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Press–Schechter, thermodynamics and gravitational clustering

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ABSTRACT

For the special case of non-linear gravitational clustering from an initially Poissonian distribution, the counts-in-cells distribution fuction is obtained from the excursion set, Press–Schechter multiplicity function. This Poisson Press–Schechter distribution function has the same form as the gravitational quasi-equilibrium counts-in-cells distribution function, predicted by the Saslaw–Hamilton thermodynamic model of non-linear gravitational clustering, which fits the observed galaxy distribution well.

By changing an *ad hoc* guess in the Saslaw–Hamilton thermodynamic model, the negative binomial distribution (which also fits relevant observations well) is derived from the thermodynamic approach. At present, analytic simplicity is the primary reason for preferring one guess over another, so the thermodynamic approach does not, at present, yield a unique prediction for the gravitational counts-in-cells distribution function. Two ways to constrain the parameter space of possible guesses are described; one of these suggests that the negative binomial is not a physically reasonable model.

One possible relation between the original Saslaw–Hamilton thermodynamic model and the Poisson Press–Schechter approach is obtained. A system of non-interacting, virialized clusters having a range of masses, the distribution of masses being given by the Poisson Press–Schechter multiplicity function, is shown to be consistent with the original Saslaw–Hamilton thermodynamic model. For this model to work, the virialized clusters must be in thermal equilibrium with each other, so that all clusters have the same temperature, independent of their masses. This last requirement and the idealization that the clusters do not interact gravitationally with each other are in contradiction with observations.

Key words: galaxies: clusters: general – galaxies: evolution – galaxies: formation – cosmology: theory – dark matter.

1 INTRODUCTION

The gravitational quasi-equilibrium distribution function (hereafter GQED),

$$f(N,\bar{n}V)=\frac{\bar{N}(1-b)}{N!}\,[\bar{N}(1-b)+Nb]^{N-1}\times\exp[-\bar{N}(1-b)-Nb],\qquad(1)$$

where $f(N,\bar{n}V)$ denotes the probability that a randomly placed cell of size V has exactly N particles, $\bar{n}$ is the average number density of particles (so that $\bar{N}=\bar{n}V$ is the average number of particles in a cell), and b is a free parameter with $0\le b\le 1$, was first proposed by Saslaw & Hamilton (1984). The GQED derives from their specific thermodynamic model of non-linear gravitational clustering. Although in their model b was supposed to be independent of scale, most comparisons of this functional form with counts-in-cells analyses of galaxy catalogues and N-body simulations show good agreement provided b is allowed to depend (quite strongly) on scale. When treated purely as a fitting function in this way, this GQED provides a good fit to counts-in-cells distribution functions calculated for a number of two- and three-dimensional galaxy catalogues (e.g. Saslaw & Crane 1991; Lahav & Saslaw 1992; Sheth, Mo & Saslaw 1994), and to simulations of gravitational clustering from Poisson initial conditions (e.g. Itoh, Inagaki & Saslaw 1993, and references therein). Despite the (many) objections that may be raised about the applicability of thermodynamic fluctuation theory when describing non-linear gravitational clustering, the reasonably, and perhaps unexpectedly, good agreement between the predicted GQED functional form and the counts-in-cells distribution functions measured in relevant N-body simulations has served to justify a continued interest in the thermodynamic quasi-equilibrium approach.

This paper undermines this 'the proof of the pudding is in the eating' (Sahni & Coles 1994) justification of the thermodynamic GQED model in two ways. Section 2 shows that the GQED is not the only counts-in-cells distribution function that can be derived from the quasi-equilibrium approach. Indeed, to specify their quasi-equilibrium model, and thus to specify the thermodynamics and the resulting GQED counts-in-cells distribution function, Saslaw & Hamilton (1984) made a guess regarding the functional form of the ratio of the gravitational energy of correlations, W, to the kinetic energy, K, associated with peculiar motions. This is equivalent to the guess (e.g. Davis & Peebles 1977; Hamilton 1988) that must be made to close and solve the BBGKY hierarchy. To date, there is little physical motivation to justify their guess (Sheth 1994), so it remains an *ad hoc* assumption in their model. Since, in their model, much of the gravitational physics is embodied in the functional form assumed for $-W/2K$, this paper explores the consequences of changing their guess. As a specific example, it derives the negative binomial distribution from this thermodynamic approach. Since the negative binomial also provides a good fit to N-body simulations and to observed galaxy catalogues (e.g. Gaztañaga 1992), it would appear that it has as much physical, gravothermal motivation as does the GQED.

Section 3 uses the results of Epstein (1983), who calculated the excursion-set mass spectrum (cf. Bond et al. 1991) of a Poisson distribution, to show that the GQED can be derived from a Press & Schechter (1974) type description of non-linear gravitational clustering. The Press–Schechter approach gives the probability that a cluster has a given number of particles: the cluster multiplicity function. If these Press–Schechter clusters are assumed to be idealized, point-sized clusters, and the clusters are assumed to have a Poisson spatial distribution, then the counts-in-cells distribution function of this set of Poisson-distributed Press–Schechter clumps is exactly the same as the GQED. In so far as the Press–Schechter approach is independent of an underlying thermodynamic theory, this derivation of the GQED is effected without any explicit consideration of thermodynamics. It also shows why the GQED functional form provides a good fit to counts-in-cells distribution functions measured in simulations of gravitational clustering from Poissonian initial conditions, but is not so accurate when the initial conditions are significantly different from Poissonian (e.g. Suto, Itoh & Inagaki 1990).

