PRESSURES FOR GEODESIC FLOWS OF RANK ONE MANIFOLDS

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ABSTRACT. We study the geodesic flow on the unit tangent bundle of a rank one manifold and we give conditions under which all classical definitions of pressure of a Hölder continuous potential coincide. We provide a large deviation statement, which allows to neglect (periodic) orbits that lack sufficient hyperbolic behavior. Our results involve conditions on the potential, that take into consideration its properties in the nonhyperbolic part of the manifold. We draw some conclusions for the construction of equilibrium states.

1. INTRODUCTION

We are interested in the thermodynamical formalism for the geodesic flow $G=(g^t)_{t\in\mathbb{R}}$ on the unit tangent bundle T^1M of a smooth compact nonpositively curved manifold M. More precisely, we shall investigate different notions of pressure, study equilibrium states, and relate them to large deviation results.

Given a continuous map $\varphi\colon T^1M\to\mathbb{R}$ (also called *potential*), the *variational pressure* of φ (with respect to the flow) is defined as

$$(1)\qquad P_{\mathcal{M}}(\varphi)\stackrel{\text{def}}{=}\sup_{\mu\in\mathcal{M}}\left(h(\mu)+\int_{T^1M}\varphi\,d\mu\right),$$

where $\mathcal{M}$ is the set of invariant probability measures under the geodesic flow $(g^t)_{t\in\mathbb{R}}$ and $h(\mu)$ is the entropy of the time-1 map g^1 with respect to the flow-invariant probability measure μ. An *equilibrium state* for φ is an invariant probability measure m_φ realizing the maximum of the pressure $P_{\mathcal{M}}(\varphi)$. A *measure of maximal entropy* is an invariant probability measure realizing the maximum of the *topological entropy* $h=h_{top}(T^1M)=P_{\mathcal{M}}(0)$.

When the manifold M has negative curvature, the geodesic flow on T^1M is Anosov. In this case, it is well known that any Hölder potential admits a unique equilibrium state and that this equilibrium state has very strong ergodic properties: it is ergodic, mixing, and possesses a local product structure [6]. The measure of maximal entropy is the equilibrium measure of any

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1991 *Mathematics Subject Classification.* Primary: 37D25, 37D35, 28D20, 37C45.

Key words and phrases. geodesic flow, rank 1 manifolds, entropy, topological pressure, Poincaré series, Lyapunov exponents.

BS thanks the ANR grant ANR JCJC-0108 GEODE for its support during the realization of this work. KG was partially supported by CNPq and DynEurBraz.

constant potential, while the Liouville measure is the unique equilibrium state of a certain Hölder continuous potential $\varphi^{(u)}$ that is defined by means of the derivative of the Jacobian of the geodesic flow restricted to the unstable foliation (see (12) for a precise definition).

In this paper we consider compact *rank one* manifolds, that is, smooth compact nonpositively curved manifolds where the curvature can vanish, but where there exists at least one *rank one vector*, that is, a vector whose geodesic orbit does not bound any (even infinitesimal) flat strip (see Section 2 for details). The *regular set* $\mathcal{R}$ is the set of all rank one vectors and the *higher rank set* $\mathcal{H}$ is its complementary set. In this situation, the above potentials and their equilibrium states behave quite differently. The potential $\varphi^{(u)}$ is still continuous, but probably not Hölder continuous in general. Besides the Liouville measure, it admits trivial equilibrium measures, supported by the periodic orbits staying in the flat part $\mathcal{H}$ of the manifold. Moreover, the ergodicity of the Liouville measure remains up to now a difficult open question. The measure of maximal entropy is known to exist (as does, in fact, the equilibrium measure of any continuous potential) and to be unique [21]. Beyond these few examples, little is known about uniqueness and further properties of equilibrium states.

Motivated by the study of equilibrium measures for (Hölder) continuous potentials, we need a complete understanding of the pressure function. Therefore, we investigate in greater detail different notions of pressure, and their relations to each other. The *pressure* of a potential measures the exponential growth rate of the complexity of the dynamics, weighted by this potential. There are several notions of pressure, that are known to coincide for sufficiently hyperbolic systems (see [27] in the case of the geodesic flow in negative curvature). In Theorem 1.1, we clarify their relations and prove that for certain continuous potentials, all existing definitions of pressure coincide also for geodesic flows of rank one surfaces.

