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PRESERVATION PROPERTIES OF STOCHASTIC ORDERS BY TRANSFORMATION TO HARRIS FAMILY

BY

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Abstract. Stochastic comparisons of lifetime characteristics of reliability systems and their components are of common use in lifetime analysis. In this paper, using Harris family distributions, we compare lifetimes of two series systems with random number of components, with respect to several types of stochastic orders. Our results happen to enfold several previous findings in this connection. We shall also show that several stochastic orders and ageing characteristics, such as IHRA, DHRA, NBU, and NWU, are inherited by transformation to Harris family. Finally, some refinements are made concerning related existing results in the literature.

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1. INTRODUCTION

Clearly, the lifetime of any reliability system depends on the lifetime of its components. Thus, in practice, to compare stochastically the lifetime of two systems, we need to compare the lifetimes of their components. The Harris family of distributions is a known family for the lifetime of a series system. It was introduced by Aly and Benkherouf [8] as a generalization of the Marshall–Olkin family. The Marshall–Olkin family of distributions is better known as the family with a tilt parameter. It was introduced by Marshall and Olkin [25] and was obtained as the proportional odds family (proportional odds model) by Kirmani and Gupta [23]. However, it was first proposed by Clayton [15].

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The aim of this paper is to focus on the Harris family and stochastically compare such lifetime systems with each other. We recall that the Harris family is constructed by combining the Harris probability generating function (pgf) introduced by Harris [21] and a baseline distribution function. More precisely, a survival function of the family is defined as

$$\bar{H}(x;\theta,k) = \left(\frac{\theta \bar{F}^k(x)}{1-\bar{\theta}\bar{F}^k(x)}\right)^{1/k}, \quad -\infty < x < \infty,\ 0<\theta<\infty,\ \bar{\theta}=1-\theta,\ k>0, \tag{1.1}$$

where $F(x)$ is called the *baseline distribution function* (df) and θ is called the *tilt parameter*. It is easily seen that hazard rates corresponding to $F(x)$ and $H(x;\theta,k)$, namely, $r_F(\cdot) = f(\cdot)/\bar{F}(\cdot)$ and $r_H(\cdot;\theta,k) = h(\cdot;\theta,k)/\bar{H}(\cdot;\theta,k)$, are related by

$$r_H(x;\theta,k) = \frac{r_F(x)}{1-\bar{\theta}\bar{F}^k(x)}, \quad -\infty < x < \infty,\ 0<\theta<\infty,\ \bar{\theta}=1-\theta,\ k>0. \tag{1.2}$$

Clearly, $r_H(x;\theta,k)$ is shifted below $(\theta \geqslant 1)$ or above $(0<\theta\leqslant 1)$ $r_F(x)$. When $k=1$, a Harris family distribution reduces to a Marshall–Olkin distribution.

In reliability terms, a random variable (rv) X, with Harris family distribution, can be considered as the lifetime of a series system with independent and identical (iid) component lifetimes $Y_1, Y_2, \ldots, Y_N$, with df's F, when the number of components, N, is itself a Harris rv independent of Y_i's.

Recently, Batsidis and Lemonte [11] discussed another method of constructing the Harris family of distributions. They revealed that the Harris family of distributions is a proportional failure rate model which is obtained from a modified Marshall–Olkin distribution. Then, they provided several results in connection with behavior of the failure rate function for the Harris family and discussed their certain stochastic orders. Al-Jarallah et al. [7] presented a proportional hazard version of the Marshall–Olkin family of distributions as $[\bar{H}(\cdot;\theta,1)]^{\gamma}$ and investigated likelihood ratio order in this model.

Our aim is to compare a Harris family distribution with its baseline distribution, with respect to several stochastic orders. Stochastic orders are important tools for comparing probability distributions and play a great role in statistical inference and applied probability. Frequently, they are applied in contexts of risk theory, reliability, survival analysis, economic and insurance. For instance, recently, Bartoszewicz and Skolimowska [10], Błażej [14] and Misra et al. [27] studied preservation of stochastic orders under weighting. Benduch-Frąszczak [13] investigated preservation of stochastic orders and the class of life distributions in the proportional odds family. Then, Maiti and Dey [24] applied the result of stochastic orders of [13] to the tilted normal distribution. Nanda and Das [29] studied stochastic orders in the Marshall–Olkin family. Aghababaei and Alamatsaz [3], Aghababaei et al. [4] and Alamatsaz and Abbasi [6] were concerned with stochastic comparisons of different distributions with their mixtures.

There is no theoretical basis for choosing the baseline distribution and its tilt parameter in a Harris family distribution. Therefore, it is important to see how a Harris family rv responds to the change of the baseline distribution and tilt parameter. This paper mainly investigates how the relations between tilt parameters or baseline distributions affect stochastic orders between two given Harris family distributions. Considering the utility desired, we are able to choose a baseline distribution and the tilt parameter.

Abbasi et al. [1] compared two Harris families with different tilt parameters using stochastic orders. In this paper, we are concerned with four types of stochastic orders: simple stochastic orders, shifted stochastic orders, proportional stochastic orders and shifted proportional stochastic orders. In Section 2, we shall summarize some useful relations among stochastic orders to be used in the sequel. In Section 3, we consider a baseline distribution and compare the two corresponding Harris family distributions, with different tilt parameters, with respect to several stochastic orders. In Section 4, it is observed that certain stochastic orders of the baseline distribution are preserved by transformation to the Harris family with the same tilt parameter and *vice versa*. Finally, in Section 5 we prove that certain ageing characteristics, such as increasing failure rate average (IFRA), decreasing failure rate average (DFRA), new better than used (NBU) and new worse than used (NWU), are preserved by transformation to the Harris family. Thus, our results enfold all findings on stochastic orders of [19], [20], and [23] as special cases. In our investigations, we also reveal that Theorem 2.2 of [20] is valid only if the support of the tilt parameter is corrected. Hence, their result in Theorem 2.3 is not true as it is.

2. STOCHASTIC ORDERS AND CLASSES OF LIFE DISTRIBUTIONS

Let X and Y be rv's with df's F and G, survival functions (sf) $\bar{F}$ and $\bar{G}$, probability density functions (pdf) f and g, hazard rate functions r_F and r_G, reversed hazard rate functions $\tilde{r}_F$ $(= f(\cdot)/F(\cdot))$ and $\tilde{r}_G$ and supports S_X and S_Y, respectively. The lower and upper bounds of supports are denoted by $l_.$ and $u_.$. In this paper, we consider $F^{-1}(u) = \inf\{x : F(x) \leqslant u\}$, which is called the *quantile function*. Also, throughout the paper, "increasing" is used in place of "non-decreasing" and "decreasing" is used in place of "non-increasing". In what follows, some known stochastic orders and classes of life distributions, used in this article, are recalled and their important properties are stated. For more details, we refer to [28] and [31].

