Presenting cyclotomic q-Schur algebras

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Ariki-Koike algebra

R : commutative ring,
$q, q^{-1}, Q_1, \cdots, Q_r \in R$; parameters.

Definition

$\mathscr{H}_{n,r}$; Ariki-Koike alg.$/_R$ ass. to $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

generators ; $T_0, T_1, \cdots, T_{n-1}$
fundamental relations ;

$$(T_0 - Q_1)(T_0 - Q_2)\cdots(T_0 - Q_r) = 0,$$
$$(T_i - q)(T_i + q^{-1}) = 0 \qquad (1 \le i \le n-1),$$
$$T_0T_1T_0T_1 = T_1T_0T_1T_0,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad (1 \le i \le n-2),$$
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Cyclotomic q-Schur algebra

$$\Lambda_{n,r} := \left\{ (\lambda^{(1)}, \cdots, \lambda^{(r)}) \;\middle|\; \begin{array}{c} \lambda^{(k)} = (\lambda^{(k)}_1, \cdots, \lambda^{(k)}_n) \in \mathbb{Z}^n_{\geq 0} \\ \sum_{k=1}^{r} \sum_{i=1}^{n} \lambda^{(k)}_i = n \end{array} \right\}.$$

Definition

$\mathscr{S}_{n,r} := \mathrm{End}_{\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}} M^{\mu} \Big)$, M^{μ} ; right $\mathscr{H}_{n,r}$-module.

R ; field. $\Rightarrow$ $\mathscr{S}_{n,r}$: quasi-hereditary algebra.

$\mathscr{S}_{n,r}$: "quasi-hereditary cover" of $\mathscr{H}_{n,r}$.

- $\mathscr{H}_{n,r} \cong \varphi_\omega \mathscr{S}_{n,r} \varphi_\omega \cong \mathrm{End}_{\mathscr{S}_{n,r}}(\mathscr{S}_{n,r}\varphi_\omega)^{\mathrm{opp}}$. $\varphi_\omega \in \mathscr{S}_{n,r}$; idempotent.
- $\mathrm{Hom}_{\mathscr{S}_{n,r}}(\mathscr{S}_{n,r}\varphi_\omega, ?)$: $\mathscr{S}_{n,r}$-mod $\to$ $\mathscr{H}_{n,r}$-mod. ; Schur functor
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where $q, Q_1, \cdots, Q_r$: indeterminate.

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$\mathscr{H}_{n,r}$: Ariki-Koike algebra $\big/_{\mathcal{K}}$ with parameters $q, Q_1, \cdots, Q_r$.

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$\mathscr{S}_{n,r}$; semisimple. $\quad \{W^\lambda \mid \lambda \in \Lambda^+_{n,r}\} = \{\text{simple } \mathscr{S}_{n,r}\text{-module}\}/_{\text{iso.}}$,

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$$U_q(\mathfrak{gl}_m) \curvearrowright V^{\otimes n} \curvearrowleft \mathscr{H}_{n,1} \qquad m = n \cdot r$$

$$\{\text{weight appearing in } V^{\otimes n}\} \overset{1:1}{\longleftrightarrow} \Big\{(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m_{\geq 0} \;\Big|\; \sum_{i=1}^{m} \lambda_i = n\Big\}$$

$$=: \Lambda_{n,1}.$$

$(V^{\otimes n})_\mu \cong M^\mu$; "permutation module" (as right $\mathscr{H}_{n,1}$-modules).

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- Doty-Giaquinto gives a presentation of $\mathscr{S}_{n,1}$ by

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Case of type A ($r = 1$)

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$Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\,\alpha_i$; root lattice,

$Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0}\,\alpha_i$.