Let us introduce briefly these pressures (see Section 3 for details). The *variational pressure* $P_{\mathcal{M}}(\varphi)$ has been defined in (1). The *topological pressure* $P_{top}(\varphi)$ is defined as the exponential growth rate of the values of φ along orbits that are separated through the dynamics. The *Gurevich pressure* (or *periodic orbits pressure*) $P_{Gur}(\varphi)$ is the exponential growth rate of the values of φ along periodic orbits of increasing period. The regular Gurevich pressure $P_{Gur,\mathcal{R}}(\varphi)$ is the exponential growth rate of the values of φ along *regular* periodic orbits of increasing period. Finally, the *critical exponent* $\delta_{\Gamma,\varphi}$ is defined by means of the fundamental group Γ of M acting by isometries on the universal cover $\widetilde{M}$ of M. Let $\widetilde{\varphi}$ be the Γ-invariant lift of φ to $T^1\widetilde{M}$. We denote by $\delta_{\Gamma,\varphi,x}$ the *critical exponent* of the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}\widetilde{\varphi}}$, where the integral of $\widetilde{\varphi}$ is taken along the geodesic path joining x to γx. Denote by $\delta_{\Gamma,\varphi}$ the supremum of these critical exponents over all $x\in\widetilde{M}$.

In [7], Burns and Gelfert show that if M is a compact rank one surface, then there exists an increasing sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of basic sets whose union is dense in T^1M and contains all regular periodic orbits and that topological pressure of appropriate φ on Λ_k converges to the pressure of φ on T^1M. We complete this result and study how closed geodesics contribute in this picture. In restriction to a basic set $\Lambda \subset T^1M$, for a Hölder potential φ all above introduced pressures of the flow restricted to Λ and, in particular Gurevich pressure, coincide and we will denote them shortly by $P(\varphi, \Lambda)$.

For $T > 0$, denote by $\Pi_{\mathcal{R}}(T-1,T)$ (resp. $\Pi_{\mathcal{R}}(T)$) the set of primitive rank one periodic orbits β in $\mathcal{R}$ of period $\ell(\beta) \in [T-1,T)$ (resp. $\ell(\beta) < T$). In the case of higher rank periodic orbits, there can exist infinitely many parallel periodic orbits of same length, and we denote by $\Pi_{\mathcal{H}}(T-1,T)$ (resp. $\Pi_{\mathcal{H}}(T)$) a set of representatives of each homotopy class of primitive periodic orbits of $\mathcal{H}$ of period $\ell(\beta) \in [T-1,T)$ (resp. $\ell(\beta) < T$). Denote by # the cardinality.

There are examples due to Gromov (see Section 3.5) where the set $\mathcal{H}$ carries positive entropy. However, it is always strictly less than the full topological entropy. More precisely, the following quantity is well-defined and positive (see Theorem 2.1 for details).

$$\varepsilon_0 \overset{\text{def}}{=} \limsup_{T\to+\infty} \frac{1}{T} \log \frac{\#\Pi_{\mathcal{R}}(T-1,T)}{\#\Pi_{\mathcal{H}}(T-1,T)} = \limsup_{T\to+\infty} \frac{1}{T} \log \frac{\#\Pi_{\mathcal{R}}(T)}{\#\Pi_{\mathcal{H}}(T)} . \tag{2}$$

The pressure of a continuous potential $\varphi\colon T^1M \to \mathbb{R}$ measures the complexity of the dynamics weighted by the potential φ, whereas this complexity, topologically, comes mainly from the hyperbolic part $\mathcal{R}$ of the manifold. Therefore, the different notions of pressures will behave reasonably, as soon as φ is not too large on $\mathcal{H}$, in comparison with the quantity ε_0 defined above. This is condition (4) for φ below, where $\mathcal{M}(\mathcal{H})$ (resp. $\mathcal{M}(\mathcal{R})$) denotes the set of invariant probability measures giving full measure to $\mathcal{H}$ (resp. $\mathcal{R}$).

