Prescribing Gaussian curvature on S^2

by

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§ 1. Introduction

On the standard two sphere $S^2=\{x\in\mathbf{R}^3|x_1^2+x_2^2+x_3^2=1\}$ with metric $ds_0^2=dx_1^2+dx_2^2+dx_3^2$, when the metric is subjected to the conformal change $ds^2=e^{2u}ds_0^2$, the Gaussian curvature of the new metric is determined by the following equation:

$$\Delta u+Ke^{2u}=1 \quad \text{on} \quad S^2 \tag{1.1}$$

where Δ denotes the Laplacian relative to the standard metric. The question raised by L. Nirenberg is: which function K can be prescribed so that (1.1) has a solution? There is an obvious necessary condition implied by integration of (1.1) over the whole sphere: (with $d\mu$ denoting the standard surface measure on S^2)

$$\int_{S^2} Ke^{2u}d\mu = 4\pi. \tag{1.2}$$

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Thus K must be positive somewhere. Some further necessary condition has been noted in Kazdan-Warner [9]. For each eigenfunction x_j with $\Delta x_j+2x_j=0$ ($j=1, 2, 3$), the Kazdan-Warner condition states that

$$\int_{S^2} \langle \nabla K, \nabla x_j \rangle e^{2u} d\mu = 0, \quad j=1,2,3. \tag{1.3}$$

Thus functions of the form $K=\psi \circ x_j$ where ψ is any monotonic function defined on $[-1,1]$ do not admit solutions. (We will give in §2 below an interpretation of the condition (1.3) in terms of conformal transformations of S^2.) When K is an even function on S^2 (i.e. K is reflection symmetric about the origin), Moser [12] proved that the functional

$$F[u] = \log \frac{1}{4\pi}\int_{S^2} K e^{2u} d\mu - \frac{1}{4\pi}\int_{S^2} |\nabla u|^2 d\mu - \frac{2}{4\pi}\int_{S^2} u\, d\mu \tag{1.4}$$

achieves its maximum on $H^{1,2}_{\text{even}}$ (the Sobolev space of even functions with first derivatives in $L^2(S^2)$) and hence its maximum u satisfy the Euler equation (1.1). The proof in [12] was based on Moser's sharp form of the Sobolev inequality [11]: Given $u \in H^{1,2}(S^2)$ (respectively $u \in H^{1,2}_{\text{even}}(S^2)$), there is a universal constant C_0 such that

$$\int_{S^2} \left(\exp \alpha (u-\bar{u})^2 \Big/ \int_{S^2} |\nabla u|^2 d\mu \right) d\mu \leq C_0 \tag{1.5}$$

for all $\alpha \leq 4\pi$ (respectively $\alpha \leq 8\pi$) where $\bar{u}=(1/4\pi)\int_{S^2} u\, d\mu$. In [4], we gave the corresponding version of Moser's result for those K satisfying a reflection symmetry about some plane (e.g. $K(x_1, x_2, x_3)=K(x_1, x_2, -x_3)$); in that case we exhibited a solution to (1.1) also satisfying reflection symmetry by solving the Neumann problem to (1.1) on the hemisphere $H=\{x \in S^2, x_3 \geq 0\}$ with boundary condition $\partial u/\partial n=0$ under the hypothesis that

$$\frac{1}{2\pi}\int_H K > \max\left(\max_{p \in \partial H} K(p), 0\right).$$

In case K possesses rotational symmetry some sufficient conditions were given in Hong [8]. For example, when K is rotational symmetric w.r.t. the x_3-axis, positive somewhere and $\max(K(0,0,1), K(0,0,-1)) \leq 0$.

In this paper, we give two sufficient conditions for existence to the equation (1.1). The first is an attempt to generalize Moser's result:

THEOREM I. *Let K be a smooth positive function with two nondegenerate local maxima (which we may assume w.l.o.g.) located at the north and south poles N, S. Let φ_t be the one-parameter group of conformal transformations given in terms of stereographic complex coordinates (with $z=\infty$ corresponding to N and $z=0$ corresponding to S) by $\varphi_t(z)=tz$, $0<t<\infty$. Assume*

$$\inf_{0<t<\infty} \frac{1}{4\pi}\int_{S^2} K\circ\varphi_t\,d\mu > \max_{\substack{\nabla K(Q)=0\\ Q\neq N,S}} K(Q). \tag{1.6}$$

Then (1.1) *admits a solution.*

Remark. (1.6) is an analytic condition about the distribution of K which can be verified for example if K has non-degenerate local maximum points at N, S and in addition the following properties: (1) K has (suitably) small variation in the region $\{|x_3|>\varepsilon\}$. (2) All other critical points of K occur in the strip $\{|x_3|<\varepsilon\}$ and there the critical values of K are significantly lower than the minimum value of K on $\{|x_3|>\varepsilon\}$. We observe that this is an open condition.

THEOREM II. *Let K be a positive smooth function with only non-degenerate critical points, and in addition $\Delta K(Q)\neq 0$ where Q is any critical point. Suppose there are at least two local maximum points of K, and at all saddle points of K, $\Delta K(Q)>0$, then K admits a solution to the equation* (1.1).

Remarks. (1) While under the earlier sufficient condition in [12], [8], [4] solutions obtained were local maxima of the functional F restricted to some suitable subspace of $H^{1,2}$, it is actually the case that when K is a positive function, no solution to (1.1) can be a local maximum (i.e. index zero solution) unless K is identically a constant. This follows from a second variation computation coupled with an eigenvalue estimate of Hersch [7]: Suppose u is a local maximum of the functional $F[u]$, then direct computation yields (for $(1/4\pi)\int_{S^2} Ke^{2u}=1$)

$$0\geq \frac{d^2}{dt^2}F[u+tv]\bigg|_{t=0} = 2\left[\frac{1}{4\pi}\int_{S^2} Ke^{2u}v^2-\left(\frac{1}{4\pi}\int_{S^2} Ke^{2u}v\right)^2\right]-\frac{1}{4\pi}\int_{S^2}|\nabla v|^2$$

for all $v\in H^{1,2}$. This implies that the first non-zero eigenvalue λ_1 of the Laplacian of the metric $Ke^{2u}ds_0^2$ satisfies $\lambda_1\geq 2$. While the estimate of Hersch [7] says $\lambda_1\leq 2$ with equality if and only if the metric $Ke^{2u}ds_0^2$ has constant curvature one. This coupled with the assumption that K, u satisfy (1.1) is equivalent to $K\equiv$constant.

(2) Based on the analysis in sections 3, 4, 5 of this present paper, we know the precise behavior of concentration near the saddle points of K where $\Delta K(Q)<0$, and the following stronger version of Theorem II will appear in a forthcoming article.

THEOREM II′. *Let K be a positive smooth function with only non-degenerate critical points, and in addition $\Delta K(Q)\neq 0$ where Q is any critical point. Suppose there are $p+1$ local maximum points of K, and q saddle points of K with $\Delta K(Q)<0$. If $q\neq p$ then K admits a solution to the equation* (1.1).

We sketch in the following the main idea of the proofs of Theorem I and II and an outline of the paper. As explained in the remark above, we should look for saddle points of the functional F. Thus we look for a max-min scheme for the functional F. Since the functional does not satisfy the Palais-Smale condition, we need to analyse when a maximizing max-min sequence fails to be compact. This is given in § 2 in the Concentration lemma (based on an idea of Aubin [1, Theorem 6]), where it is proved that for a sequence $u_j\in H^{1,2}(S^2)$ normalized by the condition $(1/4\pi)\int_{S^2} e^{2u_j}\,d\mu=1$, satisfying the bound

$$S[u_j]=\frac{1}{4\pi}\int_{S^2}|\nabla u_j|^2 d\mu+\frac{1}{2\pi}\int_{S^2} u_j\,d\mu\leqslant C$$

then either (i) u_j has bounded Dirichlet integral, hence we may extract a weakly convergent subsequence which gives a weak hence strong solution of (1.1) or (ii) on a subsequence the mass of e^{2u_j} concentrates at a single point P on the sphere in the sense of measure. Observe that a maximizing sequence u_j for a max-min problem will automatically satisfy the condition $S[u_j]\leqslant C$ (for some constant C; after normalizing the sequence by $(1/4\pi)\int_{S^2} e^{2u_j}\,d\mu=1$). Thus if we do not have convergence, we must study the phenomena of concentrating sequence. Our strategy is that when e^{2u} is sufficiently concentrated, we can compare $F[u]$ with $J[u]$ where

$$J[u]=\log\frac{1}{4\pi}\int_{S^2} e^{2u}d\mu-\frac{1}{4\pi}\int_{S^2}|\nabla u|^2 d\mu-\frac{1}{2\pi}\int_{S^2} u\,d\mu$$

whose critical points are known, i.e. $e^{2u}=\det|d\varphi|$ where φ is a conformal transformation of S^2. In fact $J[u]\leqslant 0$ with $J[u]=0$ precisely when $e^{2u}=\det|d\varphi|$. This analysis of J was due to Onofri [14] and will be recalled in § 3 where we also sharpen the estimates in § 2 in a form which is used crucially in later analysis (of the saddle points of K). Thus when u_j is a maximizing concentrating sequence, we will compare e^{2u_j} with $\det|d\varphi_j|$

where φ_j is a sequence of similarly concentrating conformal maps. We show that $2u_j$ is then close to $\log\det|d\varphi_j|$ in the sense that $S[u_j]$ is close to zero (which is the value of $S[\frac{1}{2}\log\det|d\varphi_j|]$). This approximation is done in § 4 and reduces the analysis of $F[u_j]$ on the concentrating sequence to that of the analysis of the first term $\log(1/4\pi)\int_{S^2}Ke^{2u_j}d\mu$, which is made explicit in our asymptotic formula (§ 5) for evaluating such integrals $\int_{S^2}fe^{2u}d\mu$ when e^{2u} is concentrated. We combine the foregoing analysis in § 6 to conclude that for a 1-dimensional max-min scheme concentration can only occur near a saddle point Q of K where $\Delta K(Q)<0$. We then give the proof of Theorem I and II in § 7.

We remark here that the analysis provided in § 4 and § 5 is more than sufficient to prove Theorems I and II, however for ease of future reference we give the complete analysis here.

While Theorems I and II and the previously cited work give sufficient conditions for existence of equation (1.1), there is another result of Kazdan-Warner [10] which states that for any K positive somewhere on S^2, there always exists some diffeomorphism φ so that the equation $\Delta u+K\circ\varphi e^{2u}=1$ is solvable. It is therefore of interest to find some analytic conditions on the class of functions K which is topologically simple (e.g. K has only a global maximum and a global minimum) that ensures existence of a solution of (1.1).

In related developments for the analogous equation of prescribing scalar curvature on a compact manifold M of dimension n, $n\geq 3$, the corresponding equation becomes

$$4\frac{n-1}{n-2}\Delta u+R\,u^{(n+2)/(n-2)}=R_0\,u$$

where R_0 is the scalar curvature of the underlying metric ds_0^2 and R is the prescribed scalar curvature of the conformally related metric $ds^2=u^{4/(n-2)}ds_0^2$. When R=constant, this was Yamabe's problem and is recently solved by Aubin [2] and Schoen [16]. While for the analogous problem of prescribing R on S^n ($n\geq 3$) with ds_0^2 the standard metric on S^n, Escobar and Schoen gave [5] the analogue of Moser's theorem (for even functions on S^2); Bahri and Coron [3] have announced an analogue of Theorem II′ on S^3.

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§ 2. Preliminary facts and notations

The standard 2-sphere S^2 is usually represented as $\{x\in\mathbf{R}^3 \mid |x|^2=1\}$. Relative to any orthonormal frame e_1, e_2, e_3 of $\mathbf{R}^3$ we have the Euclidean coordinates $x_i=x\cdot e_i$ and we call (0, 0, 1) (resp. (0, 0, −1)) the north pole (resp. south pole). Through the stereographic projection to the (x_1, x_2)-plane we have the complex stereographic coordinates

$$z=\frac{x_1+ix_2}{1-x_3} \tag{2.1}$$

which has inverse transformation

$$x_1=\frac{2}{1+|z|^2}\operatorname{Re} z,\quad x_2=\frac{2}{1+|z|^2}\operatorname{Im} z,\quad x_3=\frac{|z|^2-1}{|z|^2+1}. \tag{2.2}$$

The conformal transformations of S^2 are thus identified with fractional linear transformations

$$w=\frac{az+b}{cz+d},\quad ad-bc=1,\quad a, b, c, d \text{ complex numbers}$$

which form a six-dimensional Lie group. For our purpose we need the following set of conformal transformations: Given $P\in S^2$, $t\in(0,\infty)$ we choose a frame $e_1, e_2, e_3=P$, then using the stereographic coordinates with P at infinity we denote the transformation

$$\varphi_{P,t}(z)=tz. \tag{2.3}$$

Observe that $\varphi_{P,1}\equiv\mathrm{id}$ and $\varphi_{P,t}=\varphi_{-P,t^{-1}}$, hence the set of conformal transformations $\{\varphi_{P,t} \mid P\in S^2,\ t\geq 1\}$ is parametrized by $B^3\cong S^2\times[1,\infty)/S^2\times\{1\}$, where B^3 is the unit ball in $\mathbf{R}^3$ with each point $(Q, t)\in S^2\times[1,\infty)$ identified with $((t-1)/t)\,Q\in B^3$. $H^1=H^{1,2}=H^{1,2}(S^2)$ is the Sobolev space of L^2 functions on S^2 whose gradients also lie in L^2.

$$\|u\|_{1,2}=\left(\frac{1}{4\pi}\int_{S^2}(|\nabla u|^2+u^2)\,d\mu\right)^{1/2},$$

we also denote

$$\|u\|=\left(\frac{1}{4\pi}\int_{S^2}|\nabla u|^2 d\mu\right)^{1/2}.$$

We adopt the notation $⨍ f$ to mean the average integral $(1/4\pi)\int_{S^2} f\,d\mu$.

Definition. For $u \in H^{1,2}(S^2)$ let

$$S[u] = \int |\nabla u|^2 + 2\int u. \tag{2.4}$$

$$J[u] = \log \int e^{2u} - S[u]. \tag{2.5}$$

$$F[u] = F_K[u] = \log \int Ke^{2u} - S[u]. \tag{2.6}$$

The critical point of $J[u]$ satisfy the Euler equation

$$\Delta u + e^{2u} = 1 \tag{2.7}$$

where Δ denotes the Laplacian with respect to the standard metric. All solutions of (2.7) are of the form $u = \frac{1}{2}\log \det |d\varphi|$, φ a conformal map of S^2. Similarly the critical points of $F_K[u]$ satisfy the Euler equation

$$\Delta u + Ke^{2u} = 1. \tag{2.8}$$

The functional $S[u]$ enjoys the following invariance property:

Definition. Given $u \in H^{1,2}$ and φ a conformal transformation. Let

$$u_\varphi = u \circ \varphi + \tfrac{1}{2}\log \det |d\varphi|. \tag{2.9}$$

We also write $T^t(Q)(u)$ for u_φ when $\varphi = \varphi_{Q,t}$.

