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PRESCRIBING THE SYMMETRIC FUNCTION OF THE EIGENVALUES OF THE SCHOUTEN TENSOR

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ABSTRACT. In this paper we study the problem of conformally deforming a metric to a prescribed symmetric function of the eigenvalues of the Schouten tensor on compact Riemannian manifolds with boundary. We prove its solvability and the compactness of the solution set, provided the Ricci tensor is nonnegative-definite.

1. INTRODUCTION

Let (M^n, g) be a smooth, compact Riemannian manifold with totally geodesic boundary of dimension $n \geq 3$. The Schouten tensor of g is defined by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric and R are the Ricci and scalar curvatures of g, respectively.

Let $\sigma_k : R^n \to R$ be the k-th elementary symmetric function $(1 \leq k \leq n)$

$$\sigma_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k},$$

Γ_k the corresponding open, convex cone, i.e. $\Gamma_k = \{x \in R^n \mid \sum_i (x) > 0, 1 \leq i \leq k\}$. Let

$$\Sigma_\theta = \left\{x = (x_1, \cdots, x_n) \in R^n \mid \quad \min x_i + \theta \sum x_i > 0\right\}.$$

Now let us consider the general symmetric function F defined on Γ $(\Gamma_n \subset \Gamma \subset \Sigma_{\frac{1}{n-2}})$ and satisfying

(C_1) F is positive and $F = 0$ on $\partial\Gamma$;
(C_2) F is concave;
(C_3) F is invariant under exchange of variables;
(C_4) F is homogeneous of degree 1;
(C_5) $\lim_{s\to\infty} F(sx) = \infty, \quad \forall\, x \in \Gamma$;
(C_6) $F(x) \leq \varrho\sigma_1(x)$ in Γ and $F(1, \cdots, 1) = n\varrho$, where ϱ is a positive constant;
(C_7) $\frac{\partial F}{\partial x_i} \geq \Sigma\frac{F}{\sigma_1}$ for some constant $\Sigma > 0$ for all i.

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We need (C_1)–(C_4) to ensure that the elliptic equations are solvable. If F additionally satisfies (C_5), then the Liouville Theorem in [12] is applicable. Condition (C_6) says that the Newton-Maclaurin inequality with respect to the function F holds. $F = \sigma_k^{1/k}$ satisfies condition (C_7).

We denote $[g] = \{\tilde{g} \mid \tilde{g} = e^{-2u}g\}$. We call the metric $\hat{g} = e^{-2u}g$ (as well as the function u) Γ-admissible, or simply admissible, if $\hat{g} \in \left\{\tilde{g} \in [g] \mid \lambda(\tilde{g}^{-1}A_{\tilde{g}}) \in \Gamma\right\}$. Here, $\lambda(\tilde{g}^{-1}A_{\tilde{g}}) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of $\tilde{g}^{-1}A_{\tilde{g}}$.

In this paper we study the existence of some prescribing problems and the compactness of the solution set. The main result is as follows.

Theorem 1.1. *Let (M^n, g) be a compact n-dimensional Riemannian manifold with totally geodesic boundary. Let F be a symmetric function satisfying $(C_1) - (C_7)$ on Γ with $\Gamma_n \subset \Gamma \subset \Sigma_{\frac{1}{n-2}}$. If the manifold (M, g) is not conformally equivalent to a hemisphere, then for any positive function f, there exists an admissible conformal metric $\tilde{g} = e^{-2u}g$ with totally geodesic boundary satisfying*

$$F\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) = f.$$

Additionally, the set of all such solutions is compact in the C^m-topology for any $m \geq 0$.

We can get the following corollary from Theorem 1.1 immediately. That is, to find a conformal metric $\tilde{g}$ with nonnegative $\mathrm{Ric}_{\tilde{g}}$ such that

$$\det\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) = f^n, \tag{1.1}$$

where $\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}}) = (\mu_1, \cdots, \mu_n)$ are the eigenvalues of $\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}}$ and $f(x)$ is a positive function.

