Pressure Wave Propagation in an Imploding Liquid Metal Liner

Yasuyuki ITOH and Yoichi FUJII-E†

Department of Nuclear Engineering,
*Faculty of Engineering, Osaka University**

Received April 14, 1980
Revised July 21, 1980

The propagation of a pressure wave through an imploding liquid metal liner near turnaround, due to the compression of magnetic flux and plasma is considered, based on which the pressure distribution near turnaround is derived analytically under the assumption of a small Mach number M (defined by the ratio between characteristic radial velocity of liner and the velocity of sound through the medium). It is shown that under certain conditions, the rarefaction wave from the outer (free) surface causes the apparition of a negative pressure in the imploding liner, with consequent liability of cavitation. The criterion for the generation of the negative pressure is found to be $M\Lambda>0.6\sim0.7$ for the range of parameters considered (Λ: liner thickness at turnaround normalized by the inner radius).

KEYWORDS: liner fusion, liquid metal imploding liner, plasma, magnetic flux, adiabatic compression, pressure wave propagation, Mach number, negative pressure

I. INTRODUCTION

When an imploding liquid metal liner is applied to the compression of magnetic flux and fusion plasma(1)~(5), the dynamics of the system are affected by the finite compressibility of the fluid.

Papers published in the past have considered mainly (1) the energy loss due to the compression of the liner material(4)~(8), (2) the dwell enhancement near turnaround which results in an increase of fusion reaction time(7)(8).

The pressure distribution in an imploding liner has also been discussed in Ref. (8), in which numerical calculations with finite difference technique, have led to the finding that the compression wave propagating outward through the liner acts as a water hammer. The calculations, however, only covered the interval up to the point in time where the water hammer reached the outer surface of liner, and did not proceed further to consider the ensuing period of rarefaction wave.

The problem raised here is the possible apparition of negative pressure during propagation of the rarefaction wave, and consequent liability of cavitation(9)(10), to affect the entire liner motion and its geometry.

The present paper considers this possibility of negative pressure generation in an imploding liner, based on analytical treatment of compressible fluid dynamics in the region of small Mach number.

* *Yamadakami, Suita-shi, Osaka 565.*

† Present address: Institute of Plasma Physics, Nagoya University, *Furo-cho, Chikusa-ku, Nagoya 464.*

II. ANALYSIS

In this chapter, the pressure distribution in an imploding liner near turnaround is derived analytically for a small Mach number M. The Mach number in this paper is defined by

$$M=U_0/c\,, \tag{1}$$

where c is the velocity of sound in the liner material and U_0 the characteristic velocity of the liner.

In cylindrical geometry, equations of continuity and radial motion take the form[11]

$$\dot{\rho}=-\rho(ru)'/r\,, \tag{2}$$

$$\rho(\dot{u}-V^2/r)=-p'\,, \tag{3}$$

where ρ is the density, r the radius of a fluid element, u the radial velocity, V the azimuthal velocity and p the pressure, while the symbols ( )′, (˙) respectively denote

$$(\ )'=\partial/\partial r(\ ) \quad \text{and} \quad (\dot{\ })=(\partial/\partial t+u\partial/\partial r)(\ )\,.$$

Let $r_1(t)$, $r_2(t)$ be the inner and outer radii of the free surfaces of the liner, as shown in **Fig. 1**. The liner is assumed to rotate as a rigid body round an axis that does not displace, so that at turnaround there is no Rayleigh-Taylor instability of the inner surface[12]. If the initial liner thickness is sufficiently small compared with the initial radius, the angular momentum $A=rV$ per unit mass can be considered roughly constant in the radial direction. In such case, the azimuthal velocity V is determined from the law of angular momentum conservation and takes the form $V=A/r$. Near turnaround, the pressure on the liner outer surface is assumed to become negligible compared with that on the inner surface, *i.e.* $p(r_2)=0$[8], and the liner is accelerated outward by the pressure $p(r_1)=P_1$ of the compressed payload (magnetic flux+plasma).

