namely,

$$(1-\eta^2)P_\ell''^{(\alpha,\beta)}(\eta) + \big(\beta-\alpha-(\alpha+\beta+2)\eta\big)P_\ell'^{(\alpha,\beta)}(\eta) + \ell(\ell+\alpha+\beta+1)P_\ell^{(\alpha,\beta)}(\eta) = 0, \tag{4·2}$$

we have

$$\xi(\eta) \equiv \xi_\ell(\eta;\alpha,\beta) = P_\ell^{(\alpha,\beta)}(\eta), \quad \tilde{\mathcal{E}} = 4\ell(\ell+\alpha+\beta+1),$$
$$Q(\eta) = 2\left[\alpha-\beta+(\alpha+\beta+1)\,\eta\right] \tag{4·3}$$

for some parameters α and β. The form of $Q(\eta)$ gives, from Eq. (2·22),

$$W_0(x) = -\left(\alpha + \frac{1}{2}\right)\ln\sin x - \left(\beta + \frac{1}{2}\right)\ln\cos x. \qquad (4\cdot4)$$

The equation of $\mathcal{V}$ is

$$(1-\eta^2)\mathcal{V}'' + \left[-\beta + \alpha - (-\beta - \alpha + 2)\,\eta\right]\mathcal{V}' + \left(\frac{\mathcal{E}+\tilde{\mathcal{E}}}{4} + \alpha + \beta\right)\mathcal{V} = 0. \qquad (4\cdot5)$$

From Eq. (4·4) we have

$$e^{W_0} \propto (1-\eta)^{-\frac{1}{2}(\alpha+\frac{1}{2})}\,(1+\eta)^{-\frac{1}{2}(\beta+\frac{1}{2})}\,. \qquad (4\cdot6)$$

The exponents in Eq. (4·6) naturally divide the parameters α and β into four groups: (i) $\alpha > -1/2,\ \beta > -1/2$, (ii) $\alpha > -1/2,\ \beta < -1/2$, (iii) $\alpha < -1/2,\ \beta > -1/2$ and (iv) $\alpha < -1/2,\ \beta < -1/2$. Group (i) should be excluded, or $\xi_\ell(\eta;\alpha)$ will have zeros in the ordinary domain $[-1,1]$ (see Appendix A). So we shall study the other three cases. It turns out that these three cases correspond, respectively, to quantal systems related to the type J1, J2 exceptional Jacobi polynomials, and a rationally extended Jacobi system obtained from the Darboux-Pöschl-Teller system by deleting the lowest ℓ excited states according to the Crum-Adler method discussed in Ref. 21). We stress here that the actual admissible parameters α and β in each case are dictated by the final form of $\mathcal{V}$, as will be shown below.

4.2. *J1 Jacobi case*

Consider the case with parameters $\alpha > -1/2,\ \beta < -1/2$. The deforming function $\xi_\ell(\eta;\alpha,\beta)$ is given in Eq. (4·3),

$$\xi_\ell(\eta;\alpha,\beta) = P_\ell^{(\alpha,\beta)}(\eta). \qquad (4\cdot7)$$

We demand that $\xi_\ell(\eta;\alpha,\beta)$ has no zeros in the ordinary domain $[-1,1]$. From Eq. (A·13), one can easily check that this is the case if $\beta < -\ell$ for $\alpha > -1/2$. For this choice of the parameters the first term $(1-\eta)^{-\frac{1}{2}(\alpha+\frac{1}{2})}$ of Eq. (4·6) will make the eigenfunction $\phi(x)$ non-normalizable, if $\mathcal{V}$ is a polynomial.