"$\geq$" ; partial order on P defined by

$$\lambda \geq \mu \overset{\text{def}}{\Leftrightarrow} \lambda - \mu \in Q^+ \qquad ; \text{dominance order}$$

quantum group $U_q(\mathfrak{gl}_m)$

$P = \bigoplus_{i=1}^{m} \mathbb{Z}\,\varepsilon_i$; weight lattice of $\mathfrak{gl}_m$,

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Definition

$U_q = U_q(\mathfrak{gl}_m)/_{\mathcal{K}}$ is defined by

generators ; e_i, f_i $(1 \le i \le m-1)$, $K_i^{\pm}$ $(1 \le i \le m)$.
fundamental relations ;

$$K_iK_j = K_jK_i, \quad K_iK_i^- = K_i^-K_i = 1,$$
$$K_ie_jK_i^- = q^{\langle \alpha_j, h_i \rangle} e_j,$$
$$K_if_jK_i^- = q^{-\langle \alpha_j, h_i \rangle} f_j,$$
$$e_if_j - f_je_i = \delta_{ij}\left(\frac{K_iK_{i+1}^- - K_i^-K_{i+1}}{q - q^{-1}}\right),$$

$+$ q-Serre relations.

Theorem (Doty-Giaquinto)

$\mathscr{S}_{n,1}$ is isomorphic to the algebra defined by

generators ; e_i, f_i $(1 \le i \le m-1)$, $K_i^{\pm}$ $(1 \le i \le m)$
with fundamental relations of $U_q(\mathfrak{gl}_m)$

$$+\ K_1 K_2 \cdots K_m = q^n,$$

$$(K_i - 1)(K_i - q)(K_i - q^2)\cdots(K_i - q^n) = 0 \quad (1 \le i \le m).$$

${}_{\mathcal{A}}\mathscr{S}_{n,1}$ is the $\mathcal{A}$-subalgebra of $\mathscr{S}_{n,1}$ generated by $e_i^{(k)}$, $f_i^{(k)}$, $K_j^{\pm}$, $\begin{bmatrix} K_j ; 0 \\ t \end{bmatrix}$ $(1 \le i \le m-1,\ 1 \le j \le m,\ t \ge 1)$,

where $e_i^{(k)} = \dfrac{e_i^k}{[k]!}$, $f_i^{(k)} = \dfrac{f_i^k}{[k]!}$, $\begin{bmatrix} K_j ; 0 \\ t \end{bmatrix} = \displaystyle\prod_{s=1}^{t} \frac{K_i q^{-s+1} - K_i^{-1} q^{s-1}}{q^s - q^{-s}}$.

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A presentation of q-Schur algebra 2

$\mathscr{S}_{n,1}$ is also a quotient algebra of $\dot{U}_q$ (Lusztig's modified form of U_q).

Theorem (Doty-Giaquinto)

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fundamental relations ;

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda \in \Lambda_{n,1}} 1_\lambda = 1,$$

$$E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{if } \lambda + \alpha_i \notin \Lambda_{n,1}, \end{cases} \qquad 1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda_{n,1}, \\ 0 & \text{if } \lambda - \alpha_i \notin \Lambda_{n,1}, \end{cases}$$

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Schur-Weyl duality (Sakamoto-Shoji)

$$U_q(\mathfrak{g}) \curvearrowright V^{\otimes n} \curvearrowleft \mathscr{H}_{n,r}, \quad \text{where } \mathfrak{g} = \underbrace{\mathfrak{gl}_n \oplus \cdots \oplus \mathfrak{gl}_n}_{r} \subset \mathfrak{gl}_m$$

$(m = n \cdot r)$.

$\overline{\mathscr{S}}^0_{n,r}$; the image of $U_q(\mathfrak{g}) \to \mathrm{End}(V^{\otimes n})$.