PROPOSITION 2.1. $S[u] = S[u_\varphi]$.

The proof is left as an exercise in integration by parts, using the equation (2.7) for the part $\frac{1}{2}\log \det |d\varphi|$.

The implicit condition found by Kazdan-Warner [9] is an easy consequence of $S[u] = S[u_\varphi]$:

COROLLARY 2.1. *If u satisfies* (2.8) *then*

$$\int \langle \nabla K, \nabla x_j \rangle e^{2u} = 0, \quad j = 1, 2, 3. \tag{2.10}$$

Proof. u is a critical point of $F[u]$, hence

$$\frac{d}{dt}F[T^t(Q)(u)]\Big|_{t=1}=0.$$

But

$$\begin{aligned} F[T^t(Q)(u)] &= \log\int K\exp(2T^t(Q)(u))-S[T^t(Q)(u)] \\ &= \log\int K\exp(2T^t(Q)(u))-S[u] \\ &= \log\int Ke^{2u\circ\varphi_{Q,t}}\det|d\varphi_{Q,t}|-S[u] \\ &= \log\int K\circ\varphi_{Q,t}^{-1}\cdot e^{2u}-S[u]. \end{aligned}$$

Thus

$$\frac{d}{dt}F[T^t(Q)(u)]\Big|_{t=1}=\int\frac{d}{dt}(K\circ\varphi_{Q,t}^{-1})\Big|_{t=1}e^{2u}.$$

But a simple calculation shows

$$\frac{d}{dt}K\circ\varphi_{Q,t}^{-1}\Big|_{t=1}=\langle\nabla K,\nabla x_3\rangle,\quad\text{if}\quad x_3=\mathbf{x}\cdot Q.$$

This gives the desired conditions.

More generally, Kazdan and Warner [9, p. 33] found the following implicit consequence by a tricky partial integration. If $\Delta v+he^v=c$ then

$$\int e^v\nabla h\cdot\nabla x_i=(2-c)\int e^vhx_i,\quad i=1,2,3. \tag{2.11}$$

Given $u\in H^{1,2}(S^2)$, e^{2u} may be thought of as a mass distribution. So we define the center of mass of e^{2u}: $\text{C.M.}(e^{2u})=⨍\mathbf{x}e^{2u}/⨍e^{2u}$.

Definition. $$\mathcal{S}=\{u\in H^{1,2}|\text{C.M.}(e^{2u})=\mathbf{0}\}$$
$$\mathcal{S}_0=\{u\in\mathcal{S}|⨍e^{2u}=1\}$$

For each $Q\in S^2$, $0<t<\infty$,

$$\mathcal{S}_{Q,t} = \{u \in H^{1,2} | u_{\varphi_{Q,t}} \in \mathcal{S}_0\}.$$

For each $P \in S^2$, $0<\delta<1$,

$$C_{P,\delta} = \left\{ u \in H^{1,2} \middle| \frac{\text{C.M.}(e^{2u})}{|\text{C.M.}(e^{2u})|} = P;\ 1-\delta = ⨍ P \cdot \mathbf{x}\, e^{2u} \right\}.$$

We remark that for each $u \in H^{1,2}$ with $⨍ e^{2u} = 1$, we can find some $(Q, t) \in S^2 \times [1, \infty)$ with $u \in \mathcal{S}_{Q,t}$. This is an easy consequence of the fixed point theorem. A rigorous proof of the fact can be found in [14], but for our purpose later we also need the continuous dependence of the choice of (Q, t) in terms of a continuous path of u in $H^{1,2}(S^2)$. We state this as (and post-pone the proof to the appendix).

PROPOSITION 2.2. *Given a continuous map*

$$u\colon \mathbf{R}\ (\text{or } \Delta\text{: the unit disc in the complex plane}) \to H^{1,2}(S^2),$$

there is a continuous map

$$(Q, t)\colon \mathbf{R}\ (\text{or } \Delta) \to S^2 \times [1, \infty)/S^2 \times \{1\} \cong B^3;$$

so that $u(s) \in \mathcal{S}_{Q(s), t(s)}$ *for all* $s \in \mathbf{R}$ (*or* $s \in \Delta$).

We are ready to state the Concentration lemma which is the main technical result of this section.

PROPOSITION A. *Given a sequence of functions* $u_j \in H^{1,2}(S^2)$ *with* $⨍ e^{2u_j} = 1$ *and* $S[u_j] \leq C$, *then either*

(i) *there exists a constant* C' *such that* $⨍ |\nabla u_j|^2 \leq C'$

or

(ii) *a subsequence concentrates at a point* $P \in S^2$, i.e. *given* $\varepsilon>0$, $\exists N$ *large such that*

$$⨍_{B(p,\varepsilon)} e^{2u_j} \geq (1-\varepsilon), \quad \text{for} \quad j \geq N$$

where $B(P, \varepsilon)$ *is the ball in* S^2 *of radius* ε, *centered at* P.

The proof is based on the following result of Aubin [1, Theorem 6], since a sharpened version of this result (see Proposition B in § 3) will be required later on we refer the reader to § 3 for its proof.

PROPOSITION 2.3. [1]. *Suppose* $u\in H^{1,2}$ *with* $⨍ e^{2u}x_j=0$ *for* $j=1,2,3$ *then for every* $\varepsilon>0$, *there exists a constant* C_ε *with*

$$⨍ e^{2u}\leqslant C_\varepsilon \exp\left(\left(\frac{1}{2}+\varepsilon\right)⨍ |\nabla u|^2+2⨍ u\right). \tag{2.12}$$

(2.12) is often used in the following way:

COROLLARY 2.2. *Suppose* $u\in \mathcal{S}_0$ *then* $⨍|\nabla u|^2\leqslant 4(S[u]+\log C_{1/4})$ *where* $C_{1/4}$ *is the same constant as in* (2.12) *with* $\varepsilon=1/4$.

Proof of Corollary 2.2. Since $u\in\mathcal{S}_0$ we have $⨍ e^{2u}=1$, thus by (2.12) for each $\varepsilon>0$ we have

$$\log\frac{1}{C_\varepsilon}\leqslant\left(\frac{1}{2}+\varepsilon\right)⨍ |\nabla u|^2+2⨍ u.$$

Choose $\varepsilon=\frac{1}{4}$. We have

$$\begin{aligned}\frac{1}{4}⨍ |\nabla u|^2&=\left(⨍ |\nabla u|^2+2⨍ u\right)-\left(\frac{3}{4}⨍ |\nabla u|^2+2⨍ u\right)\\&\leqslant S[u]+\log C_{1/4}.\end{aligned}$$

We remark that the statement of Corollary 2.2 should be compared with the sharpened version of Corollary 3.1 in § 3. It indicates that $\mathcal{S}_0$ forms a compact family in $H^{1,2}$ in the subset where $S[u]$ stays bounded. This is a key fact which has also been used in the proof of Onofri's inequality ([14]) which we now state:

PROPOSITION 2.4. [14]. *Given* $u\in H^{1,2}(S^2)$ *then we have* $J[u]\leqslant 0$ *with the equality holding only for* $u=\frac{1}{2}\log\det|d\varphi|$ *where* φ *is a conformal map of* S^2.

Remark. The functional $J[u]$ has intrinsic geometric meaning which motivated the study in [14] of Proposition 2.4 above. Namely given $u\in H^{1,2}(S^2)$ with $⨍ e^{2u}=1$, let $ds^2=e^{2u}ds_0^2$ and denote $\tilde\Delta=e^{-2u}\Delta$ the Laplace-Beltrami operator associated to ds^2, and let $0=\tilde\lambda_0<\tilde\lambda_1\leqslant\tilde\lambda_2\leqslant\ldots\leqslant\tilde\lambda_n\to\infty$ be the spectrum of $-\tilde\Delta$ (Δ and $\{\lambda_k\}$ will denote the corresponding objects belonging to ds_0^2), then it was pointed out in [15] that the limit

$$\frac{\det \tilde{\Delta}}{\det \Delta} \equiv \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\tilde{\lambda}_k}{\lambda_k} = e^{-(1/3)S[u]} \tag{2.13}$$

exists. Proposition 2.4 states that under the normalization ($⨍ e^{2u}=1$), $S[u]$ is always ≥ 0 and is zero only when $e^{2u}ds_0^2$ is isometric to the standard metric ds_0^2. Thus the limit in (2.13) is always ≤ 1, with 1 only obtained by the standard metric (up to isometries).

By a simple change of variable, we often refer to Onofri's inequality in the form

$$⨍ e^{Cu} \leq \exp\left(\frac{C^2}{4} ⨍ |\nabla u|^2 + C ⨍ u\right) \quad \text{for any real number } C. \tag{2.14}$$

Notice also that by taking $C=2$ in (2.14), we have again that for $u \in H^{1,2}(S^2)$ with $⨍ e^{2u}=1$ then $S[u]\geq 0$.

We now finish this section by proving Proposition A.

Proof of the Concentration lemma (Proposition A). Since $u_j \in \mathcal{S}_{Q_j,t_j}$ so that $v_j = T^{t_j}(Q_j)(u_j) \in \mathcal{S}_0$ and $S[v_j]=S[u_j]$ we have $S[v_j]\leq C$. It follows from Corollary 2.2 above that $⨍|\nabla v_j|^2 \leq C'$. We have two possibilities. Either all φ_{Q_j,t_j} lie in a compact set, i.e. $t_j \leq C''$, in which case it follows easily that $⨍|\nabla u_j|^2 \leq C(C',C'')$ or the t_j do not remain bounded, in that case a subsequence still denoted u_j has $t_j \to \infty$ and $Q_j \to P$. Further since $⨍|\nabla v_j|^2 \leq C'$ a subsequence converges weakly to $v_\infty \in \mathcal{S}_0$. Since

$$⨍_{B(P,\varepsilon)} e^{2u_j} = ⨍_{\varphi^{-1}_{Q_j,t_j}(B(P,\varepsilon))} e^{2v_j}$$

the right hand side converges to

$$⨍_{\varphi^{-1}_{Q_j,t_j}(B(P,\varepsilon))} e^{2v_\infty}$$

which for j large is greater than $1-\varepsilon$, this proves the Concentration lemma.

§3. A variant of Onofri's inequality

In this section, we will prove a variant of Onofri's inequality as stated in §2. The statement of this variant is somewhat technical, but we need to use a consequence of the inequality (stated as a corollary below) in the proof of Proposition C and D in §4 and §5.

PROPOSITION B. *There exists some $a<1$ such that for all*

$$u\in\mathcal{S},\quad \fint e^{2u}\leqslant \exp\left(a\fint|\nabla u|^2+2\fint u\right).$$

Since $(1-a)\fint|\nabla u|^2=S[u]-(a\fint|\nabla u|^2+2\fint u)$, we get from Proposition B a direct consequence:

COROLLARY 3.1. *If $u\in\mathcal{S}_0$ then $\fint|\nabla u|^2\leqslant(1-a)^{-1}S[u]$.*

To prove Proposition B, we begin with the idea behind the original proof of Onofri's inequality and consider for each $a\leqslant 1$, the functional

$$J_a(u)=\log\fint e^{2u}-\left(a\fint|\nabla u|^2+2\fint u\right) \tag{3.1}$$

and let $\mathcal{M}_a=\sup_{u\in\mathcal{S}}J_a(u)$. Then by the result of Aubin (Proposition 2.3) for each $a>\frac{1}{2}$, $\mathcal{M}_a$ is achieved by some function $u_a\in\mathcal{S}_0$ which satisfies:

For each $\eta>0$, there exists a constant C_η with

$$\fint|\nabla u_a|^2\leqslant C_\eta \quad \text{for} \quad 1\geqslant a\geqslant\frac{1}{2}+\eta. \tag{3.2}$$

$$a\Delta u_a+e^{2u_a}=1+\sum_{j=1}^{3}\alpha_j^a x_j e^{2u_a} \quad \text{on } S^2 \quad \text{for some constants } \alpha_j^a,\ j=1,2,3. \tag{3.3}$$

We claim

$$u_a\equiv 0 \quad \text{for } a \text{ sufficiently close to 1.} \tag{3.4}$$

Assuming (3.4), it is then obvious that $\mathcal{M}_a=0$ for a sufficiently close to 1 and the assertion in Proposition B follows.

Proof of (3.4). We will first establish a general lemma.

LEMMA 3.1. *Suppose $u\in\mathcal{S}$ satisfies the equation*

$$a\Delta u+e^{2u}=1+\sum_{j=1}^{3}\alpha_j x_j e^{2u}, \quad \text{on } S^2$$

for some constants α_j $(j=1,2,3)$ and some $a\leqslant 1$. Then $\alpha_j=0$ for $j=1,2,3$.

Proof of Lemma 3.1. Applying the Kazdan-Warner condition (2.11) with $v=2u$, $c=2/a$, $h=(2/a)(1-\Sigma_{i=1}^3 \alpha_i x_i)$, we get for each $j=1,2,3$,

$$-\frac{2}{a}\int e^{2u}\nabla\left(\sum_{i=1}^{3}\alpha_i x_i\right)\cdot\nabla x_j\,d\mu=\left(2-\frac{2}{a}\right)\cdot\frac{2}{a}\int e^{2u}\left(1-\sum_{i=1}^{3}\alpha_i x_i\right)x_j\,d\mu$$
$$=-\left(2-\frac{2}{a}\right)\cdot\frac{2}{a}\int e^{2u}\left(\sum_{i=1}^{3}\alpha_i x_i\right)x_j\,d\mu. \tag{3.5}$$

Multiplying (3.5) by α_j and sum over $j=1,2,3$, we get

$$-\frac{2}{a}\int e^{2u}\left|\nabla\left(\sum_{i=1}^{3}\alpha_i x_i\right)\right|^2 d\mu=-\frac{2}{a}\left(2-\frac{2}{a}\right)\int e^{2u}\left|\sum_{i=1}^{3}\alpha_i x_i\right|^2 d\mu. \tag{3.6}$$

When $a<1$, the left hand side of (3.6) is always negative while the right hand side is always positive (or zero when $a=1$) unless $\Sigma_{i=1}^3 \alpha_i x_i \equiv 0$, i.e. $\alpha_i=0$ for all $i=1,2,3$, which finishes the proof of the lemma.