Since $\mathrm{Ric}_{\tilde{g}} = (n-2)A_{\tilde{g}} + \sigma_1(\lambda(\tilde{g},^{-1} A_{\tilde{g}}))\tilde{g}$ if we define $F(\lambda) = \sigma_n^{1/n}\left((n-2)\lambda + (\sum_{i=1}^n \lambda_i)\right)$ and $\Gamma = \{\lambda \mid F(\lambda) > 0\}$, then

$$\begin{aligned}
& \det{}^{1/n}\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) \\
= \;& \sigma_n^{1/n}\Big(\mu\Big(g^{-1}\Big[(n-2)(du \otimes du - |\nabla u|^2 g) + (n-2)\nabla^2 u + \triangle u g + \mathrm{Ric}_g\Big]\Big)\Big) \\
= \;& F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \tfrac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big),
\end{aligned}$$

where $\mu = (n-2)\lambda + \sum_{i=1}^n \lambda_i$. From the definition, it is easy to verify that F satisfies $(C_1) - (C_5)$, since

$$\frac{\partial F}{\partial \lambda_i} = \frac{\partial\left(\sigma_n^{1/n}\right)}{\partial \mu_s}\left(1 + (n-2)\delta_i^s\right)$$

and

$$\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} = \left(1 + (n-2)\delta_i^s\right)\frac{\partial^2\left(\sigma_n^{1/n}\right)}{\partial \mu_s \partial \mu_t}\left(1 + (n-2)\delta_j^t\right).$$

Moreover, from

$$\begin{aligned}
F\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) &= \sigma_n^{1/n}\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) \\
&\leq \frac{1}{n}\sigma_1\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) \\
&= \frac{2n-2}{n}\sigma_1\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right),
\end{aligned}$$

we know that F satisfies (C_6) with $\varrho = \frac{2n-2}{n}$. Thus (1.1) turns out to be a proper equation with respect to the Schouten tensor. Furthermore, as in [6] and [15], by use of the volume comparison theorem, the C^0 estimate of the solutions of such an equation can be derived if $\mathrm{Ric} \geq 0$. In other words, the condition $\Gamma \subset \Sigma_{\frac{1}{n-2}}$ ensures that the volume comparison theorem is applicable, where the eigenvalues λ of the Schouten tensor satisfy $(n-2)\lambda + \sum_{i=1}^{n} \lambda_i \geq 0$ if and only if the eigenvalues of the Ricci tensor $\mu \geq 0$. Similarly, on the manifold with totally geodesic boundary, based on the boundary C^1, C^2 estimates with the Neumann boundary condition for the general symmetric function ([2] or [9], etc.), we can get

Corollary 1.2. *Let (M, g) be a compact n-dimensional Riemannian manifold with totally geodesic boundary and with the Ricci tensor semi-positive-definite. If it is not conformally equivalent to a hemisphere, then for any positive function f, there exists a conformal metric $\tilde{g} = e^{-2u} g$ with totally geodesic boundary and $\mathrm{Ric}_{\tilde{g}} \geq 0$ and*

$$\det\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) = f^n.$$

Additionally, the set of all such solutions is compact in the C^m-topology for any $m \geq 0$.

Remark 1.3. The conformal problem with respect to the Ricci tensor has been studied extensively. In [13] and [8], the authors studied the negative Ricci curvature and proved that there exists a conformal metric $\tilde{g}$ with negative Ricci tensor $\mathrm{Ric}_{\tilde{g}}$ such that

$$\det\left(\mu(\tilde{g}^{-1}\mathrm{Ric}_{\tilde{g}})\right) = const.$$

When the Ricci tensor is positive-definite, in [5], Guan and Wang derived a conformal metric with a constant smallest eigenvalue of the Ricci tensor. In [15], Trudinger and Wang proved the prescribing problem of a positive Ricci tensor on a closed manifold.

This paper is organized as follows. We begin with some preliminaries in Section 2. In Section 3, we will discuss the deformation and a priori estimates. The proof of Theorem 1.1 is in Section 4.