This prompted us to try $\mathcal{V} = (1-\eta)^\gamma U(\eta)$ where γ is a real parameter and $U(\eta)$ a function of η. From Eq. (4·5) we find that $U(\eta)$ satisfies

$$(1-\eta^2)U'' + (-2\gamma - \beta + \alpha - (2\gamma - \beta - \alpha + 2)\eta)\,U' + \left(\frac{\mathcal{E}+\tilde{\mathcal{E}}}{4} + \alpha + \beta + \gamma(\gamma+\beta-\alpha-1) + 2\gamma(\gamma-\alpha)\frac{\eta}{1-\eta}\right)U = 0. \qquad (4\cdot8)$$

If $\gamma = 0,\ \alpha$, the coefficient of U in the last term of the above equation can be reduced to a constant, so that Eq. (4·8) can be compared with the Jacobi differential equation (4·2). As $\gamma = 0$ does not solve our original problem with normalizability of the wave function, so we shall take $\gamma = \alpha$. This leads to

$$(1-\eta^2)U'' + (-\beta - \alpha - (-\beta + \alpha + 2)\eta)\,U' + \left(\frac{\mathcal{E}+\tilde{\mathcal{E}}}{4} + \beta(\alpha+1)\right)U = 0. \qquad (4\cdot9)$$

Comparing this equation with Eq. (4·2), we conclude that

$$U(\eta;\alpha,\beta)=P_n^{(\alpha,-\beta)}(\eta),$$
$$\mathcal{E}\equiv\mathcal{E}_n=4\left[n(n+\alpha-\beta+1)-\ell(\ell+\alpha+\beta+1)-\beta(\alpha+1)\right]. \qquad (4·10)$$

Putting all these results into $F(\eta)$ and $G(\eta)$ gives

$$p(\eta)\equiv p_{\ell,n}(\eta;\alpha,\beta)=4(1-\eta)^{\alpha+1}P_{\ell,n}(\eta;\alpha,\beta),$$
$$P_{\ell,n}(\eta;\alpha,\beta)\equiv\left\{(1+\eta)P_n^{(\alpha,-\beta)}(\eta)\xi_\ell'+\left[\beta P_n^{(\alpha,-\beta)}(\eta)-(1+\eta)P_n'^{(\alpha,-\beta)}(\eta)\right]\xi_\ell\right\}$$
$$=(1+\eta)P_n^{(\alpha,-\beta)}(\eta)\xi_\ell'-(n-\beta))P_n^{(\alpha+1,-\beta-1)}(\eta)\xi_\ell. \qquad (4·11)$$

We have made use of Eq. (A·9) to arrive at the last line of (4·11). By Eq. (2·29), the orthogonality relations of $P_{\ell,n}(\eta;\alpha)$'s are

$$\int_{-1}^{1}d\eta\,\frac{(1-\eta)^{(\alpha+1)}(1+\eta)^{-(\beta+1)}}{\xi_\ell^2}P_{\ell,n}(\eta;\alpha,\beta)P_{\ell,m}(\eta;\alpha,\beta)\propto\delta_{nm}. \qquad (4·12)$$

The exactly solvable potential is given by Eq. (2·23) with $W_0(x)$ and $\xi_\ell(\eta;\alpha,\beta)$ given by Eqs. (4·4) and (4·3), respectively. The eigenvalues $\mathcal{E}_n$ are given in Eq. (4·10). The complete eigenfunctions are

$$\phi_{\ell,n}(x;\alpha,\beta)\propto\frac{(1-\eta)^{\frac{1}{2}(\alpha+\frac{3}{2})}\,(1+\eta)^{-\frac{1}{2}(\beta+\frac{1}{2})}}{\xi_\ell}P_{\ell,n}(\eta;\alpha,\beta),$$
$$\alpha>-1/2,\ \beta<-\ell. \qquad (4·13)$$

Using the identity (A·9) one can show easily that

$$P_{\ell,n}(\eta;\alpha,\beta)=(\ell+\beta)P_n^{(\alpha,-\beta)}(\eta)\xi_\ell(\eta;\alpha+1,\beta-1)$$
$$-(1+\eta)P_n'^{(\alpha,-\beta)}(\eta)\xi_\ell(\eta;\alpha,\beta). \qquad (4·14)$$

Up to a multiplicative constant, this is just the HOS form of the J1 Jacobi polynomial presented in Ref. 5), with the substitution $\alpha=g+\ell-3/2$ and $\beta=-h-\ell-\frac{1}{2}$. It is easy to show that $V(x)$ and $\mathcal{E}_n$ are equivalent to those for J1 Jacobi case given in Refs. 4), 5), 11) with these values of α and β.