Applying Lemma 3.1 to the functions u_a ($a\leq 1$), we get

$$a\Delta u_a+e^{2u_a}=1. \tag{3.3$'$}$$

We will now use (3.3)′ to derive some pointwise estimates of u_a.

LEMMA 3.2. u_a *($a\leq 1$) satisfies*

(a) $\displaystyle⨍ e^{4(u_a-⨍ u_a)}=1+o(1)$ *as* $a\to 1$,

(b) $\displaystyle⨍ u_a=o(1)$ *as* $a\to 1$,

(c) *actually,* $u_a(\xi)=o(1)$ *for all* $\xi\in S^2$ *as* $a\to 1$.

Proof. (a) Assuming the contrary, there will be an $\varepsilon>0$ and a sequence $a_k\to 1$ with $v_k=u_{a_k}-⨍ u_{a_k}$ satisfying $⨍ e^{4v_k}\geq 1+\varepsilon$ as $k\to\infty$. From (3.2), there is some $v\in H^1$ with $v_k\to v$ weakly in H^1. Thus by an argument in Moser [12], $⨍ e^{cv_k}\to ⨍ e^{cv}$ for any real number c, in particular $c=2,4$, and also $v_k\in\mathscr{S}$ implies $v\in\mathscr{S}$. Thus

$$J[v]=J_1[v]=\log⨍ e^{2v}-\left(⨍|\nabla v|^2+2⨍ v\right)$$
$$=\log⨍ e^{2v}-\left(⨍|\nabla v|^2\right)$$

$$\begin{aligned}
&\geqslant \limsup J_1(v_k)\\
&= \limsup \left(J_{a_k}(v_k) - (1-a_k) \fint |\nabla v_k|^2 \right)\\
&= \limsup \left(\mathcal{M}_{a_k} - (1-a_k) \fint |\nabla v_k|^2 \right)\\
&\geqslant 0.
\end{aligned}$$

On the other hand $J_1[v] \leqslant \mathcal{M}_1 = 0$ by Onofri's inequality. Thus $J_1[v]=0$ and hence v is a solution of the equation $\Delta v + e^{2v}/\fint e^{2v} = 1$. This together with the fact that $v \in \mathcal{S}$ with $\fint v = 0$ implies $v \equiv 0$, which contradicts our assumption that

$$\fint e^{4v} = \lim_k \fint e^{4v_k} \geqslant 1+\varepsilon$$

and establishes (a).

(b) is an easy consequence of (a): we simply notice that $\fint e^{2u_a} = 1$ and hence $\fint e^{4u_a} \geqslant 1$ and $\fint u_a \leqslant 0$.

(c) To see this, we apply Green's function to the equation (3.3)′ and obtain for all $\xi \in S^2$, $G(\xi, P)$ the Green's function on S^2,

$$\begin{aligned}
-u_a(\xi) + \fint u_a &= \fint_{S^2} \Delta u_a(P)\, G(\xi, P)\, d\mu(P)\\
&= \frac{1}{a} \fint (1 - e^{2u_a(P)})\, G(\xi, P)\, d\mu(P).
\end{aligned}$$

Thus

$$\begin{aligned}
\left| u_a(\xi) - \fint u_a \right| &\leqslant \frac{1}{a} \left(\fint (1-e^{2u_a})^2 \right)^{1/2} \left(\int |G(\xi, P)|^2 d\mu(P) \right)^{1/2}\\
&\leqslant \mathrm{const} \cdot \frac{1}{a} \left(\fint e^{4u_a} - 1 \right)^{1/2}
\end{aligned}$$

and the claimed estimate (c) follows from (a) and (b).

We will now finish the proof of (3.4) by using the fact that $u_a \in \mathcal{S}_0$ and that the next eigenvalue of the Laplacian operator in S^2 greater than 2 is 6. Thus

$$6 \fint (e^{2u_a} - 1)^2 \leqslant \fint |\nabla (e^{2u_a} - 1)|^2$$

$$\begin{aligned}
&= 4\int |\nabla u_a|^2 e^{4u_a} \\
&= -\int (\Delta u_a)\, e^{4u_a} \\
&= -\frac{1}{a}\int (1-e^{2u_a})\, e^{4u_a} \quad \text{(by (3.3)}'\text{)} \\
&= \frac{1}{a}\int (e^{2u_a}-1)(e^{4u_a}-1) \\
&= \frac{1}{a}\int (e^{2u_a}-1)^2(e^{2u_a}+1) \\
&= \frac{2+(o(1))}{a}\int (e^{2u_a}-1)^2 \quad \text{as} \quad a\to 1
\end{aligned}$$

where the last step follows from (c) in Lemma 3.2. Thus as $a\to 1$, $⨍(e^{2u_a}-1)^2=0$, i.e. $u_a\equiv 0$ as $a\to 1$, which finishes the proof of (3.4) and hence the proof of Proposition B.

Remark. In view of Aubin's inequality (2.2) and (3.2) above, it may be interesting to see if Proposition B holds with $a=\frac{1}{2}$.

§4. A Lifting lemma: comparing $F[u]$ with $J[u]$

When a function $u\in H^{1,2}$ or a parametrized family of functions $u_s\in H^{1,2}$ with $⨍ e^{2u_s}=1$ is sufficiently concentrated near a point $P\in S^2$, namely $u\in \mathscr{S}_{Q,t}$ with $t\geq t_0$, we can compare the functional $F[u]$ with $J[u]$. In fact we will compare $F'[u]$ with $J'[u]$ to construct a continuous lifting process which increases the value of $F[u]$ and $J[u]$ simultaneously until $S[u]$ becomes suitably small while leaving fixed the class $\mathscr{S}_{Q,t}$ to which u or u_s belongs. We formulate this process as the following Lifting lemma.

PROPOSITION C. *Given u_s a continuous family in $H^{1,2}$, where $u_s\in \mathscr{S}_{Q_s,t_s}$ with t_s large and $S[u_s]\leq c_1$, there exists a continuous path $u_{s,\gamma}$, $\gamma\in[0,\gamma_0]$ with $u_{s,0}=u_s$, $u_{s,\gamma}\in \mathscr{S}_{Q_s,t_s}$ for all $\gamma\in[0,\gamma_0]$ such that $J[u_{s,\gamma}]$, $F[u_{s,\gamma}]$ both are increasing in γ and $S[u_{s,\gamma_0}]=O(t_s^{-1}(\log t_s)^2)$.*

The basic concepts behind the proof of Proposition C are the facts that $J'[u]=0$ only when $S[u]=0$ and also that the functional value of $J[u]$ remains unchanged when one changes u to $T'(P)(u)$. Thus when the center of e^{2u} is sufficiently close to the boundary of the unit ball **B**, we anticipate $F[u]$ to behave like $J[u]$. To make these statements more precise, we will break the proof of Proposition C into several lemmas. The first lemma is a technical one, which lists some properties of functions in $\mathcal{S}_0$ (recall $\mathcal{S}_0=\{u\in H^1, ⨍ e^{2u}=1, ⨍ e^{2u}x_j=0, j=1,2,3\}$) which we will use in the proof of Lemmas 4.2 and 4.3.

LEMMA 4.1. *Suppose* $u\in\mathcal{S}_0$. *Then*

$$\left|⨍ ux_j\right| \leq -⨍ u \leq \frac{1}{2}⨍ |\nabla u|^2, \quad j=1,2,3 \tag{4.1}$$

Suppose $S[u]\leq C_1$, *then*

$$\begin{gathered}\text{the matrix } \Lambda(u)=(\Lambda_{ij}(u)) \quad \text{with} \quad \Lambda_{ij}(u)=⨍ e^{2u}x_ix_j, \quad i,j=1,2,3 \\ \text{has lowest eigenvalue } \geq C(C_1)>0.\end{gathered} \tag{4.2}$$

Proof. The first inequality in (4.1) is an easy consequence of the following inequalities: (with $a_j=⨍ ux_j$; $\bar{u}=⨍ u$)

$$1=⨍ e^{2u}(1-x_j)\geq e^{2⨍ u(1-x_j)}=e^{2(\bar{u}-a_j)}$$

$$1=⨍ e^{2u}(1+x_j)\geq e^{2⨍ u(1+x_j)}=e^{2(\bar{u}+a_j)}.$$

Thus $\bar{u}-a_j\leq 0$, $\bar{u}+a_j\leq 0$, i.e. $|a_j|\leq -\bar{u}$ for each $j=1,2,3$. The fact $-\bar{u}\leq\frac{1}{2}⨍|\nabla u|^2$ is the content of Onofri's inequality. To prove (4.2), for each unit vector $\mathbf{C}=(C_1,C_2,C_3)$ we have

$$\langle\Lambda(u)\mathbf{C},\mathbf{C}\rangle=⨍ e^{2u}\left|\sum C_ix_i\right|^2=⨍ e^{2u}\langle x,x\rangle \quad \text{where} \quad x=\sum_{i=1}^{3}C_jx_j.$$

Since

$$\begin{aligned}⨍ e^{2u}\langle x,x\rangle &\geq \left(⨍ \langle x,x\rangle\right)^2\left(⨍ e^{-2u}\langle x,x\rangle\right)^{-1}\\ &=\left(\frac{1}{3}\right)^2\left(⨍ e^{-2u}\langle x,x\rangle\right)^{-1}\end{aligned}$$

to see that the eigenvalue of $\Lambda(u)$ are bounded from below for $u \in \mathcal{S}_0$, it suffices to prove that the eigenvalue of $\Lambda(-u)$ are bounded from above. We can see this latter fact by applying again the Onofri's inequality and Corollary 3.1:

$$\begin{aligned} ⨍ e^{-2u}\langle x,x\rangle &\leq \left(⨍ e^{-4u}\right)^{1/2}\left(⨍ \langle x,x\rangle^2\right)^{1/2} \\ &\lesssim e^{-2⨍u+2⨍|\nabla u|^2} \\ &\lesssim e^{4⨍|\nabla u|^2} \lesssim e^{\frac{4}{1-a}S[u]} \end{aligned}$$

LEMMA 4.2. *For each constants $C_1, C_2>0$, there exists some constant $C(C_1,C_2)>0$ with*

$$\inf_{\substack{C_2\leq S[u]\leq C_1 \\ u\in\mathcal{S}_0}} \sup_{v\in\mathcal{H}(u)} \frac{J'[u](v)}{\|v\|} \geq C(C_1,C_2)>0. \tag{4.3}$$

where

$$\mathcal{H}(u)=\left\{v\in H^1,\ ⨍ v=0,\ ⨍ e^{2u}vx_j=0 \ for\ j=1,2,3\right\}.$$

And if u depends continuously ($H^{1,2}$ topology) on some parameter s, v satisfying (4.3) *can be chosen continuously as well.*

Proof. We will argue by contradiction. Suppose (4.3) fails, then there exists some sequence $\{u_n\}$, ε_n with $u_n\in\mathcal{S}_0$, $C_2\leq S[u_n]\leq C_1$, $\varepsilon_n\to 0$ as $n\to\infty$ such that for all $v_n\in\mathcal{H}(u_n)$ we have

$$J'[u_n](v_n)\leq\varepsilon_n\|v_n\|. \tag{4.4}$$

Since $u_n\in\mathcal{S}_0$ we have

$$⨍ |\nabla u_n|^2 \leq \frac{1}{1-a}S[u_n]\leq\frac{C_1}{1-a}.$$

Thus some subsequence of $\{u_n\}$, which we will denote by $\{u_n\}$, again will have a weak limit in $H^{1,2}$ to some $u\in\mathcal{S}_0$ with $S[u]\leq C_1$. We now claim

$$J'[u](v)=0 \quad \text{for all} \quad v\in\mathcal{H}(u). \tag{4.5}$$

To see (4.5), fixe $v\in\mathcal{H}(u)$ with $\|v\|_2=1$, let $\gamma_n^j=⨍ e^{2u_n}vx_j$, $j=1,2,3$, $\boldsymbol{\gamma}_n=(\gamma_n^1,\gamma_n^2,\gamma_n^3)$ and choose $\boldsymbol{\beta}_n$ to satisfy $\Lambda(u_n)\boldsymbol{\beta}_n=\boldsymbol{\gamma}_n$. Then $v_n=v-\Sigma_{j=1}^3\beta_n^j x_j$ is in $\mathcal{H}(u_n)$ and we have

$$\int |\nabla v_n|^2 = \int |\nabla v|^2 - 4\sum_{j=1}^{3} \beta_n^j b_j + \frac{2}{3}\sum_{j=1}^{3} (\beta_n^j)^2 \tag{4.6}$$

where $b_j = ⨍ v x_j$ (thus $|b_j| \leq \frac{1}{2}\|v\|^2 = \frac{1}{2}$) and

$$\begin{aligned} J'[u_n](v) &= J'[u_n](v_n) - 2\sum_{j=1}^{3} \beta_n^j ⨍ \nabla u_n \cdot \nabla x_j \\ &= J'[u_n](v_n) - 4\sum_{j=1}^{3} \beta_n^j \alpha_n^j \end{aligned} \tag{4.7}$$

where $\alpha_n^j = ⨍ u_n x_j$.

We now notice that by (4.2) we have ($\boldsymbol{\gamma} = (⨍ e^{2u} v x_j) = 0$)

$$\boldsymbol{\beta}_n = (\Lambda(u_n))^{-1} \boldsymbol{\gamma}_n \to (\Lambda(u))^{-1} \boldsymbol{\gamma} = 0 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

We also have from (4.1) and Corollary 3.1

$$|\alpha_n^j| = \left| ⨍ u_n x_j \right| \leq \frac{1}{2} \|u_n\|^2 \leq \frac{1}{2} \frac{1}{1-a} S[u_n] \leq \frac{C_1}{2(1-a)}. \tag{4.9}$$

Substituting (4.8), (4.9) and (4.4) into (4.6), (4.7) we conclude that $\|v_n\|^2 \to 1$ and $J'[u_n](v) \to 0$ as $n \to \infty$. Since $J'[u_n](v) \to J'[u](v)$ from the definition of weak convergence, we have proved that $J'[u](v) = 0$ for all $v \in \mathscr{S}(u)$ as claimed in (4.5).

Since

$$\begin{aligned} \frac{1}{2} J'[u](v) &= ⨍ e^{2u} v - \left(⨍ \nabla u \nabla v + ⨍ v \right) \\ &= ⨍ (e^{2u} + \Delta u - 1) v, \end{aligned}$$

an immediate consequence of (4.5) is that u satisfies

$$\Delta u + e^{2u} = 1 + \sum_{j=1}^{3} d_j x_j e^{2u}$$

for some coefficients d_j, $j = 1, 2, 3$. Since $u \in \mathscr{S}_0$, we apply Lemma 3.1 to conclude $d_j = 0$ for all $j = 1, 2, 3$. Thus $\Delta u + e^{2u} = 1$ and we have (since $u \in \mathscr{S}_0$) $u \equiv 0$.