Therefore, until the thermodynamic approach is able to provide some physical motivation for the correct, gravothermodynamic choice of $-W/2K$, so that its functional form is no longer *ad hoc*, or until some physical relation between thermodynamics and the Press–Schechter approach has been derived, this paper serves to undermine the thermodynamic approach. Section 4 summarizes the results and then derives and discusses a relation between the thermodynamic and Press–Schechter approaches. The thermodynamic model of non-interacting, virialized clumps that results differs in an important way from the usual Press–Schechter models of virialized clumps. It assumes that all clusters have the same temperature, independent of clump mass. This idealization, and the assumption that different clumps do not interact gravitationally, are in conflict with observations.

2 THERMODYNAMICS AND GRAVITY

2.1 The GQED model

Standard statistical thermodynamic results (e.g. Hill 1956) show that, for a system of identical particles interacting via a pairwise attractive Coulomb potential, the internal energy, U, and the pressure, P, are given by

$$U=\frac{3\bar{N}T}{2}(1-2b), \qquad \text{and} \qquad P=\frac{\bar{N}T}{V}(1-b), \tag{2}$$

where T is the temperature of the system, $\bar{N}=\bar{n}V$ is the average number of particles in a member of the grand canonical ensemble of systems, each of size V, and $b=-W/2K$ is a measure of the ratio of the interaction energy and the kinetic energy due to peculiar velocities (note that $K=3NT/2$). Without additional physical information, the functional form of b is undetermined, except by a dimensional argument (Landau & Lifshitz 1959, p. 93) which shows that, for the inverse-square law force, b must be some function of the combination $(\bar{n}T^{-3})$. Otherwise, the choice for the exact functional form for $b(\bar{n}T^{-3})$ is not constrained. Note that the thermodynamic description of the system is completely specified only after the choice for $b(\bar{n}T^{-3})$ has been made. Also note that equation (2) for U and P is exact, and so specifying the functional form for $b(\bar{n}T^{-3})$ corresponds to specifying what are known as the virial coefficients. [The virial expansion for an imperfect gas is defined by writing the pressure as a power series in the density, i.e. $P/T=\Sigma_j B_j \bar{n}^j$, with $B_1=1$, or, alternatively, $P/T=\bar{n}-\Sigma_j(j/j+1)\beta_j\bar{n}^{j+1}$, with $j\geq 1$ for both expansions. The B_j coefficients are known as virial coefficients, and the β_j values are called irreducible cluster integrals (cf. Hill 1956). As their name suggests, the β_j values are easily related to volume integrals of the j-point correlation functions.]

When $b=0$ the system described above is an ideal gas, and the system approaches the virial limit as $b\to 1$. Therefore Saslaw & Hamilton (1984) chose

$$b=\frac{b_0\bar{n}T^{-3}}{1+b_0\bar{n}T^{-3}}, \tag{3}$$

with b_0 independent of $\bar{n}$ and of T, since this is the simplest analytical form that has the appropriate ideal gas and virialized limits, and since it is the first Padé approximation to a more general function of $(\bar{n}T^{-3})$. With this choice for b the virial coefficients are $B_j=(-b_0T^{-3})^{j-1}$, and the irreducible cluster integrals are $\beta_j=-(j+1)(-b_0T^{-3})^j/j$. Also, the Euler relation, $TS=U+PV-\bar{N}\mu$, with equation (2) and the standard thermodynamic relation between the entropy, S, and P and U, and with equation (3) for b, requires that $b_0T^{-3/2}e^{\mu/T}=be^{-b}$.

The counts-in-cells distribution, $f(N,\bar{n}V)$, is obtained from the generating function of its cumulants using thermodynamic fluctuation theory:

$$K(t)\equiv\ln g(e^t)\equiv\ln\left(\sum_{N=0}^{\infty}f(N,\bar{n}V)\,e^{tN}\right)$$
$$=\sum_{k=1}^{\infty}\frac{t^k}{k!}\left[\frac{\partial^k\ln\Xi}{\partial(\mu/T)^k}\right]_{T,V}, \tag{4}$$

where $\ln \Xi \equiv PV/T = \bar{N}(1-b)$ (e.g. Green & Callen 1951; Callen 1985). Using Lagrange's theorem on the inversion of series (Whittaker & Watson 1927), with the relation implied by equation (3) for $b(\bar{n}T^{-3})$, i.e. $b_0 T^{-3/2} e^{\mu/T} = b e^{-b}$, and $\bar{N}(1-b) = T^3 V b / b_0$, we find

$$K(t) = \frac{T^3 V}{b_0} \sum_{k=1}^{\infty} \frac{t^k}{k!} \left[\frac{\partial^k b}{\partial(\mu/T)^k} \right]_{T,V}$$

$$= \frac{T^3 V}{b_0} \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{\partial^k}{\partial(\mu/T)^k} \sum_{N=1}^{\infty} \frac{N^{N-1}}{N!} \left(\frac{b_0 e^{\mu/T}}{T^{3/2}} \right)^N$$

$$= \frac{\bar{N}(1-b)}{b} \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{N=1}^{\infty} \frac{N^{N-1} N^k}{N!} \left(\frac{b_0 e^{\mu/T}}{T^{3/2}} \right)^N$$

$$= \bar{N}(1-b) \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{N=1}^{\infty} \frac{N^k (Nb)^{N-1} e^{-Nb}}{N!}$$

$$= \bar{N}(1-b) \sum_{N=1}^{\infty} \frac{(Nb)^{N-1} e^{-Nb}}{N!} (e^{tN} - 1). \tag{5}$$