As $\ell\to 0$, the system reduces to the trigonometric Darboux-Pöschl-Teller potential, where α and β can now take the values $\alpha>-3/2,\ \beta<-1/2$.

4.3. *J2 Jacobi case*

One can proceed in a similar manner to construct the exactly solvable systems with $\alpha<-1/2,\ \beta>-1/2$. This turns out to lead to the system involving the J2 Jacobi polynomials.

We shall not bore the readers with similar details here. Instead, we point out that it is easier to obtain the system by symmetry consideration. One notes that under the parity transformation $\eta\to-\eta$, together with interchange $\alpha\leftrightarrow\beta$, Eqs. (4·1) (with $Q(\eta)$ given by (4·3)) and (4·5) are invariant in form. This is in complete accordance with the parity property of the Jacobi polynomials, namely

$$P_n^{(\alpha,\beta)}(-\eta)=(-1)^nP_n^{(\beta,\alpha)}(\eta). \qquad (4·15)$$

This implies that the J2 Jacobi system is simply the mirror image of the J1 Jacobi system, and thus it can be obtained from the J1 case by taking the above transformations.

4.4. *Rationally extended Jacobi case*

Let $\alpha,\ \beta < -1/2$. In this case, the factors in Eq. (4·6) cause no problem with normalizability of the wave function even if $\mathcal{V}(\eta)$ is a polynomial. For $\xi_\ell(\eta;\alpha,\beta)$ to be nodeless in the ordinary domain $[-1,1]$, we must choose α and β such that the conditions in (A·13) are satisfied. For example, if $\ell = 1$, one can have $\alpha < -1,\ -1 < \beta < -1/2$, or $\beta < -1,\ -1 < \alpha < -1/2$. For $\ell = 2$, we have $\alpha,\ \beta < -2$, or $-2 < \alpha,\ \beta < -1$. For ℓ odd, we must have $\alpha \neq \beta$, or $\xi(\eta)$ will have a zero at $\eta = 0$ in view of Eq. (4·15).

Comparing Eqs. (4·5) and (4·2), one obtains

$$\begin{aligned} \mathcal{V}(\eta) &= P_n^{(-\alpha,-\beta)}(\eta), \\ \mathcal{E} \equiv \mathcal{E}_n &= 4\left[n(n-\alpha-\beta+1) - \ell(\ell+\alpha+\beta+1) - \alpha - \beta\right]. \end{aligned} \tag{4·16}$$

From $F(\eta)$ and $G(\eta)$ we get

$$\begin{aligned} p(\eta) \equiv P_{\ell,n}(\eta;\alpha,\beta) \equiv 4\Big\{&(1-\eta^2)P_n^{(-\alpha,-\beta)}(\eta)\xi'_\ell \\ &+ \left[(\beta-\alpha-(\beta+\alpha)\eta)\,P_n^{(-\alpha,-\beta)}(\eta) - (1-\eta^2)P_n'^{(-\alpha,-\beta)}(\eta)\right]\xi_\ell\Big\}. \end{aligned} \tag{4·17}$$

Again, by applying the identity (A·7) and (A·9), one can reduce $P_{\ell,n}(\eta;\alpha,\beta)$ to

$$\begin{aligned} &P_{\ell,n}(\eta;\alpha,\beta) \\ &\quad = 4\Big\{(\ell+\beta)(1-\eta)P_n^{(-\alpha,-\beta)}(\eta)\xi_\ell(\eta;\alpha+1,\beta-1) \\ &\qquad + (n-\alpha)(1+\eta)P_n^{(-\alpha-1,-\beta+1)}(\eta)\xi_\ell(\eta;\alpha,\beta)\Big\}. \end{aligned} \tag{4·18}$$

One notes that $P_{\ell,n}(\eta;\alpha,\beta)$ is a polynomial of degree $\ell+n+1$, and has $n+1$ nodes. Thus the wave function with $P_{\ell,0}(\eta;\alpha,\beta)$ has one node, and does not correspond to the ground state. In fact, in this case the ground state wave function is given by Eq. (2·10) with $p(\eta) = 1$ and $\mathcal{E} = 0$, since $\phi_0(x)$ is normalizable. To ensure that the energies of the excited states are positive, i.e., $\mathcal{E}_n > 0$ for $n = 0,1,2,\ldots$, one must have, besides the constraints stated at the beginning of this subsection, the condition $\alpha+\beta < -\ell$, which can be easily checked from the form of $\mathcal{E}$ in Eq. (4·16).