We will now see that $u \equiv 0$ contradicts our assumption that $S[u_n] \geq C_1$ by the following reasoning: Applying (4.4) and similar arguments as in (4.5)–(4.9), we obtain $|J'[u_n]| \to 0 = |J'[0]|$. Thus

$$\begin{aligned}|J'[u_n](u_n)| &\leqslant |J'[u_n]|\,\|u_n\| \leqslant |J'[u_n]| \left(\frac{1}{1-a}\right)^{1/2} (S[u_n])^{1/2} \\ &\leqslant |J'[u_n]| \left(\frac{C_1}{1-a}\right)^{1/2} \to 0 \quad \text{as} \quad n\to\infty. \\ 2J'[u_n](u_n) &= \fint |\nabla u_n|^2 - \fint e^{2u_n}(u_n-\bar{u}_n) \\ &\to \fint |\nabla u_n|^2 - \fint e^{2u}(u-\bar{u}) \\ &\to \fint |\nabla u_n|^2 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus $\fint|\nabla u_n|^2\to 0$ as $n\to\infty$. Since $\fint|\nabla u_n|^2 \geqslant S[u_n] \geqslant C_1$ for all u_n with $\fint e^{2u_n}=1$. We have obtained a contradiction and thus established (4.3) in Lemma 4.2.

The continuous dependence assertion follows from the following observations. Firstly $\mathcal{S}(u)$ depends continuously on u, since for all $\varphi \in H^{1,2}$ we have

$$\left|\fint (e^{2u}-e^{2(u+\delta u)})\,\varphi\right| \leqslant \left(\fint e^{2pu}\right)^{1/p} \left(\fint |e^{2\delta u}-1|^q\right)^{1/q} \left(\fint \varphi^r\right)^{1/r} \quad \text{for all} \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,$$

and the middle factor can be estimated by

$$\begin{aligned}\left(\fint |e^{2\delta u}-1|^q\right) &\leqslant \fint (2|\delta u| e^{2\delta u})^q \\ &\leqslant \left(\fint (2|\delta u|)^2\right)^{1/2} \left(\fint e^{4q\delta u}\right)^{1/2} \to 0 \quad \text{as} \quad \|\delta u\|_{1,2}\to 0.\end{aligned}$$

while the remaining factors stay bounded.

Therefore as u_s varies continuously in the parameter s, $\sup_{v\in\mathcal{S}(u_s)} J'[u]\,v/\|v\|$ also varies continuously in s. By taking the projection of v_{s_0} (which achieves $\sup_{v\in\mathcal{S}(u_{s_0})} J'[u]\,v/\|v\|$) into the nearby $\mathcal{S}(u_s)$; we can achieve in a neighborhood of s a continuous choice v_s such that

$$\frac{J'[u_s](v_s)}{\|v_s\|} \geqslant \frac{1}{2} \sup_{v\in\mathcal{S}(u_s)} \frac{J'[u_s](v)}{\|v\|}.$$

A partition of unity then gives the desired continuous choice of v_s such that

$$\frac{J'[u_s](v_s)}{\|v_s\|} \geqslant \frac{1}{2} \sup_{v\in\mathcal{S}(u_s)} \frac{J'[u_s](v)}{\|v\|} \quad \text{for all } s.$$

The next lemma establishes the complimentary case of Lemma 4.2, i.e. the case when $S[u]$ is small.

LEMMA 4.3. *There exists a constant $\eta>0$ whenever $u\in\mathcal{S}_0$ with $S[u]\leq\eta$. Then*

$$\sup_{v\in\mathcal{S}(u)} J'[u](v)/\|v\| \geq C(\eta)(S[u])^{1/2} \quad \text{for some} \quad C(\eta)>0. \tag{4.3)'$$

Remark. Actually in the case of (4.3)′, the function v can be expressed explicitly in terms of u (as is apparent in the proof of the lemma), hence clearly depends continuously on u.

Proof. We will make the explicit choice of v namely $v=-(u-\bar{u}-\Sigma_{j=1}^{3}\beta_j x_j)$ where $\boldsymbol{\beta}=(\beta_j)$ satisfies $\Lambda(u)\boldsymbol{\beta}=\boldsymbol{\gamma}$ with $\gamma_j=⨍ e^{2u}ux_j$. To see that v satisfies (4.3)′, we write

$$\|v\|^2=⨍|\nabla v|^2=⨍|\nabla u|^2-4\sum_{j=1}^{3}\beta_j\alpha_j+\frac{2}{3}\sum_{j=1}^{3}\beta_j^2 \tag{4.6)'$$

where $\alpha_j=⨍ ux_j$ and

$$2J'[u](v)=⨍|\nabla u|^2-2\sum_{j=1}^{3}\beta_j\alpha_j+⨍ e^{2u}(v-\bar{v}). \tag{4.7)'$$

To see that $J'[u](v)$ is bounded from below, we first observe that for $u\in\mathcal{S}_0$,

$$-⨍ e^{2u}(v-\bar{v})=⨍ e^{2u}(u-\bar{u})$$
$$=⨍ e^{2u}(\tilde{u}-\bar{\tilde{u}})$$

where $\tilde{u}=u-3\,\Sigma\alpha_j x_j$ with $\tilde{u}$ satisfying $⨍\tilde{u}\,x_j=0$. Thus

$$⨍(\tilde{u}-\bar{\tilde{u}})^2\leq\frac{1}{6}⨍|\nabla\tilde{u}|^2=\frac{1}{6}⨍|\nabla u|^2,$$

and

$$⨍ e^{2u}(\tilde{u}-\bar{\tilde{u}})=⨍(e^{2u}-1)(\tilde{u}-\bar{\tilde{u}})\leq\left(⨍(e^{2u}-1)^2\right)^{1/2}\left(⨍(\tilde{u}-\bar{\tilde{u}})^2\right)^{1/2}$$
$$\leq\left(e^{2⨍|\nabla u|^2+2S[u]}-1\right)^{1/2}\frac{1}{\sqrt{6}}\left(⨍|\nabla u|^2\right)^{1/2} \quad \text{(Onofri's inequality)} \tag{4.10}$$

$$\leq \frac{2}{\sqrt{6}}\left(\fint |\nabla u|^2\right) e^{(4/(1-a))S[u]}.$$

We now begin to estimate the coefficients α_j, β_j. To simplify notations, we will write $S=S[u]$. Notice that it follows from Corollary 3.1 that $\fint|\nabla u|^2 \sim S[u]$ for $u \in \mathcal{S}_0$. Thus by (4.1) $|\alpha_j|=O(S)$. To estimate β_j, we notice that

$$\Lambda_{ij}(u) = \fint e^{2u}x_i x_j = \fint (e^{2u}-1)\,x_i x_j \quad \text{if} \quad i \neq j,$$

$$\Lambda_{i,i}(u) = \fint e^{2u}x_i^2 = \fint (e^{2u}-1)\,x_i^2 + \frac{1}{3}.$$

Hence similar estimates as in (4.10) indicate that $\Lambda_{ij}(u)=O(S^{1/2})$ when $i \neq j$, $\Lambda_{ii}(u)=\frac{1}{3}+O(S^{1/2})$ for all $i,j=1,2,3$, and $\boldsymbol{\beta}=\Lambda^{-1}\boldsymbol{\gamma}=(3+O(S^{1/2}))\,\boldsymbol{\gamma}$. Thus to estimate β_j it suffices to estimate γ_j. To do so, we rewrite

$$\begin{aligned}
\gamma_j = \alpha_j+\gamma_j-\alpha_j &= \alpha_j + \fint (e^{2u}-1)\,u x_j \\
&= \alpha_j + \fint (e^{2u}-1)\,(u-\bar{u})\,x_j \\
&\leq \alpha_j + \left(\fint (e^{2u}-1)^2\right)^{1/2} \left(\fint (u-\bar{u})^2\right)^{1/2} \\
&= \alpha_j + O(S).
\end{aligned}$$

And conclude that $|\gamma_j| \leq |\alpha_j| + O(S) = O(S)$. It now follows from (4.6)′, (4.7)′ and (4.10) that

$$\|v\|^2 = \fint |\nabla u|^2 + O(S^2) \tag{4.6''}$$

and

$$2J'[u](v) \geq \left(1 - \frac{2}{\sqrt{6}} e^{(4/(1-a))S[u]}\right) \fint |\nabla u|^2 - O(S^2).$$

It follows that when $S=S[u]$ is sufficiently small we have

$$J'[u](v) \geq \frac{1}{2}\left(1 - \frac{\sqrt{5}}{\sqrt{6}}\right) \fint |\nabla u|^2$$

$$\geqslant \frac{1}{3}\left(1-\frac{\sqrt{5}}{\sqrt{6}}\right)(S[u])^{1/2}\|v\|.$$

We have thus finished the proof of Lemma 4.3.
As immediate consequences of Lemma 4.2, 4.3, we have

COROLLARY 4.1. *Suppose* $u\in\mathcal{S}_0$ *with* $S[u]\leqslant c_1$. *Then there exists some positive constant* $C(c_1)$ *and some* $v_u\in H^1$ *with* $\|v_u\|\leqslant 1$, $⨍ v_u=0$ *and*

$$J'[u](v_u)\geqslant C(c_1)(S[u])^{1/2}, \quad \text{and} \quad ⨍ e^{2u}v_u x_j=0 \quad \text{for all} \quad j=1,2,3.$$

COROLLARY 4.2. *Suppose* $u\in\mathcal{S}_{Q,t}$ *with* $S[u]\leqslant c_1$. *Then there exists some constant* $C(c_1)$ *and some* $v_u\in H^1$ *with* $\|v_u\|\leqslant 1$ *and*

(a) $$J'[u](v_u)\geqslant C(c_1)(S[u])^{1/2}$$

(b) $$\left|⨍ \nabla u\nabla v_u+⨍ v_u\right|\leqslant\frac{1}{1-a}(S[u])^{1/2}$$

(c) $$\frac{d}{ds}\left(⨍ e^{2T^t(Q)(u+sv_u)}x_j\right)\Bigg|_{s=0}=0 \quad \text{for all} \quad j=1,2,3.$$

Remark. v_u can be chosen to depend continuously on u.

Proof. Given $u\in\mathcal{S}_{Q,t}$ denote $\varphi=\varphi_{Q,t}$, then $u_\varphi=T^t(Q)(u)\in\mathcal{S}_0$. Choose $v_u=v_{u_\varphi}\circ\varphi^{-1}$, where the pair $(u_\varphi, v_{u_\varphi})$ satisfies the condition in Corollary 4.1. Then

(a) $$J'[u](v_u)=J'[u_\varphi](v_{u_\varphi})\geqslant C(c_1)S[u_\varphi]^{1/2}=C(c_1)S[u]^{1/2}.$$

(b)
$$\begin{aligned}⨍ \nabla u\cdot\nabla v_u+⨍ v_u&=⨍ \nabla u_\varphi\cdot\nabla v_{u_\varphi}+⨍ v_{u_\varphi}=⨍ \nabla u_\varphi\cdot\nabla v_{u_\varphi}\\&\leqslant\left(⨍ |\nabla u_\varphi|^2\right)^{1/2}\left(⨍ |\nabla v_{u_\varphi}|^2\right)^{1/2}\leqslant\left(\frac{1}{1-a}S[u_\varphi]\right)^{1/2}\\&=\left(\frac{1}{1-a}S[u]\right)^{1/2}.\end{aligned}$$

(c)
$$\begin{aligned}\frac{d}{ds}⨍ e^{2T^t(Q)(u+sv_u)}x_j\Big|_{s=0}&=\left(\frac{d}{ds}⨍ e^{2(u_\varphi+sv_u\circ\varphi)}x_j\right)\Bigg|_{s=0}=2⨍ e^{2u_\varphi}(v_u\circ\varphi)x_j\\&=2⨍ e^{2u_\varphi}v_{u_\varphi}x_j=0 \text{ by Corollary 4.1.}\end{aligned}$$

We will now begin to estimate the difference between $J'[u](v_u)$ and $F'[u](v_u)$ when $u\in\mathcal{S}_{Q,t}$ with $t\to\infty$. For computational purpose, we will now adopt the coordinate system in the plane through the stereographic projection treating Q as north pole. Denote $Q=(0,0,1)$. Then through the projection π, a point $\xi=(x_1,x_2,x_3)$ in S^2 corresponds to $z=(x, y)$ in the plane with

$$x_1=\frac{2x}{1+|z|^2},\quad x_2=\frac{2y}{1+|z|^2},\quad x_3=\frac{|z|^2-1}{|z|^2+1}.$$

Using these notations, we first list a preliminary estimate of $⨍ fe^{2u}$ (which we will sharpen later in § 5) when $u\in\mathcal{S}_{Q,t}$, f some $\mathscr{C}^2$ function.

Lemma 4.4. *Suppose $u\in\mathcal{S}_{Q,t}$ with $S[u]\leqslant c_1$ and f some C^2 function defined on S^2, then there exists some neighborhood $N(Q)=N(Q,f)$ such that when $t\to\infty$.*

$$⨍_{N(Q)}|f-f(Q)|\,e^{2u}=O(t^{-1/2}),\quad ⨍_{N(Q)^c}e^{2u}=O(t^{-1/2}).\tag{4.11}$$

Proof. Assume w.l.o.g. that $Q=(0,0,1)$, the north pole. Assume also that in a neighborhood of Q, say $|z|\geqslant M$ we have

$$f(\xi)=f(Q)+ax_1+bx_2+O(|x_1|^2+|x_2|^2)\quad\text{for}\quad \xi=(x_1,x_2,x_3)\in S^2.\tag{4.12}$$

We now choose $N(Q)=\{\xi'\in S^2, \pi(\xi')=z', |z'|\geqslant t^{\alpha}M\}$ for some $\alpha>0$ chosen later. Then

$$\begin{aligned}⨍_{N(Q)}|f-f(Q)|(\xi')\,e^{2u(\xi')}d\mu(\xi')&=⨍_{\varphi_{Q,t}^{-1}(N(Q))}|f-f(Q)|\circ\varphi_{Q,t}(\xi)\exp(2T'(Q)(u)(\xi))\,d\mu(\xi)\\&=\int_{|z|\geqslant t^{\alpha-1}M}|f(zt)-f(Q)|\exp(2T'(Q)(u)(z))\,dA(z)\end{aligned}$$

where

$$dA(z)=\frac{1}{\pi}\frac{d|z|^2}{(1+|z|^2)^2}\,d\theta$$

is the area form on the plane.