A. ***Usual stochastic orders***

(a) X is *statistically* smaller than Y ($X \leqslant_{st} Y$) if $\bar{F}(x) \leqslant \bar{G}(x)$ for all $x \in (-\infty, \infty)$.

(b) X is smaller than Y in the *likelihood ratio order*, denoted by $X \leqslant_{lr} Y$, if $g(x)/f(x)$ increases in x over the $S_X \cup S_Y$.

(c) X is smaller than Y in the *hazard rate order*, denoted by $X \leqslant_{hr} Y$, if $r_F(x) \geqslant r_G(x)$ for all $x \in (-\infty, \infty)$.

(d) X is smaller than Y in the *reversed hazard rate order*, denoted by $X \leqslant_{rh} Y$, if $\tilde{r}_F(x) \leqslant \tilde{r}_G(x)$ for all $x \in (-\infty, \infty)$.

(e) X is smaller than Y in the *expectation order*, denoted by $X \leqslant_E Y$, if $E(X) \leqslant E(Y)$, where expectations are assumed to exist.

(f) The *mean residual life* (mrl) function of X is defined as $m(t) = E(X - t|\, X > t)$ for $t < t^*$, where $t^* = \sup\{t : \bar{F}(t) > 0\}$. If m and m^* are mrl functions of X and Y, respectively, then X is smaller than Y in the *mrl order*, denoted by $X \leqslant_{mrl} Y$, if $m(t) \leqslant m^*(t)$ for all t or, equivalently, if $\int_t^\infty \bar{F}(u)du / \int_t^\infty \bar{G}(u)du$ decreases in t, when defined.

(g) X is smaller than Y in the *convex order*, denoted by $X \leqslant_{cx} Y$, if for every real-valued convex function $\phi(\cdot)$ defined on the real line, $E\big(\phi(X)\big) \leqslant E\big(\phi(Y)\big)$.

(h) For non-negative rv's, X is smaller than Y in the *Lorenz order*, denoted by $X \leqslant_{Lorenz} Y$, if $L_X(p) \geqslant L_Y(p)$ for all $p \in [0, 1]$, where

$$L_X(p) = \frac{\int_0^p F^{-1}(u)du}{\int_0^1 F^{-1}(u)du}, \qquad 0 \leqslant p \leqslant 1,$$

is the Lorenz curve of X.

(i) Zimmer et al. [32] defined the *log-odds function* of an rv X by

$$LO_X(t) = \ln \frac{F_X}{\bar{F}_X}$$

and introduced a new time-to-failure model based on the log-odds ratio (LOR) function. The *LOR function* of an rv X is defined by

$$LOR_X(t) = \frac{d}{dt} LO_X(t) = \frac{f(t)}{F(t)\bar{F}(t)} = \frac{r_X(t)}{F(t)}.$$

We say that X is smaller than Y in the *LOR order*, denoted by $X \leqslant_{LOR} Y$, if $l_X \leqslant l_Y, u_X \leqslant u_Y$ and $LOR_X(t) \geqslant LOR_Y(t)$ for all $t \in (l_Y, u_X)$.

(j) X is smaller than Y in the *dispersive order*, denoted by $X \leqslant_{disp} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leqslant \beta < 1$, or, equivalently, if $G^{-1}F(x) - x$ increases in x.

(k) X is smaller than Y in the *convex transform order*, denoted by $X \leqslant_c Y$, if $G^{-1}F(x)$ is convex in $x \in S_X$.

(l) For non-negative rv's, X is smaller than Y in the *star order*, denoted by $X \leqslant_* Y$, if $G^{-1}F(x)/x$ increases in $x \geqslant 0$.

(m) For non-negative rv's, X is smaller than Y in the *super-additive order*, denoted by $X \leqslant_{su} Y$, if $G^{-1}F(t+u) \geqslant G^{-1}F(t) + G^{-1}F(u)$ for $t \geqslant 0$, $u \geqslant 0$.

(n) X is smaller than Y in the *ageing intensity order*, denoted by $X \leqslant_{AI} Y$, if for all $x \geqslant 0$,

$$\frac{1}{r_F(x)} \int_0^x r_F(u)du \leqslant \frac{1}{r_G(x)} \int_0^x r_G(u)du.$$

B. ***Shifted stochastic orders***

(o) X is smaller than Y in the *up likelihood ratio order*, denoted by $X \leqslant_{lr\uparrow} Y$, if $[X - t \mid X > t] \leqslant_{lr} Y$ for all $t \geqslant 0$ or, equivalently, if $g(x)/f(t+x)$ increases in $x \in [l_Y, u_X - t]$.

(p) X is smaller than Y in the *down likelihood ratio order*, denoted by $X \leqslant_{lr\downarrow} Y$, if $X \leqslant_{lr} [Y - t \mid Y > t]$ for all $x \geqslant 0$ or, equivalently, if $g(t+x)/f(x)$ increases in $x \in [l_X, u_Y - t]$.

(q) X is smaller than Y in the *up hazard rate order* (*up reversed hazard rate order*), denoted by $X \leqslant_{hr\uparrow} (\leqslant_{rh\uparrow}) Y$, if for all $t \geqslant 0$, $[X - t \mid X > t] \leqslant_{hr} (\leqslant_{rh}) Y$ or, equivalently, if $\bar{G}(x)/\bar{F}(t+x)$ $(G(x)/F(t+x))$ increases in $x \in (-\infty, u_Y)$ for all $t \geqslant 0$.

(r) X is smaller than Y in the *down hazard rate order* (*down reversed hazard rate order*), denoted by $X \leqslant_{hr\downarrow} (\leqslant_{rh\downarrow}) Y$, if for all $t \geqslant 0$, $X \leqslant_{hr} (\leqslant_{rh}) [Y - t \mid Y > t]$ or, equivalently, if $\bar{G}(t+x)/\bar{F}(x)$ $(G(t+x)/F(x))$ increases in $x \geqslant 0$ for all $t \geqslant 0$.

C. ***Proportional stochastic orders*.** Belzunce et al. [12] and Ramos Romero and Sordo Díaz [30] have introduced the proportional likelihood ratio, proportional hazard rate and proportional reversed hazard rate orders as follows. Let X and Y be continuous and non-negative rv's. Then

(s) X is smaller than Y in the *proportional likelihood ratio order* (plr) (*proportional hazard rate order* (phr), *proportional reversed hazard rate order* (prh)), denoted by $X \leqslant_{plr} (\leqslant_{phr}, \leqslant_{prh}) Y$, if for all $0 < \lambda \leqslant 1$, $\lambda X \leqslant_{lr} (\leqslant_{hr}, \leqslant_{rh}) Y$ or, equivalently, if $g(\lambda x)/f(x)$ $(\bar{G}(\lambda x)/\bar{F}(x), G(\lambda x)/F(x))$ increases in x for all $0 < \lambda \leqslant 1$.