Relations between $\overline{\mathscr{S}}^0_{n,r}$ and $\mathscr{S}_{n,r}$ (Sawada-Shoji)

$$\begin{array}{ccc} & \mathscr{S}^0_{n,r} & \hookrightarrow \mathscr{S}_{n,r} \\ & \Big\downarrow & \\ U_q(\mathfrak{g}) \twoheadrightarrow & \overline{\mathscr{S}}^0_{n,r} \cong \displaystyle\bigoplus_{\substack{(n_1,\cdots,n_r) \\ n_1+\cdots+n_r=n}} \mathscr{S}_{n_1,1} \otimes \cdots \otimes \mathscr{S}_{n_r,1} & \end{array}$$

Motivations (for the case of $r \geq 2$)

Schur-Weyl duality (Sakamoto-Shoji)

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Motivations (for the case of $r \geq 2$)

generalization to $\mathbf{p} = (r_1, \cdots, r_g)$ s.t. $r_1 + \cdots + r_g = r$ (Shoji-Wada)

$$\begin{array}{ccc} \mathscr{S}^{\mathbf{p}}_{n,r} & \hookrightarrow & \mathscr{S}_{n,r} \\ \downarrow & & \\ \overline{\mathscr{S}}^{\mathbf{p}}_{n,r} \cong \bigoplus_{\substack{(n_1,\cdots,n_g) \\ n_1+\cdots+n_g=n}} \mathscr{S}_{n_1,r_1} \otimes \cdots \otimes \mathscr{S}_{n_g,r_g} & & \end{array}$$

The case of $\mathbf{p} = (1, \cdots, 1) \Rightarrow$ (Sawada-Shoji)

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$$\begin{array}{ccc} \mathscr{S}^{\mathbf{p}}_{n,r} & \hookrightarrow & \mathscr{S}_{n,r} \\ \downarrow\!\!\downarrow & & \end{array}$$

$$^{\exists}\widetilde{U}_q(\mathfrak{g}^{\mathbf{p}}) \twoheadrightarrow \overline{\mathscr{S}}^{\mathbf{p}}_{n,r} \cong \bigoplus_{\substack{(n_1,\cdots,n_g) \\ n_1+\cdots+n_g=n}} \mathscr{S}_{n_1,r_1} \otimes \cdots \otimes \mathscr{S}_{n_g,r_g}$$

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$\mathfrak{g}^{p} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_g} \subset \mathfrak{gl}_m$, where $m_k = n \cdot r_k$.

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The case of $\mathbf{p} = (r)$

$$^{\exists}\widetilde{U}_q(\mathfrak{gl}_m) \twoheadrightarrow \mathscr{S}_{n,r}$$

Theorem (Parshall-Wang (case of type A, $r = 1$))

$\exists\, \mathscr{S}^{\geq 0}_{n,1},\ \exists\, \mathscr{S}^{\leq 0}_{n,1} \subset \mathscr{S}_{n,1}$; "Borel subalgebra" s.t. $\mathscr{S}_{n,1} = \mathscr{S}^{\leq 0}_{n,1} \cdot \mathscr{S}^{\geq 0}_{n,1}$.

$\rho|_{\mathcal{B}^+} : \mathcal{B}^+ \twoheadrightarrow \mathscr{S}^{\geq 0}_{n,1}$,

$\mathcal{B}^+ := \langle e_i, K_j \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m \rangle_{alg.} \subset U_q(\mathfrak{gl}_m)$,

$\rho|_{\mathcal{B}^-} : \mathcal{B}^- \twoheadrightarrow \mathscr{S}^{\leq 0}_{n,1}$,

$\mathcal{B}^- := \langle f_i, K_j \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m \rangle_{alg.} \subset U_q(\mathfrak{gl}_m)$

Theorem (Du-Rui (case of $r \geq 2$))

$\exists\, \mathscr{S}^{\geq 0}_{n,r},\ \exists\, \mathscr{S}^{\leq 0}_{n,r} \subset \mathscr{S}_{n,r}$; "Borel subalgebra" s.t. $\mathscr{S}_{n,r} = \mathscr{S}^{\leq 0}_{n,r} \cdot \mathscr{S}^{\geq 0}_{n,r}$.