Equations (4) and (5) show that equation (3) for $b(\bar{n}T^{-3})$ implies that the generating function for the counts-in-cells distribution is

$$g(s) = \exp\left\{ -\bar{N}(1-b) \left[1 - \sum_{N=1}^{\infty} \frac{(Nb)^{N-1} e^{-Nb}}{N!} s^N \right] \right\}. \tag{6}$$

Now, the generating function of a Poisson distribution with density λ is $e^{-\lambda(1-s)}$. Equation (6) is similar to the Poisson generating function, except that s is replaced by the generating function of

$$\eta(N, b) = \frac{(Nb)^{N-1} e^{-Nb}}{N!}, \tag{7}$$

the Borel distribution (e.g. Moran 1984). That is, equation (6) has the compound Poisson form, so that it can be understood as describing a distribution of randomly distributed point-shaped clusters (i.e. idealized clusters having no spatial extent), where the probability, $\eta(N,b)$, that a randomly placed cluster has exactly N particles is given by the Borel distribution of equation (7).

Equations (4) and (6) show that $f(N, \bar{n}V)$ is obtained by computing the Nth derivative of $g(s)$ with respect to s, and evaluating it at $s=0$. This shows that, with equation (3) for $b(\bar{n}T^{-3})$, the counts-in-cells distribution function is given by equation (1). This distribution function is derived using a completely different method from, but has the same functional form as, the $f(N, \bar{n}V)$ distribution function derived by Saslaw & Hamilton (1984). Also, equation (6) for the generating function has the same form as, but was obtained in a different way from, that obtained by Hamilton, Saslaw & Thuan (1985) and Saslaw (1989).

2.2 The negative binomial

As emphasized above, for particles interacting with an inverse-square law force, most of the physics in the thermodynamic equilibrium state (if it exists) is described by the functional form assumed for b. Since assuming some functional form for b is equivalent to making some specific assumptions that, essentially, enable one to close the BBGKY hierarchy, then, while the physical reason for these assumptions remains unclear, this Saslaw–Hamilton solution of the BBGKY hierarchy remains incompletely understood. In particular, so long as the primary motivating factor in choosing equation (3) for $b(\bar{n}T^{-3})$ is analytic simplicity, one's understanding of the gravitational physics remains incomplete.

The following example illustrates this problem. Recall that $\ln \Xi = \bar{P}V/T = \bar{N}(1-b)$, whatever the functional form of b. If, instead of equation (3), we assume that

$$b = \sum_{N=1}^{\infty} \left[\frac{q_0^N e^{N\mu/T}}{N(N+1)} \right], \tag{8}$$

with q_0 being some function of T but not of $\bar{n}$, then repeating the analysis above, and writing $q_0 e^{\mu/T} = Q$ and $\alpha = Q/(1-Q)$, we show that the first few cumulants of this distribution [obtained by taking derivatives of $\ln \Xi = \bar{N}(1-b)$ with respect to μ/T, while holding T and V constant] are

$$\kappa_1 = \bar{N},$$

$$\kappa_2 = \bar{N}(1+\alpha),$$

$$\kappa_3 = \bar{N}(1+3\alpha+2\alpha^2),$$

$$\kappa_4 = \bar{N}(1+7\alpha+12\alpha^2+6\alpha^3), \tag{9}$$

and that the generating function of the counts-in-cells distribution function is

$$g(s) = \exp\left\{ \frac{-\bar{N}(1-Q)\ln(1-Q)}{Q} \left[\frac{\ln(1-Qs)}{\ln(1-Q)} - 1 \right] \right\}$$

$$= \left(\frac{1-Qs}{1-Q} \right)^{-\bar{N}(1-Q)/Q}. \tag{10}$$

Equation (10) is the generating function of a negative binomial distribution (e.g. Moran 1984). This shows that, if b is given by equation (8), rather than by equation (3), then the counts-in-cells distribution function that is 'predicted' by gravitational thermodynamics is a negative binomial.

It is possible to rearrange equation (8) to obtain b as a function of $\bar{n}T^{-3}$. The Euler relation shows that $q_0 = b_0 e / T^{3/2}$, so that $Q = b_0 \bar{n} T^{-3} / (1 + b_0 \bar{n} T^{-3})$, and

$$b = 1 + \left[\frac{(1-Q)\ln(1-Q)}{Q} \right] = 1 - \frac{\ln(1 + b_0 \bar{n} T^{-3})}{b_0 \bar{n} T^{-3}}. \tag{11}$$

Notice that, when $b_0 \bar{n} T^{-3}$ is large, equation (11) shows that $b \to 1$, and, when $b_0 \bar{n} T^{-3} \ll 1$, a Taylor series expansion shows that $b \to 0$. Although equation (11) is not as simple as the Saslaw–Hamilton guess (equation 3), it has the required physical limits. In this sense, the negative binomial distribution has as much physical motivation as does the GQED.