D. ***Shifted proportional stochastic orders*.** Jarrahiferiz et al. [22] have introduced the shifted proportional likelihood ratio order and shifted proportional hazard rate order for continuous and non-negative rv's as follows:

(t) X is smaller than Y in the *up proportional likelihood ratio order*, denoted by $X \leqslant_{plr\uparrow} Y$, if $[X - t \mid X > t] \leqslant_{plr} Y$ or, equivalently, $g(\lambda x)/f(t+x)$ is increasing in $x \in (l_X - t, u_X - t) \cup (l_Y/\lambda, u_Y/\lambda)$ for all $t \geqslant 0$ and $0 < \lambda \leqslant 1$.

(u) X is smaller than Y in the *down proportional likelihood ratio order*, denoted by $X \leqslant_{plr\downarrow} Y$, if $X \leqslant_{plr} [Y - t \mid Y > t]$ or, equivalently, if $g(\lambda x + t)f(x)$ is increasing in $x \geqslant 0$ for all $t \geqslant 0$ and $0 < \lambda \leqslant 1$.

(v) X is smaller than Y in the *up proportional hazard rate order*, denoted by $X \leqslant_{phr\uparrow} Y$, if $[X - t \mid X > t] \leqslant_{phr} Y$ or, equivalently, if $\bar{G}(\lambda x)/\bar{F}(t+x)$ is increasing in $x \in (0, u_Y/\lambda)$ for all $t \geqslant 0$ and $0 < \lambda \leqslant 1$.

(w) X is smaller than Y in the *down proportional hazard rate order*, denoted by $X \leqslant_{phr\downarrow} Y$, if $X \leqslant_{phr} [Y - t \mid Y > t]$ or, equivalently, if $\bar{G}(\lambda x + t)/\bar{F}(x)$ is increasing in $x \geqslant 0$ for all $t \geqslant 0$ and $0 < \lambda \leqslant 1$.

E. ***Classes of life distributions***

(a) X has the *increasing likelihood ratio* (ILR) (*increasing failure rate* (IFR), *increasing reversed failure rate* ($IRFR$)) *property*, denoted by $X \in ILR$ $(IFR, IRFR)$, if

$$X \leqslant_{lr\uparrow} (\leqslant_{hr\uparrow}, \leqslant_{rh\uparrow})\ X$$

or, equivalently, if $f(x)/f(x+t)$ $(\bar{F}(x)/\bar{F}(x+t), F(x)/F(x+t))$ increases in x for any $t \geqslant 0$ and X has the *decreasing likelihood ratio* (DLR) (*decreasing failure rate* (DFR), *decreasing reversed failure rate* ($DRFR$)) *property*, denoted by $X \in DLR$ $(DFR, DRFR)$, if $X \leqslant_{lr\downarrow} (\leqslant_{hr\downarrow}, \leqslant_{rh\downarrow})\ X$ or, equivalently, if $f(x+t)/f(x)$ $(\bar{F}(x+t)/\bar{F}(x), F(x+t)/F(x))$ increases in x for any $t \geqslant 0$.

(b) X has the *increasing proportional likelihood ratio* ($IPLR$) (*increasing proportional failure rate* ($IPFR$), *increasing proportional reversed failure rate* ($IPRF$)) *property*, denoted by $X \in IPLR$ $(IPFR, IPRF)$, if $X \leqslant_{plr} (\leqslant_{phr}, \leqslant_{prh})\ X$ or, equivalently, if $f(\lambda x)/f(x)$ $(\bar{F}(\lambda x)/\bar{F}(x), F(\lambda x)/F(x))$ increases in x for all $0 < \lambda \leqslant 1$.

(c) X has the *up increasing proportional likelihood ratio* ($UIPLR$) (*up increasing proportional failure rate* ($UIPFR$)) *property*, denoted by

$$X \in UIPLR\ (UIPFR),$$

if $X \leqslant_{plr\uparrow} (\leqslant_{phr\uparrow})\ X$ or, equivalently, if $f(\lambda x)/f(x+t)$ $(\bar{F}(\lambda x)/\bar{F}(x+t))$ increases in x for all $0 < \lambda \leqslant 1$ and $t \geqslant 0$ and X has the *down increasing proportional likelihood ratio* ($DIPLR$) (*down increasing proportional failure rate* ($DIPFR$)) *property*, denoted by $X \in DIPLR$ $(DIPFR)$, if $X \leqslant_{plr\downarrow} (\leqslant_{phr\downarrow})\ X$ or, equivalently, if $f(\lambda x + t)/f(x)$ $(\bar{F}(\lambda x + t)/\bar{F}(x))$ increases in x for all $0 < \lambda \leqslant 1$ and $t \geqslant 0$.

TABLE 1. Some useful relations among various types of stochastic orders.

$\leqslant_{lr}$	$\Rightarrow$	$\leqslant_{hr}$	$\Rightarrow$	$\leqslant_{st}$		$\leqslant_{hr\uparrow}$	$\Rightarrow$	$\leqslant_{hr}$	$\Rightarrow$	$\leqslant_{st}$		
$\Uparrow$						$\Uparrow$						
$\leqslant_{plr\uparrow}$	$\Rightarrow$	$\leqslant_{prh\uparrow}$	$\Rightarrow$	$\leqslant_{rh\uparrow}$	$\Leftarrow$	$\leqslant_{lr\uparrow}$	$\Leftarrow$	$\leqslant_{plr\uparrow}$	$\Rightarrow$	$\leqslant_{phr\uparrow}$	$\Rightarrow$	$\leqslant_{hr\uparrow}$
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\leqslant_{plr}$	$\Rightarrow$	$\leqslant_{prh}$	$\Rightarrow$	$\leqslant_{rh}$	$\Leftarrow$	$\leqslant_{lr}$	$\Leftarrow$	$\leqslant_{plr}$	$\Rightarrow$	$\leqslant_{phr}$	$\Rightarrow$	$\leqslant_{hr}$
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$
$\leqslant_{plr\downarrow}$		$\leqslant_{prh\downarrow}$	$\Rightarrow$	$\leqslant_{rh\downarrow}$		$\leqslant_{lr\downarrow}$	$\Leftarrow$	$\leqslant_{plr\downarrow}$	$\Rightarrow$	$\leqslant_{phr\downarrow}$	$\Rightarrow$	$\leqslant_{hr\downarrow}$
				$\Downarrow$		$\Downarrow$						
		$\leqslant_{st}$	$\Leftarrow$	$\leqslant_{rh}$		$\leqslant_{hr\downarrow}$						

(d) A non-negative rv X has IFRA (DFRA) if $\left(-\frac{1}{t}\right)\ln\bar{F}(t)$ is increasing (decreasing) in $t \geqslant 0$.

(e) A non-negative rv X is NBU (NWU) if $\bar{F}(t+u) \leqslant (\geqslant)\ \bar{F}(t)\bar{F}(u)$ for $t \geqslant 0$ and $u \geqslant 0$.