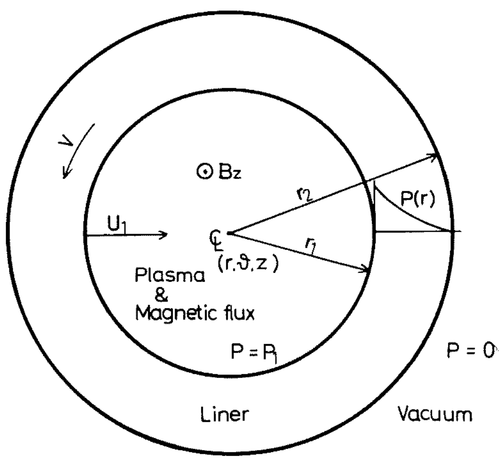


Fig. 1 Geometry of model

In the case of an incompressible liner, *i.e.* $(ru)'=0$, Eq. (3) can be easily solved, upon which the pressure distribution in the rotating liner is obtained in the form[7]

$$p=P_M+P_1L\,, \tag{4}$$

where $$P_M=\frac{1}{2}\rho(U_1^2+V_1^2)[1-r_1^2/r^2-(1-r_1^2/r_2^2)(1-L)]\,, \tag{5}$$

$$L=\ln(r_2/r)/\ln(r_2/r_1)\,, \tag{6}$$

and U_1, V_1 are the radial and azimuthal velocities of the inner surface. In the right-hand side of Eq. (4), the first term is the pressure due to liner motion and hence independent of boundary pressure, *i.e.* $P_M(r_1)=P_M(r_2)=0$.

The trajectory of an incompressible liner near turnaround is approximated by[13]

$$r_1^2(t)=r_{1m}^2(1+t^2/\tau^2)\,, \tag{7}$$

$$r^2(t)-r_1^2(t)=\text{const.}, \tag{8}$$

where the instant of turnaround is taken as the initial point in time and 2τ the dwell time of the liner near turnaround.

The payload pressure $P_1(t)$ is assumed to satisfy the adiabatic law(8)

$$P_1(t)=P_{1m}(r_{1m}/r_1)^{2\gamma}\,, \tag{9}$$

where the suffix m denotes quantities at turnaround, and γ is the ratio of specific heat of the payload, and ranges from 5/3 to 2(8). The major part of the fusion reaction in a duty cycle is considered to occur during the dwell time 2τ, since at $|t|=\tau$ the payload pressure is reduced to $2^{-\gamma}P_{1m}$.

The half dwell time τ is determined from energy conservation:

$$\pi\rho\int_{r_1}^{r_2} r(U^2+V^2)dr+\pi P_1 r_1^2/(\gamma-1)\simeq\text{const.}, \tag{10}$$

where U is the radial velocity of the incompressible liner. From Eq. (7),

$$U=(t/r)U_0^2\,, \qquad U_0=r_{1m}/\tau\,, \tag{11}$$

and from Eqs. (7), (10) and (11),

$$\tau^2\simeq(P_{1m}/\rho/r_{1m}^2)^{-1}\ln(r_{2m}/r_{1m})\,. \tag{12}$$

Letting U_0 represent the characteristic velocity of the liner near turnaround, the Mach number

$$M=(r_{1m}/\tau)/c\simeq(P_{1m}/\rho c^2)^{1/2}[\ln(r_{2m}/r_{1m})]^{-1/2}\,. \tag{13}$$

When the transit time $(r_{2m}-r_{1m})/c$ of pressure wave in the liner increases to the point where it becomes comparable to the half dwell time, the compressibility can no longer be neglected in calculating the pressure distribution.

For a small Mach number, the radial velocity u can be considered to roughly equal that of the incompressible liner, and hence(8),

$$u=U+\tilde{U}\,, \tag{14}$$

where $\tilde{U}$ is the deviation due to the compressibility.

The general equation of state for a compressible material is(14)

$$\rho\partial p/\partial\rho=K=\rho c^2\,, \tag{15}$$

where K is the bulk modulus. The energy of compression is given for a unit volume by $p^2/2K$, which is of the order of $\rho\tilde{U}^2/2$, which would indicate that the ratio $\tilde{U}/U$ should be of the same order as the Mach number. From Eq. (15) and the mass conservation law $(rU)'=0$, Eq. (2) of continuity becomes(14)

$$\dot{\tilde{P}}=-K[(r\tilde{U})'/r+\dot{P}_M/K]\,, \tag{16}$$

where $\tilde{P}$ is defined by

$$\tilde{P}=p-P_M\,. \tag{17}$$

The second term on the right-hand side of Eq. (16) is estimated from Eqs. (5) and (11) to be $\dot{P}_M/K\sim M^2U/r$, while $(r\tilde{U})'/r\sim MU/r$. Therefore, for a small Mach number, Eq. (16)