On the other hand, note that $\bar{N}^N \bar{\xi}_N$ is the Nth factorial cumulant of the distribution function, and that scale-invariant distributions are described by $\bar{\xi}_N = S_N \bar{\xi}_2^{N-1}$, with all S_N values independent of scale (e.g. Fry 1985). Equation (10), with the definition of the factorial cumulants, shows that the factorial cumulant generating function for a negative binomial distribution is

$$\ln g(1+t) \equiv \sum_{k=1}^{\infty} \frac{t^k}{k!}\mu_{[k]} = \frac{-\bar{N}(1-Q)}{Q}\ln\left(1-\frac{Qt}{1-Q}\right)$$

$$= \sum_{k=1}^{\infty} \frac{t^k}{k!}\bar{N}(k-1)!\left(\frac{Q}{1-Q}\right)^{k-1}, \qquad (12)$$

where $\mu_{[k]}$ denotes the kth factorial cumulant. Since

$$S_N \equiv \frac{\bar{\xi}_N}{\bar{\xi}_2^{N-1}} = \frac{\mu_{[N]}}{\bar{N}^N}\left(\frac{\bar{N}^2}{\mu_{[2]}}\right)^{N-1} = (N-1)! \qquad (13)$$

is independent of the cell size V, this shows that the negative binomial is a member of the class of scale-invariant distributions.

So, if b is given by equation (11), rather than by equation (3), then the quasi-equilibrium thermodynamic description implies that scale invariance is a characteristic signature of gravitational clustering. If b is given, instead, by equation (3), a similar analysis of the factorial cumulants shows that this is not the case (Saslaw & Sheth 1993). Therefore, while there remains no physical justification for preferring equation (3) to equation (11) (or vice versa) when calculating the correct description of gravitational thermodynamics, the thermodynamic model can be used to argue that scale invariance is a characteristic signature of gravitational clustering, and that the converse is also true!

Although the GQED and the negative binomial counts-in-cells distributions look rather similar for a range of parameter choices, so that both provide adequate descriptions of the observations presently available (e.g. Sheth, Mo & Saslaw 1994), the thermodynamics described by the two guesses (equations 3 and 11, respectively), for the functional form of b, may be quite different. Fig. 1 shows $b(b_0\bar{n}T^{-3})$ for the GQED (solid line) and for the negative binomial (dashed line). The curves (described by equations 3 and 11) are relatively similar. However, using equation (2) for P, and noting that b is a function of $(b_0\bar{n}T^{-3})$, it can be seen that it is equally permissible to write b as a function of the combination (b_0PT^{-4}). Then, equation (2), with equations (3) and (11), shows that $b=y$ and $b=1-[y/(e^y-1)]$, where $y=b_0PT^{-4}$, for the GQED (dot-dashed line) and the negative binomial (dotted line) models, respectively. These two curves are quite different; whereas both curves show $b=0$ when $b_0PT^{-4}=0$, they differ markedly as $b\to 1$. Indeed, whereas b_0PT^{-4} satisfies the requirement that $0\le b_0PT^{-4}\le 1$ for $0\le b\le 1$ in the GQED model, it varies over a larger range ($b_0PT^{-4}\to\infty$ as $b\to 1$) in the negative binomial model. To date, there is no physical theory for preferring one description of 'gravitational thermodynamics' to the other, since there is no theory for the appropriate relation between P and T when, for example, $b\to 1$. For instance, when $b\to 1$ in the GQED model, then $P\propto T^4$. It is not clear why gravitational thermodynamics should require this relation, which looks like the relation between the pressure and the temperature for the blackbody relation (Stefan's Law), to hold in virial equilibrium.

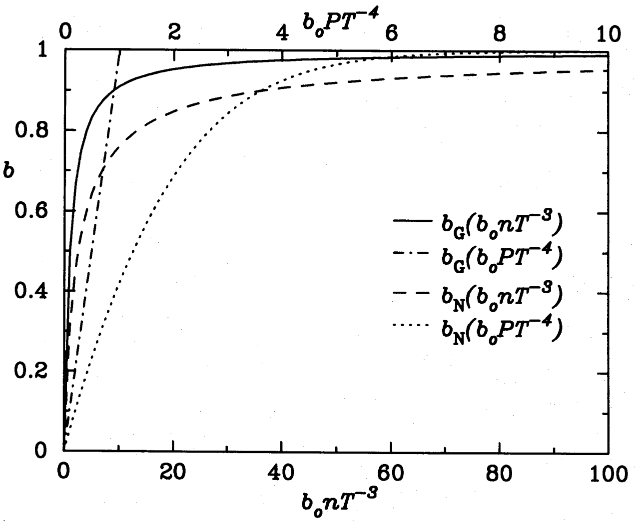

Figure 1. b as a function of $b_0\bar{n}T^{-3}$ and of b_0PT^{-4} for the GQED (solid and dot-dashed curves) and the negative binomial (dashed and dotted curves) models.

While there is, to date, no physical theory that enables us to treat either of these guesses as the correct gravothermodynamic functional form for $b(\bar{n}T^{-3})$, or, equivalently, for $b(PT^{-4})$, there may be other ways to constrain the parameter space of possible guesses. For example, it may be reasonable to require that $b(\bar{n}T^{-3})$ should be a single-valued, monotonic function of nT^{-3}, and that $b(PT^{-4})$ should also be a single-valued, monotonic function of PT^{-4}. Fig. 1 shows that both the GQED and the negative binomial models satisfy this requirement. Additionally, the cosmic energy equation

$$\frac{d}{dt}(K+W)+\frac{\dot{R}}{R}(2K+W)=0 \qquad (14)$$

(Irvine 1961; Layzer 1963), where R is the scale factor of the Universe and W and K denote the potential energy due to the gravitationally induced correlations and the kinetic energy of peculiar motions, respectively, strongly suggests that the Universe expands adiabatically: that is, $dU+PdV=0$, with U and P given by equation (2), and with $b=-W/2K$. This is most easily solved by writing U as a function of T and P (which is possible because b is a function of PT^{-4}), and by noting that $dV/V=3dR/R$. It may be physically reasonable to require $b=-W/2K$ to increase monotonically as the Universe expands; this would provide further constraints on the allowed functional forms for $b(\bar{n}T^{-3})$ [and, equivalently, for $b(PT^{-4})$].