From (4.12), on the range $|z|\geqslant t^{\alpha-1}M$ we have the pointwise estimate

$$\begin{aligned}|f(zt)-f(Q)|&=O\left(\frac{|zt|}{1+|zt|^2}\right)+O\left(\frac{|zt|^2}{(1+|zt|^2)^2}\right)\\&\leqslant O\left(\frac{1}{|zt|}\right)+O\left(\frac{1}{|zt|^2}\right)=O\left(\frac{1}{t^{\alpha}}\right).\end{aligned}\tag{4.13}$$

Thus

$$\int_{N(Q)} |f-f(Q)|\, e^{2u} \leq O\left(\frac{1}{t^{\alpha}}\right) \int \exp(2T^{t}(Q)(u)) = O\left(\frac{1}{t^{\alpha}}\right). \tag{4.14}$$

On the other hand, when $\xi' \in (N(Q))^{c}$, applying the same change of variable as before, and noticing that $u_{\varphi_{Q,t}} \in \mathcal{S}_0$, we have

$$\begin{aligned} \int_{N(Q)^{c}} e^{2u} &= \int_{|z|\leq t^{\alpha-1}M} \exp(2T^{t}(Q)(u)(z))\, dA(z) \\ &= \int_{|z|\leq t^{\alpha-1}} (\exp(2T^{t}(Q)(u))-1)\, dA(z) + \int_{|z|\leq t^{\alpha-1}M} dA(z) \\ &\leq \left| \int (\exp(2T^{t}(Q)(u)(\xi)-1)^{2} \right|^{\frac{1}{2}} \left(\int_{|z|\leq t^{\alpha-1}M} dA(z) \right)^{\frac{1}{2}} \\ &\quad + \int_{|z|\leq t^{\alpha-1}M} dA(z) \\ &= O\left(S[u]e^{\frac{4}{1-\alpha}S[u]}\right)^{1/2} O(t^{\alpha-1}) + O(t^{2(\alpha-1)}) \\ &= O(t^{\alpha-1}). \end{aligned} \tag{4.15}$$

From (4.14), (4.15), it is clear that the choice of $\alpha=\frac{1}{2}$ would satisfy (4.11).

LEMMA 4.5. *Suppose $u \in \mathcal{S}_{Q,t}$ with $S[u] \leq c_1$, then as $t \to \infty$ we have for all $v \in H^1$*

$$|F'[u](v)-J'[u](v)| \leq \delta_1 \left(S[u] + \left| \int \nabla u \nabla v + \int v \right| + \frac{1}{4} \int |\nabla v|^2 \right) + \delta_2 \tag{4.16}$$

where $\delta_1 = O(t^{-1/2})$, $\delta_2 = O(t^{-1/2}\log t)$.

Proof. Fix $v \in H^1$ and denote $A = \int Ke^{2u}v$, $B = \int Ke^{2u}$, $A_1 = K(Q)\int e^{2u}v$, $B_1 = K(Q)\int e^{2u} = K(Q)$. Then

$$\frac{1}{2}(F'[u](v)-J'[u](v)) = \frac{A}{B} - \frac{A_1}{B_1} = \frac{1}{BB_1}[(A-A_1)B_1 + A_1(B_1-B)].$$

Thus

$$|F'[u](v)-J'[u](v)| \leq 2\left|\frac{A_1}{B_1}\right| \frac{|B_1-B|}{|B|} + \frac{2}{|B|}|A_1-A|.$$

Since $|B|\geqslant\inf K$, to obtain (4.16), it suffices to estimate $|A_1|$, $|B_1-B|$, and $|A_1-A|$. For A_1 we apply Onofri's inequality to get, if $A_1\geqslant 0$,

$$\begin{aligned} A_1 &= \fint e^{2u}v\leqslant \log\fint e^{2u}e^{v}\leqslant \frac{1}{4}\fint |\nabla(2u+v)|^2+\fint(2u+v) \\ &= S[u]+\left(\fint \nabla u\nabla v+\fint v\right)+\frac{1}{4}\fint|\nabla v|^2. \end{aligned} \tag{4.17$'$}$$

If $A_1\leqslant 0$, we apply the above argument to $-v$ and get

$$|A_1|\leqslant S[u]+\left|\fint \nabla u\nabla v+\fint v\right|+\frac{1}{4}\fint|\nabla v|^2. \tag{4.17}$$

For the term B_1-B, we apply Lemma 4.4 with $f=K$, $N(Q)=N(Q,K)$, then from (4.11) we have

$$|B_1-B|\leqslant \fint_{N(Q)}|K-K(Q)|\,e^{2u}+2\|K\|_\infty\fint_{N(Q)^c}e^{2u}=O(t^{-1/2}). \tag{4.18}$$

To estimate A_1-A, we apply Lemma 4.4 again with $N(Q)=N(Q,K)$ and notice that from (4.13)

$$\delta=\sup_{tz\in N(Q)}|K(tz)-K(Q)|$$

that $\delta=O(t^{-1/2})$. Thus if we rewrite

$$A_1-A=\fint(K-K(Q))\,e^{2u}v=\fint(K-K(Q))\circ\varphi_{Q,t}(\xi)\exp(2T'(Q)(u)(\xi))\,v\circ\varphi_{Q,t}(\xi)\,d\mu(\xi)$$

and denote $\bar u(z)=T'(Q)(u)(\xi)$, $(\pi(\xi)=z)$, $\bar v(z)=v\circ\varphi_{Q,t}(\xi)$. Then $A_1-A=\mathrm{I}+\mathrm{II}+\mathrm{III}$, where

$$\begin{aligned} \mathrm{I} &= \int_{tz\in N(Q)}(K(tz)-K(Q)+2\delta)\,e^{2\bar u(z)}\bar v(z)\,dA(z) \\ \mathrm{II} &= \int_{tz\in N(Q)^c}(K(tz)-K(Q)+2\delta)\,e^{2\bar u(z)}\bar v(z)\,dA(z) \\ \mathrm{III} &= -2\delta\fint e^{2u}v. \end{aligned}$$

Denote $h(z)=K(tz)-K(Q)+2\delta$, then the choice of δ implies that $h>0$ on $t^{-1}N(Q)$ with $\|h\|_\infty\leqslant 4\delta$ on $t^{-1}N(Q)$. Thus

$$\mathrm{I} \leqslant \int_{tz \in N(Q)} he^{2\tilde{u}} dA(z) \log \frac{\int_{tz \in N(Q)} he^{2\tilde{u}} e^{\tilde{v}} dA(z)}{\int_{tz \in N(Q)} he^{2\tilde{u}} dA(z)} = C_\delta^{(1)} \log \frac{1}{C_\delta^{(1)}} + C_\delta^{(1)} \log \left(4\delta \int_{\mathbf{R}^2} e^{2\tilde{u}} e^{\tilde{v}} dA(z) \right)$$

where

$$C_\delta^{(1)} = \int_{tz \in N(Q)} he^{2\tilde{u}} dA(z) \leqslant 4\delta = O(t^{-1/2}).$$

And $⨍ e^{2\tilde{u}} e^{\tilde{v}} = ⨍ e^{2u} e^{v}$. Applying the same estimate as in (4.17)′ we conclude that

$$\mathrm{I} \leqslant C_\delta^{(1)} \log \frac{1}{C_\delta^{(1)}} + C_\delta^{(1)} \left(S[u] + ⨍ \nabla u \cdot \nabla v + ⨍ v + \frac{1}{4} ⨍ |\nabla u|^2 \right).$$

Applying similar estimates to $-v$, we get

$$|\mathrm{I}| \leqslant C_\delta^{(1)} \log \frac{1}{C_\delta^{(1)}} + C_\delta^{(1)} \left(S[u] + \left| ⨍ \nabla u \cdot \nabla v + ⨍ v \right| + \frac{1}{4} ⨍ |\nabla v|^2 \right).$$

We may apply the same technique, as in the estimation of I, to estimate II, using $e^{2\tilde{u}}$ as weight, and observe that from (4.15), (4.11) we have

$$C_\delta^{(2)} = \int_{zt \in (N(Q))^c} e^{2\tilde{u}(z)} dA(z) = ⨍_{(N(Q))^c} e^{2u(\xi)} d\mu(\xi) = O(t^{-1/2}).$$

Thus

$$|\mathrm{II}| \leqslant C_\delta^{(2)} \log \frac{1}{C_\delta^{(2)}} + 4\|K\|_\infty C_\delta^{(2)} + C_\delta^{(2)} \left(S[u] + \left| ⨍ \nabla u \cdot \nabla v + ⨍ v \right| + \frac{1}{4} ⨍ |\nabla v|^2 \right).$$

A direct application of (4.17) also gives

$$|\mathrm{III}| \leqslant 2\delta \left(S[u] + \left| ⨍ \nabla u \cdot \nabla v + ⨍ v \right| + \frac{1}{4} ⨍ |\nabla v|^2 \right).$$

Combining the estimates in I, II, III we obtain

$$|A - A_1| \leqslant \delta_2 \left[S[u] + \left| ⨍ \nabla u \cdot \nabla v + ⨍ v \right| + \frac{1}{4} ⨍ |\nabla v|^2 \right] + \delta_3 \tag{4.19}$$

where $\delta_2 = O(t^{-1/2})$, $\delta_3 = O(t^{-1/2} \log t)$.

Combining (4.17), (4.18), (4.19), we obtain the estimate (4.16) as claimed in the lemma.

COROLLARY 4.3. *Given $u \in \mathcal{S}_{Q,t}$ with $t \to \infty$ and with $S[u] \leqslant c_1$, then there exists some $v_u \in H^{1,2}$ with $\|v_u\| \leqslant 1$ and which satisfies:*

$$J'[u](v_u) \geqslant C(c_1)(S[u])^{1/2} \tag{4.20}$$

$$F'[u](v_u) \geqslant C(c_1)(S[u])^{1/2} - O(t^{-1/2}\log t) \tag{4.21}$$

$$\frac{d}{ds}\left(\fint \exp 2T'(Q)(u+sv_u)x_j\right)\Bigg|_{s=0} = 0 \quad \text{for all} \quad j=1,2,3. \tag{4.22}$$

Proof. Choose v_u associated with $u \in \mathcal{S}_{Q,t}$ as in the statement of Corollary 4.2 then (4.20), (4.22) are automatically satisfied. To see (4.21), we apply condition (b) and Lemma 4.5 to the pair (u, v_u) then

$$\begin{aligned} F'[u](v_u) &\geqslant J'[u](v_u) - \delta_1\left(S[u] + \left|\fint (\nabla u)\nabla v_u + \fint v_u\right| + \frac{1}{4}\fint |\nabla v_u|^2\right) - \delta_2 \\ &\geqslant J'[u](v_u) - \delta_1\left(c_1 + c_1^{1/2}\frac{1}{1-a} + \frac{1}{4}\right) - \delta_2 \\ &= J'[u](v_u) - O(t^{-1/2}\log t) \\ &\geqslant C(c_1(S[u])^{1/2} - O(t^{-1/2}\log t) \end{aligned}$$

which establishes (4.21).

We are now ready to prove Proposition C.

Proof. Suppose $u_0 \in \mathcal{S}_{Q,t}$ with t large and $S[u_0] \leqslant c_1$, we will first prove a version of the proposition which corresponds to lifting above the point u_0. To do this we apply Corollary 4.3 to obtain some $v_{u_0} \in H^1$ satisfying (4.20), (4.21) and (4.22). We can now construct the path u_γ by solving the ordinary differential equation $du_\gamma/d\gamma = v_{u_\gamma}$ with u_0 given and normalize the solution by $\fint e^{2u_\gamma} = 1$ for all γ. It then follows from (4.22) that for all $j=1,2,3$,

$$\begin{aligned} \frac{d}{d\gamma}\fint \exp(2T'(Q)(u_\gamma))x_j &= 2\fint \exp(2T'(Q)(u_\gamma))\, v_{u_\gamma} \circ \varphi_{Q,t} x_j \\ &= \frac{d}{ds}\fint \exp(2T'(Q)(u_\gamma + sv_{u_\gamma}))x_j\Big|_{s=0} \\ &= 0. \end{aligned}$$

Then

$$\int \exp(2T'(Q)(u_\gamma))x_i = 0 = \int \exp(2T'(Q)(u_0)) \quad \text{for all } \gamma.$$

We may now apply (4.20), (4.21) to the pair u_γ, v_{u_γ} to conclude that $J[u_\gamma]$, $F[u_\gamma]$ both are increasing functions of γ and we can continue this lifting process till we reach the point where $S[u_{\gamma_0}] = -J[u_{\gamma_0}] = O(t^{-1}(\log t)^2)$.

If u varies continuously in the parameter, then according to Lemmas 4.2, 4.3 and Corollaries 4.1, 4.2 we have continuous dependence of v_u on u, hence we have continuous dependence of the O.D.E. solution $du_\gamma/d\gamma = v_{u_\gamma}$ on the initial data u.

§ 5. An asymptotic formula

In this section we will derive an asymptotic formula (Proposition D) which will be used in the analysis of the concentrated masses. This formula is a sharpened version of the corresponding estimates (4.11) in Lemma 4.4. Again, we will break the derivation of the formula into several technical lemmas.