The functions $P_{\ell,n}(\eta;\alpha,\beta)$ $(n = 0,1,2,\ldots)$, together with $p(\eta) = 1$, form a complete set and are orthogonal with respect to the weight function

$$\frac{(1-\eta)^{-(\alpha+1)}(1+\eta)^{-(\beta+1)}}{\xi_\ell^2}. \tag{4·19}$$

The complete eigenfunctions are given by

$$\begin{aligned} \phi_0(x;\alpha,\beta) &\propto \frac{(1-\eta)^{-\frac{1}{2}(\alpha+\frac{1}{2})}\,(1+\eta)^{-\frac{1}{2}(\beta+\frac{1}{2})}}{\xi_\ell}, \\ \phi_{\ell,n}(x;\alpha,\beta) &\propto \frac{(1-\eta)^{-\frac{1}{2}(\alpha+\frac{1}{2})}\,(1+\eta)^{-\frac{1}{2}(\beta+\frac{1}{2})}}{\xi_\ell}P_{\ell,n}(\eta;\alpha,\beta). \end{aligned} \tag{4·20}$$

The exactly solvable potential is given by Eq. (2·23) with $W_0(x)$ and $\xi_\ell(\eta;\alpha,\beta)$ given by Eqs. (4·4) and (4·3), respectively. Since the polynomials in the eigenfunctions start with degree zero, the polynomials $P_{\ell,n}(\eta;\alpha,\beta)$ cannot be considered as exceptional. In fact, this system corresponds to the system discussed in Ref. 21), which is obtained from the Darboux-Pöschl-Teller system by deleting the lowest ℓ excited states according to the Crum-Adler method. It belongs to the same class of rationally extended exactly solvable systems discussed in Refs. 22) and 23). It is easy to show, using the identities in Appendix A.2, that the polynomials in Eq. (4·18) are proportional to those given by Eq. (A.33) of Ref. 21) for the Jacobi case.

§5. Summary

We have demonstrated how all the quantal systems related to the exceptional Laguerre and Jacobi polynomials can be constructed in a direct and systematic way. In this approach one does not need to rely on the requirement of shape invariance and the Darboux-Crum transformation. Even the prepotential need not be assumed a priori. The prepotential, the deforming function, the potential, the eigenfunctions and eigenvalues are all derived within the same framework. It is worth to note that the main part of the eigenfunctions, which are the exceptional orthogonal polynomials, can be expressed as bilinear combination of the deformation function $\xi(\eta)$ and its derivative $\xi'(\eta)$. However, they are equivalent to the forms given in Ref. 5). We have also derived easily a rationally extended Jacobi model obtained from the Darboux-Pöschl-Teller system by deleting the lowest ℓ excited states according to the Crum-Adler method discussed in Ref. 21).

We have not discussed the related hyperbolic Darboux-Pöschl-Teller systems (which are of J2 type). They can be generated in the same way by choosing the appropriate sinusoidal coordinates. They can also be obtained from the trigonometric case by suitable analytic continuation.

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Appendix A
—— *Useful Identities* ——

In this Appendix, we collect some useful identities satisfied by the Laguerre and Jacobi polynomials which are used in the main text.