$\exists \mathcal{F}^{\geq 0}$; $\mathscr{S}^{\geq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\geq 0}_{n,r}$,

$\exists \mathcal{F}^{\leq 0}$; $\mathscr{S}^{\leq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\leq 0}_{n,r}$.

$$\begin{array}{cc} \mathscr{S}^{\leq 0}_{n,r} & \mathscr{S}^{\geq 0}_{n,r} \\ \wr\Vert & \wr\Vert \\ \mathscr{S}^{\leq 0}_{n,1} & \mathscr{S}^{\geq 0}_{n,1} \\ \uparrow\uparrow & \uparrow\uparrow \\ \mathcal{B}^- & \mathcal{B}^+ \end{array}$$

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$^\exists \mathscr{S}_{n,1}^{\geq 0}$, $^\exists \mathscr{S}_{n,1}^{\leq 0} \subset \mathscr{S}_{n,1}$; "Borel subalgebra" s.t. $\mathscr{S}_{n,1} = \mathscr{S}_{n,1}^{\leq 0} \cdot \mathscr{S}_{n,1}^{\geq 0}$.

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Theorem (Du-Rui (case of $r \geq 2$))

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$^\exists \mathcal{F}^{\geq 0}$; $\mathscr{S}_{n,1}^{\geq 0} \xrightarrow{\sim} \mathscr{S}_{n,r}^{\geq 0}$,

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$$\mathscr{S}_{n,r}^{\leq 0} \cong \mathscr{S}_{n,1}^{\leq 0} \twoheadleftarrow \mathcal{B}^-, \qquad \mathscr{S}_{n,r}^{\geq 0} \cong \mathscr{S}_{n,1}^{\geq 0} \twoheadleftarrow \mathcal{B}^+$$

Borel subalgebras of $\mathscr{S}_{n,r}$

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$$\begin{array}{cc} \wr\Vert & \wr\Vert \\ \mathscr{S}_{n,1}^{\leq 0} & \mathscr{S}_{n,1}^{\geq 0} \\ \uparrow & \uparrow \\ \mathcal{B}^- & \mathcal{B}^+ \end{array}$$

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Theorem (Du-Rui (case of $r \geq 2$))

$\exists\, \mathscr{S}^{\geq 0}_{n,r}$, $\exists\, \mathscr{S}^{\leq 0}_{n,r} \subset \mathscr{S}_{n,r}$; "Borel subalgebra" s.t. $\mathscr{S}_{n,r} = \mathscr{S}^{\leq 0}_{n,r} \cdot \mathscr{S}^{\geq 0}_{n,r}$.

$$\begin{array}{ccc} & \cong & \cong \\ & \mathscr{S}^{\leq 0}_{n,1} & \mathscr{S}^{\geq 0}_{n,1} \\ & \uparrow & \uparrow \\ & \mathcal{B}^- & \mathcal{B}^+ \end{array}$$

$\exists \mathcal{F}^{\geq 0}$; $\mathscr{S}^{\geq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\geq 0}_{n,r}$,

$\exists \mathcal{F}^{\leq 0}$; $\mathscr{S}^{\leq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\leq 0}_{n,r}$.

Theorem (Parshall-Wang (case of type A, $r = 1$))

$^\exists \mathscr{S}_{n,1}^{\geq 0}$, $^\exists \mathscr{S}_{n,1}^{\leq 0} \subset \mathscr{S}_{n,1}$; "*Borel subalgebra*" s.t. $\mathscr{S}_{n,1} = \mathscr{S}_{n,1}^{\leq 0} \cdot \mathscr{S}_{n,1}^{\geq 0}$.