2. Preliminaries

We first introduce Fermi coordinates in a boundary neighborhood. In these local coordinates, we take the geodesic in the inner normal direction $\nu = \frac{\partial}{\partial x^n}$ parameterized by arc length, and $\left(x^1, ..., x^{n-1}\right)$ forms a local chart on the boundary where $x^n = 0$. The metric can be expressed as

$$g = g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2.$$

The Greek letters $\alpha, \beta, \gamma, ...$ stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma, ... \leq n-1$, while the Latin letters $i, j, k, ...$ stand for the full indices, $1 \leq i, j, k, ... \leq n$ (see [4] and [1]).

We denote the functions, tensors and covariant differentiations with respect to the induced metric on the boundary by a *bar* (e.g. $\bar{\Gamma}^\alpha_{\beta\gamma}$, $\bar{R}_{\alpha\beta}$). Then the Christoffel symbols on the boundary satisfy

$$\bar{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\right) = \Gamma^\gamma_{\alpha\beta}$$

and $\Gamma^n_{nn} = 0$, $\Gamma^\alpha_{nn} = 0$, $\Gamma^n_{n\alpha} = 0$.

Let us denote $\frac{\partial}{\partial x^i}$ by ∂_i. The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \tau g_{\alpha\beta}$, where τ is a function defined on ∂M. Since the boundary ∂M is connected, by the Schur Theorem, $\tau = const$. A totally geodesic boundary is umbilic with $\tau = 0$.

Thus $\Gamma^n_{\alpha\beta}|_{\partial M} = L_{\alpha\beta} = \tau g_{\alpha\beta}$ and $\Gamma^\alpha_{n\beta}|_{\partial M} = -L_{\alpha\gamma}g^{\gamma\beta} = -\tau\delta^\beta_\alpha$.

Under the conformal metric $\tilde{g} = e^{-2u}g$, the functions, tensors and the covariant differentiations with respect to $\tilde{g}$ are denoted by a *tilde* (e.g. $A_{\tilde{g}}$, $\tilde{L}_{\alpha\beta}$).

Let $[g]$ be the set of metrics conformal to g. For $\tilde{g} = e^{-2u}g \in [g]$, we consider the equation

$$F\left(\lambda(\tilde{g}^{-1}A_{\tilde{g}})\right) = f. \tag{2.1}$$

The Schouten tensor transforms according to the formula

$$A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g,$$

where ∇u and $\nabla^2 u$ denote the gradient and Hessian of u with respect to g. Consequently, (2.1) is equivalent to

$$F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x)e^{-2u}.$$

Then the second fundamental form satisfies

$$\tilde{L}_{\alpha\beta}e^u = \frac{\partial u}{\partial \nu}g_{\alpha\beta} + L_{\alpha\beta}.$$

Note that the umbilicity is conformally invariant. When the boundary is umbilic, the above formula becomes

$$\tilde{\tau}e^{-u} = \frac{\partial u}{\partial \nu} + \tau,$$

where $\tilde{L}_{\alpha\beta} = \tilde{\tau}\tilde{g}_{\alpha\beta}$.

Therefore, whereas the initial metric g on the manifold M is with totally geodesic boundary ∂M, the boundary of the manifold M with conformal metric $\tilde{g} = e^{-2u}g$ is still totally geodesic if and only if $\frac{\partial u}{\partial \nu} = 0$.

Therefore, in order to prove Theorem 1.1, we need to find admissible solutions of the following equation:

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g\Big]\Big)\Big) = f(x)e^{-2u} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{2.2}$$

3. Deformation, C^1 and C^2 estimates

To prove the existence of a solution to the equation (2.2), we employ the following deformation, which is defined in [7]:

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\varsigma(1-\psi(t))g + \psi(t)A_g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) & \\ = \psi(t)f(x)e^{-2u} + (1-t)(\int e^{-(n+1)u})^{2/n+1} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{3.1}$$

where $\psi \in C^1[0,1]$ satisfies $0 \le \psi(t) \le 1, \psi(0) = 0$, $\psi(t) = 1$ for $t \ge \frac{1}{2}$, and $\varsigma = (n\varrho)^{-1}\mathrm{vol}(M_g)^{\frac{2}{n+1}}$, where $F(1,\cdots,1) = n\varrho$.