can be rewritten

$$\ddot{\tilde{p}} \simeq -(K/r)[(r\tilde{U})^{\cdot}]' . \tag{18}$$

Since $\dot{U}=-(P_M+P_1L)'/\rho$, the first order equation of motion is found from Eqs. (3) and (17) to be

$$\dot{\tilde{U}}=-\rho^{-1}(\tilde{P}-P_1L)' . \tag{19}$$

By combining Eqs. (18) and (19), we obtain the pressure wave equation

$$c^{-2}\partial^2\tilde{P}/\partial t^2 \simeq r^{-1}(r\tilde{P}')' , \tag{20}$$

where the terms

$$\rho\tilde{U}U/r\tilde{P}' \sim \rho\tilde{U}U/\tilde{P} \sim M , \qquad r\rho'/\rho \sim \tilde{P}/K \sim M^2 \quad \text{and}$$

$$c^{-2}[\ddot{\tilde{P}}-\partial^2\tilde{P}/\partial t^2] \lesssim 2M(\partial\tilde{P}/\partial t)'/c+M^2(\tilde{P}''+r^{-1}\tilde{P}')$$

have been neglected. The initial and the boundary conditions of Eq. (20) are, respectively,

$$P(r,\ 0)=0 , \tag{21}$$

$$\tilde{P}(r_1,\ t)=P_1(t) , \qquad \tilde{P}(r_2,\ t)=0 . \tag{22}$$

If we define the quantity $\Pi(r,\ t)$ by

$$\Pi(r,\ t)=R(r)\tilde{P}(r,\ t) , \tag{23}$$

where

$$R^{-1}(r)=(r_2-r)^{-1}\int_r^{r_2} r^{-1}dr=(r_2-r)^{-1}\ln(r_2/r) , \tag{24}$$

the relation

$$R^{-1}\Pi''=\tilde{P}''+\frac{1}{r}\tilde{P}'(1-\zeta) , \tag{25}$$

is obtained, wherein

$$\zeta=\left[1-2\left(\frac{1}{\ln(r_2/r)}-\frac{r}{r_2-r}\right)\right]\left(1-\frac{\tilde{P}/\tilde{P}'}{L/L'}\right) . \tag{26}$$

Since

$$1-2\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$$

$$=\frac{1}{6}\left[x-\frac{1}{2}x^2+\frac{1}{3}\left(1-\frac{1}{20}\right)x^3-\frac{1}{4}\left(1-\frac{1}{10}\right)x^4+\frac{1}{5}\left(1-\frac{145}{1{,}008}\right)x^5-\cdots\right] \simeq \frac{1}{6}\ln(1+x) ,$$

Eq. (26) can be rewritten

$$\zeta \simeq \frac{1}{6}\ln(r_2/r)\left(1-\frac{\tilde{P}/\tilde{P}'}{L/L'}\right) ,$$

where $\tilde{P} \to P_1L$ for $M \to 0$ from Eqs. (4) and (17), and $\frac{1}{6}\ln(r_2/r) \lesssim 0.1$ for $r \gtrsim r_2/2$. The boundary condition $\tilde{P}(r_1)=P_1$ justifies considering the pressure distribution $\tilde{P}$ near the inner surface to be similar to that of the incompressible liner P_1L when the Mach number is small. Hence, the quantity ζ in Eq. (25) is small compared to unity, and Eq. (20) is simplified into a slab-type wave equation

$$c^{-2}\partial^2\Pi/\partial t^2 \simeq \Pi''. \tag{27}$$

The initial and boundary conditions then become

$$\Pi(r, 0)=0 \tag{28}$$

and

$$\Pi(r_1, t)=R(r_1)P_1(t), \qquad \Pi(r_2, t)=0. \tag{29}$$

It should be noted that Eq. (27) gives $\tilde{P}=P_1L$ for $c\to\infty$, which agree with the value for an incompressible liner.

The APPENDIX contains a detailed analysis of the solution of Eq. (20) under the conditions specified by Eqs. (21) and (22). The result proves that Eq. (27) is valid only when

$$M\Lambda \lesssim 1, \tag{30}$$

where $\Lambda \equiv (r_{2m}-r_{1m})/r_{1m}$.

For $M\Lambda \gg 1$, $R(r)$ in Eq. (23) must be replaced by $\sqrt{r}$ as shown in APPENDIX, the function $R(r)$ or $\sqrt{r}$ characterizing the cylindrical wave, while r the spherical wave (the quantity $r\tilde{P}$ transforms a spherical wave equation into that of slab type).