$\rho|_{\mathcal{B}^+} : \mathcal{B}^+ \twoheadrightarrow \mathscr{S}_{n,1}^{\geq 0}$,
$\mathcal{B}^+ := \langle e_i, K_j \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m \rangle_{alg.} \subset U_q(\mathfrak{gl}_m)$,

$\rho|_{\mathcal{B}^-} : \mathcal{B}^- \twoheadrightarrow \mathscr{S}_{n,1}^{\leq 0}$,
$\mathcal{B}^- := \langle f_i, K_j \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m \rangle_{alg.} \subset U_q(\mathfrak{gl}_m)$

Theorem (Du-Rui (case of $r \geq 2$))

$^\exists \mathscr{S}_{n,r}^{\geq 0}$, $^\exists \mathscr{S}_{n,r}^{\leq 0} \subset \mathscr{S}_{n,r}$; "*Borel subalgebra*" s.t. $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}^{\leq 0} \cdot \mathscr{S}_{n,r}^{\geq 0}$.

$$\mathscr{S}_{n,r}^{\leq 0} \cong \mathscr{S}_{n,1}^{\leq 0} \twoheadleftarrow \mathcal{B}^-, \qquad \mathscr{S}_{n,r}^{\geq 0} \cong \mathscr{S}_{n,1}^{\geq 0} \twoheadleftarrow \mathcal{B}^+$$

$^\exists \mathcal{F}^{\geq 0}$; $\mathscr{S}_{n,1}^{\geq 0} \xrightarrow{\sim} \mathscr{S}_{n,r}^{\geq 0}$,

$^\exists \mathcal{F}^{\leq 0}$; $\mathscr{S}_{n,1}^{\leq 0} \xrightarrow{\sim} \mathscr{S}_{n,r}^{\leq 0}$.

Definition

$U_q = U_q(\mathfrak{gl}_m)/_{\mathcal{K}}$ is defined by

generators ; $e_i,\ f_i\ (1 \le i \le m-1),\ K_i^{\pm}\ (1 \le i \le m)$.
fundamental relations ;

$$\begin{aligned} K_iK_j &= K_jK_i, \quad K_iK_i^- = K_i^-K_i = 1, \\ K_ie_jK_i^- &= q^{\langle \alpha_j, h_i \rangle} e_j, \\ K_if_jK_i^- &= q^{-\langle \alpha_j, h_i \rangle} f_j, \\ e_if_j - f_je_i &= \delta_{ij}\left(\frac{K_iK_{i+1}^- - K_i^-K_{i+1}}{q-q^{-1}}\right), \end{aligned}$$

$+\ q$-Serre relations.

Definition

$\widetilde{U}_q = \widetilde{U}_q(\mathfrak{gl}_m)/_{\mathcal{K}}$ is defined by

generators ; e_i, f_i $(1 \le i \le m-1)$, $K_i^{\pm}$ $(1 \le i \le m)$, τ_i $(1 \le i \le m-1)$.
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$$K_iK_j = K_jK_i, \quad K_iK_i^- = K_i^-K_i = 1,$$
$$K_ie_jK_i^- = q^{\langle \alpha_j, h_i \rangle} e_j,$$
$$K_if_jK_i^- = q^{-\langle \alpha_j, h_i \rangle} f_j,$$
$$K_i\tau_jK_i^- = \tau_j$$
$$e_if_j - f_je_i = \delta_{ij}\ \tau_i,$$
$$+\ q\text{-Serre relations.}$$

Proposition

Let $\widetilde{I} := \langle \tau_i - \frac{K_i K_{i+1}^- - K_i^- K_{i+1}^-}{q - q^{-1}} \mid 1 \leq i \leq m \rangle_{ideal} \subset \widetilde{U}_q$,

we have $\widetilde{U}_q / \widetilde{I} \cong U_q$.

Corollary

Let $\widetilde{\mathcal{B}}^+ := \langle e_i, K_j \mid 1 \leq i \leq m-1, 1 \leq j \leq m \rangle_{alg.} \subset \widetilde{U}_q(\mathfrak{gl}_m)$,

$\widetilde{\mathcal{B}}^- := \langle f_i, K_j \mid 1 \leq i \leq m-1, 1 \leq j \leq m \rangle_{alg.} \subset \widetilde{U}_q(\mathfrak{gl}_m)$,

we have $\widetilde{\mathcal{B}}^\pm \cong \mathcal{B}^\pm$.