To illustrate this, Fig. 2 shows $d\ln R/db$, calculated subject to the cosmic energy constraint that $dU+PdV=0$, as a function of b, for the GQED model [dashed line; calculated using the GQED result in Saslaw 1992, $d\ln R/db=(1+6b)/8b(1-b)$], and for the negative binomial model [solid line; $d\ln R/db=(1-b)^{-1}-\{(1-2b)(1-b)^{-2}/8[1-(1-b)e^y]\}$, with $y=b_0PT^{-4}$ as before]. The requirement that b increase monotonically as the Universe expands means that $d\ln R/db$ must always be positive. Fig. 2 shows that, whereas the negative binomial model violates this requirement, the GQED does not. If this requirement that b increase monotonically is, indeed, physically reasonable, then Fig. 2 suggests that, whereas the GQED remains a physically viable model, the negative binomial is not.

Thus this subsection serves to argue that it is important to constrain the parameter space of possible guesses for the

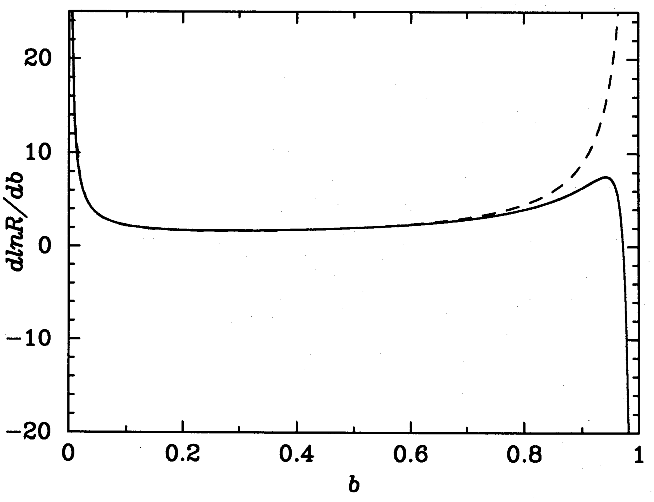

Figure 2. d ln $R/\mathrm{d}b$, calculated subject to the cosmic energy constraint that $\mathrm{d}U+P\,\mathrm{d}V=0$, as a function of b for the GQED (dashed line) and for the negative binomial (solid line) models. The requirement that b increase monotonically as the Universe expands means that d ln $R/\mathrm{d}b$ must always be positive.

functional form of $b(\bar{n}T^{-3})$ [and, equivalently, of $b(PT^{-4})$], and provides two examples of constraints that may be physically reasonable. It also shows that, to date, this parameter space of guesses is not adequately constrained, so that the thermodynamic model developed by Saslaw and collaborators does not make a unique prediction as to the correct gravothermal counts-in-cells distribution function.

There is one final remark concerning the applicability of this thermodynamic approach. It has long been known that for a gravitating system there is no true equilibrium state (e.g. Padmanabhan 1990, and references therein). To apply thermodynamics to a gravitating system, therefore, it is necessary to modify the theory slightly so that certain regions of phase space are rendered inaccessible and, in effect, this is what the assumed *ad hoc* functional forms for b (equations 3 and 11) accomplish. Since the observations are as well described by the negative binomial distribution function as they are by the GQED, it appears that the determination of the functional form for b that is required by gravity remains an open question. This section provides two 'guesses' for this functional form: equations (3) and (11); it is interesting that both forms are in good agreement with observations of galaxy positions and with simulations of gravitational clustering.

3 PRESS–SCHECHTER AND THE POISSON DISTRIBUTION

The previous section shows that, to date, the thermodynamic approach does not provide a unique description of gravitational clustering: the functional form of b, which specifies the thermodynamics, is determined *ad hoc*. Nevertheless, it is interesting that the quasi-equilibrium approach is able to derive a functional form, the GQED of equation (1), that fits the observations and relevant N-body simulations very well. This section provides an alternative derivation of the GQED functional form that makes no explicit consideration of thermodynamics.

Epstein (1983) shows that the probability that a given particle of a Poisson distribution is at the centre of a cluster of size k, at an overdensity threshold Δ, is

$$F(k,\Delta)=\frac{\Delta}{(1+\Delta)}\left(\frac{k}{1+\Delta}\right)^{k-1}\frac{\mathrm{e}^{-k/(1+\Delta)}}{(k+1)!}. \qquad (15)$$

This expression is the discrete analogue of the multiplicity function (including the *ad hoc* factor of 2) obtained by Press & Schechter (1974) for non-linear clustering from an initially Gaussian field. This multiplicity function has an interesting relation to the GQED distribution.