The solution of Eq. (27) is

$$\Pi(r, t)=\sum_{n=0}^{\infty}[\Pi_1(t-\tau_n^+)-\Pi_1(t-\tau_n^-)], \tag{31}$$

where

$$\Pi_1(r, t)=R(r_1)P_1(t), \tag{32}$$

$$\tau_n^{\pm}=[\pm(r-r_2)+(2n+1)(r_2-r_1)]/c. \tag{33}$$

Equation (31) reveals that the history of the inner boundary pressure $P_1(t)$ affects the pressure of a fluid element of radius r after a time delay $\tau_n^{\pm}$ during which the pressure wave propagates through the liner conserving the quantity $\Pi_1(t-\tau_n^{\pm})$. In the same equation, τ_n^+ is the time necessary for the pressure wave to propagate from r_1 to r after reflecting $2n$ times at the inner and outer surfaces, and τ_n^- the corresponding time after reflecting $2n+1$ times (see Fig. 2). The transit time $\tau_n^{\pm}$ calculated with Eq. (33), however, does not give the true value of $\tau_n^{\pm}$, since the radial motion of the liner has the effect of changing the liner thickness. It is only when this radial velocity is negligible compared with that of sound that the actual transit time $\tau_n^{\pm}$ can be calculated in the straightforward manner indicated in **Fig. 2.**

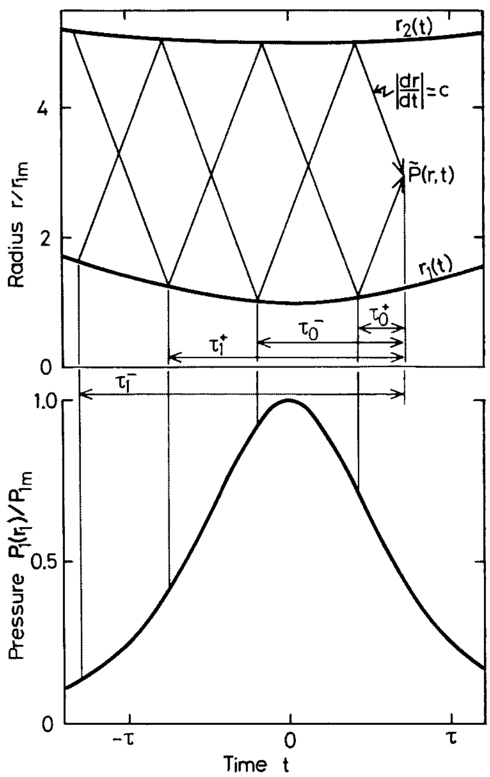


Fig. 2 Trajectories of pressure waves and of inner and outer surfaces of liner near turnaround

Thus, provided that the Mach number is small and granted the condition $M\Lambda \lesssim 1$, an imploding liner near turnaround has a pressure distribution expressed by

$$p(r,\ t)=P_M(r,\ t)+R^{-1}(r)\sum_{n=0}^{\infty}[\Pi_1(t-\tau_n^+)-\Pi_1(t-\tau_n^-)]\ , \quad (34)$$

where P_M, R and Π_1 are obtained from Eqs. (5), (24) and (32). The normalized pressure

$$\hat{p}(\hat{r},\ \hat{t})=p/P_{1m}\ , \qquad (\hat{r}=r/r_{1m},\ \ \hat{t}=t/\tau)\ ,$$

is determined by parameters M, Λ, γ and $\hat{\Omega}=\Omega\tau$, where $\Omega=A/r_{1m}^2$ is the angular velocity of the inner surface at turnaround.

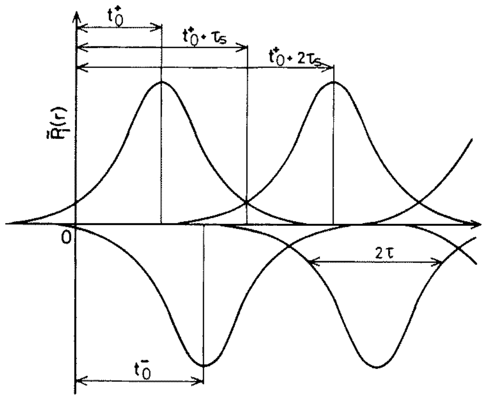
III. Discussion

The compression and expansion undergone by the payload causes the compression wave to propagate through the liner from inner foward outer surface. At the outer free surface, the compression wave is reflected in the form of a rarefaction wave, which is further reflected at the inner surface as a compression wave. This alteration between compression and rarefaction waves is illustrated in **Fig. 3** for a fluid element of radius r, where the instant of turnaround is adopted for the origin of time, and where

$$t_0^+=(r-r_1)/c\ , \quad t_0^-\simeq\tau_s+(r_2-r)/c\ ,$$

$$\tau_s=(r_{2m}-r_{1m})/c\ . \quad (35)$$

The pressure $\tilde{P}(r,\ t)$ is determined by superposing the instantaneous pressure $\tilde{P}_i$ of these waves. If the superposed sum of compression waves is smaller than the corresponding sum for the rarefaction waves,