In Table 1, we summarize some useful relationships among several stochastic orders to be used in the sequel.

3. STOCHASTIC COMPARISON

Assume that the baseline df $F(x)$ in (1.1) is absolutely continuous with pdf $f(x)$. Then, the pdf and df associated with $\bar{H}(x;\theta,k)$ in (1.1) are given by

$$h(x;\theta,k) = \frac{\theta^{1/k} f(x)}{\left(1-\bar{\theta}\bar{F}^k(x)\right)^{1+1/k}}, \quad -\infty < x < \infty,\ 0 < \theta < \infty,\ \bar{\theta} = 1-\theta,\ k > 0, \tag{3.1}$$

and

$$H(x;\theta,k) = 1 - \left[\frac{\theta\bar{F}^k(x)}{\left(1-\bar{\theta}\bar{F}^k(x)\right)}\right]^{1/k}, \quad -\infty < x < \infty,\ 0 < \theta < \infty,\ \bar{\theta} = 1-\theta,\ k > 0, \tag{3.2}$$

respectively.

Batsidis and Lemonte [11] in their Proposition 2 compared a Harris family distribution with its corresponding baseline distribution with respect to several stochastic and shifted stochastic orders. In the following theorem, we compare two Harris families with respect to their tilt parameter θ.

THEOREM 3.1. *Let X, Y_1 and Y_2 be continuous and non-negative rv's corresponding to survival functions $\bar{F}(\cdot)$, $\bar{H}(\cdot;\theta_1,k_1)$ and $\bar{H}(\cdot;\theta_2,k_2)$, respectively. Moreover, let $\{0<\theta_1 \leqslant 1, \theta_2 \geqslant 1\}$. Then:*

(i) *If $X \in UIPLR$ $(IPLR, ILR)$, then $Y_1 \leqslant_{plr\uparrow} (\leqslant_{plr}, \leqslant_{lr\uparrow})\ Y_2$.*

(ii) *If $X \in DIPLR$ (DLR), then $Y_1 \leqslant_{plr\downarrow} (\leqslant_{lr\downarrow})\ Y_2$.*

(iii) *If $X \in UIPFR$ $(IPFR, IFR)$, then $Y_1 \leqslant_{phr\uparrow} (\leqslant_{phr}, \leqslant_{hr\uparrow})\ Y_2$.*

(iv) *If $X \in DIPFR$ (DFR), then $Y_1 \leqslant_{phr\downarrow} (\leqslant_{hr\downarrow})\ Y_2$.*

Proof. We give the proof for the first part. Proofs of other parts are similar and thus omitted. Let $\{0<\theta_1 \leqslant 1, \theta_2 \geqslant 1\}$ and $X \in UIPLR$. For $Y_1 \leqslant_{plr\uparrow} Y_2$, it is sufficient to show that

$$\frac{h(\lambda x;\theta_2,k_2)}{h(x+t;\theta_1,k_1)} = \frac{\theta_2^{1/k_2}}{\theta_1^{1/k_1}}\frac{f(\lambda x)}{f(x+t)}\left[\frac{\left(1-\bar{\theta}_1\bar{F}^{k_1}(x+t)\right)^{1/k_1+1}}{\left(1-\bar{\theta}_2\bar{F}^{k_2}(\lambda x)\right)^{1/k_2+1}}\right]$$

is increasing in x for any $0 < \lambda \leqslant 1$, $t \geqslant 0$ and $k_1, k_2 > 0$. Since $X \in UIPLR$, $f(\lambda x)/f(x+t)$ is increasing in x for any $0 < \lambda \leqslant 1$ and $t \geqslant 0$. Also the term in the brackets is increasing in x because

$$\frac{d}{dx}\left[\frac{\left(1-\bar{\theta}_1\bar{F}^{k_1}(x+t)\right)^{1/k_1+1}}{\left(1-\bar{\theta}_2\bar{F}^{k_2}(\lambda x)\right)^{1/k_2+1}}\right] = \left[\frac{\left(1-\bar{\theta}_1\bar{F}^{k_1}(x+t)\right)^{1/k_1+1}}{\left(1-\bar{\theta}_2\bar{F}^{k_2}(\lambda x)\right)^{1/k_2+1}}\right]$$
$$\times\left[\frac{\bar{\theta}_1(k_1+1)f(x+t)\bar{F}^{(k_1-1)}(x+t)}{1-\bar{\theta}_1\bar{F}^{k_1}(x+t)} - \frac{\lambda\bar{\theta}_2(k_2+1)f(\lambda x)\bar{F}^{(k_2-1)}(\lambda x)}{1-\bar{\theta}_2\bar{F}^{k_2}(\lambda x)}\right]$$

is non-negative provided that $\{0 < \theta_1 \leqslant 1, \theta_2 \geqslant 1\}$. Thus, we have the assertion. Our proof above also yields $Y_1 \leqslant_{plr} Y_2$, by putting $t = 0$, and $Y_1 \leqslant_{lr\uparrow} Y_2$, by letting $\lambda = 1$. ■

THEOREM 3.2. *Let Y_1 and Y_2 be rv's corresponding to the df's $H(\cdot;\theta_1,k_1)$ and $H(\cdot;\theta_2,k_2)$, respectively. If $\{0 < \theta_1 \leqslant 1, \theta_2 \geqslant 1\}$ or $\{0 < \theta_1 \leqslant \theta_2, k_1 = k_2 = k\}$, then $Y_1 \leqslant_{lr} Y_2$.*

Proof. $Y_1 \leqslant_{lr} Y_2$ is equivalent to $h(x;\theta_1,k_1)/h(x;\theta_2,k_2)$ being decreasing in x. But, by equation (3.1), we have

$$\frac{h(x;\theta_1,k_1)}{h(x;\theta_2,k_2)} = \left(\frac{\theta_1^{1/k_1}}{\theta_2^{1/k_2}}\right)\frac{\left[1-\bar{\theta}_2\bar{F}^{k_2}(x)\right]^{1/k_2+1}}{\left[1-\bar{\theta}_1\bar{F}^{k_1}(x)\right]^{1/k_1+1}}.$$

Thus, for any $k_1 > 0$ and $k_2 > 0$ we obtain

$$\frac{d}{dx}\left[\frac{h(x;\theta_1,k_1)}{h(x;\theta_2,k_2)}\right]$$
$$= \frac{h(x;\theta_1,k_1)}{h(x;\theta_2,k_2)}f(x)\left[\frac{(k_2+1)\bar{\theta}_2\bar{F}^{k_2-1}(x)}{1-\bar{\theta}_2\bar{F}^{k_2}(x)} - \frac{(k_1+1)\bar{\theta}_1\bar{F}^{k_1-1}(x)}{1-\bar{\theta}_1\bar{F}^{k_1}(x)}\right]$$

which is non-positive if $\{0 < \theta_1 \leqslant 1, \theta_2 \geqslant 1\}$.