A.1. *Laguerre polynomials*

Some useful relations among Laguerre polynomials are:

$$\frac{d}{d\eta}L_\ell^{(\alpha)}(\eta) = -L_{\ell-1}^{(\alpha+1)}(\eta), \tag{A·1}$$

$$L_\ell^{(\alpha)}(\eta) + L_{\ell-1}^{(\alpha+1)}(\eta) = L_\ell^{(\alpha+1)}(\eta), \tag{A·2}$$

$$\eta L_{\ell-1}^{(\alpha+2)}(\eta) - (\alpha+1)\, L_{\ell-1}^{(\alpha+1)}(\eta) = -\ell L_\ell^{(\alpha)}(\eta). \tag{A·3}$$

According to the Theorem 6.73 of Ref. 24), for an arbitrary real number $\alpha \neq -1, -2, \ldots, -\ell$, the number of the positive zeros of $L_\ell^{(\alpha)}(\eta)$ is ℓ if $\alpha > -1$; it is $\ell + [\alpha] + 1$ if $-\ell < \alpha < -1$; it is 0 if $\alpha < -\ell$. Here $[a]$ denotes the integral part of a. Furthermore, $\eta = 0$ is a zero when and only when $\alpha = -1, -2, \ldots, -\ell$.

A.2. *Jacobi polynomials*

Some useful relations among the Jacobi polynomial are:

$$\frac{d}{d\eta} P_\ell^{(\alpha,\beta)}(\eta) = \frac{\ell+\alpha+\beta+1}{2} P_{\ell-1}^{(\alpha+1,\beta+1)}(\eta), \tag{A·4}$$

$$\begin{aligned} 2(\beta+1)P_\ell^{(\alpha-1,\beta+1)}(\eta) &+ (\ell+\alpha+\beta+1)(\eta+1)P_{\ell-1}^{(\alpha,\beta+2)}(\eta) \\ &= 2(\ell+\beta+1)P_\ell^{(\alpha,\beta)}(\eta), \end{aligned} \tag{A·5}$$

$$\begin{aligned} (\ell+\alpha)P_\ell^{(\alpha-1,\beta+1)}(\eta) &- \alpha P_\ell^{(\alpha,\beta)}(\eta) \\ &= \frac{1}{2}(\ell+\alpha+\beta+1)(\eta-1)P_{\ell-1}^{(\alpha+1,\beta+1)}(\eta), \end{aligned} \tag{A·6}$$

$$(n+\alpha)(1+\eta)P_{\ell-1}^{(\alpha,\beta+1)}(\eta) - \beta(1-\eta)P_{\ell-1}^{(\alpha+1,\beta)}(\eta) = 2\ell P_\ell^{(\alpha,\beta-1)}(\eta). \tag{A·7}$$

Using Eqs. (A·4) and (A·6) to eliminate the $P_{\ell-1}^{(\alpha+1,\beta+1)}(\eta)$ term gives

$$(1-\eta)\frac{d}{d\eta} P_\ell^{(\alpha,\beta)}(\eta) = \alpha P_\ell^{(\alpha,\beta)}(\eta) - (\ell+\alpha)P_\ell^{(\alpha-1,\beta+1)}(\eta). \tag{A·8}$$

Combining Eqs. (A·4) and (A·5) to eliminate the $P_{\ell-1}^{(\alpha,\beta+2)}(\eta)$ term gives

$$(1+\eta)\frac{d}{d\eta} P_\ell^{(\alpha-1,\beta+1)}(\eta) = -(\beta+1)P_\ell^{(\alpha-1,\beta+1)}(\eta) + (\ell+\beta+1)P_\ell^{(\alpha,\beta)}(\eta). \tag{A·9}$$

Setting $\alpha \to \alpha+1,\ \beta \to \beta-1$, we get

$$(1+\eta)\frac{d}{d\eta} P_\ell^{(\alpha,\beta)}(\eta) = (\ell+\beta)P_\ell^{(\alpha+1,\beta-1)}(\eta) - \beta P_\ell^{(\alpha,\beta)}(\eta). \tag{A·10}$$

According to the Theorem 6.72 of Ref. 24), for arbitrary real values of α and β, the number of zeros of $P_\ell^{(\alpha,\beta)}(\eta)$ in $(-1,1)$ is

$$N_1(\alpha,\beta) = \begin{cases} 2\left[\frac{X+1}{2}\right], & \text{if } (-1)^\ell \begin{pmatrix} \ell+\alpha \\ \ell \end{pmatrix} \begin{pmatrix} \ell+\beta \\ \ell \end{pmatrix} > 0; \\ 2\left[\frac{X}{2}\right]+1, & \text{if } (-1)^\ell \begin{pmatrix} \ell+\alpha \\ \ell \end{pmatrix} \begin{pmatrix} \ell+\beta \\ \ell \end{pmatrix} < 0. \end{cases} \tag{A·11}$$