Similarly to [7], at $t = 1$, (3.1) becomes (2.2), while at $t = 0$, it becomes

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) = (\int e^{-(n+1)u})^{\frac{2}{n+1}} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

We can show that the above equation has a unique solution $u(x) \equiv 0$. In fact, it is obvious that $u \equiv 0$ is a solution. Now we are going to prove its uniqueness.

At the maximum point x_0 of u, no matter if x_0 is an interior or a boundary point, we always have that $\nabla u|_{x_0} = 0$ and $\nabla^2 u|_{x_0}$ is nonpositive-definite. In fact if x_0 is an interior point, it is clear; if x_0 is a boundary point, we have $\frac{\partial u}{\partial \nu}|_{\partial M} = 0$ by equation (3.1) and $\frac{\partial u}{\partial x^\alpha}|_{x_0} = 0$, where $\{x^\alpha\}_{1 \le \alpha \le n-1}$ are local coordinates on the boundary ∂M around x_0. Therefore $\nabla^2 u|_{x_0}$ is nonpositive-definite. Now at x_0 we have

$$\begin{aligned} \mathrm{vol}(M_g)^{\frac{2}{n+1}} &= \varsigma \cdot n\varrho = \varsigma F(\lambda(g^{-1} \cdot g)) \\ &\ge F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &= (\int e^{-(n+1)u})^{\frac{2}{n+1}}. \end{aligned}$$

Similarly, at the minimum point of u, we can get $\varsigma \cdot n\varrho \le (\int e^{-(n+1)u})^{\frac{2}{n+1}}$. As a result, we have $\mathrm{vol}(M_g)^{\frac{2}{n+1}} = \varsigma \cdot n\varrho = (\int e^{-(n+1)u})^{\frac{2}{n+1}}$.

By (C_6), we know that $F \le \varrho\sigma_1$. Hence,

$$\begin{aligned} \varsigma \cdot n\varrho &= F\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &\le \varrho\, \sigma_1\Big(\lambda\Big(g^{-1}\Big[\varsigma g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &= \varrho\, \Big(n\varsigma + \triangle u + (1 - \frac{n}{2})|\nabla u|^2\Big). \end{aligned}$$

Then

$$(\frac{n}{2} - 1)\int_M |\nabla u|^2 \le \int_M \triangle u = \int_{\partial M} \frac{\partial u}{\partial \nu} = 0,$$

and $u \equiv const. = 0$.

Thus the operator

$$\begin{aligned} \Psi_t[u] &= F\Big(\lambda\Big(g^{-1}\Big[\varsigma(1 - \psi(t))g + \psi(t)A_g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) \\ &\quad -\psi(t)f(x)e^{-2u} - (1-t)(\int e^{-(n+1)u})^{\frac{2}{n+1}} \end{aligned}$$

satisfies the Leray-Schauder degree $\deg(\Psi_0, \mathcal{O}_0, 0) \ne 0$ at $t = 0$, where the Leray-Schauder degree is defined by [11] (see [2] for the boundary case) and $\mathcal{O}_0$ is a neighborhood of the zero solution in $\{u \in C^{4,\alpha}(M) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M\}$. Thus, whereas we obtain the homotopy-invariance of the degree, we can derive that the Leray-Schauder degree is nonzero at $t = 1$. This shows that equation (2.2) is solvable.

The C^1 and C^2 estimates of the solutions to (3.1) have been proved in [9]. We may obtain

Lemma 3.1. *For any fixed* $0 < \delta < 1$, *there is a constant* $C = C(\delta, n, g, f)$ *such that any solution of* (3.1) *with* $t \in [0, 1-\delta]$ *satisfies* $\|u\|_{C^{4,\alpha}} \le C$.

So without loss of generality, we may assume that u_{t_i} tends to $-\infty$ at $t_i \to 1$, where u_{t_i} is the solution of (3.1) at $t = t_i$, which will be denoted by u_i in what follows. Thus equation (3.1) turns out to be

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[A_g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big]\Big)\Big) & \text{in } M, \\ = (1-t)o + f(x)e^{-2u} & \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{3.2}$$

where u is assumed to be admissible and $o \geq 0$ is a constant.