Theorem 1.1. *Let M be a smooth compact rank one manifold and let $\varphi\colon T^1M \to \mathbb{R}$ be a continuous map. Then the following inequalities hold*

$$P_{Gur,\mathcal{R}}(\varphi) \le P_{Gur}(\varphi) \le \delta_{\Gamma,\varphi} \le P_{top}(\varphi) = P_{\mathcal{M}}(\varphi) . \tag{3}$$

Moreover, if the potential φ satisfies

$$\max_{\mu\in\mathcal{M}(\mathcal{H})} \int \varphi \, d\mu < \inf_{\mu\in\mathcal{M}(\mathcal{R})} \int \varphi \, d\mu + \varepsilon_0 , \tag{4}$$

then we have

$$P_{Gur,\mathcal{R}}(\varphi) = P_{Gur}(\varphi) \tag{5}$$

If M is a surface and φ is Hölder continuous and such that $\varphi|_{\mathcal{H}}$ is constant then all notions of pressure coincide:

$$P_{Gur,\mathcal{R}}(\varphi) = P_{Gur}(\varphi) = \delta_{\Gamma,\varphi} = P_{top}(\varphi) = \sup_{k\ge1} P_{top}(\varphi, \Lambda_k) = P_{\mathcal{M}}(\varphi) . \tag{6}$$

Moreover, even if (4) is probably not optimal, it is certainly relevant. Indeed, we provide in Section 3.5 an example where (4) is not satisfied, (5) holds, but (6) does not hold, because of a strict inequality $P_{Gur}(\varphi) < P_{\mathcal{M}}(\varphi)$.

Observe that (4) holds for any non-negative potential φ which vanishes on the higher rank set $\mathcal{H}$. In particular, the potential $\varphi^{(u)}$ as well as any constant potential satisfy (4). In the case of the zero potential, (5) is due to Knieper [21] (see Theorem 2.1). Equality (5) says that good potentials do not see singular periodic orbits, "good" meaning in particular those that are positive and smaller on $\mathcal{H}$ than the difference ε_0 of growth rates of regular and singular periodic orbits.

Our second result shows another way to avoid zero curvature, by considering only periodic orbits that are sufficiently hyperbolic from the point of view of their Lyapounov exponent. To obtain this, we apply *large deviation techniques*. Denote by $\chi(\beta)$ the smallest positive Lyapunov exponent of a periodic orbit β, by $\Gamma_\delta \subset \Gamma$ the subset of elements whose associated periodic orbit has positive Lyapunov exponents greater than δ, and by ν_β the invariant measure of mass $\ell(\beta)$ supported by the periodic orbit $\beta \in \Pi$.

Following a strategy in [32], we obtain the following result.

Theorem 1.2. *Let M be a compact rank one surface. If $\varphi\colon T^1M \to \mathbb{R}$ is Hölder continuous and such that $\varphi|_{\mathcal{H}}$ is constant and satisfies*

$$\alpha(\varphi) \overset{\text{def}}{=} P_{\mathcal{M}}(\varphi) - \max_{T^1M} \varphi > 0, \tag{7}$$

then for all $\delta \in (0, \alpha(\varphi))$, we have

1. $\displaystyle \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\beta\in\Pi(T-1,T),\, \chi(\beta)>\alpha(\varphi)-\delta} e^{\int \varphi\, d\nu_\beta} = P_{\mathcal{M}}(\varphi),$
2. *for all $x \in \widetilde{M}$, we have*
$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\gamma\in\Gamma_{\alpha-\delta},\, T-1\le d(x,\gamma x)<T} e^{\int_x^{\gamma x} \widetilde{\varphi}} = P_{\mathcal{M}}(\varphi).$$

In Section 3.5, we develop an example which contradicts assumptions and conclusions of this theorem.

After some preliminaries in Section 2, we prove Theorem 1.1 in Section 3. In Section 4 we deal with equilibrium states. Section 5 studies the conditions for the potentials used in our main theorems. Theorem 1.2 is shown in Section 6.

2. Geometric preliminaries

More details for the material in this section can be found in Ballmann [3], Knieper [22], and Eberlein [11].

2.1. **Periodic orbits of the geodesic flow.** Let M be a smooth compact nonpositively curved manifold. The rank of a vector $v \in T^1M$ is the dimension of the set of parallel Jacobi fields along the geodesic defined by v.

It is at least one. If this geodesic bounds a euclidean (flat) strip isometric to $[0,\varepsilon]\times\mathbb{R}$ for some $\varepsilon>0$, then this rank is at least 2. On the other hand, vectors whose geodesic eventually enters the negatively curved part of the manifold have rank one. The manifold M is a *rank one manifold* if it admits at least one rank one vector. The geodesic flow acts on the unit tangent bundle T^1M.