Here

$$X \equiv E\left[\frac{1}{2}\left(|2\ell+\alpha+\beta+1| - |\alpha| - |\beta| + 1\right)\right], \tag{A·12}$$

where $E(u)$ is Klein's symbol defined by

$$E(u)=\begin{cases} 0, & u\le 0;\\ [u] & u>0,\quad u \ \text{non-integral};\\ u-1 & u=1,2,3,\ldots \quad . \end{cases}$$

From this theorem, we conclude that the conditions for $P_\ell^{(\alpha,\beta)}(\eta)$ to have no zeros in the ordinary domain $(-1,1)$ are

$$\begin{aligned} & |2\ell+\alpha+\beta+1|-|\alpha|-|\beta|+1\le 0,\\ \text{and}\quad & (-1)^\ell\begin{pmatrix}\ell+\alpha\\ \ell\end{pmatrix}\begin{pmatrix}\ell+\beta\\ \ell\end{pmatrix}>0. \end{aligned} \tag{A·13}$$

It is noted that $\eta=+1(-1)$ is a zero of $P_\ell^{(\alpha,\beta)}(\eta)$ if and only if $\alpha(\beta)=-1,-2,\ldots,-\ell$ with multiplicity $|\alpha|$ ($|\beta|$).

Appendix B
—— *L1 Laguerre Case* ——

As with the L2 Laguerre case discussed in §3, let us take $\eta(x)=x^2$. But now the negative signs in Eqs. (2·20) and (2·21) will be taken leading to $c_2(\eta)=-4\eta$ and $c_1=-2(1-Q(\eta))$.

B.1. *W_0, ξ and $\tilde{\mathcal{E}}$*

Equation determining ξ is

$$-\eta\xi''(\eta)-\frac{1}{2}\left(1-Q(\eta)\right)\xi'(\eta)+\frac{\tilde{\mathcal{E}}}{4}\xi(\eta)=0. \tag{B·1}$$

We shall take $\mathcal{E}>0$, otherwise the problem reduces to the L2 Laguerre case discussed in §3. The first term of Eq. (B·1) differs in sign from that of the the Laguerre equation (3·2). Suppose we make a parity change $\eta\to-\eta$ in Eq. (B·1), then we will have

$$\eta\xi''(-\eta)+\frac{1}{2}\left(1-Q(-\eta)\right)\xi'(-\eta)+\frac{\tilde{\mathcal{E}}}{4}\xi(-\eta)=0. \tag{B·2}$$

This equation has the form of Eq. (3·2), provided that

$$\xi(-\eta)\equiv\xi_\ell(-\eta;\alpha)=L_\ell^{(\alpha)}(\eta),\quad \tilde{\mathcal{E}}=4\ell,\quad Q(-\eta)=2\left(\eta-\alpha-\frac{1}{2}\right) \tag{B·3}$$

for some parameter α. This means

$$\xi_\ell(\eta;\alpha)=L_\ell^{(\alpha)}(-\eta), \tag{B·4}$$

and

$$Q(\eta)=-2\left(\eta+\alpha+\frac{1}{2}\right). \tag{B·5}$$

For $\xi_\ell(\eta;\alpha)$ not to have zeros in the ordinary domain $[0,\infty)$, we must have $\alpha > -1$ at the least (precise bound will be determined later). The form of $Q(\eta)$ then leads to

$$W_0(x) = -\frac{x^2}{2} - \left(\alpha + \frac{1}{2}\right)\ln x. \tag{B·6}$$

As before we ignore the constant of integration.