For $k_1 = k_2 = k$, by equation (3.1), we have

$$\frac{h(x;\theta_1,k)}{h(x;\theta_2,k)} = \left(\frac{\theta_1}{\theta_2}\right)^{1/k}\left[\frac{1-\bar{\theta}_2\bar{F}^{k}(x)}{1-\bar{\theta}_1\bar{F}^{k}(x)}\right]^{1+1/k}.$$

Thus, for all $k > 0$ we obtain

$$\frac{d}{dx}\left[\frac{h(x;\theta_1,k)}{h(x;\theta_2,k)}\right] = C(x;k,\theta_1,\theta_2)\left[\frac{1-\bar{\theta}_2\bar{F}^{k}(x)}{1-\bar{\theta}_1\bar{F}^{k}(x)}\right]^{1/k}\frac{\bar{\theta}_2-\bar{\theta}_1}{\left(1-\bar{\theta}_1\bar{F}^{k}(x)\right)^2},$$

where $C(x;k,\theta_1,\theta_2) = (\theta_1/\theta_2)^{1/k}(1+k)f(x)\bar{F}^{k-1}(x) \geqslant 0$ is non-positive if $\theta_1 \leqslant \theta_2$. This completes the proof. ■

By Theorem 3.2 and Table 1, we immediately obtain

COROLLARY 3.1. *Let* Y_1 *and* Y_2 *be rv's corresponding to df's* $H(\cdot;\theta_1,k_1)$ *and* $H(\cdot;\theta_2,k_2)$, *respectively. If* $\{0<\theta_1\leqslant 1,\theta_2\geqslant 1\}$ *or* $\{0<\theta_1\leqslant\theta_2,k_1=k_2=k\}$, *then* $Y_1\leqslant_{hr}(\leqslant_{rh},\leqslant_{st},\leqslant_E)\,Y_2$.

REMARK 3.1. *It is worth mentioning that, in view of our Theorem* 3.2, *Theorem* 2.3 *of* [20] *concerning the Marshall–Olkin family is not valid unless* $\theta_1\geqslant\theta_2$ *is replaced by* $\theta_2\geqslant\theta_1$.

REMARK 3.2. *Our results in Theorem* 3.2 *can be viewed as extensions of those of Theorem* 3 *of* [13], *Theorem* 4 *of* [16] *and Proposition* 1 *of* [17], *where they consider the special case of* $k=1$, *i.e., the Marshall–Olkin family. Furthermore, our result in Corollary* 3.1 *for* $k=1$ *was proved by Benduch-Frąszczak* [13] *in Corollary* 2.

In the following theorem we study ageing intensity orders between rv's Y_1 and Y_2 corresponding to df's $H(\cdot;\theta_1,k)$ and $H(\cdot;\theta_2,k)$, respectively.

THEOREM 3.3. *Let* Y_1 *and* Y_2 *be rv's corresponding to Harris family df's* $H(\cdot;\theta_1,k)$ *and* $H(\cdot;\theta_2,k)$, *respectively. Then* $Y_1\leqslant_{AI}Y_2$ *provided that* $\theta_1>\theta_2$.

Proof. $Y_1\leqslant_{AI}Y_2$ if and only if, for all $x>0$, we have

$$\frac{1}{r_H(x;\theta_1,k)}\int_0^x r_H(u;\theta_1,k)du\leqslant\frac{1}{r_H(x;\theta_2,k)}\int_0^x r_H(u;\theta_2,k)du,\qquad k>0,$$

or, by equation (1.2),

$$\frac{1-\bar\theta_1\bar F^k(x)}{r_F(x)}\int_0^1\frac{r_F(u)}{1-\bar\theta_1\bar F^k(u)}du\leqslant\frac{1-\bar\theta_2\bar F^k(x)}{r_F(x)}\int_0^x\frac{r_F(u)}{1-\bar\theta_2\bar F^k(u)}du,\qquad k>0,$$

which is equivalent to

$$\int_0^x r_F(u)\left[\frac{1-\bar\theta_1\bar F^k(x)}{1-\bar\theta_1\bar F^k(u)}-\frac{1-\bar\theta_2\bar F^k(x)}{1-\bar\theta_2\bar F^k(u)}\right]du\geqslant 0,\qquad k>0.$$

But this is true if $\theta_1>\theta_2$ because

$$\frac{d}{d\theta}\left(\frac{1-\bar\theta\bar F^k(x)}{1-\bar\theta\bar F^k(u)}\right)=\frac{\bar F^k(x)-\bar F^k(u)}{\left(1-\bar\theta\bar F^k(u)\right)^2}\leqslant 0,$$

or if $\left(1-\bar\theta\bar F^k(x)\right)/\left(1-\bar\theta\bar F^k(u)\right)$ is decreasing in θ when $0<u<x$. Thus, we have the result. ∎

4. PRESERVATION OF STOCHASTIC ORDERS BY HARRIS FAMILY WITH THE SAME TILT PARAMETERS

Let X_1 and X_2 be two rv's with df's F_1 and F_2 and pdf's f_1 and f_2, respectively. Suppose that Y_1 and Y_2 are their corresponding Harris family rv's, i.e., the df's F_1 and F_2 with baseline, respectively. In this section, we shall study several stochastic order preservations of the baseline distribution by its corresponding Harris family.

Kirmani and Gupta [23] have shown that usual stochastic, hazard rate, convex transform, super-additive and star orders are preserved by transformation to proportional odds ratio (Marshall–Olkin) family. In what follows, their results are generalized to Harris family, i.e., for any $k > 0$ in equation (1.1). In fact, more generally, we have the following necessary and sufficient property.

THEOREM 4.1. *$X_1 \leqslant_{st} X_2$ if and only if $Y_1 \leqslant_{st} Y_2$.*

Proof. It is true by Theorem 3.1 of [1] when $\alpha = \beta$. ∎

Since the Harris family of distributions coincides with weighted distributions, with weight $\omega(x) = \theta^{1/k}/\big(1 - \bar{\theta}\bar{F}^k(x)\big)^{1/k+1}$, by Theorem 9(a) of [9] we conclude that the hazard rate order is preserved by transformation to the Harris family. The following theorem also provides a both-sided preservation for different types of hazard rate orders. That is, by comparing lifetimes of two given systems, we can detect which one is made of better quality components. But, in these cases, the range of the tilt parameter values plays a restrictive role.

THEOREM 4.2. (i) *Assume that $\theta \geqslant 1$. If $X_1 \leqslant_{phr\uparrow} (\leqslant_{phr}, \leqslant_{hr\uparrow}, \leqslant_{hr}) X_2$, then $Y_1 \leqslant_{phr\uparrow} (\leqslant_{phr}, \leqslant_{hr\uparrow}, \leqslant_{hr})\, Y_2$.*

(ii) *Assume that $0 < \theta \leqslant 1$. If $Y_1 \leqslant_{phr\uparrow} (\leqslant_{phr}, \leqslant_{hr\uparrow}, \leqslant_{hr})\, Y_2$, then $X_1 \leqslant_{phr\uparrow} (\leqslant_{phr}, \leqslant_{hr\uparrow}, \leqslant_{hr})\, X_2$.*

Proof. (i) It is true by Theorem 3.2(i) of [1] when $\alpha = \beta \geqslant 1$.