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Let $\widetilde{I} := \langle \tau_i - \frac{K_i K_{i+1}^{-} - K_i^{-} K_{i+1}^{-}}{q - q^{-1}} \mid 1 \leq i \leq m \rangle_{ideal} \subset \widetilde{U}_q$,

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Corollary

Let $\widetilde{\mathcal{B}}^{+} := \langle e_i, K_j \mid 1 \leq i \leq m-1,\ 1 \leq j \leq m \rangle_{alg.} \subset \widetilde{U}_q(\mathfrak{gl}_m)$,

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Notations

Recall $m = n \cdot r$.

$$\{1, \cdots, m\} \overset{1:1}{\longleftrightarrow} \Gamma := \{(i,k) \mid 1 \le i \le n,\ 1 \le k \le r\}, \quad \Gamma' := \Gamma \setminus \{(n,r)\}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ n(k-1)+i & \longleftrightarrow & (i,k) \end{array}$$

$$P = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i \overset{1:1}{\longleftrightarrow} P = \bigoplus_{(i,k)\in\Gamma} \mathbb{Z}\varepsilon_{(i,k)}$$

$$\begin{array}{ccc} \bigcup & & \bigcup \\ \Lambda_{n,1} & \overset{1:1}{\longleftrightarrow} & \Lambda_{n,r} \end{array}$$

$$Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i = \bigoplus_{(i,k)\in\Gamma'} \mathbb{Z}\alpha_{(i,k)}$$

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Surjection $\widetilde{U}_q \twoheadrightarrow \mathscr{S}_{n,r}$

$$\mathscr{S}_{n,r} = \mathscr{S}^{\leq 0}_{n,r} \cdot \mathscr{S}^{\geq 0}_{n,r}.$$

$$\mathcal{F}^{\geq 0} : \mathscr{S}^{\geq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\geq 0}_{n,r}, \qquad \widetilde{\mathcal{B}}^{+} \xrightarrow{\sim} \mathcal{B}^{+} \overset{\rho}{\twoheadrightarrow} \mathscr{S}^{\geq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\geq 0}_{n,r}.$$

$$\mathcal{F}^{\leq 0} : \mathscr{S}^{\leq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\leq 0}_{n,r}, \qquad \widetilde{\mathcal{B}}^{-} \xrightarrow{\sim} \mathcal{B}^{-} \overset{\rho}{\twoheadrightarrow} \mathscr{S}^{\leq 0}_{n,1} \xrightarrow{\sim} \mathscr{S}^{\leq 0}_{n,r}.$$

Proposition

$^{\exists}\widetilde{\rho} : \widetilde{U}_q \longrightarrow\!\!\!\!\to \mathscr{S}_{n,r}$ s.t.

- $\widetilde{\rho}(e_{(i,k)}) = \varphi^{+}_{(i,k)} := \mathcal{F}^{\geq 0} \circ \rho(e_{(i,k)}) \qquad ((i,k) \in \Gamma'),$
- $\widetilde{\rho}(f_{(i,k)}) = \varphi^{-}_{(i,k)} := \mathcal{F}^{\leq 0} \circ \rho(f_{(i,k)}) \qquad ((i,k) \in \Gamma'),$
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$$\mathscr{S}_{n,r} = \mathrm{End}_{\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu\in\Lambda_{n,r}} M^{\mu}\Big) \cong \bigoplus_{\mu,\nu\in\Lambda_{n,r}} \mathrm{Hom}_{\mathscr{H}_{n,r}}(M^{\mu}, M^{\nu})$$

$\kappa^{\pm}_{(i,k)}$: multiplying $q^{\pm\mu_i^{(k)}}$ on M^{μ} $(\mu \in \Lambda_{n,r})$,

$\varphi^{+}_{(i,k)}|_{M^{\mu}}$: $M^{\mu} \to M^{\mu+\alpha_{(i,k)}}$,

$\varphi^{-}_{(i,k)}|_{M^{\mu}}$: $M^{\mu} \to M^{\mu-\alpha_{(i,k)}}$.