A number of analytic properties of the GQED distribution function (equation 1) are derived in Hamilton, Saslaw & Thuan (1985), Saslaw (1989), and Sheth (1994). Of particular importance in the present work is the cluster decomposition of the GQED. When b is independent of scale, the results of the previous section show that this GQED can be understood as describing a distribution of randomly distributed point-shaped clusters (i.e. idealized clusters having no spatial extent; the extension to cases where the Poisson-distributed custers have some non-trivial shape and density profile is developed in Sheth & Saslaw 1994). For the GQED, the probability, $\eta(N,b)$, that a randomly placed cluster has exactly N particles is given by the Borel distribution of equation (7).

Since the first moment of the Borel distribution is $1/(1-b)$ (e.g. Moran 1984), this Borel decomposition of the GQED shows that, when the number density of particles is $\bar{n}$, $\bar{n}(1-b)\,\eta(N,b)$, with $\eta(N,b)$ given by equation (7), gives an expression for the number density of (Poisson-distributed) clusters of size N. Multiplication of $\bar{n}(1-b)\,\eta(N,b)$ by N, the number of particles in a cluster of size N, and division by the number density of particles, $\bar{n}$, gives the probability, for a GQED, that a particle is a member of an N-sized cluster. Setting

$$b=1/(1+\Delta), \qquad (16)$$

it can be seen that the Borel cluster decomposition of the GQED is essentially the same as the cluster multiplicity function provided by equation (15). Since in the Borel decomposition of the GQED the clusters have a Poisson spatial distribution, if the Press–Schechter clumps described by equation (15) are assumed to also have a Poisson spatial distribution then it is clear that this result is, in effect, a derivation of the GQED functional form that is effected without any explicit consideration of thermodynamic fluctuation theory.

In so far as the Press–Schechter theory describes non-linear gravitational clustering in a density field that is initially Gaussian (e.g. Lacey & Cole 1994), the GQED functional form should describe clustering in a discrete distribution that is initially Poissonian. Thus this derivation of the Borel distribution (i.e. essentially equation 15) shows why the GQED functional form fits the N-body simulations of gravitational clustering from an initially Poissonian distribution so well. It also shows that the GQED functional form may be expected to describe gravitational clustering from Poissonian initial conditions regardless of whether or not the thermodynamic theory developed by Saslaw & Hamilton (1984) applies.

This result also clarifies some results regarding N-body simulations of gravitational clustering from initial conditions that are significantly different from Poissonian. Whereas Saslaw (1985) and Suto, Itoh & Inagaki (1990) argue that the better agreement between the GQED functional form and counts-in-cells distribution functions for simulations of clustering from Poissonian initial conditions, as compared to non-Poissonian initial conditions, is due to 'thermodynamically unrelaxed structure' in the non-Poissonian simulations, this section shows explicitly that the lack of agreement is almost certainly due to the intimate connection between the GQED functional form and the Poisson distribution.

4 DISCUSSION AND AN EXTENSION

By changing an *ad hoc* guess in the Saslaw–Hamilton thermodynamic model, the negative binomial distribution (which also fits relevant observations well) was derived from the thermodynamic approach (Section 2). There is, at present, no reason (other than analytic simplicity) to prefer one guess over another and so the thermodynamic approach does not, at present, yield a unique prediction for the gravitational counts-in-cells distribution function. Two ways to constrain the parameter space of possible guesses were described. One of these, namely the requirement that the ratio of potential correlation energy to (twice the) kinetic energy of peculiar motions increase monotonically as the Universe expands adiabatically, suggests that the negative binomial is not a physically reasonable model. On the other hand, the original Saslaw–Hamilton GQED model remains physically viable.

However, the GQED functional form can be derived from a Press–Schechter-type approach to describing non-linear gravitational clustering from an initially Poissonian distribution (Section 3). The demonstrated lack of a unique prediction, and the alternative, Press–Schechter, derivation (in Section 3) of the GQED functional form, coupled with the absence, at present, of a relation between the thermodynamic and Press–Schechter approaches, when compounded with the many questions that may be raised about the applicability of thermodynamic fluctuation theory to non-linear gravitational clustering, and when contrasted with the many successes of the Press–Schechter approach, may serve to undermine continued interest in the thermodynamic approach.

That there may be some underlying connection between the thermodynamic and the Press–Schechter approaches is suggested by the following toy model. Consider the set of Poisson-distributed Borel clusters that are obtained from the Poisson Press–Schechter analysis. Let the number density of clumps be $\bar{n}(1-b)$ when the number density of particles is $\bar{n}$. Assume that these clumps are virialized, and let K_i and W_i denote the kinetic and potential energies associated with the ith clump. Also, let $\mathscr{K}$ denote the kinetic energy associated with the peculiar velocities of the clumps within a cell of size V, characterized by a temperature T. Since the clumps are Poisson-distributed, assume that there is no energy associated with the interaction between clumps. Now calculate the total energy for this system.