Furthermore, we can get a more exact estimate on the geodesic ball $B(x, r) = \{y \in M \mid \mathrm{dist}(x, y) < r\}$:

Lemma 3.2 ([9])**.** *Let $u \in C^4(M)$ be a k-admissible solution of* (3.1) *in $B(x, r)$ and $0 \leq r < 1$. Then there is a constant $C = C(n, g, f)$ such that*

$$\left(|\nabla^2 u| + |\nabla u|^2\right)(x') \leq C\Big(r^{-2} + \exp\left(-2 \inf_{B(x, 2\sqrt{10}r)} u\right)\Big) \tag{3.3}$$

for all $x' \in B(x, r)$.

4. Proof of Theorem 1.1

We call $\{u_k\}$ the blow-up sequence and $\bar{x} \in M$ the blow-up point if $u_k(x_{0,k}) \to -\infty$ as $x_{0,k} \to \bar{x}$, where $\{x_{0,k}\} \subset M$. Now let $\{u_k\}$ be a blow-up solution of (3.2) with the blow-up point $\bar{x}$.

First of all, we would like to prove that $\bar{x}$ can be approximated by local minimum points of u_k. Let $v_k = e^{-(n-2)/2u_k}$, denote $v_k(x_{0,k})^{\frac{1}{n-2}}$ by $R_{0,k}$ and $\frac{1}{1-e^{-1/2}}$ by A_0.

Lemma 4.1. *In each geodesic ball $B(x_{0,k}, A_0 R_{0,k}^{-1}) \subset M$ we may find a local maximum point of v_k, denoted by x_k. Furthermore,*

$$v_k(x_k) = \sup_{B(x_k, v_k(x_k)^{-\frac{1}{n-2}})} v_k.$$

Proof. Let $e^{u_k(x_{0,k})} = \varepsilon_{0,k}$. We define a mapping:

$$\begin{aligned} \mathcal{U}_{0,k} : \mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2}) \subset T_{x_{0,k}}(M) \quad &\to \quad B(x_{0,k}, {\varepsilon_{0,k}}^{1/2}), \\ y \quad &\longmapsto \quad \exp_{x_{0,k}}(\varepsilon_{0,k} y), \end{aligned}$$

where the metric on the tangent space is $\check{g}_k = \varepsilon_{0,k}^{-2} \mathcal{U}_{0,k}^* g$ and $\mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2})$ is a geodesic ball. Moreover, consider a sequence of functions $\mu_{0,k}(y) = u_k(\mathcal{U}_{0,k}(y)) - \log \varepsilon_{0,k}$. We may derive an equation that $\mu_{0,k}(y)$ satisfies. In fact, we have

$$\begin{cases} F\Big(\lambda\Big(\check{g}_k^{-1}\Big[A_{\check{g}_k} + \nabla^2 \mu_{0,k} + d\mu_{0,k} \otimes d\mu_{0,k} - \frac{1}{2}|\nabla \mu_{0,k}|^2 \check{g}_k\Big]\Big)\Big) \\ = \varepsilon_{0,k}^2 (1-t)o + f(\mathcal{U}_{0,k}(y))e^{-2\mu_{0,k}} \quad \text{in } \mathcal{B}(0, \varepsilon_{0,k}^{-1/2}), \\ \frac{\partial \mu_{0,k}}{\partial x^n} = 0 \quad \text{on } \mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2}) \cap \{x^n = 0\}, \end{cases}$$

where $\mu_{0,k}$ is admissible and o is a nonnegative constant.