We denote by $\Gamma=\pi_1(M)$ the fundamental group of M and by $\widetilde{M}$ its universal cover, so that M identifies with $\widetilde{M}/\Gamma$. The set of conjugacy classes of elements $\gamma\in\Gamma$ is in one-to-one correspondence with the set of free homotopy classes of (oriented) loops in M. In such a homotopy class, there exists a closed geodesic on M which is length-minimizing. When such a geodesic is regular, it is unique. Otherwise, such minimizing geodesics are all parallel and with same length, and we choose one closed geodesic among them.

Each oriented closed geodesic (chosen as above in the singular case) lifts to T^1M in a unique way into a periodic orbit of the geodesic flow. Let Π denote the subset of primitive periodic orbits (that is, not an iterate of another periodic orbit). For all $\beta\in\Pi$, we denote by $\ell(\beta)$ its period (that is the length of the associated closed geodesic in M) and ν_β the Lebesgue measure along the orbit β. Denote by $\Pi(T)\subset\Pi$ the subset of orbits of period smaller than T and by $\Pi(T_1,T_2)$ the subset of orbits whose period belongs to the interval (T_1,T_2). In a similar way, denote by $\Pi_{\mathcal{R}},\Pi_{\mathcal{R}}(T),\Pi_{\mathcal{R}}(T,T')$ (resp. $\Pi_{\mathcal{H}},\Pi_{\mathcal{H}}(T),\Pi_{\mathcal{H}}(T,T')$) the corresponding subsets of regular (resp. singular) periodic orbits.

In the case of zero potentials, Knieper [21, (Theorem 1.1, Corollary 1.2, and Proposition 6.3)] proved the following result.

Theorem 2.1 (Knieper [21])**.** *Let M be a smooth compact rank one manifold. There exists $\varepsilon>0$, such that for T large enough,*

$$\#\Pi_{\mathcal{R}}(T)>e^{\varepsilon T}\#\Pi_{\mathcal{H}}(T)\geq 0\,. \tag{8}$$

Moreover,

$$\lim_{T\to\infty}\frac{1}{T}\log\#\Pi_{\mathcal{R}}(T)=h_{top}(T^1M)\,. \tag{9}$$

When M is a surface, then $\limsup_{T\to+\infty}\frac{1}{T}\log\#\Pi_{\mathcal{H}}(T)=0$.

The last statement in Theorem 2.1 is not true in higher dimensions as Gromov [15] provides an example of a compact rank one manifold in which the number of closed singular geodesics grows exponentially. Theorem 2.1 implies $\limsup_{T\to+\infty}\frac{1}{T}\log\#\Pi(T)>0$, whence

$$\limsup_{T\to+\infty}\frac{1}{T}\log\#\Pi(T)=\limsup_{T\to+\infty}\frac{1}{T}\log\#\Pi(T-1,T)\,. \tag{10}$$

Corresponding equalities are true for $\Pi_{\mathcal{R}}$ as well as $\Pi_{\mathcal{H}}$ instead of Π.

In the proof of Theorem 1.2, we will need the following classical observations in nonpositive curvature. Any two unit speed geodesics $\beta_1,\beta_2\colon[0,T]\to$

$\widetilde{M}$ satisfy for $0 \le t \le T$

$$d(\beta_1(t), \beta_2(t)) \le d(\beta_1(0), \beta_2(0)) + d(\beta_1(T), \beta_2(T)) .$$

Moreover, for every $\eta > 0$ there exist $\rho > 0$ and $T_0 > 0$ such that for any two unit speed geodesics $\beta_1, \beta_2 \colon [0,T] \to \widetilde{M}$ with $T > T_0$ satisfying $d(\dot\beta_1(0), \dot\beta_2(0))$, $d(\dot\beta_1(T), \dot\beta_2(T)) \le \rho$ we have $d(\dot\beta_1(t), \dot\beta_2(t)) \le \eta$ for all $t \in [0,T]$, where we denote by d the Sasaki distance on T^1M (see next section). Further, for every $x \in M$ there is exactly one geodesic arc $\beta_\gamma \colon [0,T] \to M$, for some $T > 0$, with $\beta_\gamma(0) = x = \beta_\gamma(T)$ in each (non-trivial) homotopy class $\gamma \in \Gamma$.