If equipartition of energy holds, and since there are, on average, $\bar{N}(1-b)$ clumps in a cell of size V, where $\bar{N}\equiv\bar{n}V$, then, on average, the kinetic energy associated with a cell is $3\bar{N}T(1-b)/2\equiv\mathscr{K}$. Since each clump is virialized, $W_i=-2K_i$. If the mean square velocity of the particles is $\langle v^2\rangle=3T$, then the kinetic energy of a clump with N particles is given by treating one of the particles as the centre, and summing over the energy due to the motion of the other $N-1$ particles towards it. This prescription means that, on average, the energy in a cell of size V that is contributed by the internal energy of the clumps in it is $\bar{N}(1-b)\langle W+K\rangle=\bar{N}(1-b)\langle -K\rangle$, where

$$\langle K\rangle=\frac{\langle v^2\rangle}{2}\sum_{N=1}^{\infty}(N-1)\,\eta(N,b)=\frac{3Tb}{2(1-b)}. \tag{17}$$

So, the average total energy, U, in a cell of size V is the sum of the internal energy due to the clumps, $\bar{N}(1-b)\langle -K\rangle=-3\bar{N}Tb/2$, and the kinetic energy due to the random motions of the clumps, $\mathscr{K}=3\bar{N}T(1-b)/2$. So, $U=3\bar{N}T(1-2b)/2$, and is identical to equation (2). Moreover, since in this model there are, on average, $\bar{N}(1-b)$ independent objects in a cell of size V, the pressure is given by the relation $PV/T=\bar{N}(1-b)$. This expression is also identical to that in equation (2). This suggests that, in essence, the thermodynamic model that is specified by equations (2) and (3) corresponds to the treatment of the Borel clumps of the Poisson Press–Schechter analysis as a mixture of non-interacting virialized clumps that are in thermal equilibrium.

Sheth (1995) uses equations (1) and (16), and a linear theory analysis of the variance on large scales (e.g. Peebles 1980, section 70; Zhan 1989), to calculate the evolution of the overdensity threshold, Δ, for this set of Poisson Press–Schechter clumps. His analysis shows that, at late times, particularly in less dense cosmologies, the evolution of Δ slows appreciably. If these clumps are virialized in the manner assumed by the model in the previous paragraphs, then the evolution, as the Universe expands, of the ratio $-W/2K_{\rm tot}$ for this system is also easy to understand.

Consider a cosmology that initially has a Poisson distribution of particles that are at rest. Assume that, at least at early times, $\Omega\approx 1$. Provided $\Omega\approx 1$, gravity will be able to build up correlations that cause the Press–Schechter clumps to form, and the peculiar velocities it induces on the particles means that the Borel Press–Schechter clumps into which they form will also have some net motions. If the clumps are virialized, then $-W/2K_{\rm tot}=K/(K+\mathscr{K})$, where $\mathscr{K}$ is the contribution to the kinetic energy that is due to the peculiar motions of the clumps as above, and K was denoted by $\bar{N}(1-b)\langle K\rangle$ earlier. The evolution of W and $K_{\rm tot}$ is coupled, as shown by the cosmic energy equation (equation 14). Without the relation to adiabatic expansion (cf. the discussion preceding Fig. 2), the Press–Schechter approach cannot describe the effects of this coupling.

However, in the absence of correlations, the kinetic energy of peculiar velocities decreases as R^{-2}. (To see this, let $W=0$ in equation 14.) In the model, however, the clumps are virialized, so that W is certainly not zero. On the other hand, since the model assumes that the clumps are independent, this decay of peculiar velocities will be a good approximation for the behaviour of $\mathscr{K}$, provided $\Omega<1$, since then the time-scale for clustering to develop is less than the expansion time-scale (e.g. Saslaw 1992). This condition obtains at late times in cosmologies that are less dense than critical. Thus, at late times in less dense cosmologies, $\mathscr{K}\to 0$, so $-W/2K_{\rm tot}\to 1$, whatever the value of Δ. Therefore if this model of virialized

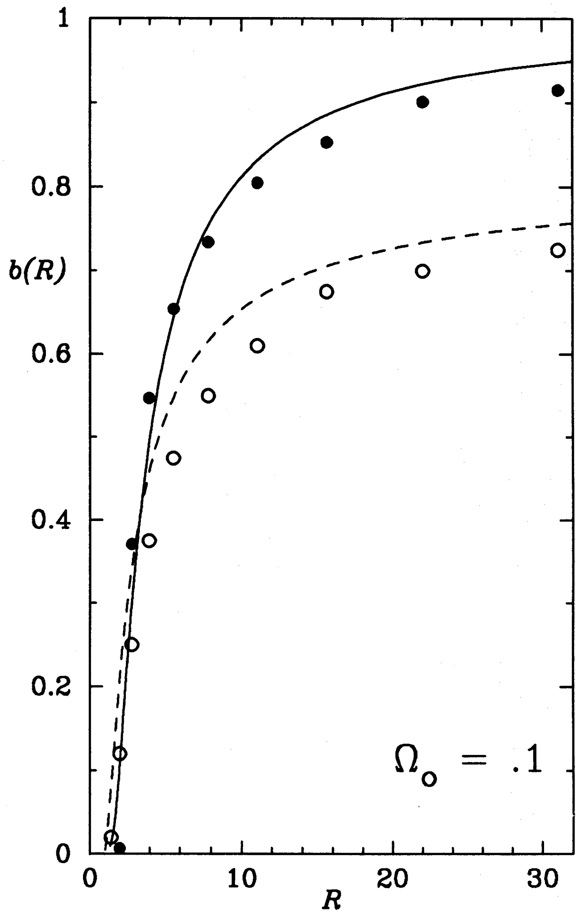

Figure 3. Linear (dashed line) and adiabatic (solid line) evolution of $1/(1+\Delta)$ (open circles) and $-W/2K_{\rm tot}$ (filled circles) for a cosmology with $\Omega=0.1$ when $R=26$. To specify the adiabat, $R_*=2.38$.