Let us begin with the easy case, $u_k\,(x) \geq u_k(x_{0,k}) - 1$ in $B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$. In this case, $0 \leq e^{-\frac{n-2}{2}\mu_{0,k}} \leq e^{\frac{n-2}{2}}$ in $\mathcal{B}(0, {\varepsilon_{0,k}}^{-1/2})$. Hence, $\mu_{0,k}$ converges in C^3 to μ_∞ with $0 \leq e^{-\frac{n-2}{2}\mu_\infty} \leq e^{\frac{n-2}{2}}$ on $\mathbb{R}^n$, and the limit function μ_∞ satisfies

$$F\Big(\lambda\Big(\delta^{-1}\Big[\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 \delta)\Big]\Big)\Big) = f(\bar{x})e^{-2u}.$$

Then by the Liouville Theorem [12], we know that 0 is the locally minimum point of $\mu_{0,k}$. Rescaling back, we see that $x_{0,k}$ is the locally minimum point of u_k in $B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$.

The alternative case is that there exists $x_{1,k} \in B(x_{0,k}, {\varepsilon_{0,k}}^{1/2})$ such that $u_k(x_{1,k}) < u_k(x_{0,k}) - 1$. Then we may consider the lower bound of u_k in $B(x_{1,k}, {\varepsilon_{1,k}}^{1/2})$, where $\varepsilon_{1,k} = e^{u_k(x_{1,k})} < e^{-1}\varepsilon_{0,k}$. If $u_k \geq u_k(x_{1,k}) - 1$ in $B(x_{1,k}, {\varepsilon_{1,k}}^{1/2})$, then $\mu_{1,k}(y) = u_k(\mathcal{U}_{1,k}(y)) - \log \varepsilon_{1,k} > -1$, where

$$\mathcal{U}_{1,k} : y \to \exp_{x_{1,k}}(\varepsilon_{1,k} y)$$

and $x_{1,k}$ is a locally minimum point of u_k.

Otherwise, we may repeat the previous proceedings with $u_k(x_{j,k}) < u_k(x_{j-1,k}) - 1$ $(x_{j,k} \in B(x_{j-1,k}, {\varepsilon_{j-1,k}}^{1/2}))$, $\varepsilon_{j,k} = e^{u_k(x_{j,k})} < e^{-1}\varepsilon_{j-1,k}$ and $\mu_{j,k}(y) = u_k(\mathcal{U}_{j,k}(y)) - \log \varepsilon_{j,k}$, where

$$\mathcal{U}_{j,k} : y \to \exp_{x_{j,k}}(\varepsilon_{j,k} y).$$

For any given k, as $u_k \in C^\infty(M)$, there exists $j(k) \in \mathbb{N}$, $j(k) < \infty$ such that $u_k(x_{j(k),k}) < u_k(x_{j(k)-1,k}) - 1$ and $u_k \geq u_k(x_{j(k),k}) - 1$ in $B(x_{j(k),k}, {\varepsilon_{j(k),k}}^{1/2})$. Hence, we can find a locally minimum point of the u_k in $B(x_{j(k),k}, {\varepsilon_{j(k),k}}^{1/2}) \subset B(x_{0,k}, A_0{\varepsilon_{0,k}}^{1/2})$. This completes the proof (see Lemma 3.2 in [15] for more details). □

Now we consider the rescaled sequence $w_k = u_k - \sup_M u_k$. Suppose x_k^0 is the maximum point of u_k. Since $e^{-2\sup u_k} f(x_k^0) = e^{-2u_k(x_k)} f(x_k^0) \leq C(\triangle u_k + A_g)(x_k^0) \leq C$, then $\bar{x} = \lim x_k$ is the blow-up point with respect to w_k as well. It is obvious that w_k satisfies the equation

$$\begin{cases} F\Big(\lambda\Big(g^{-1}\Big[A_g + \nabla^2 w_k + dw_k \otimes dw_k - \frac{1}{2}|\nabla w_k|^2 g\Big]\Big)\Big) & \\ = (1-t)o + f(x)e^{-2\sup_M u_k} e^{-2w_k} & \text{in } M, \\ \frac{\partial w_k}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

where w_k is admissible and $o \geq 0$ is a constant.