(ii) For the up proportional hazard rate order, let $Y_1 \leqslant_{phr\uparrow} Y_2$. So, for all x, $t \geqslant 0$ and $0 < \lambda \leqslant 1$ we have $r_{H_1}(x+t; \theta, k) \geqslant \lambda r_{H_2}(\lambda x; \theta, k)$. So, by equation (1.2), we have

$$\frac{r_{F_1}(x+t)}{\lambda r_{F_2}(\lambda x)} \geqslant \frac{1 - \bar{\theta}\bar{F}_1^k(x+t)}{1 - \bar{\theta}\bar{F}_2^k(\lambda x)}. \tag{4.1}$$

Since the hazard rate order is implied by the up proportional hazard rate order (Table 1) and the simple stochastic order is implied by the hazard rate order, for any x and all $k > 0$ we have $\bar{H}_1^k(x) \leqslant \bar{H}_2^k(x)$. Also, by Theorem 4.1, $\bar{F}_1^k(x) \leqslant \bar{F}_2^k(x)$. Further, the survival function is decreasing, so for all $0 < \lambda \leqslant 1$, $t \geqslant 0$, $k > 0$ and x, we get

$$\bar{F}_1^k(x+t) \leqslant \bar{F}_1^k(x) \leqslant \bar{F}_2^k(x) \leqslant \bar{F}_2^k(\lambda x).$$

Thus, when $0 < \theta < 1$, we have

$$-\bar{\theta}\bar{F}_1^k(x+t) \geqslant -\bar{\theta}\bar{F}_2^k(\lambda x) \Longrightarrow 1-\bar{\theta}\bar{F}_1^k(x+t) \geqslant 1-\bar{\theta}\bar{F}_2^k(\lambda x).$$

Consequently, the right-hand side of inequality (4.1) is greater than one, which implies $r_{F_1}(x+t) \geqslant \lambda r_{F_2}(\lambda x)$, i.e., $X_1 \leqslant_{phr\uparrow} X_2$, as required.

With proper choices of t or λ, i.e. $t = 0$ or $\lambda = 1$, or both, proofs for the other parts are immediate. ∎

By using the counterexample 3.2 of [1], the following counterexample shows that the up hazard rate order is not preserved by transformation to the Harris family, when $0 < \theta < 1$.

COUNTEREXAMPLE 4.1. Let X_1 and X_2 be two rv's having the Erlang distributions with survival functions $\bar{F}_1(x) = (1+2x)e^{-2x}$, $\bar{F}_2(x) = (x+1)e^{-x}$ and hazard rates $r_{F_1}(x) = 4x/(1+2x)$, $r_{F_2}(x) = x/(x+1)$, for $x > 0$, respectively. So, $X_1 \leqslant_{hr\uparrow} X_2$. However, Figure 1 shows that for some $0 < \theta < 1$, $t > 0$ and some $x > 0$, $r_{H_1}(x+t;\theta,k) \ngeq r_{H_2}(x;\theta,k)$ or, equivalently, $\bar{H}_2(x;\theta,k)/\bar{H}_1(x+t;\theta,k)$ is not increasing in x, i.e., the up hazard rate order is not preserved by transformation to the Harris family when $0 < \theta < 1$.

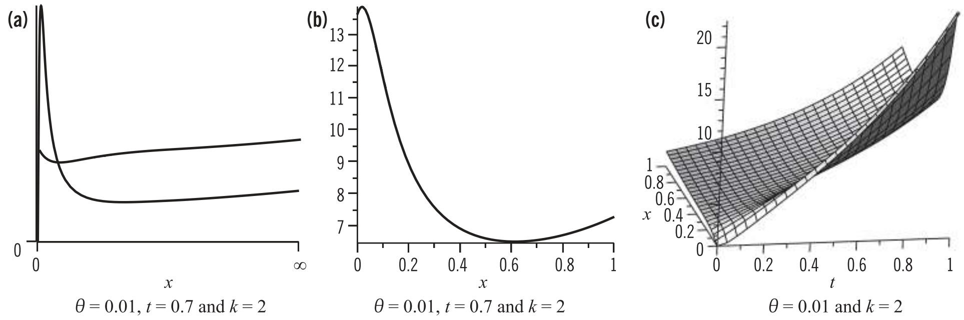

FIGURE 1. (a) showing that $r_{H_1}(x+t;\theta,k) \ngeq r_{H_2}(x;\theta,k)$, and (b) and (c) showing that $\bar{H}_2(x;\theta,k)/\bar{H}_1(x+t;\theta,k)$ is not increasing in x.

COROLLARY 4.1. *Let X_1 and X_2 be two rv's with mean residual life (mrl) functions m_1 and m_2 and Harris family rv's Y_1 and Y_2 having mrl functions m_1^* and m_2^*, respectively, such that $m_1(t)/m_2(t)$ increases in t. If $X_1 \leqslant_{mrl} X_2$, then $Y_1 \leqslant_{mrl} Y_2$ provided that $\theta \geqslant 1$. The orders are reversed if $m_1^*(t)/m_2^*(t)$ increases in t and $0 < \theta \leqslant 1$.*

P r o o f. By Theorem 2.A.2 of [31], the assertion holds because if $X_1 \leqslant_{mrl} X_2$ and $m_1(t)/m_2(t)$ increases in t, then $X_1 \leqslant_{hr} X_2$. Thus, by Theorem 4.2(i) we can conclude that $Y_1 \leqslant_{hr} Y_2$. But by the sufficiency of the hazard rate order for mrl order (Theorem 1.D.1 of [31]), this implies that $Y_1 \leqslant_{mrl} Y_2$. ∎

REMARK 4.1. *Note that for the special case when $k = 1$, the log-odds function of an rv X is equal to the log-odds function of the corresponding Harris family rv Y. Consequently, the log-odds ratio order is also preserved by transformation to the Marshall–Olkin family.*

For the ageing intensity order, we have the following

THEOREM 4.3. *Assume that X_1 and X_2 are non-negative rv's. For all $k > 0$, if $X_1 \leqslant_{AI} X_2$ and $X_1 \leqslant_{hr} X_2$, then $Y_1 \leqslant_{AI} Y_2$ provided that $\theta > 1$.*