2.2. Stable and unstable bundles, Lyapunov exponents and the potential $\varphi^{(u)}$.

We refer particularly to [3, Chapter IV] for this subsection. Given a vector $v \in T_pM$, we identify T_vTM with $T_pM \oplus T_pM$ via the isomorphism

$$\Psi \colon \xi \mapsto (d\pi(\xi), C(\xi)),$$

where $\pi \colon TM \to M$ denotes the canonical projection and $C \colon TTM \to TM$ denotes the connection map defined by the Levi Civita connection. Under this isomorphism we have $T_vT^1M \simeq T_pM \oplus v^\perp$, where $v^\perp$ is the subspace of T_pM orthogonal to v. The vector field that generates the geodesic flow is $V \colon v \mapsto (v, 0)$. The Riemannian metric on M lifts to the *Sasaki metric* on TM defined by

$$\langle\langle\, \xi, \eta \,\rangle\rangle_v = \langle d\pi_v(\xi), d\pi_v(\eta)\rangle_{\pi(v)} + \langle C_v(\xi), C_v(\eta)\rangle_{\pi(v)}.$$

Roughly speaking, two vectors in T^1M are close with respect to the distance induced by the Sasaki metric if their orbits under the geodesic flow stay close during a fixed interval of time.

A *Jacobi field* J (see for example [14, III.C]) along a geodesic γ is a vector field along γ which satisfies the Jacobi equation

$$J''(t) + R(J(t), \dot\gamma(t))\dot\gamma(t) = 0, \tag{11}$$

where R denotes the Riemannian curvature tensor of M and $'$ denotes covariant differentiation along γ.

A Jacobi field J along a geodesic γ with $\dot\gamma(0) = v$ is uniquely determined by its initial conditions $(J(0), J'(0)) \in T_pM \times T_pM$, for $p = \pi(v)$. Moreover, the set of Jacobi fields along a geodesic γ such that $J(0)$ and $J'(0)$ are orthogonal to $v = \dot\gamma(0)$ is exactly the set of Jacobi fields such that for all t, $J(t)$ is normal to $\dot\gamma(t)$ [14, Theorem 3.43]. They are called the *orthogonal Jacobi fields*.

Given a vector $v \in TM$, a Jacobi field along the geodesic determined by v is uniquely determined by $(J(0), J'(0)) \in T_\pi(v)M \oplus T_\pi(v)M \simeq T_vTM$. An orthogonal Jacobi field along the geodesic determined by v is uniquely determined by $(J(0), J'(0)) \in v^\perp \oplus v^\perp \subset T_\pi(v)M \oplus v^\perp \simeq T_vT^1M$. Therefore, the set of orthogonal Jacobi fields can be identified with the subbundle of TT^1M whose fiber over v is $v^\perp \oplus v^\perp \in T_pM \oplus T_pM \simeq T_vT^1M$. This fiber is the orthogonal complement in $T_vT^1M \simeq T_pM \oplus v^\perp$ of the subspace spanned by the vector field $(v,0)$ that generates the geodesic flow.

Jacobi fields give a geometric description of the derivative of the geodesic flow. Given $\xi \in v^\perp \oplus v^\perp \subset T_vT^1M$, denote by J_ξ the unique Jacobi field along γ_v with initial conditions $J_\xi(0) = d\pi_v(\xi)$ and $J'_\xi(0) = C_v(\xi)$, or equivalently $(J_\xi(0), J'_\xi(0)) = \Psi(\xi)$. Then, $\Psi(dg^t_v(\xi))$ equals $(J_\xi(t), J'_\xi(t))$.

Orthogonal stable (unstable) Jacobi fields provide a convenient geometric way of describing the vector bundles that by Oseledec theorem correspond to non-positive (non-negative) Lyapunov exponents of the geodesic flow on the unit tangent bundle. As curvature is nonpositive, the function $t \mapsto \|J(t)\|$ is convex [3, IV, Lemma 2.3]. An (orthogonal) Jacobi field J along a geodesic is called *stable* (resp. *unstable*) if $\|J(t)\|$ is bounded for all $t \geq 0$ (resp. bounded for all $t \leq 0$). Let J^s (resp. J^u) denote the set of stable (resp. unstable) orthogonal Jacobi fields and introduce the subspaces

$$F^s_v \stackrel{\text{def}}{=} \{\xi \in T_vT^1M \colon J_\xi \in J^s\}, \quad F^u_v \stackrel{\text{def}}{=} \{\xi \in T_vT^1M \colon J_\xi \in J^u\}.$$

Each such subspace in $v^\perp \times v^\perp$ has dimension $n-1$. The distributions $F^s \colon v \in T^1M \mapsto F^s_v \subset T_vT^1M$ and $F^u \colon v \in T^1M \mapsto F^u_v \subset T_vT^1M$ obtained in this way are invariant and continuous (but rarely have higher regularity). The subbundle F^s_v (resp. F^u_v) coincides with the space of vectors $\xi \in v^\perp \times v^\perp \subset T_vT^1M$ such that $\|dg^t_v(\xi)\|$ is uniformly bounded for all $t \geq 0$ (resp. bounded for all $t \leq 0$).