By virtue of Lemma 4.1, we may assume that $\bar{x} = \lim x_k$, where $\{x_k\}$ are locally minimum points of u_k. Hence $\{x_k\}$ are also locally minimum points of w_k and

$$w_k(x_k) = \inf_{B(x_k, e^{\frac{1}{2}w_k(x_k)})} w_k.$$

Note that F satisfies $(C_1) - (C_6)$ and the w_k are Γ admissible, where $\Gamma \subset \Sigma_{\frac{1}{n-2}}$. Hence w_k are subharmonic and satisfy

$$W + \frac{1}{n-2}\sigma_1(W) g \geq 0, \tag{4.1}$$

where $W = \nabla^2 w_k + dw_k \otimes dw_k - \frac{1}{2}|\nabla w_k|^2 g + A_g$. We need the idea of the minimal radial functions of w in $B_R(x_0)$ ([15]):

$$\widehat{w}(x) = \sup\{w(y) : y \in \partial B_r(x_0), r = d(x, x_0) \leq R\},$$

and we denote $\nabla^2 \widehat{w} + d\widehat{w} \otimes d\widehat{w} - \frac{1}{2}|\nabla \widehat{w}|^2 g + A_g$ by $\widehat{W}$. Now we are ready to prove the following.

Propostion 4.2. *Let u_j be a blow-up sequence of solutions to* (3.2)*. Then $w_j = u_j - \sup_M u_j$ converges in $w^{1,p}$ (for any $1 < p < \frac{n}{n-1}$) to an admissible function w. Moreover, if $\bar{x}$ is a blow-up point of w, then near $\bar{x}$,*

$$w(x) = 2\log d(x, \bar{x}) + o(1), \tag{4.2}$$

where $d(x, \bar{x})$ denotes the geodesic distance from x to $\bar{x}$ with respect to the metric g. Furthermore, each blow-up point is isolated.

Proof. Since a similar proposition on a manifold without boundary has appeared in [15], we only focus on the differences.

Step 1. We may get admissible solutions on the doubled manifold. Glue two copies of (M, g) together along the totally geodesic boundary and denote the doubled manifold by $\check{M}$. With the given smooth Riemannian metric g on M, there is a standard metric $\check{g}$ on $\check{M}$ induced from g. When ∂M is totally geodesic in (M, g), $\check{g}$ is $C^{2,1}$ on $\check{M}$ (see [3]).

We can extend w_k to a $C^2(\check{M})$ function $\check{w}_k$ as follows: Near the boundary we take Fermi coordinates, where $\check{w}_k$ is then defined as

$$\check{w}_k(x_1, \cdots, x_n) = \begin{cases} w_k(x_1, \cdots, x_n), & x_n \geq 0, \\ w_k(x_1, \cdots, -x_n), & x_n \leq 0. \end{cases}$$

Since $\frac{\partial w_k}{\partial \nu} = 0$, it is easy to verify by definition that $\check{w}_k \in C^2(\check{M})$. As a matter of fact,

$$\begin{aligned} & \lim_{x_n \to 0^+} \frac{\partial \check{w}_k}{\partial x^n}(x_1, \cdots, x_n) = \frac{\partial w_k}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) \\ = \quad & 0 = -\frac{\partial w_k}{\partial x^n}(x_1, \cdots, x_{n-1}, 0) = \lim_{x_n \to 0^-} \frac{\partial \check{w}_k}{\partial x^n}(x_1, \cdots, x_n) \end{aligned}$$

and

$$\lim_{x_n \to 0^+} \frac{\partial^2 \check{w}_k}{\partial (x^n)^2}(x_1, \cdots, x_n) = \lim_{x_n \to 0^-} \frac{\partial^2 \check{w}_k}{\partial (x^n)^2}(x_1, \cdots, x_n).$$

Thus from the admissible property of w_k we know that $\check{w}_k$ is also admissible and satisfies (4.1).