LEMMA 5.1. *Suppose f is a C^2 function defined on S^2. Then for $t \to \infty$*

$$\int f \circ \varphi_{Q,t} = f(Q) + 2\Delta f(Q)(t^{-2}\log t) + O(t^{-2}). \tag{5.1}$$

Proof. We will use the plane coordinates deriving from the stereographic projection treating Q as north pole as explained before in Lemma 4.4. Using the Taylor series expansion of f around $Q = (0, 0, 1)$, we have

$$f(x_1, x_2) = f(Q) + ax_1 + bx_2 + Ax_1^2 + Bx_1x_2 + Cx_2^2 + O(|x_1|^3 + |x_2|^3) \tag{5.2}$$

which holds in a neighbourhood of Q say $\tilde{N}(Q) = \{z \in \mathbf{C}, |z| \geq M\}$.
Now we let R be the region in $\mathbf{C}$ such that $(\varphi_{Q,t}(R)) = \tilde{N}(Q)$, i.e. $R = \{z \in \mathbf{C}, |z| \geq M/t\}$ then

$$\int_{R^c} dA(z) = \frac{1}{2}\int_0^{M/t} \frac{d|z|^2}{(1+|z|^2)^2} = \frac{1}{2}\frac{M^2}{t^2+M^2} = O\left(\frac{1}{t^2}\right) \quad \text{as} \quad t \to \infty \tag{5.3}$$

and

$$\int_{S^2} f \circ \varphi_{Q,t} = \int_{\mathbf{C}^*} f(tz)\, dA(z) = \int_R f(tz)\, dA(z) + \int_{R^c} f(tz)\, dA(z). \tag{5.4}$$

In the region R we apply (5.2) and notice that since

$$x_1 = \frac{2\,\mathrm{Re}\, z}{1+|z|^2} = \frac{2\cos\theta|z|}{1+|z|^2}, \quad x_2 = \frac{2\,\mathrm{Im}\, z}{1+|z|^2} = \frac{2\sin\theta|z|}{1+|z|^2},$$

and R is a symmetric region w.r.t the x_1 and x_2 coordinates, we have

$$\int_R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(tz)\, dA(z) = 0$$

$$\int_R x_1 x_2 (tz)\, dA(z) = 0.$$

Thus

$$\begin{aligned}\int_R f(tz)\, dA(z) &= \int_R f(Q)\, dA(z) + A\int_R x_1^2(tz)\, dA(z) + C\int_R x_2^2(tz)\, dA(z) + O\left(\int_R \frac{|tz|^3}{(1+|tz|^2)^3}\, dA(z)\right) \\ &= f(Q) + O(t^{-2}) + A\int_R x_1^2(tz)\, dA(z) + C\int_R x_2^2(tz)\, dA(z) + O(t^3) \quad \text{(from (5.3))}.\end{aligned} \tag{5.5}$$

And

$$\int_R x_1^2(tz)\, dA(z) = 4\int_{|z|=M/t}^{\infty} \frac{t^2|z|^3}{(1+t^2|z|^2)^2}\frac{d|z|}{(1+|z|^2)^2} = 4\mathscr{I}_t.$$

Similarly we have also

$$\int_R x_2^2(tz)\, dA(z) = 4\mathscr{I}_t.$$

Now an explicit calculation yields that $\mathscr{I}_t = t^{-2}\log t + O(t^{-2})$.

Combining the estimates in (5.6), (5.5), (5.3) into (5.4), we get the desired estimate (5.1) as in the statement of the lemma.

LEMMA 5.2. *Suppose $g \geqslant 0$ is a bounded function defined on S^2, and $w \in \mathscr{S}_0$ with $S[w] \leqslant c_1$ for each $\varepsilon > 0$ we have, setting $c_a = (1-a)/2$*

$$\textit{where} \quad \begin{cases} L_\varepsilon \leqslant \displaystyle⨍ g e^{2w} \leqslant U_\varepsilon \\ U_\varepsilon = e^{2(\bar{w}+\varepsilon)}\left(⨍ g\right) + \|g\|_\infty O(e^{-\varepsilon^2 c_a/S[w]}) \\ L_\varepsilon = e^{2(\bar{w}-\varepsilon)}\left(⨍ g\right) - \|g\|_\infty O(e^{-\varepsilon^2 c_a/S[w]}) \end{cases}$$

Proof. For the fixed function $w \in \mathcal{S}_0$, $\varepsilon>0$ denote $A_\varepsilon=\{\xi \in S^2, |w(\xi)-\bar{w}|\geq\varepsilon\}$. Then by the inequality of Moser (1.5),

$$|A_\varepsilon| \leq C_0\, e^{-\varepsilon^2/\fint|\nabla w|^2}$$

Thus for each positive function g we have

$$\begin{aligned}\fint ge^{2w} &= \fint_{|w-\bar{w}|\leq\varepsilon} ge^{2w}+\fint_{|w-\bar{w}|\geq\varepsilon} ge^{2w}\\ &\leq e^{2(\bar{w}+\varepsilon)}\fint g+\|g\|_\infty\left(\fint e^{4w}\right)^{1/2}|A_\varepsilon|^{1/2}\\ &\leq e^{2(\bar{w}+\varepsilon)}\fint g+C_0\|g\|_\infty\, e^{2\fint|\nabla w|^2+2S[w]}e^{-\varepsilon^2/2\fint|\nabla w|^2}\\ &\leq e^{2(\bar{w}+\varepsilon)}\fint g+C_0\|g\|_\infty\, e^{\frac{4}{1-a}S[w]}e^{-\varepsilon^2c_a/S[w]}=U_\varepsilon \quad \text{(by Corollary 3.1).}\end{aligned}$$

Similarly, we may obtain the lower estimate L_ε of $\fint ge^{2w}$.

We will now apply the estimates in the lemma above to evaluate the center of mass of e^{2u} with $u \in \mathcal{S}_{Q,t}$.

LEMMA 5.3. *Suppose $u\in\mathcal{S}_{Q,t}$ and $S[u]=O(t^{-\alpha})$ for some $\alpha>0$ and for t sufficiently large assuming $Q=(0,0,1)$, we have*

$$\fint x_i e^{2u}=O(t^{-(1+\alpha')}),\quad \textit{for } i=1,2,\quad \textit{for any } \alpha'<\alpha/2 \tag{5.7}$$

$$\fint x_3 e^{2u}=1-4t^{-2}\log t+O(t^{-2}) \tag{5.8}$$

$$\fint x_i^2 e^{2u}=4t^{-2}\log t+O(t^{-2}),\quad \textit{for}\quad i=1,2 \tag{5.9}$$

$$\fint x_1x_2 e^{2u}=O(t^{-2}). \tag{5.10}$$

Proof. For the given $u\in\mathcal{S}_{Q,t}$ denote $w=u_\varphi$, $\varphi=\varphi_{Q,t}$. Then $w\in\mathcal{S}_0$ with $S[w]=S[u]\leq c_1$. It follows from Lemma 5.2 that for $i=1,2$,

$$e^{2(\bar{w}-\varepsilon)}\left(\fint_{x_i\geq0} x_i\circ\varphi\right)-O(e^{-\varepsilon^2c_a/S[w]})\leq\fint_{x_i\geq0} x_i\circ\varphi e^{2w}\leq e^{2(\bar{w}+\varepsilon)}\left(\fint_{x_i\geq0} x_i\circ\varphi\right)+O(e^{-\varepsilon^2c_a/S[w]}). \tag{5.10}$$

Similarly we have estimates for $\fint_{x_i\le 0} x_i\circ\varphi$. Since $\fint x_i\circ\varphi=0$, we have

$$\fint_{x_i\ge 0} x_i\circ\varphi=-\fint_{x_i\le 0} x_i\circ\varphi$$

and by an explicit computation

$$\fint_{x_i\ge 0} x_i\circ\varphi=O(t^{-1}).$$

Thus

$$\begin{aligned}\fint x_i e^{2u}=\fint (x_i\circ\varphi)\,e^{2w}&=\fint_{x_i\ge 0} x_i\circ\varphi\,e^{2w}+\fint_{x_i\le 0} x_i\circ\varphi\,e^{2w}\\ &\le (e^{2(\bar w+\varepsilon)}-e^{2(\bar w-\varepsilon)})\left(\fint_{x_i\ge 0} x_i\circ\varphi\right)+O(e^{-\varepsilon^2 c_a/S[w]})\\ &\le \varepsilon\,O\left(\frac{1}{t}\right)+O(e^{-\varepsilon^2c_a/S[w]})\quad(\bar w\le 0).\end{aligned}$$

Similarly we have

$$\begin{aligned}\fint x_i e^{2u}&\ge (e^{2(\bar w-\varepsilon)}-e^{2(\bar w+\varepsilon)})\fint_{x_i\ge 0}(x_i\circ\varphi)-O(e^{-\varepsilon^2c_a/S[w]})\\ &\ge(-\varepsilon)\,O\left(\frac{1}{t}\right)-O(e^{-\varepsilon^2c_a/S[w]}).\end{aligned}$$

Since $S[w]=S[u]=O(t^{-a})$, we may pick ε with $\varepsilon^2\sim\log t/t^a$ such that

$$e^{-\varepsilon^2c_a/S[w]}=O(t^{-2});$$

with this choice of ε, (5.7) follows.

For the terms $\fint x_i^2e^{2u}$, $i=1,2$, we apply Lemma 5.2 directly and obtain

$$\begin{aligned}e^{2(\bar w-\varepsilon)}\left(\fint x_i^2\circ\varphi\right)-O(e^{-\varepsilon^2c_a/S[w]})\le\fint x_i^2e^{2u}&=\fint x_i^2\circ\varphi\,e^{2w}\\ &\le e^{2(\bar w+\varepsilon)}\left(\fint x_i^2\circ\varphi\right)+O(e^{-\varepsilon^2c_a/S[w]}).\end{aligned}$$

Since $\fint x_i^2\circ\varphi=4t^{-2}\log t+O(t^{-2})$ by Lemma 5.1, we observe that

$$e^{-\frac{a}{1-a}S[w]}\le e^{2\bar w}\le e^{S[w]}.$$

Thus the same choice of ε as before gives (5.9).
For (5.8), since $x_3=(|z|^2-1)/(|z|^2+1)$, a computation indicates that

$$\int x_3 \circ \varphi_{Q,t} = 1-4t^{-2}\ln t+O(t^{-2}).$$

Thus we may apply Lemma 5.2 to the function $1-x_3$, similarly as we did for x_i^2 $(i=1,2)$, and obtain (5.8).
Formula (5.10) can be verified similarly as (5.7) based on the information that

$$\int x_1 x_2 \circ \varphi = 0 \quad \text{and} \quad \int_{x_1 x_2 \geq 0} x_1 x_2 \circ \varphi = O(t^{-2}\log t)$$

with the same choice of ε $(\varepsilon^2 \sim (\log t/t^{\alpha}))$ as before.

Now we are ready to prove Proposition D. We first simplify some notations. For the given $u \in \mathscr{S}_{Q,t}$ with $S[u]=O(t^{-\alpha})$, we assume w.l.o.g. that $Q=(0,0,1)$ and (x_1,x_2,x_3) the coordinate system with Q as north pole. We also denote $P=(p_1,p_2,p_3)\in S^2$, the projection of the center of mass of e^{2u} on the sphere, i.e.

$$p_i = \int e^{2u}x_i \Big/ \left(\sum_{i=1}^{3}\left(\int e^{2u}x_i\right)^2\right)^{1/2} \quad \text{for} \quad i=1,2,3.$$

Then by estimates (5.7), (5.8) in Lemma 5.3 we have $p_1=O(t^{-1-\alpha'})$, $p_2=O(t^{-1-\alpha'})$ $(\alpha'<\alpha/2)$ while $p_3=1-4t^{-2}\ln t+O(t^{-2})$. Denote by (y_1,y_2,y_3) the coordinate system in S^2 treating P as north pole. And to simplify notation, we may rotate coordinates in the (x_1,x_2)-plane and assume w.l.o.g. that $P=(p_1,p_2,p_3)$ with $p_1=O(t^{-1})$, $p_2=0$, p_3 unchanged as before. Then in the new coordinate system we have

$$\begin{aligned} y_1 &= \mathbf{x}\cdot(p_3,0,-p_1) = p_3x_1-p_1x_3 \\ y_2 &= x_2 \\ y_3 &= \mathbf{x}\cdot P = p_1x_1+p_3x_3. \end{aligned} \tag{5.11}$$

To compute $\int fe^{2u}$ for a general $\mathscr{C}^2$ function, we now expand f in a Taylor series in a neighborhood of P say

$$f(y_1,y_2,y_3) = f(p)+\tilde{a}y_1+\tilde{b}y_2+\tilde{A}y_1^2+\tilde{B}y_1y_2+\tilde{C}y_2^2+O(|y_1|^3+|y_2|^3) \tag{5.12}$$

for (y_1, y_2, y_3) in the same neighborhood $\tilde{N}(Q)=\{z\in\mathbf{C},\ |z|\geqslant M\}$ as in the expansion (5.2) before. Denote again $R=\{z\in\mathbf{C},\ |z|\geqslant M/t\}$. Then

$$\int fe^{2u}=\int f\circ\varphi_{Q,t}\,e^{2w}=\int_R f\circ\varphi_{Q,t}\,e^{2w}dA(z)+\int_{R^c} f\circ\varphi_{Q,t}\,e^{2w}$$

Applying (5.3) and Lemma 5.2 to the function $g=X_{R^c}$ we have (adopting the same argument as in Lemma 5.3)

$$\int_{R^c} f\circ\varphi_{Q,t}e^{2w}\leqslant\|f\|_\infty\int_{R^c} e^{2w}=O(t^{-2}). \tag{5.13}$$

In the region R, we apply the expansion (5.12) and notice that by our choice of the coordinate system (y_1, y_2, y_3) we have

$$\int y_i\circ\varphi_{Q,t}\,e^{2w}=\int y_i e^{2u}=0 \quad\text{for}\quad i=1,2.$$

Thus

$$\begin{aligned}\int_R f(tz)\,e^{2w}dA(z)=f(P)+\tilde{A}\int_R y_1^2\circ\varphi_{Q,t}\,e^{2w}dA(z)+\tilde{C}\int_R y_2^2\circ\varphi_{Q,t}\,e^{2w}dA(z)\\+\tilde{B}\int_R y_1y_2\circ\varphi_{Q,t}\,e^{2w}dA(z)+O(t^{-2}).\end{aligned} \tag{5.14}$$

Now, applying (5.11), we have

$$\begin{aligned}\int_R y_1^2\circ\varphi_{Q,t}\,e^{2w}dA(z)&=\int_R (p_3x_1-p_1x_3)^2\circ\varphi_{Q,t}(z)\,e^{2w}dA(z)\\&=p_3^2\int_R x_1^2\circ\varphi_{Q,t}\,e^{2w}dA-2p_1p_3\int_R x_1x_3\circ\varphi_{Q,t}e^{2w}dA(z)\\&\quad+p_1^2\int_R x_3^2\circ\varphi_{Q,t}e^{2w}dA.\end{aligned}$$

Applying the estimates in (5.9), (5.7) and (5.13), and noticing that $x_3^2=1-x_1^2-x_2^2$, $x_1x_3=-x_1(1-x_3)+x_1$, we have

$$\int_R y_1^2\circ\varphi_{Q,t}\,e^{2w}dA=4t^{-2}\log t+O(t^{-2}). \tag{5.15}$$

Applying (5.9) and (5.13) directly we have

$$\int_R y_2^2 \circ \varphi_{Q,t}\, e^{2w} dA = \int_R x_2^2 \circ \varphi_{Q,t}\, e^{2w} dA = 4t^{-2}\log t + O(t^{-2}). \tag{5.16}$$

For the cross term $y_1 y_2$ we have from (5.11)

$$\int_R y_1 y_2 \circ \varphi_{Q,t}\, e^{2w} dA = p_3 \int_R x_1 x_2 \circ \varphi_{Q,t}\, e^{2w} dA - p_1 \int_R x_2 x_3 \circ \varphi_{Q,t}\, dA.$$

We can apply (5.10) to estimate the term involving $x_1 x_2$, and estimate the term involving $x_2 x_3$ in the same way as we treated $x_1 x_3$ before. We get the conclusion that

$$\int_R y_1 y_2 \circ \varphi_{Q,t}\, e^{2w} dA = O(t^{-2}). \tag{5.17}$$

Combining (5.15), (5.16), (5.17) and (5.14) we have obtained the formula

$$\int f e^{2u} = f(P) + 2\Delta f(P)\,(t^{-2}\log t) + O(t^{-2})$$

as desired in Proposition D below.