A vector ξ belongs to $F^s_v \cap F^u_v$ if and only if $t \mapsto \|J_\xi(t)\|$ is constant (as a convex bounded map), or in other words iff the function $t \mapsto \|dg^t_v(\xi)\|$ is constant. One says in this case that J_ξ is a *parallel Jacobi field* along γ_v. When M is a surface, then $F^s_v \cap F^u_v$ is nontrivial if and only if $F^s_v = F^u_v$, that is if and only if the sectional curvature along γ_v is everywhere zero. In general, both subbundles will have nonzero intersection at some vectors $v \in T^1M$. In fact, the geodesic flow is Anosov precisely if and only if the intersection is zero at *every* vector [10].

Orthogonal Jacobi fields provide a *continuous* vector bundle that defines the following *continuous* potential which is of great importance for many thermodynamic properties of the flow. Let $\mathrm{Jac}(dg^t|_{F^u_v})$ be the Jacobian of the linear map $dg^t_v \colon F^u_v \to F^u_{g^t(v)}$ and consider the geometric potential defined by

$$\varphi^{(u)}(v) \stackrel{\text{def}}{=} -\frac{d}{dt}\,\mathrm{Jac}(dg^t|_{F^u_v})|_{t=0} = -\lim_{t\to 0}\frac{1}{t}\log \mathrm{Jac}(dg^t|_{F^u_v}), \tag{12}$$

which is well-defined and depends differentiably on F^u_v and hence continuously on v. Moreover, in restriction to each basic set, the map $v \to F^u_v$ is Hölder continuous, and hence the restriction of $\varphi^{(u)}$ to such a basic set is also Hölder continuous. Indeed, since in the uniformly hyperbolic case F^s_v (resp. F^u_v) coincides with the stable (unstable) subspace in the hyperbolic splitting of the tangent bundle, by [18, Theorem 19.1.6] these spaces vary

Hölder continuously in v. Further, $\varphi^{(u)}$ vanishes on $\mathcal{H}$ because the norm of any unstable Jacobi field is constant along geodesics in $\mathcal{H}$.

The Lyapunov exponents of the geodesic flow are well defined for all *Lyapunov regular* vectors v. The set of Lyapunov regular vectors is of full measure with respect to any invariant probability measure (see for example the appendix of [18]). For such a *Lyapunov regular* vector v, classical computations give

$$\sum_{i\colon \lambda_i(v)\geq 0} \lambda_i(v) = \lim_{T\to\infty} \frac{1}{T} \log \operatorname{Jac}(dg^T|_{F^u_v}) = \lim_{T\to\infty} -\frac{1}{T}\int_0^T \varphi^{(u)}(g^t(v))\,dt.$$

Ruelle's inequality [34] asserts that for all invariant probability measures μ, we have

$$h(\mu) \leq \int_{T^1M} \sum_{i\colon \lambda_i(v)\geq 0} \lambda_i(v)\,d\mu\,,$$

where $h(\mu)$ is the entropy of the measure μ with respect to the time one of the geodesic flow. With the above and (1), it ensures that for all invariant probability measures $\mu \in \mathcal{M}$ we have

$$P_{\mathcal{M}}(\varphi^{(u)}) = \sup_{\mu\in\mathcal{M}} \left(h(\mu) - \int_{T^1M} \sum_{i:\lambda_i(v)\geq 0} \lambda_i(v)\,d\mu \right) \leq 0.$$

Let $\widetilde{m}$ be the restriction of the Liouville measure to the invariant open set $\mathcal{R}$ normalized into a probability measure. In all known examples $\widetilde{m}$ coincides with the Liouville measure, but this has not been proved in general. Ergodicity of $\widetilde{m}$ was proved in [28]. By Pesin's formula [29] $h(\widetilde{m}) = -\int_{T^1M}\varphi^{(u)}\,d\widetilde{m}$, so that

$$P_{\mathcal{M}}(\varphi^{(u)}) = 0, \tag{13}$$

and the Liouville measure $\widetilde{m}$ is an equilibrium state for $\varphi^{(u)}$. Observe however that when the manifold has flat strips, the potential $\varphi^{(u)}$ vanishes on the flat strips. In particular, if there are periodic orbits in the flat strips, the normalized Lebesgue measure on any such higher rank periodic orbit is also an equilibrium state for $\varphi^{(u)}$.