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$\mathcal{S}_{n,r}$ is an asso. algebra $/_{\mathcal{K}}$ defined by

generators ; $E_{(i,k)}, F_{(i,k)}$ $((i,k) \in \Gamma')$, 1_λ $(\lambda \in \Lambda_{n,r})$.

fundamental relations ;

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \qquad \sum_{\lambda \in \Lambda_{n,r}} 1_\lambda = 1,$$

$$E_{(i,k)} 1_\lambda = \begin{cases} 1_{\lambda+\alpha_{(i,k)}} E_{(i,k)} & \text{if } \lambda + \alpha_{(i,k)} \in \Lambda_{n,r}, \\ 0 & \text{if } \lambda + \alpha_{(i,k)} \notin \Lambda_{n,r}, \end{cases}$$

$$1_\lambda E_{(i,k)} = \begin{cases} E_{(i,k)} 1_{\lambda-\alpha_{(i,k)}} & \text{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}, \\ 0 & \text{if } \lambda - \alpha_{(i,k)} \notin \Lambda_{n,r}, \end{cases}$$

$$F_{(i,k)} 1_\lambda = \begin{cases} 1_{\lambda-\alpha_{(i,k)}} F_{(i,k)} & \text{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}, \\ 0 & \text{if } \lambda - \alpha_{(i,k)} \notin \Lambda_{n,r}, \end{cases}$$

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where

$$\eta^{\lambda}_{(i,k)} = \begin{cases} \Big(-Q_{k+1}[\lambda_n^{(k)} - \lambda_1^{(k+1)}] \\ \quad + q^{\lambda_n^{(k)} - \lambda_1^{(k+1)}} \big(q^{-1} g^{\lambda}_{(n,k)}(F,E) - q g^{\lambda}_{(1,k+1)}(F,E) \big) \Big) 1_{\lambda} & \text{if } i = n, \\ [\lambda_i^{(k)} - \lambda_{i+1}^{(k)}] 1_{\lambda} & \text{if } i \neq n. \end{cases}$$

$$g^{\lambda}_{(n,k)}(F,E) = \sum_{\substack{(i_1,\cdots,i_l) \\ (j_1,\cdots,j_l)}} r^{(i_1,\cdots,i_l)}_{(j_1,\cdots,j_l)} F_{i_1} \cdots F_{i_l} E_{j_1} \cdots E_{j_l} \quad (r^{(i_1,\cdots,i_l)}_{(j_1,\cdots,j_l)} \in \mathcal{K})$$

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Proposition

$^{\exists}\Psi : \widetilde{U}_q \longrightarrow\!\!\!\!\rightarrow \mathcal{S}_{n,r}$ s.t.

$$\Psi(e_{(i,k)}) = E_{(i,k)}, \quad \Psi(f_{(i,k)}) = F_{(i,k)}, \quad \Psi(K^{\pm}_{(i,k)}) = \sum_{\lambda \in \Lambda_{n,r}} q^{\pm \lambda_i^{(k)}} 1_\lambda,$$

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For $\lambda \in \Lambda_{n,1} \overset{1:1}{\longleftrightarrow} \Lambda_{n,r}$,

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$\widetilde{I}_{n,r}$; two-sided ideal of $\widetilde{U}_q$ generated by

$$\tau_{(i,k)} - \eta_{(i,k)} \qquad ((i,k) \in \Gamma')$$
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$$(K_i - 1)(K_i - q)(K_i - q^2) \cdots (K_i - q^n) \qquad (1 \le i \le m)$$

$U_{n,r} := \widetilde{U}_q / \widetilde{I}_{n,r}$.