B.2. $p(\eta)$, $\phi(\eta)$ *and* $\mathcal{E}$

Consider $\exp(W_0) \propto \exp(-x^2/2)x^{-(\alpha+\frac{1}{2})}$ ($\alpha > -1$ at the least). Contrary to the L1 case, this time it is the term $x^{-(\alpha+\frac{1}{2})}$ that could cause $\phi(x)$ non-normalizable (when $\alpha > -1/2$) if $\mathcal{V}(\eta)$ is a polynomial in η. So we try $\mathcal{V} = \eta^\beta U(\eta)$ where β is a real parameter and $U(\eta)$ a function of η. From Eq. (2·18) we get

$$\eta U'' + (2\beta - \alpha + 1 - \eta)\,U' + \left(\frac{\beta(\beta-\alpha)}{\eta} + \frac{\mathcal{E} - \tilde{\mathcal{E}}}{4} - \beta - 1\right)U = 0. \tag{B·7}$$

If $\beta = 0,\ \alpha$, the η-dependent term in the last term of the above equation can be eliminated, and Eq. (B·7) can be reduced to the Laguerre equation (3·2). As $\beta = 0$ does not solve our original problem with normalizability of the wave function, we shall take $\beta = \alpha$. This leads to

$$U(\eta) = L_n^{(\alpha)}(\eta), \quad \mathcal{E} \equiv \mathcal{E}_n = 4(n + \alpha + \ell + 1). \tag{B·8}$$

Putting all these results into $F(\eta)$ and $G(\eta)$ gives

$$\begin{aligned} p(\eta) \equiv p_{\ell,n}(\eta) &= -4\eta^{\alpha+1}P_{\ell,n}(\eta;\alpha), \\ P_{\ell,n}(\eta;\alpha) &\equiv L_n^{(\alpha)}\xi_\ell' + \left(L_n^{(\alpha)} - L_n'^{(\alpha)}\right)\xi_\ell \\ &= L_n^{(\alpha)}\xi_\ell' + L_n^{(\alpha+1)}\xi_\ell, \end{aligned} \tag{B·9}$$

where use has been made of Eqs. (A·1) and (A·2) to get the last line. $P_{\ell,n}(\eta;\alpha)$ is a polynomial of degree $\ell + n$. It will be shown below that it is just the L1 type exceptional Laguerre polynomial. It is also easy to check that $P_{\ell,n}(\eta;\alpha)$'s are orthogonal with respect to the weight function

$$\frac{e^{-\eta}\eta^{(\alpha+1)}}{\xi_\ell^2}. \tag{B·10}$$

The exactly solvable potential is given by Eq. (2·23) with $W_0(x)$ and $\xi_\ell(\eta;\alpha)$ given by Eqs. (B·6) and (B·4), respectively. The eigenvalues are $\mathcal{E}_n = 4(n+\alpha+\ell+1)$. It is easy to show that $V(x)$ is equivalent to the potential for L1 Laguerre case in Refs. 4), 5), 11) with $\alpha = g + \ell - 3/2$ ($g > 0$). Particularly, it is exactly equal to the form of potential in Eq. (2.20) of Ref. 11). The complete eigenfunctions are

$$\phi_{\ell,n}(x;\alpha) \propto \frac{e^{-\frac{x^2}{2}}x^{(\alpha+\frac{3}{2})}}{\xi_\ell}P_{\ell,n}(\eta;\alpha), \quad \alpha > -\frac{3}{2}. \tag{B·11}$$

As in the L2 case, this system reduces to the radial oscillator system in the limit $\ell \to 0$.

B.3. *Reducing $P_{\ell,n}(\eta;\alpha)$ to HOS form*

Using Eqs. (A·1) and (A·2), we have

$$\xi'_\ell(\eta;\alpha) = L_\ell^{(\alpha+1)}(-\eta) - L_\ell^{(\alpha)}(-\eta). \tag{B·12}$$

Then it is easy to check that

$$P_{\ell,n}(\eta;\alpha) = L_n^{(\alpha)}(\eta)\xi_\ell(\eta;\alpha+1) - L_n'^{(\alpha)}(\eta)\xi_\ell(\eta;\alpha), \tag{B·13}$$

$$\xi_\ell(\eta;\alpha+1) \equiv L_\ell^{(\alpha+1)}(-\eta). \tag{B·14}$$

This is, up to a multiplicative constant, the HOS form of the L1 Laguerre polynomial, with the substitution $\alpha = g + \ell - 3/2$.

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