$\tilde{P}_i$: Instantaneous pressure

Fig. 3 Transition of compression and rarefaction waves observed at fluid element with radius r

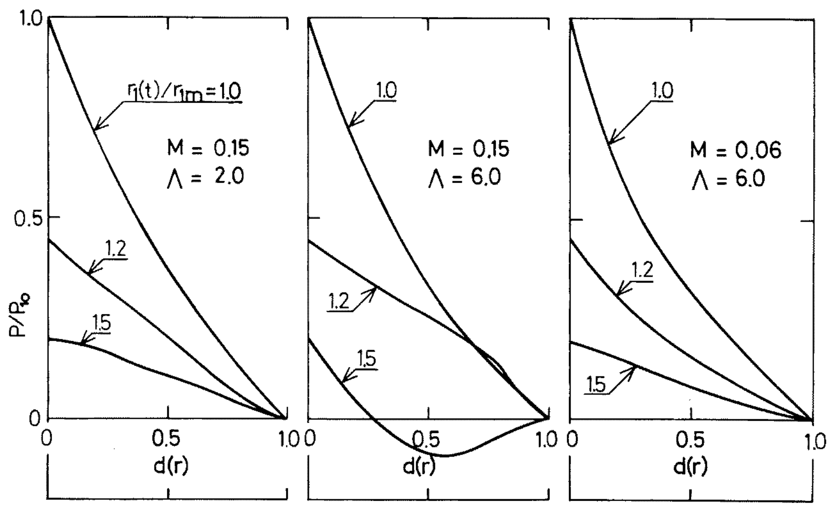


Fig. 4 Pressure distribution in liner after turnaround $d(r)=(r-r_1)/(r_2-r_1)$

the resulting pressure in the fluid becomes negative. The calculated pressure distribution after turnaround is presented in **Fig. 4.** It is revealed that the pressure could become negative under certain conditions of liner implosion, and that its occurrence would tend to be promoted by increasing Mach number and liner thickness.

In the case of a static liner $(U=V=0)$, the most favorable condition for the generation of negative pressure is found from Fig. 3 to be

$$t \simeq t_0^- \simeq t_0^+ + \tau_s, \tag{36}$$

in which case the pressure corresponds to the superposition of the maximum value of the sum of $\tilde{P}_i$ for rarefaction waves on the minimum value of that for compression waves. Under this condition, the criterion for $p(r, t)=\tilde{P}(r, t)<0$ becomes, from Eqs. (7), (9) and (34),

$$\sum_{n=0}^{\infty} [1+(2n-1)^2\hat{\tau}_s^2]^{-\gamma} < \sum_{n=0}^{\infty} [1+(2n\hat{\tau}_s)^2]^{-\gamma}, \tag{37}$$

where $\hat{\tau}_s=\tau_s/\tau=M\Lambda$.

By neglecting the terms of $n \geqq 2$ in Eq. (37), the criterion is approximated by

$$M\Lambda > (2^{1/\gamma}-1)^{1/2}. \tag{38}$$

Figure 5 presents, as function of the thickness Λ normalized with liner motion taken into account, the minimum value Q of the quantity $M\Lambda$ at which negative pressure is induced. From this figure, the criterion for $p(r, t)<0$ could also be approximated by

$$M\Lambda > Q(\gamma, \hat{\Omega}), \tag{39}$$

where $Q(2, 0) \simeq 0.60$ and $Q(5/3, 0) \simeq 0.64$, in the present range of Λ. Liner rotation $(\hat{\Omega} \neq 0)$ produces an increase of P_M, as seen from Eq. (5), which has the effect of raising the minimum value Q. This effect, however, is found to be small, the increment being only 10% even when $\hat{\Omega} \sim 1$, *i.e.* when the kinetic energy of the liner rotation is comparable to the compressed payload energy at turnaroud.