Finally, in the particular case when M is a surface, there exists at most one positive Lyapunov exponent $\chi(v) = -\varphi^{(u)}(v)$.

3. Pressure and equilibrium states

We introduce first several notions of pressure as well as the Poincaré series; in Section 3.6 we prove Theorem 1.1. In this section, let $\varphi\colon T^1M \to \mathbb{R}$ be a continuous map.

3.1. **Variational and topological pressure.** The *variational pressure* of φ (with respect to the flow) was defined in (1) as

$$P_{\mathcal{M}}(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h(\mu) + \int_{T^1 M} \varphi \, d\mu \right).$$

For $\varepsilon > 0$ and $T > 0$, a set $E \subset T^1 M$ is (ε, T)-*separated* if for all $v, w \in E$, $v \neq w$, we have $\max_{0 \leq t \leq T} d(g^t v, g^t w) \geq \varepsilon$. Define

$$Z_{\varphi}^{sep}(T, \varepsilon) = \sup_{E} \sum_{v \in E} e^{\int_0^T \varphi(g^t v) \, dt},$$

where the supremum is taken over all (ε, T)-separated sets E. The *topological pressure* (which, in fact, should rather be called the *metric pressure*) of φ with respect to the geodesic flow is defined to be the following limit

$$P_{top}(\varphi) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log Z_{\varphi}^{sep}(T, \varepsilon).$$

Bowen and Ruelle [6] observed that this definition is equivalent to defining $P_{top}(\varphi)$ as the topological pressure of the function $\varphi^1 \colon v \mapsto \int_0^1 \varphi(g^t(v)) \, dt$ with respect to the time-1 map g^1 of the flow (see [35]). The variational principle [35, Theorem 9.10] ensures that

$$P_{top}(\varphi) = P_{\mathcal{M}}(\varphi).$$

The *topological entropy* of the geodesic flow, denoted for simplicity $h = h_{top}(T^1 M)$, is the topological pressure of the potential $\varphi = 0$. The topological pressure (resp. entropy) of the geodesic flow restricted to any compact set $\Lambda \subset T^1 M$ is denoted by $P_{top}(\varphi, \Lambda)$ (resp. $h_{top}(\Lambda)$).

3.2. **Gurevic Pressure.** The *Gurevic pressure*, or *periodic orbit pressure*, is defined as follows:

$$P_{Gur}(\varphi) \stackrel{\text{def}}{=} \limsup_{T \to +\infty} \frac{1}{T} \log \sum_{\beta \in \Pi(T-1, T)} e^{\int \varphi \, d\nu_\beta}. \tag{14}$$

Recall that $\Pi(T-1, T)$ is the set of *primitive periodic orbits* of length between $T-1$ and T. In a similar way, we define the *regular Gurevic pressure* as

$$P_{Gur,\mathcal{R}}(\varphi) \stackrel{\text{def}}{=} \limsup_{T \to +\infty} \frac{1}{T} \log \sum_{\beta \in \Pi_{\mathcal{R}}(T-1, T)} e^{\int \varphi \, d\nu_\beta}. \tag{15}$$

It is clear that $P_{Gur,\mathcal{R}}(\varphi) \leq P_{Gur}(\varphi)$. Knieper [21] proved the following fact.