We may now summarize what we have proved above in the following:

PROPOSITION D. *Suppose $u \in \mathcal{S}_{Q,t}$ with $S[u]=O(t^{-\alpha})$ for $\alpha>0$ and t sufficiently large, then $u \in C_{P,\delta}$ where $\delta=4t^{-2}\ln t+O(t^{-2})$ and $|P-Q|=O(t^{-1})$, and for every $\mathcal{C}^2$ function f defined on S^2 we have*

$$\int f e^{2u} = f(P) + 2\Delta f(P)\, t^{-2}\log t + O(t^{-2}) = f(P) + \frac{1}{2}\Delta f(P)\,\delta + O(\delta/\log 1/\delta). \tag{5.18}$$

We also want to remark that we can run above parameter changes (from $\mathcal{S}_{Q,t}$ to $C_{P,\delta}$) backwards, and obtain:

COROLLARY 5.1. *Suppose $u \in C_{P,\delta}$ with $S[u]=O(\delta^{\beta})$ for some $\beta>0$ and δ sufficiently small. Then $u \in \mathcal{S}_{Q,t}$ where $\delta=4t^{-2}\log t+O(t^{-2})$ and $|P-Q|=O(t^{-1})$ and (5.18) holds for any $\mathcal{C}^2$ function f.*

Proof. We will use the same notation as in the proof of Proposition D and denote $Q=(0,0,1)$, $P=(p_1,p_2,p_3)$ with the coordinate system $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)$

related as in (5.11) as before. Then following the computations in (5.7), (5.8) and observe that for $S[u]=S[T'(Q)(u)]=O(\delta^\beta)$ we can choose ε small with

$$e^{-\varepsilon^2 c_a/S[u]} \leq \delta^2.$$

Thus (5.7), (5.8) appear in the form

$$\int x_i e^{2u} = O(t^{-1})+O(\delta^2) \quad \text{for} \quad i=1,2 \tag{5.7$'$}$$

$$\int x_3 e^{2u} = 1-4t^{-2}\log t+O(t^{-2})+O(\delta^2). \tag{5.8$'$}$$

Since $(1-\delta)^2=\Sigma_{i=1}^3 (\int x_i e^{2u})^2$, we have $\delta=4t^{-2}\log t+O(t^{-2})$. Substituting (5.7)′, (5.8)′ into (5.11) we get

$$p_1=(1-\delta)\left(\int x_1 e^{2u}\right)\Big/\left(\left(\int x_1 e^{2u}\right)^2+\left(\int x_2 e^{2u}\right)^2\right) = O(t^{-1}).$$

Since $p_2=0$ by our choice of coordinate system we conclude that $|Q-P|=O(t^{-1})$, which finishes our proof of the corollary.

As it turns out, sometimes it is more convenient to express the asymptotic formula (5.18) in terms of the (Q, t) parametes of a function $u\in \mathcal{S}_{Q,t}$. The only disadvantage in using these parameters is that if we expand $f\in \mathcal{C}^2(S^2)$ in terms of its Taylor series expansion in a neighborhood of $Q=(0,0,1)$, we have

$$f(x_1,x_2,x_3)=f(Q)+ax_1+bx_2+Ax_1^2+Bx_1x_2+Cx_2^2+O(|x|^3). \tag{5.12$'$}$$

Then $\int x_i e^{2u}\neq 0$ for $i=1,2$, hence in the formula (5.18)′ we pick up some $|\nabla f(Q)|$ term. We may estimate this linear term as:

$$\begin{aligned}\int x_i e^{2u} &= \int x_i\circ\varphi_{Q,t}\exp(2T'(Q)(u))\\ &= \int x_i\circ\varphi_{Q,t}\exp(2T'(Q)(u))\\ &\leq (O(t^{-2}\log t))^{1/2}(S[u])^{1/2}e^{\frac{2}{1-a}S[u]}\end{aligned}$$

using the fact that $u_{\varphi_{Q,t}}\in\mathcal{S}_0$ for $i=1,2$. Thus we have

COROLLARY. *Suppose* $u\in \mathcal{S}_{Q,t}$ *with* $S[u]=O(t^{-\alpha})$ *for some* $\alpha>0$, *and t large. Then for any* $f\in \mathscr{C}^2(S^2)$ *we have*

$$\int fe^{2u}=f(Q)+2\Delta f(Q)\,t^{-2}\log t+O(t^{-2})+O(|\nabla f(Q)|(t^{-2}\log t)^{1/2}(S[u])^{1/2}). \qquad (5.18)'$$

§ 6. Analysis of concentration near critical points

In this section we apply § 4 and § 5 to analyze the phenomenon of concentration near critical points of K. We will consider the concentration which occurs in the variational scheme Var (P_α, P_β), which we now define.

Given two points P_α, P_β on S^2, we formulate the one-dimensional scheme Var (P_α, P_β) as follows. Let $\mathscr{P}(P_\alpha, P_\beta)=\Big\{ u\colon(-\infty,\infty)\to H^{1,2}(S^2)$, u_p: $-\infty<p<\infty$ is a continuous one parameter family of functions in $H^{1,2}(S^2)$ with $⨍ e^{2u_p}=1$ and satisfy:

(1) $$S[u_p]\to 0 \text{ as } |p|\to\infty$$

(2) $$\lim_{p\to-\infty}⨍ \mathbf{x}e^{2u_p}=P_\alpha,\ \lim_{p\to+\infty}⨍ \mathbf{x}e^{2u_p}=P_\beta\Big\}.$$

Let

$$c=\sup_{u\in\mathscr{P}(P_\alpha,P_\beta)}\min_p F[u_p].$$

Given a maximizing path $u^{(k)}$ which assumes its minimum at p_k denoted by $u^{(k)}_{p_k}$, then if $\{u^k_{p_k}\}$ converges weakly in H^1, then the limit u will weakly satisfy the Euler equation (1.1). Consequently by the regularity theory for elliptic equations u will be a strong solution of (1.1). In the proofs of Theorem I and II in § 7, we will show that such a scheme converges for suitable choice of P_α, P_β.

We first remark that in the scheme Var(P_α, P_β), we may restrict the class of competing paths in such a way that if concentration occurs along the paths then the functional S must be small at such points so that we may apply the asymptotic formulas of § 5. More precisely we define the class of paths

$$\mathscr{P}'_{t_0}(P_\alpha,P_\beta)=\{u\in\mathscr{P}(P_\alpha,P_\beta)\,|\,u_p\in\mathcal{S}_{Q,t},\,t\geqslant 2t_0\Rightarrow S[u_p]=O(t^{-1}(\log t)^2)\}.$$

Choose t_0 and constant C large so that the lifting Proposition C holds for all $u \in \mathcal{S}_{Q,t}$, $t \geq t_0$. For each $u_s \in \mathcal{S}_{Q,t}$, there exists $u_{s,\tau}$, $0 \leq \tau \leq \tau(u_s)$ continuous in τ with $u_{s,0} = u_s$, $u_{s,\tau} \in \mathcal{S}_{Q,t}$, $F[u_{s,\tau}]$ and $J[u_{s,\tau}]$ both monotone increasing in τ such that at $\tau = \tau(u_s)$, $S[u_{s,\tau}] = O(t^{-1}(\log t)^2)$. Let $\varrho(t) = \min[1, (t-t_0)/t_0]$ for $t \in [t_0, \infty)$. For $u_s \in \mathcal{S}_{Q,t}$ let

$$u'_s = \begin{cases} u_{s,\varrho(t)\tau(u_s)} & \text{if } t \geq t_0 \\ u_s & \text{if } t < t_0 \end{cases}.$$

Then $u'_s \in \mathcal{P}'_{t_0}(P_\alpha, P_\beta)$. While $F[u'_s] \geq F[u_s]$. Hence it follows that

$$\sup_{u \in \mathcal{P}'_{t_0}} \min_s F[u_s] = \sup_{u \in \mathcal{P}} \min_s F[u_s].$$

In view of the equality above, we will assume that all paths u in the scheme belong to the lifted path class $\mathcal{P}'_{t_0}(P_\alpha, P_\beta)$.

Assuming u_k is an unbounded sequence in $H^{1,2}$ and a max-min sequence for the scheme $\operatorname{Var}(P_\alpha, P_\beta)$, it then follows from Proposition A (the Concentration lemma) that the masses $e^{2u_k}(u_k = u_{p_k}^{(k)})$ converges (perhaps on a subsequence) to a delta function concentrated at $P_\infty \in S^2$. Our first proposition says that in this case we may assume without loss of generality that P_∞ is a critical point of K. We state this as

Proposition E. *For the variational scheme* $\operatorname{Var}(P_\alpha, P_\beta)$, *if the maximizing sequence of minima* $\{u_k\}$ *does not converge, then we can construct another maximizing sequence of minima* $\{v_k\}$ *for the scheme (if necessary) such that* e^{2v_k} *concentrates at a critical point of* K.

Proof of Proposition E. Our first observation is that for the max-min sequence $\{u_k\}$ we have $u_k \in \mathcal{S}_{Q_k,t_k}$ with $P_k \to P_\infty \in S^2$, and $t_k \to \infty$, since we choose to work in path class $\mathcal{P}'_{t_0}$, this means $S[u_k] \to 0$. We construct a competing max-min sequence of paths by replacing our given sequence of path over the intervals where the (Q, t) parameters fall in the region $D = \{|\nabla K(Q)|^2 \geq ct^{-1}; t \geq t_0\}$ by paths obtained from the given one using the following flow Ψ_s in $H^{1,2}(S^2)$ associated to the gradient field ∇K on S^2.

$$\frac{d}{ds} u_s = \frac{d}{d\tau}\bigg|_{\tau=0} u_{s,\tau}; \quad u_{s,\tau} = (u_s)_{R(Q,-\tau\theta)}$$

where $u_s \in \mathcal{S}_{Q_s, t_s}$, $R(Q, \theta')$=rotation in the plane spann $(Q, K\nabla K(Q))$, with angle of rotation θ'; $\theta = \varrho(Q, t)\,|\nabla K(Q)|$, ϱ is a cut off function with support in $\mathcal{D}$.

Since R is a rotation, $u_R \in \mathcal{S}_{R^{-1}Q, t}$ for $u \in \mathcal{S}_{Q, t}$. Thus the flow Ψ_s does not change the t-parameter of a function while it rotates the Q parameter along the gradient line of K. It follows from the asymptotic formula that along the flow $dF[u_s]/ds$ is positive, hence F is increasing. Since the gradient flow ∇K has critical points as limiting values, it follows that our modified sequence of paths concentrates at a critical point of K.

Turning our attention to the situation where a maximizing sequence of minima $u_{p_k}^{(k)}$ concentrates at a critical point P_∞ of K, the next Proposition F rules out the possibilities that P_∞ can be (a) a local maximum (b) a local minimum and finally (c) a saddle point Q with $\Delta K(Q)>0$. This is accomplished with the asymptotic formula (Proposition D) which can be applied to evaluate the functional F on very concentrated mass distributions e^{2u}.

PROPOSITION F. *In the problem* $\operatorname{Var}(P_\alpha, P_\beta)$ *where* P_α, P_β *are local maxima of* K, *if a maximizing sequence of paths* $u^{(k)} \in \mathcal{P}'_{t_0}(P_\alpha, P_\beta)$ *has minima* $e^{2u_{p_k}^{(k)}}$ *concentrating at a critical point* P_∞, *then w.l.o.g. we may assume that*

(a) P_∞ *cannot be a local maximum of* K

(b) P_∞ *cannot be a local minimum or a saddle point of* K *where* $\Delta K(P_\infty)>0$.

Proof of Proposition F. We begin with a simple consequence of the asymptotic formula: under the hypothesis of the proposition we have

$$\begin{aligned} \sup\min F[u_p] &= \lim_{k\to\infty} F[u_{p_k}^{(k)}] \\ &= \lim_{k\to\infty} \log \int K e^{2u_{p_k}^{(k)}} - S[u_{p_k}^{(k)}] \\ &= \lim_{k\to\infty} \log \int K e^{2u_{p_k}^{(k)}} \quad \text{because } u^{(k)} \in \mathcal{P}'_{t_0} \qquad (6.4) \\ &= \lim_{k\to\infty} \log\,[K(P_k)+O(\delta_k)] \quad \text{where } u_{p_k}^{(k)} \in C_{p_k, \delta_k} \\ &= \log K(P_\infty). \end{aligned}$$

We choose coordinates x_1, x_2, x_3 so that $P_\infty=(0,0,1)$.

For assertion (a) observe that for any path $u_p \in \mathcal{P}'_{t_0}$, the path of the center of mass

$p \mapsto \text{C.M.}(e^{2u_p}) = \fint \mathbf{x} e^{2u_p}$ is continuous in p. Hence if for some p, $\text{C.M.}(e^{2u_p})$ is very close to P_∞, the path $\text{C.M.}(e^{2u_p})$ must hit the disk $x_3 = 1-\varepsilon_0$, for some small ε_0 and for some $p=p_0$. Since $u_{p_0} \in \mathcal{P}'_{t_0}$ we apply the asymptotic formula to estimate $F[u_{p_0}]$:

$$\begin{aligned} F[u_{p_0}] &= \log \fint K e^{2u_{p_0}} - S[u_{p_0}] \\ &\leq \log \fint K e^{2u_{p_0}} \\ &\leq \log \left[K(P_0) + \frac{1}{2} \Delta K(P_0)\, \delta_0 + o(\delta_0) \right], \quad \text{where } u_{p_0} \in C_{P_0, \delta_0}. \end{aligned}$$

Since $|P_0 - P_\infty| \leq \sqrt{\varepsilon_0}$, $\Delta K(P_0) \leq \frac{1}{2} \Delta K(P_\infty) < 0$ we find

$$K(P_\infty) - K(P_0) - \frac{1}{2} \Delta K(P_0)\, \delta_0 \geq -\frac{1}{4} \Delta K(P_\infty) \cdot (|P_0 - P_\infty|^2 + \delta_0) \geq -\frac{1}{8} \Delta K(P_\infty)\, \varepsilon_0.$$

Thus $F[u_{P_0}] \leq \log K(P_\infty) - C\varepsilon_0$, which contradicts (6.4).