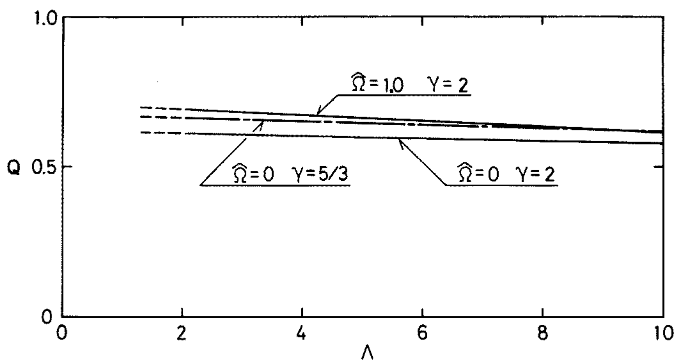


Fig. 5 Minimum value Q of $M\Lambda$ for generation of negative pressure, as function of Λ

The apparition of negative pressure would introduce the liability of cavitation occurrence, which will affect the whole liner motion and its geometry after turnaround.

We know from Eq. (36) that, for $M\Lambda \simeq Q$, the negative pressure would first appear in the fluid element within the radius $r \simeq (r_2-r_1)/2$, *i.e.* at the center of the liner. The time t^* at which the negative pressure begins to appear would then be $(3/2)\tau_s$ after turnaround. With increasing $\tau_s/\tau(=M\Lambda)$, the time t^* approaches τ_s. This can be easily understood when we consider the case where an impulsive transient pressure is applied at the inner surface; the negative pressure would then appear immediately after the reflection at the outer surface. The inner radius r_1 at time t^* is related to $M\Lambda$ in **Fig. 6,** where it is seen that the minimum value of $r_1(t^*)/r_{1m}$ is roughly 1.3~1.4, when the payload pressure is reduced to 30~40% of the turnaround value.

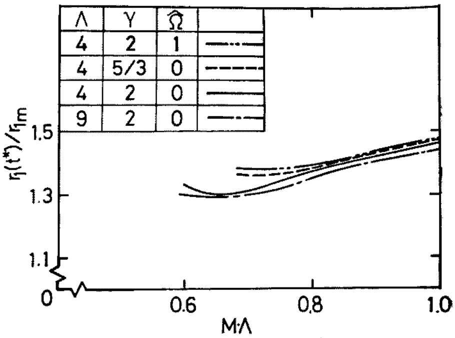


Fig. 6 Inner radius r_1 at time t^* when negative pressure begins to appear

The foregoing observation would indicate that the bulk of the fusion reaction in a cycle would be left unaffected even if the liner is disrupted by cavitation immediately after turnaround.

There still remains the problem, however, that in order to enhance the reactor plant efficiency, the kinetic energy of the liner must be recovered during its expansion (*cf.* MHD conversion similarly to the case of θ-pinched plasma(15)). For this reason, the criterion $M\Lambda<Q$ requires to be satisfied if the occurrence of negative pressure affects the liner motion significantly.

IV. CONCLUSIONS

The pressure distribution in an imploding liner near turnaround has been obtained analytically for a small Mach number M under the condition $M\Lambda\lesssim1$. The results indicate possible apparition of negative pressure in the liner under certain conditions. Such negative pressure is liable to induce cavitation, which could basically affect the entire liner motion and its geometry after turnaround.

The occurrence of negative pressure, however, has been shown to be preventable by the condition $M\Lambda<Q(\gamma, \hat{\Omega})$, with $0.6\lesssim Q\lesssim0.64$ for the present range of Λ with irrotational liner. Liner rotation has the effect of slightly increasing Q.

[NOMENCLATURE]

A: Angular momentum per unit mass
c: Sound velocity
K: Bulk modulus
L: Defined by Eq. (6)
M: Mach number
p: Pressure
P_M: Pressure due to motion
$\tilde{P}=p-P_M$
P_j: Boundary pressure ($j=1, 2$)
r: Radius
R: Defined by Eq. (23)
t: Time
t^*: Time when negative pressure begins to appear
u: Radial velocity
U_0: Characteristic velocity of liner
U: Radial velocity of incompressible liner
$\tilde{U}$: Deviation of radial velocity due to compressibility
V: Azimuthal velocity
Q: Minimum value of $M\Lambda$ for generation of negative pressure
γ: Ratio of specific heats of payload
Λ: Liner thickness at turnaround normalized by r_{1m}
$\Pi=R\tilde{P}$
ρ: Liner density
τ: Half dwell time defined in Eq. (7)
$\tau_s=(r_{2m}-r_{1m})/c$: Transit time of pressure wave in liner
Ω: Angular velocity of inner surface at turnaround
$\hat{\Omega}=\Omega\tau$