Press–Schechter clumps is correct then, in general, the relation $-W/2K_{\rm tot}\geq 1/(1+\Delta)$ should always hold.

To illustrate this, Fig. 3 shows the evolution of $-W/2K_{\rm tot}$ (filled circles) and of $1/(1+\Delta)$ (open circles, determined from the variance of particle counts in large cells), and the corresponding adiabatic (solid curve) and linear theory (dashed curves) descriptions of the evolution, measured in N-body simulations, of an $\Omega_0=0.1$ cosmology. The simulations have cold, Poissonian initial conditions, from which the subsequent motions of $N_{\rm tot}=4000$ identical particles were integrated directly. They were kindly made available by Dr M. Itoh and Prof. S. Inagaki [see Itoh, Inagaki & Saslaw (1988), and Sheth & Saslaw (1995) for details, and also for results regarding other cosmologies]. In the text, b was used to denote both the physical quantity $-W/2K_{\rm tot}$ and the statistical quantity (since it is obtained from the variance) $1/(1+\Delta)$. Fig. 3 shows how the evolution of these two quantities differs. Whereas $-W/2K_{\rm tot}\approx 1/(1+\Delta)$ at early times when $\Omega\approx 1$, at late times $-W/2K_{\rm tot}\geq 1/(1+\Delta)$, as expected.

Fig. 3 suggests that, at least for low-density cosmologies in which the initial distribution is cold and Poissonian, this model of non-interacting, virialized, Poisson Press–Schechter clumps provides a useful description of the clustering. However, for this model, which relates the Press–Schechter to the thermodynamic approach, to work, all clumps were assumed to have the same temperature. That is, in the model all particles have the same mean square velocity, whatever the mass of their host clump. This differs from the way in which circular velocities are usually assigned to Press–Schechter clumps by, e.g., White & Frenk (1991). The idealization that different clumps do not interact gravitationally, and this assumption that the clumps are in thermal equilibrium, and so all have the same temperature, are in conflict with observations (e.g. White & Frenk 1991, and references therein).

If this isothermal idealization is wrong, then why does the expression for adiabatic expansion (solid line), which depends crucially on the accuracy and applicability of the thermodynamic approach, fit the measured values of $-W/2K$ (filled circles) so well (cf. Fig. 3)? At any epoch, in these $\Omega_0=0.1$ simulations the average clump size is $M_{\rm clump}\approx(1-b)^{-1}$, where $b=1/(1+\Delta)$ is given by the dashed line, or by the open circles. The root-mean-square deviation around this mean is $\sqrt{b/(1-b)^3}$. Provided $b\lesssim 0.4$, this rms dispersion is somewhat less than $\sqrt{1/(1-b)}$, the square root of the mean. Thus, for small values of b, the distribution of clump sizes around the average mass is small. Even when $b\approx 0.6$ (as it is at late times in the low-density simulations), the distribution of clump sizes around the average mass remains relatively small, and the number of clumps that are more massive than the average, $N_{\rm tot}(1-b)\sum_{M>M_{\rm clump}}\eta(M,b)$, is also small. This occurs particularly because $N_{\rm tot}$ in these simulations is only $\sim 10^4$. Therefore in low-density cosmologies in which b remains relatively small, the approximation that all clumps have the same temperature independent of clump mass is relatively accurate, since most clumps have approximately the same mass anyway.

At late times in more dense cosmologies, $b=1/(1+\Delta)$ is greater than in less dense cosmologies, so the distribution of clump sizes around the average, as well as the number of clumps that are more massive than the average, is larger. Therefore in simulations of denser cosmologies the adiabatic expansion expression should fit the measured values of $-W/2K$ less well than in Fig. 3. This qualitative behaviour is consistent with the evolution of $-W/2K$ measured in an ensemble of simulations of an $\Omega_0=1$ cosmology [Sheth & Saslaw (1995); see their fig. 5, although, for reasons they discuss, $-W/2K$ in more dense cosmologies is more difficult to measure reliably].

Therefore if there is a relation between the Press–Schechter and the thermodynamic approaches, and if it has the form described in the previous paragraphs, then the two idealizations just described have strong implications for the viability of the thermodynamic approach. They suggest that, in the general case that is suggested by observations, in which clumps interact gravitationally and so are correlated with each other, and in which different clumps have different temperatures, the thermodynamic model will, at best, only provide an approximate description of the clustering process.

Despite the various idealizations that are inherent in the thermodynamic model and that are contradicted by observations, we end with a caveat. In contrast to the Press–Schechter model, the particular thermodynamic model (equation 3) from which the GQED can be derived, when combined with the adiabatic expansion interpretation of the Irvine–Layzer cosmic energy equation, makes specific predictions about the temporal evolution of the gravitational correlation and kinetic energies (cf. essentially, the information in Fig. 2, and the solid curve in Fig. 3). These predictions about the dynamics of clustering are, at present, outside the scope of the Press–Schechter approach, and appear to be in good agreement with *N*-body simulations (Fig. 3; Saslaw 1992; Sheth & Saslaw 1995). Moreover, whereas the Press–Schechter mass function predictions are in good agreement with *N*-body simulations in an average sense, they are not so good on a particle-by-particle basis (Lacey & Cole 1994). This is just what would be expected if the thermodynamic approach were, indeed, correct. It may therefore be of interest to extend the simple thermodynamic approach considered in this paper to describe a more general system in which virialized clumps have a range of temperatures, and where the gravitational interaction between different clumps is not neglected.

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