Step 2. We can find convergent "minimal radial functions" on the doubled manifold. Inequality (4.1) says that $\check{w}_k$ is subharmonic. From Corollary 2.1 in [15], $\{\check{w}_k\}$ converges to a subharmonic function $\check{w}$ in $W^{1,p}$ (for any $1 < p < \frac{n}{n-1}$). By Corollary 2.2 in [15], the corresponding minimal radial functions $\widehat{\check{w}}_k$ also converge to $\widehat{\check{w}}$. Noting that the minimal radial functions depend only on the distance to the center, by Corollary 2.1 and Corollary 2.2 in [15], we may obtain

$$\widehat{\check{w}}\,(r) = \lim_{k \to \infty} \widehat{\check{w}}_k\,(r)\,, \tag{4.3}$$

where

$$\widehat{\check{w}}_k(r) = \sup\{\check{w}_k(y) : y \in \partial B_r(x_k)\}$$

and

$$\widehat{\check{w}}(r) = \sup\{\check{w}(y) : y \in \partial B_r(\bar{x})\}.$$

On the one hand, based on (4.3) and (4.1), we can get the following estimates:

$$\widehat{\check{w}}(x) \leq 2\log d(x, \bar{x}) + C. \tag{4.4}$$

In fact, we may assume that $\widehat{\breve{w}}_k(r) = \breve{w}_k(x_r)$, $x_r = (0, \cdots, 0, r)$, and $|A_g| \le Cr/2$. The $\widehat{\breve{w}}_k$ are still admissible and satisfy inequality (4.1). Thus

$$\begin{aligned} 0 &\le \left((n-2)\widehat{W}_{nn} + \Sigma_i \widehat{W}_{ii}\right)(x_r) \\ &\le (n-1)\left(\widehat{\breve{w}}_k'' + (\widehat{\breve{w}}_k')^2 - \frac{g_{nn}}{2}(\widehat{\breve{w}}_k')^2 + Cr/2\right) \\ &\quad + \sum_{i=1}^{n-1}\left((\frac{1}{r} + C)\widehat{\breve{w}}_k' - \frac{g_{ii}}{2}(\widehat{\breve{w}}_k')^2 + Cr/2\right) \\ &\le (n-1)\left(\widehat{\breve{w}}_k'' + \frac{1}{r}\widehat{\breve{w}}_k' + C(\widehat{\breve{w}}_k' + r)\right), \end{aligned}$$

where the last inequality comes from $\sum_i g_{ii} \ge n$. Hence,

$$\left(\log(r\widehat{\breve{w}}_k' + r^2)\right)' + C \ge 0.$$

By taking a limit we get (4.4).

On the other hand, let $\widehat{\breve{v}}_k = e^{-(n-2)/2\widehat{\breve{w}}_k}$. From $\triangle\widehat{\breve{v}}_k \le C\widehat{\breve{v}}_k r$, we get

$$[r^{n-1}\widehat{\breve{v}}_k']' \le Cr^n\widehat{\breve{v}}_k.$$

Thus, by a direct calculation, we know that

$$\widehat{\breve{w}}(x) \ge 2\log d(x, \bar{x}) + o(1).$$

Therefore

$$\widehat{\breve{w}}(x) = 2\log d(x, \bar{x}) + o(1). \tag{4.5}$$

Then the comparison principle helps us to deduce (4.2) from (4.5). Roughly speaking, since $\breve{w}$ equals $\widehat{\breve{w}}$ at some points, the comparison principle implies that they are equal everywhere. That is,

$$\breve{w}(x) = 2\log d(x, \bar{x}) + o(1).$$

(For more details, one may consult section 3 of [15].) □

Proof of Theorem 1.1. As the proof of Proposition 4.2, we glue two copies of (M, g) together. Denote the doubled manifold and functions by a "check" (e.g. $\breve{M}, \breve{w}$). Since the Ricci curvature $\mathrm{Ric}_{e^{-2\breve{w}}g}$ is still nonnegative, by (4.5) and the Volume Comparison Theorem, there is at most one end away from the blow-up points. The metric $e^{-2\breve{w}}g$ is in fact a Euclidean one (see section 7 of [6] for details); namely, (M, g) is conformally equivalent to the unit half sphere, which contradicts the assumption in Theorem 1.1. Therefore there is a unform L^∞ bound for solutions. So the set of solutions is compact. This completes the proof of Theorem 1.1. □

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