(Subscripts)
$j=1, 2$: Quantities at inner and outer surfaces, respectively
m: Quantities at turnaround

Symbols

$(\dot{\ }) = (\partial/\partial t + u\partial/\partial r)(\ )$, $(\ )' = \partial/\partial r(\ )$

ACKNOWLEDGMENT

The authors wish to thank Dr. K. Miyazaki for his instructive discussions. The authors are indepted also to Mr. T. Kanagawa for his assistance in the numerical computation.

(Text edited grammatically by Mr. M. Yoshida.)

—REFERENCES—

(1) BORIS, J.P., *et al.*: *Proc. 5th Eur. Conf. Controlled Fusion Plasma Physics, 1972*, p. 20 (1972).
(2) BARCILON, A., *et al.*: "*Plasma Physics and Controlled Nuclear Fusion Research*", Vol. 2, p. 567 (1975).
(3) BOOK, D.L., HAMMER, D.A., TURCHI, P.J.: *Nucl. Fusion*, **18**, 159 (1978).
(4) ROBSON, A.E.: *NRL Memorandum Rep.* 2616, (1973).
(5) GERWIN, R.A., MALONE, R.C.: *Nucl. Fusion*, **19**, 155 (1979).
(6) SOMON, J.P., JABLON, C.: *ibid.*, **16**, 1040 (1976).
(7) YAMAGISHI, T., SCHAFFER, M.J.: *GA-A* 14792, (1978).
(8) BOOK, D.L., TURCHI, P.J.: *Phys. Fluids*, 22, 68 (1979).
(9) KATOH, Y.: "*Cavitation*", (in Japanese), (1978), Maki Shoten.
(10) YAMAZAKI, T.: "*Cavitation*", (in Japanese), (1979), Nikkan Kogyo Shinbun.
(11) BIRD, R.B., *et al.*: "*Transport Phenomena*", 187 (1960), John Wiley and Sons.
(12) BARCILON, A., BOOK, D.L., COOPER, A.L.: *Phys. Fluids*, **17**, 1707 (1974).
(13) COOPER, A.L., BOOK, D.L.: *ibid.*, 21, 34 (1978).
(14) MIYAZAKI, K., FUJII-E, Y.: *J. Nucl. Sci. Technol.*, 14[5], 327 (1977).
(15) MILEY, G.H.: "*Fusion Energy Conversion*", p. 153 (1976), ANS.

[APPENDIX]

Here, the solution of Eq. (19) is derived under the conditions specified by Eqs. (20) and (21).

Under the initial condition Eq. (20), Eq. (19) becomes, by Laplace transformation,

$$\tilde{P}''(r,\ s)+\frac{1}{r}\tilde{P}'(r,\ s)-(s/c)^2\tilde{P}(r,\ s)=0\,, \tag{A1}$$

where s is the operator of the Laplace transformation. The solution of Eq. (A1) is obtained with the boundary conditions Eq. (21):

$$\tilde{P}(r,\ s)=\frac{X(z_2,\ z)}{X(z_2,\ z_1)}P_1(s)\,, \tag{A2}$$

where $X(z_2, z)=I_0(z_2)K_0(z)-K_0(z_2)I_0(z)$

$z=sr/c\,,\ z_j=sr_j/c\ \ (\text{j}=1, 2)$,

while I_0 is the modified Bessel function of the first kind of order zero, and K_0 the second kind.

The function $X(z+\delta z,\ z)$ is Taylor expanded in the power of

$$\delta z=(r_2-r)s/c=(r_2-r)z/r:$$

$$\delta z\cdot X(z+\delta z,\ z)$$

$$=\sum_{n=1}^{\infty}(n\,!)^{-1}[K_0(z)I_0^{(n)}(z)-I_0(z)K_0^{(n)}(z)]\delta z^{n+1}$$

$$=\delta z\left[\alpha-\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3-\frac{1}{4}\alpha^4+\frac{1}{5}\alpha^5-\frac{1}{6}\alpha^6+\frac{1}{7}\alpha^7-\frac{1}{8}\alpha^8+\frac{1}{9}\alpha^9-\cdots\right]$$