Proof. Let $k > 0$ and $\theta > 1$. $Y_1 \leqslant_{AI} Y_2$ if and only if

$$\frac{1}{r_{H_1}(x;\theta,k)} \int_0^x r_{H_1}(u;\theta,k)du \leqslant \frac{1}{r_{H_2}(x;\theta,k)} \int_0^x r_{H_2}(u;\theta,k)du$$

or

$$\frac{1-\bar{\theta}\bar{F}_1^k(x)}{r_{F_1}(x)} \int_0^x r_{H_1}(u;\theta,k)du \leqslant \frac{1-\bar{\theta}\bar{F}_2^k(x)}{r_{F_2}(x)} \int_0^x r_{H_2}(u;\theta,k)du$$

But we have

$$\int_0^x r_H(u;\theta,k)du = -\ln \bar{H}(x;\theta,k) = -\ln \bar{F}(x) + \frac{1}{k}\ln\left(\frac{1-\bar{\theta}\bar{F}^k(x)}{\theta}\right).$$

So, we should show that

$$(4.2)\quad \left(1-\bar{\theta}\bar{F}_1^k(x)\right)\left[\frac{-\ln \bar{F}_1(x)}{r_{F_1}(x)} + \frac{1}{k}\frac{\ln\left(\left(1-\bar{\theta}\bar{F}_1^k(x)\right)/\theta\right)}{r_{F_1}(x)}\right] \leqslant \left(1-\bar{\theta}\bar{F}_2^k(x)\right)\left[\frac{-\ln \bar{F}_2(x)}{r_{F_2}(x)} + \frac{1}{k}\frac{\ln\left(\left(1-\bar{\theta}\bar{F}_2^k(x)\right)/\theta\right)}{r_{F_2}(x)}\right].$$

Since $X_1 \leqslant_{AI} X_2$, we also have

$$\frac{1}{r_{F_1}(x)} \int_0^x r_{F_1}(u)du \leqslant \frac{1}{r_{F_2}(x)} \int_0^x r_{F_2}(u)du$$

or

$$\frac{1}{r_{F_1}(x)} \int_0^x \frac{f_1(u)}{\bar{F}_1(u)}du \leqslant \frac{1}{r_{F_2}(x)} \int_0^x \frac{f_2(u)}{\bar{F}_2(u)}du.$$

Equivalently, we have

$$(4.3)\quad \frac{-\ln \bar{F}_1(x)}{r_{F_1}(x)} \leqslant \frac{-\ln \bar{F}_2(x)}{r_{F_2}(x)}.$$

On the other hand, if $X_1 \leqslant_{hr} X_2$, for all x we have $1/r_{F_1}(x) \leqslant 1/r_{F_2}(x)$ and also $X_1 \leqslant_{st} X_2$. Thus, $\bar{F}_1^k(x) \leqslant \bar{F}_2^k(x)$. So, since $\theta > 1$, we have

$$\frac{1-\bar{\theta}\bar{F}_1^k(x)}{\theta} \leqslant \frac{1-\bar{\theta}\bar{F}_2^k(x)}{\theta}.$$

Hence, we can conclude that

$$\text{(4.4)} \qquad \frac{\ln\big(\big(1-\bar{\theta}\bar{F}_1^k(x)\big)/\theta\big)}{r_{F_1}(x)} \leqslant \frac{\ln\big(\big(1-\bar{\theta}\bar{F}_2^k(x)\big)/\theta\big)}{r_{F_2}(x)}.$$

Now, adding up inequalities (4.3) and (4.4) and multiplying the left-hand side by $\big(1-\bar{\theta}\bar{F}_1^k(x)\big)$ and the right-hand side by $\big(1-\bar{\theta}\bar{F}_2^k(x)\big)$, we obtain inequality (4.2). This completes the proof. ■

In the next lemma we need inverses of the df and survival function of a Harris family distribution. It is easy to verify that equations (1.1) and (3.2) lead to

$$\text{(4.5)} \qquad \bar{H}^{-1}(p;\theta,k) = \bar{F}^{-1}\left(\frac{p^k}{\theta+\bar{\theta}p^k}\right)^{1/k}, \quad 0 < p < 1,$$

and

$$\text{(4.6)} \qquad H^{-1}(p;\theta,k) = F^{-1}\left(1-\left[\frac{(1-p)^k}{\theta+\bar{\theta}(1-p)^k}\right]^{1/k}\right), \quad 0 < p < 1.$$

Equation (4.6) was observed by Batsidis and Lemonte [11].

LEMMA 4.1. *If $H_1(x) \equiv H_1(x;\theta,k)$ and $H_2(x) \equiv H_2(x;\theta,k)$ are two Harris family df's with baseline df's F_1 and F_2, respectively, then $H_2^{-1}\big(H_1(x)\big) = F_2^{-1}\big(F_1(x)\big)$ for all x.*

P r o o f. This result can be obtained by using the assumed form of H_1 together with H_2^{-1}, which follows from equation (4.6). For any $k > 0$ and $\theta > 0$, we have

$$\text{(4.7)} \qquad H_2^{-1}\big(H_1(x)\big) = F_2^{-1}\left(1-\left[\frac{\big(1-H_1(x)\big)^k}{\theta+\bar{\theta}\big(1-H_1(x)\big)^k}\right]^{1/k}\right).$$

Thus, substituting $H_1(x)$ of equation (3.2) into (4.7), we obtain the lemma. ■

Without any restriction on the tilt parameter θ, we have

THEOREM 4.4. *The following orders are preserved by transformation from a baseline distribution to its corresponding Harris family and vice versa.*

(i) *convex transform order*,

(ii) *star order*,

(iii) *supper-additive order*,

(iv) *dispersive order.*

Proof. (i) $X_1 \leqslant_c X_2$ holds if $F_2^{-1}F_1(x)$ is convex in $x \in S_{X_1}$. Thus, by Lemma 4.1, $H_2^{-1}\big(H_1(x)\big)$ is also convex in $x \in S_{Y_1}$. So $Y_1 \leqslant_c Y_2$.

(ii) $X_1 \leqslant_* X_2$ holds if $F_2^{-1}F_1(x)/x$ increases in $x \geqslant 0$. Thus, by Lemma 4.1, $H_2^{-1}\big(H_1(x)\big)/x$ also increases in $x \geqslant 0$. So $Y_1 \leqslant_* Y_2$.

(iii) $X_1 \leqslant_{su} X_2$ if $F_2^{-1}F_1(t+u) \geqslant F_2^{-1}F_1(t) + F_2^{-1}F_1(u)$ for $t \geqslant 0$ and $u \geqslant 0$. Thus, by Lemma 4.1, $H_2^{-1}H_1(t+u) \geqslant H_2^{-1}H_1(t) + H_2^{-1}H_1(u)$ for $t \geqslant 0$ and $u \geqslant 0$. So $Y_1 \leqslant_{su} Y_2$.

(iv)$X_1 \leqslant_{disp} X_2$ if $F_2^{-1}F_1(x) - x$ increases in x. Thus, by Lemma 4.1, it follows that $H_2^{-1}\big(H_1(x)\big) - x$ also increases in x. So $Y_1 \leqslant_{disp} Y_2$.