For assertion (b) we will construct a flow Φ_s which will yield a competing sequence of paths which have minima achieved at functions $\hat{u}^k_{p_k}$ not concentrating at any critical points Q with $\Delta K(Q) > 0$. Given $P \in S^2$, let $\varphi_{P,\tau}$ be the conformal transformation given in stereographic complex coordinates z with $z(P) = \infty$, $z(-P) = 0$ defined by $\varphi_{P,\tau}(z) = \tau z$. Choose ε small enough so that in each ε disk $B(Q, \varepsilon)$ centered at any critical point Q of K with $\Delta K(Q) > 0$ we have $\Delta K(P) \geq M > 0$ for all $P \in B(Q, \varepsilon)$. Choose a smooth function ϱ, defined on $S^2 \times (0, 1]$, $0 \leq \varrho < 1$ with

$$\operatorname{supp} \varrho \subset \bigcup_{\substack{Q \text{ critical} \\ \Delta K(Q) > 0}} B(Q, \varepsilon) \times (0, \delta_0]$$

where $\delta_0 = 4 t_0^{-2} \log t_0$, $\varrho \equiv 1$ on

$$\bigcup_{\substack{Q \text{ critical} \\ \Delta K(Q) > 0}} B(Q, \varepsilon/2) \times (0, \delta_0/2].$$

Define the flow $\Phi_s(u) = u_s$ by the o.d.e.

$$\dot{u}_s = \varrho(P_s, \delta_s) \frac{d}{d\tau}\bigg|_{\tau=1} T^\tau(P_s)(u_s), \quad \text{where} \quad u_s \in C_{P_s, \delta_s}.$$

Claim (6.5). For $u \in C(P, \delta)$ with

$$(P,\delta) \in \bigcup_{\substack{Q \text{ critical} \\ \Delta K(Q)>0}} B(Q,\varepsilon)\times(0,\delta_0]$$

we have $u_s \in C(P_s, \delta_s)$ where δ_s increases as s increases.

To prove claim (6.5) choose coordinates x_1, x_2, x_3 so that $P=(0,0,1)$. Then

$$\begin{aligned}
\frac{d}{ds}\text{C.M.}(e^{2u_s}) &= \frac{d}{ds}\int \mathbf{x}e^{2u_s} \\
&= \int \mathbf{x}\frac{d}{ds}e^{2u_s} = 2\int \mathbf{x}\varrho(P_s,\delta_s)\cdot e^{2u_s}\frac{d}{d\tau}\Big|_{\tau=1} T^{\tau}(P_s)(u_s) \\
&= \int \mathbf{x}\varrho(P_s,\delta_s)\frac{d}{d\tau}\Big|_{\tau=1} \exp(T^{\tau}(P_s)(u_s)) \\
&= \varrho(P_s,\delta_s)\int \frac{d}{d\tau}\Big|_{\tau=1} (\mathbf{x}\circ\varphi_{P_s,\tau^{-1}})\, e^{2u_s} = -\varrho(P_s,\delta_s)\int \langle \nabla\mathbf{x}, \nabla x\cdot P_s\rangle\, e^{2u_s}.
\end{aligned}$$

Observing that $|\nabla x_3|^2 = 1-x_3^2$, $\langle\nabla x_i, \nabla x_3\rangle = -x_i x_3$ for $i=1,2$,

$$\Delta(|\nabla x_3|^2) = -2+6x_3^2, \quad \Delta\langle\nabla x_i, \nabla x_3\rangle = 6x_i x_3,$$

we apply the asymptotic formula (Proposition D) to the integral to find

$$\int |\nabla x_3|^2 e^{2u_s} = 2\,\delta_s + o(\delta_s)$$

$$\int \langle\nabla x_i, \nabla x_3\rangle\, e^{2u_s} = o(\delta_s) \quad \text{for} \quad i=1,2.$$

Thus it follows that

$$\frac{d}{ds}\langle \text{C.M.}(e^{2u_s}), \text{C.M.}(e^{2u_s})\rangle < 0,$$

verifying the claim (6.5).

Claim (6.6). The flow Φ_s increases the value of the functional F:

$$\frac{d}{ds}F[u_s] = \frac{d}{ds}\log\int Ke^{2u_s} - \frac{d}{ds}S[u_s]$$

$$= \left(\int Ke^{2u_s}\right)^{-1} \varrho(P_s, \delta_s) \int \frac{d}{d\tau}\Big|_{\tau=1} K \circ \varphi^{-1}_{P_s,\tau} e^{2u_s}$$

$$= -\left(\int Ke^{2u_s}\right)^{-1} \varrho(P_s, \delta_s) \int \langle \nabla K, \nabla x \cdot P_s \rangle e^{2u_s}$$

$$= \left(\int Ke^{2u_s}\right)^{-1} \varrho(P_s, \delta_s) \left[-\frac{\Delta \langle \nabla K, \nabla x \cdot P_s \rangle}{2} (P_s) \delta_s + o(\delta_s) \right].$$

Taking the Taylor expansion of K around $Q=(0,0,1)$ where $P_s=(\alpha, 0, \sqrt{1-\alpha^2}) \in B(Q, \varepsilon)$,

$$K(x_1, x_2) = K(Q) + Ax_1^2 + Bx_1x_2 + Cx_2^2 + O(|x|^3).$$

We have

$$\langle \mathbf{x} \cdot P_s \rangle = \alpha x_1 + \sqrt{1-\alpha^2}\, x_3 = \alpha x_1 + \sqrt{1-\alpha^2} \sqrt{1-x_1^2-x_2^2}$$

and

$$\Delta \langle \nabla K, \nabla x \cdot P_s \rangle (P_s) = -2\Delta K(P_s) + O(\alpha).$$

Thus we find for $u_s \in C_{P_s, \delta_s}$ with $P_s \in B(Q, \varepsilon)$

$$\frac{d}{ds} F[u_s] = \left(\int Ke^{2u_s}\right)^{-1} \varrho(P_s, \delta_s) [(\Delta(K)(P_s) - \varepsilon) \delta_s + o(\delta_s)] \geqslant 0.$$

Otherwise $du_s/ds=0$ hence $dF[u_s]/ds=0$, verifying claim (6.6). Thus to finish the assertion (b) we apply the flow Φ_s to a maximizing sequence whose minima concentrates at a critical point Q with $\Delta K(Q)>0$, then for large values of s, we obtain a competing sequence which do not concentrate at Q in fact not at any such Q because of claim (6.5). This finishes the proof of Proposition F.

§ 7. Proof of Theorems I and II

In this section we will use the analysis in § 6 to prove Theorems I and II.

Proof of Theorem I. The first observation is: Suppose K is a function which allows a solution u for the equation (1.1), then so is $K \circ \varphi$ with u_φ as a solution for any conformal transformation φ of S^2. Thus we may assume w.l.o.g. using conformal transformation that the given local maxima of K are located at the north and south poles of the sphere which we denote by N, S respectively.

We now consider the variational scheme $\mathrm{Var}(N, S)$ introduced in § 6, and let $c = \max_u \min_{p \in (-\infty, \infty)} F[u_p]$. Let $u^{(k)}$ denote a maximizing family of paths with $\min_p F[u_p^{(k)}] = F[u_{p_k}^{(k)}]$; we abbreviate $u_{p_k}^{(k)}$ by u_k. Normalize u_k by $⨍ e^{2u_k} = 1$, and apply the Concentration lemma (Proposition A) to the sequence $\{u_k\}$. If $⨍ |\nabla u_k|^2$ stay bounded, then $u_k \to u$ weakly in $H^{1,2}$, and the function u would be a weak solution, hence (e.g. [12]) a strong solution of (1.1). Thus we assume u_k has a subsequence which we also denote by u_i which is a concentrated sequence with its mass e^{2u_i} converging to a point $P_\infty \in S^2$. Assuming $u_k \in \mathcal{S}_{Q_k, t_k}$ and assume w.l.o.g. (via Lemma 6.1) that $Q_k \to P_\infty$, and $t_k \to \infty$ as $k \to \infty$, and since $u^{(k)} \in \mathcal{P}'_{t_0}(N, S)$ we get $S[u_k] = O(t_k^{-1} (\log t_k)^2)$ after applying Proposition C and

$$c = \log K(P_\infty). \tag{7.1}$$

We now apply Proposition E in § 6 and conclude that we may assume w.l.o.g. that P_∞ is a critical point of K. Next, we apply part (i) of proposition F in § 6 to conclude, since N, S both are local maximum points of K, that

$$c < \max(\log K(N), \log K(S)). \tag{7.2}$$

On the other hand, the assumption (1.6) of Theorem I indicates that for the test functions $u_t = \frac{1}{2} \log |d\varphi_t|$ with $\varphi_t(z) = tz$ in the stereographic projection coordinates based at N, we have

$$\begin{aligned} F[u_t] &= \log ⨍ K e^{2u_t} - S[u_t] \\ &\leq \log ⨍ K |d\varphi_t| \\ &= \log ⨍ K \circ \varphi_t^{-1} = \log ⨍ K \circ \varphi_{t^{-1}}. \end{aligned}$$

Thus by our assumption (1.6) in Theorem I,

$$\inf_{0 < t < \infty} F[u_t] > \sup_{\substack{\nabla K(Q) = 0 \\ Q \neq N, S}} \log(K(Q))$$

which implies (by definition) that

$$c > \sup_{\substack{\nabla K(Q) = 0 \\ Q \neq N, S}} \log(K(Q)).$$

We draw from (7.1), (7.2), (7.3) a contradiction, and conclude that the original max-min sequence $\{u_k\}$ for the scheme $\mathrm{Var}(N, S)$ must converge to a solution u which satisfies (1.1). We have thus finished the proof of Theorem I.

Proof of Theorem II. Choose any two local maxima P_1, P_2 of K and do the variational scheme $\mathrm{Var}(P_1, P_2)$. It follows from our study of the concentration phenomenon that a maximizing sequence of minima must converge to a solution.

Appendix. Proof of Proposition 2.2

First we will recall the proof (cf. [14]) that given $u \in H^{1,2}(S^2)$ with $⨍ e^{2u}=1$, there exists some conformal transformation $\varphi_{Q,t}$ so that the center of mass of $\exp(2T'(Q)(u))$ is at the origin. To see this, we consider the map

$$X: B^3 \simeq \{\varphi_{Q,t} \mid Q \in S^2, 1 \leq t < \infty\} \to \mathbf{R}^3 \quad \text{given by} \quad X(\varphi_{Q,t}) = ⨍ \mathbf{x} \circ \varphi_{Q,t} e^{2u}.$$

This is obviously a continuous map, with the continuous boundary value

$$\lim_{t \to \infty} X(\Phi_{Q,t}) = Q.$$

Thus the Brouwer degree theorem gives the existence of some (Q, t) with the required property

$$⨍ \mathbf{x} \exp(2T'(Q)(u)) = ⨍ \mathbf{x} \circ \varphi_{Q,t}^{-1} e^{2u} = 0.$$

Next to produce a continuously varying set of φ_{Q_s,t_s} when $u_s \in H^{1,2}(S^2)$ depends continuously on the parameter s, we will first prove the existence of a continuously varying φ_s with

$$⨍ \mathbf{x} e^{2(u_s)_{\varphi_s}} = 0$$

where φ_s is a general conformal map of S^2, not necessarily of the form $\varphi_{Q,t}$. To see this, we will apply the Implicit function theorem. Denote by G the full group of conformal maps of S^2 onto itself. Given $u \in H^{1,2}$ consider the map $X: G \to \mathbf{R}^3$ defined by $X(g) = ⨍ (\mathbf{x} \circ g)\, e^{2u}$ for all $g \in G$ the same map as before. We claim that the differential of the map X evaluated at the identity map $g=1$ has full rank equal to 3. To see this, we let e_1, e_2, e_3 denote an orthonormal frame in $\mathbf{R}^3$ and $x_i=x\cdot e_i$, $i=1,2,3$. Then

$$\frac{d}{dt}\bigg|_{t=1}\int x_j \circ \varphi_{e_i,t}\, e^{2u}$$

gives a linearly independent tangent vector to G at $g=1$. Thus the differential dX of the map X at $g=1$ expressed in the coordinates x_i has the matrix $dX|_{g=1}=(\Lambda, B)$ where $\Lambda=(\Lambda_{ij})^3_{i,j=1}$ with $\Lambda_{ij}=⨍\langle \nabla x_i \cdot \nabla x_j\rangle\, e^{2u}$ (B another 3×3 matrix) with rank $(dX|_{g=1})=3$ as claimed. Thus the ordinary differential equation

$$\frac{d}{ds}\bigg|_{s=0}⨍(\mathbf{x}\circ\varphi_s)\, e^{2u_s} = ⨍\frac{d}{ds}\bigg|_{s=0}(x\circ\varphi_s)\, e^{2u} + 2⨍\mathbf{x}e^{2u}\frac{du_s}{ds}\bigg|_{s=0}$$

with $\varphi_0=\varphi$, $u_0=u$ satisfying $⨍e^{2u}\mathbf{x}=0$ is always solvable for some $\varphi_s\in G$ with

$$\frac{d}{ds}\bigg|_{s=0}⨍(\mathbf{x}\circ\varphi_s)\, e^{2u_s}=0.$$

Continuing the flow φ_s, we get a continuous family of $\varphi_s\in G$ with $⨍(\mathbf{x}\circ\varphi_s)\, e^{2u_s}=\mathbf{0}$ for given continuous family u_s.

Finally we show that the conformal map $\varphi_s \in G$ chosen above may in fact be chosen of the form $\varphi_s=\varphi_{Q_s,t_s}$. For this we observe that for every rotation R of $\mathbf{R}^3$ we have

$$⨍(\mathbf{x}\circ\varphi\circ R)\, e^{2u}=0 \quad \text{if} \quad ⨍(\mathbf{x}\circ\varphi)\, e^{2u}=0.$$

Hence we may appeal to the following basic fact about Lie-groups:

Polar decomposition [13]. Given $\varphi\in G=$ the conformal group of S^2. Then φ may be uniquely written as $\varphi=PR$, where R is a rotation of S^2 and P is a positive hermitian matrix (which corresponds to $P=\varphi_{Q,t}$ for some $Q\in S^2$, $1\le t<\infty$). Furthermore, the choices of R and P depend continuously on φ.

Choose P_s corresponding to φ_s in the polar decomposition. Then $P_s=\varphi^{-1}_{Q_s,t_s}$ for a continuous map of Q_s, t_s with

$$⨍\mathbf{x}\circ\exp(2T^{t_s}(Q_s)(u_s)) = ⨍\mathbf{x}\circ P_s\, e^{2u_s}=0.$$

This finishes the proof of Proposition 2.2.

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