Proposition

$\mathcal{S}_{n,r} \cong U_{n,r}$ *as algebras.*

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Presentation of cyclotomic q-Schur algebras

$$\begin{array}{ccc} \widetilde{U}_q & \xrightarrow{\widetilde{\rho}} & \mathscr{S}_{n,r} \\ \Psi \downarrow & \nearrow \Phi & \\ U_{n,r} \cong \mathcal{S}_{n,r} & & \end{array}$$

Theorem

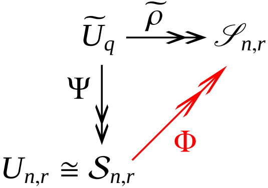
$\mathscr{S}_{n,r}$ is isomorphic to $\mathcal{S}_{n,r} \cong U_{n,r}$.

Moreover, ${}_{\mathcal{A}}\mathscr{S}_{(n,r)}$ is isomorphic to the $\mathcal{A}$-subalgebra of $\mathcal{S}_{n,r}$ generated by

$E^{(t)}_{(i,k)}, F^{(t)}_{(i,k)}$ $((i,k) \in \Gamma', \ t \geq 1)$ *and* 1_λ $(\lambda \in \Lambda_{n,r})$.

${}_{\mathcal{A}}\mathscr{S}_{n,r}$ is also isomorphic to the $\mathcal{A}$-subalgebra of $U_{n,r}$ generated by

$e^{(t)}_i, \ f^{(t)}_i, K_j$ *and* $\begin{bmatrix} K_j; 0 \\ t \end{bmatrix}$ $(1 \leq i \leq m-1, \ 1 \leq j \leq m, \ t \geq 1)$.



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$$\begin{array}{ccc} \widetilde{U}_q & \xrightarrow{\widetilde{\rho}} & \mathscr{S}_{n,r} \\ \Psi \downarrow & \nearrow \Phi & \\ U_{n,r} \cong \mathcal{S}_{n,r} & & \end{array}$$

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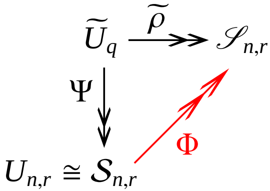

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$\Delta(\lambda) := \mathcal{S}^-_{n,r} \cdot 1_\lambda + \mathcal{S}_{n,r}(> \lambda) \subset \mathcal{S}_{n,r}(\geq \lambda) / \mathcal{S}_{n,r}(> \lambda)$; left $\mathcal{S}_{n,r}$-module.

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Proposition

1. $\Delta(\lambda) \otimes_{\mathcal{K}} \Delta^\sharp(\lambda) \twoheadrightarrow \mathcal{S}_{n,r}(\geq \lambda)/\mathcal{S}_{n,r}(> \lambda)$ *as* $(\mathcal{S}_{n,r}, \mathcal{S}_{n,r})$*-bimodules.*
2. $\Delta(\lambda)$ $(\lambda \in \Lambda^+_{n,r})$ *has the unique top* $L(\lambda)$.
3. $\{L(\lambda) \mid \lambda \in \Lambda^+_{n,r}\} = \{$*simple (left)* $\mathcal{S}_{n,r}$*-module*$\}/_{iso.}$

$\Lambda^+_{n,r} = \{\lambda_{\langle 1\rangle}, \lambda_{\langle 2\rangle}, \cdots, \lambda_{\langle t\rangle}\}$ s.t. $\lambda_{\langle i\rangle} < \lambda_{\langle j\rangle} \Rightarrow i < j$.

$\mathcal{S}_{n,r} = \mathcal{S}_{n,r}(\lambda_{\langle 1\rangle}) \supsetneqq \mathcal{S}_{n,r}(\lambda_{\langle 2\rangle}) \supsetneqq \cdots \supsetneqq \mathcal{S}_{n,r}(\lambda_{\langle t\rangle}) \supsetneqq 0$

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Sketch of proof of Theorem ($\mathcal{S}_{n,r} \cong \mathscr{S}_{n,r}$)

$$\begin{array}{ccc} \widetilde{U}_q & \xrightarrow{\widetilde{\rho}} & \mathscr{S}_{n,r} \\ \Psi \downarrow & \nearrow \Phi & \\ \mathcal{S}_{n,r} & & \end{array}$$

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By induction on $\Lambda^+_{n,r}$, we have

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