$$+\frac{\delta z^3}{3!}\left[\alpha-\frac{1}{2}\alpha^2+\frac{1}{3}\left(1+\frac{1}{20}\right)\alpha^3-\frac{1}{4}\left(1+\frac{1}{10}\right)\alpha^4+\frac{1}{5}\left(1+\frac{1}{7}\right)\alpha^5\right.$$
$$\left.-\frac{1}{6}\left(1+\frac{5}{28}\right)\alpha^6+\frac{1}{7}\left(1+\frac{5}{24}\right)\alpha^7-\cdots\right]$$
$$+\frac{\delta z^5}{5!}\left[\alpha-\frac{1}{2}\alpha^2+\frac{1}{3}\left(1+\frac{1}{14}\right)\alpha^3-\frac{1}{4}\left(1+\frac{1}{7}\right)\alpha^4+\frac{1}{5}\left(1+\frac{23}{112}\right)\alpha^5-\cdots\right]$$
$$+\frac{\delta z^7}{7!}\left[\alpha-\frac{1}{2}\alpha^2+\frac{1}{3}\left(1+\frac{1}{12}\right)\alpha^3-\cdots\right]$$
$$+\frac{\delta z^9}{9!}[\alpha-\cdots]+\cdots$$
$$\cong\frac{\alpha}{\sqrt{1+\alpha}}\sum_{n=0}^{\infty}\frac{\delta z^{2n+1}}{(2n+1)!}-3F(\alpha)G(\delta z), \qquad \text{(A3)}$$

where $\alpha=\delta z/z=(r_2-r)/r$

$$F(\alpha)=\frac{1}{24}\alpha^3-\frac{1}{16}\alpha^4+\frac{47}{640}\alpha^5+\frac{61}{768}\alpha^6+\cdots=\frac{\alpha}{\sqrt{1+\alpha}}-\ln(1+\alpha)$$
$$G(\delta z)=\sum_{n=1}^{\infty}\frac{1}{2n+1}\cdot\frac{\delta z^{2n+1}}{(2n-1)!}=\delta z^{-1}\cosh\delta z-\delta z^{-2}\sinh\delta z\,.$$

Finally,

$$I_0(z+\delta z)K_0(z)-K_0(z+\delta z)I_0(z)$$
$$\cong\frac{\alpha}{\sqrt{1+\alpha}}\cdot\frac{\sinh\delta z}{\delta z}[1-3f(\alpha)(\delta z^{-1}\coth\delta z-\delta z^{-2})], \qquad \text{(A4)}$$

where $f(\alpha)=1-\sqrt{1+\alpha}\ln(1+\alpha)/\alpha$.

Since, in the present instance, $|s|^{-1}$ is of the order of the half dwell time τ, $|\delta z|$ can be estimated by

$$|\delta z|=(r_2-r)s/c\sim(r_2-r)/c\tau\lesssim(r_{2m}-r_{1m})/c\tau=M\Lambda\,,$$

where $\Lambda=(r_{2m}-r_{1m})/r_{1m}$, $M=(r_{1m}/\tau)/c$.

When $\delta z\sim M\Lambda\lesssim1$, $3(\delta z^{-1}\coth\delta z-\delta z^{-2})$ in Eq. (A4) is nearly equal to unity, and $X(z+\delta z, z)$ can be estimated by

$$X(z+\delta z,\ z)\simeq\ln(1+\alpha)\sinh\delta z/\delta z\,. \qquad \text{(A5)}$$

On the other hand, for $\delta z\sim M\Lambda\gg1$, Eq. (A5) gives

$$X(z+\delta z,\ z)\simeq\frac{\alpha}{\sqrt{1+\alpha}}\sinh\delta z/\delta z\,. \qquad \text{(A6)}$$

By applying Eq. (A5) to Eq. (A2) and performing inverse Laplace transformation, the pressure $\tilde{P}(r, t)$ for $M\Lambda\lesssim1$ is found to be

$$\Pi(r,\ t)=\sum_{n=0}^{\infty}[\Pi_1(t-\tau_n^+)-\Pi_1(t-\tau_n^-)], \qquad \text{(A7)}$$

where $\Pi(r, t)=R(r)\tilde{P}(r, t)$, $R=(r_2-r)/\ln(r_2/r)$

$\tau_n^{\pm}=[\pm(r-r_2)+(2n+1)(r_2-r_1)]/c\le t$, $\Pi_1=\Pi(r_1, t)$.

When $M\Lambda\gg1$, $R(r)$ must be replaced by $\sqrt{r}$, as seen from Eqs. (A5) and (A6).