Proofs of converse transformations are similar. ■

COROLLARY 4.2. *If $X_1 \leqslant_{Lorenz} X_2$, then $Y_1 \leqslant_{Lorenz} Y_2$ provided that the function $F_2^{-1}(x)/F_1^{-1}(x)$ is increasing for all $x > 0$.*

Proof. If $F_2^{-1}(x)/F_1^{-1}(x)$ is increasing for all $x > 0$, then, clearly, it follows that $F_2^{-1}F_1(x)/x$ is also increasing for all $x > 0$. Thus, $X_1 \leqslant_* X_2$ and the Lorenz order is implied by the star order (cf. [9], p. 90), i.e., $X_1 \leqslant_{Lorenz} X_2$. Since, by Theorem 4.4, the star order is preserved by transformation to the Harris family, we have $Y_1 \leqslant_* Y_2$, which yields the Lorenz order, as required. ■

REMARK 4.2. *The usual stochastic, hazard rate, convex transform and star orders are preserved by transformation to the frailty family (proportional hazard family,* cf. [26], p. 240) *and to Marshall–Olkin family* (cf. [23]). *By combining these facts with Remark* 1 *of* [11], *it follows that such orders are also preserved under transformation to the Harris family.*

5. AGEING PROPERTIES

In the investigations pertaining to ageing concepts, the problem is to examine how a component or system improves or deteriorates with age. In the reliability context, life distributions are classified into different classes based on the monotonic behavior of the failure rate and mean residual life functions. The works [2], [5] and [18] proceed in this direction. Batsidis and Lemonte [11] showed that IFR and DFR properties are preserved by transformation to the Harris family. In what follows, we shall show that the ageing characteristics, i.e., IFRA, DFRA, NBU and NWU, are also preserved by transformation to the Harris family. First, we need to recall some results.

PROPOSITION 5.1 ([26], p. 182). *The following two statements are equivalent:*

(i) *X has IFRA (DFRA),*

(ii) *$X \leqslant_* (\geqslant_*)\ X_1$, where X_1 has an exponential distribution.*

PROPOSITION 5.2 ([26], p. 182). *The following two statements are equivalent*:
(i) X *is NBU* (NWU),
(ii) $X \leqslant_{su} (\geqslant_{su})\, X_1$, *where* X_1 *has an exponential distribution.*

In the following corollary we shall investigate preservation of the IFRA, DFRA, NBU and NWU characteristics by transformation to the Harris family.

COROLLARY 5.1. *Let* $\theta > 1$ $(0 < \theta < 1)$.
(i) *The IFRA* $(DFRA)$ *characteristic is preserved by transformation to the Harris family.*
(ii) *The NBU* (NWU) *characteristic is preserved by transformation to the Harris family.*

P r o o f. (i) Assume that an rv X has the IFRA (DFRA) property and that X_1 is an rv with survival function $\bar{F}_1(x) = e^{-x}$ for $x \geqslant 0$. We transform $\bar{F}_1(\cdot)$ to the Harris family as follows:

$$\bar{H}_1(x; \theta, k) = \frac{\theta^{1/k} e^{-x}}{(1 - \bar{\theta} e^{-kx})^{1/k}}, \quad x \geqslant 0.$$

Let Y and Y_1 be the corresponding Harris family rv's with survival functions $\bar{H}(\cdot; \theta, k)$ and $\bar{H}_1(\cdot; \theta, k)$, respectively. By Proposition 5.1, we get $X \leqslant_* (\geqslant_*)\, X_1$. But, by Theorem 4.4(ii), the star order is preserved by transformation to the Harris family, so we have $Y \leqslant_* (\geqslant_*)\, Y_1$. Thus, by [10], for $\theta > 1$ $(0 < \theta < 1)$, Y_1 has the IFR (DFR) property. Moreover, the IFR (DFR) property implies the IFRA (DFRA) property (cf. [26], p. 181). Thus, Y_1 has IFRA (DFRA), and so, by Proposition 5.1, $Y_1 \leqslant_* (\geqslant_*)\, X_1$. From the transitivity property of partial order, we obtain $Y \leqslant_* (\geqslant_*)\, X_1$. Thus, by Proposition 5.1, Y has the IFRA (DFRA) property.

(ii) Let an rv X with survival function $\bar{F}(\cdot)$ be NBU (NWU) and X_1 be an rv with survival function $\bar{F}_1(x) = e^{-x}$ for $x \geqslant 0$. We transform $\bar{F}_1(\cdot)$ to the Harris family as follows:

$$\bar{H}_1(x; \theta, k) = \frac{\theta^{1/k} e^{-x}}{(1 - \bar{\theta} e^{-kx})^{1/k}}, \quad x \geqslant 0.$$

Let Y and Y_1 be rv's with survival functions defined in (1.1) and $\bar{H}_1(\cdot; \theta, k)$, respectively. By Proposition 5.2, $X \leqslant_{su} (\geqslant_{su})\, X_1$, but by Theorem 4.4(iii), the super-additive order is preserved by transformation to the Harris family. Thus, we have $Y \leqslant_{su} (\geqslant_{su})\, Y_1$. For $\theta > 1$ $(0 < \theta < 1)$, it can be easily shown that Y_1 is NBU (NWU). Then, by Proposition 5.2, $Y_1 \leqslant_{su} (\geqslant_{su}) X_1$. Due to the transitivity property of partial order, this implies that $Y \leqslant_{su} (\geqslant_{su})\, X_1$. Thus, by Proposition 2, Y is NBU (NWU). ∎

REMARK 5.1. *Since the Harris family of distributions coincides with weighted distributions with weight*

$$\omega(x) = \frac{\theta^{1/k}}{\left(1 - \bar{\theta}\bar{F}^k(x)\right)^{1/k+1}},$$

the above corollary is a consequence of Theorem 3 *of* [10] *and Theorem* 3 *of* [14]. *Note that, by Theorem* 3 *of* [10], *for the IFRA and NBU characteristics we should let* $\omega(x)\bar{F}(x)$ *be increasing in* x, *but in our Corollary* 5.1 *we have a larger class of* θ *values with no restriction on* k *and* x.

6. DISCUSSION AND CONCLUSION

The hazard and lifetime in a series system with variable number of components, model (1.1), are functions of a tilt parameter. So, a proper choice of the range of values of this parameter plays an important role in optimization of the systems lifetime. In Section 3, we indicated how a lower risk (hazard rate order), longer lifetime (usual stochastic order), higher likelihood ratio (likelihood ratio order), etc. can be achieved by a system comparing to its components by a proper choice of the tilt parameter values. In Section 3, we also discussed how one can distinct the optimum case of two systems using their tilt parameters. Section 4 determined when a stochastic order between components is preserved by their corresponding systems and, more interestingly, *vice versa* for the cases in which components are not observable. Finally, in Section 5, we revealed when the ageing properties IFRA, DFRA, NBU and NWU of components are transferred to their corresponding systems.

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