Lemma 1. *For all $\varepsilon > 0$ smaller than the injectivity radius of M the set $\Pi(T)$ is (ε, T)-separated.*

For any continuous potential φ, we deduce that

$$P_{Gur}(\varphi) \leq P_{top}(\varphi).$$

When φ is the constant function $\varphi = 0$, Theorem 2.1 gives

$$P_{Gur,\mathcal{R}}(0) = P_{Gur}(0) = P_{top}(0) = h. \tag{16}$$

3.3. **Pressure on basic sets.** A flow-invariant set Λ is *hyperbolic* if the tangent bundle restricted to Λ can be written as the Whitney sum of dg^t-invariant subbundles $T_\Lambda T^1M = E^s \oplus E \oplus E^u$ where E is the one-dimensional bundle tangent to the flow, and there are positive constants c, α such that $\|dg^t_v(\xi)\| \le ce^{-\alpha t}\|\xi\|$ for $\xi \in E^s_v$, $t \ge 0$ and $\|dg^{-t}_v(\xi)\| \le ce^{-\alpha t}\|\xi\|$ for $\xi \in E^u_v$, $t \ge 0$. Such a set $\Lambda \subset T^1M$ is *locally maximal* if there exists a neighborhood U of Λ such that $\Lambda = \bigcap_{t\in\mathbb{R}} g^t(\overline{U})$. The flow $G|_\Lambda$ is *topologically transitive* if for all nonempty open sets U and V intersecting Λ there exists $t \in \mathbb{R}$ such that $g^t(U) \cap V \cap \Lambda \neq \emptyset$. A *basic set* is a compact locally maximal hyperbolic set on which the flow is transitive.

In restriction to any basic set Λ, in restriction to which the geodesic flow is topologically mixing, for $\varphi\colon \Lambda \to \mathbb{R}$ Hölder continuous, all above introduced pressures of the flow coincide. The proof of this classical fact (see [18, 18.5.1 and 20.3.3] for diffeomorphisms and [13, Lemma 2.8] for flows) uses the *specification property*. This property holds for geodesic flows of compact negatively curved manifolds [9]. On rank one manifolds, in restriction to any basic set Λ, by [5, (3.2)] exactly one of the following two distinct cases is true: (a) $g_{|\Lambda}$ is a time τ-suspension of an axiom A$*$ homeomorphism or, (b) Λ is C-dense (the unstable manifold of every periodic point in Λ is dense in Λ). In case (a) the suspended homeomorphism is topologically mixing and verifies the corresponding pressure identities and the pressure of the suspension flow with *constant* time ceiling function does not alter these identities for the flow. In case (b), by [5, (3.8)] the flow satisfies the specification property. We will then denote this quantity shortly by

$$P(\varphi,\Lambda) \overset{\text{def}}{=} P_{Gur}(\varphi,\Lambda) = P_{top}(\varphi,\Lambda) = P_{\mathcal{M}}(\varphi,\Lambda) \tag{17}$$

Observe further that $P(\varphi,\Lambda_1) \le P(\varphi,\Lambda_2)$ if $\Lambda_1 \subset \Lambda_2$.

When M is a rank one *surface*, by [7, Theorem 1.5] there is a family of basic sets $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathcal{R}$ such that $\bigcup_k \Lambda_k$ is dense in T^1M and that for any basic set $\Lambda \subset \mathcal{R}$, $\Lambda \neq \mathcal{R}$ there exists $k \ge 1$ such that $\Lambda \subset \Lambda_k$. Moreover, by construction, any regular periodic orbit is eventually contained in Λ_k for k large enough. Observe however that the singular periodic orbits are not contained in these Λ_k, so that $P_{Gur}(\varphi,\Lambda_k) = P_{Gur,\mathcal{R}}(\varphi,\Lambda_k)$. Finally, by [7, Theorem 6.2], if $\mathcal{H} \neq \emptyset$ and $\varphi|_{\mathcal{H}}$ is constant then we have

$$P_{top}(\varphi) = \sup_{k\ge1} P(\varphi,\Lambda_k) = \lim_{k\to\infty} P(\varphi,\Lambda_k) \tag{18}$$

3.4. **Critical Exponent of Poincaré series.** For a given $x \in \widetilde{M}$, and $\gamma \in \Gamma$, denote by $\int_x^{\gamma x} \widetilde{\varphi}$ the integral of the Γ-invariant lift $\widetilde{\varphi}$ of φ to $T^1\widetilde{M}$ along the unique lift to $T^1\widetilde{M}$ of the geodesic segment joining x to γx. The *Poincaré series* associated to Γ, φ, x, and $s \in \mathbb{R}$ is defined by

$$Z_{\Gamma,\varphi,x,s} \overset{\text{def}}{=} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x} \widetilde{\varphi} - sd(x,\gamma x)} \,.$$

The *critical exponent* $\delta_{\Gamma,\varphi,x}$ is defined by the fact that the series diverges when $s < \delta_{\Gamma,\varphi,x}$ and converges when $s > \delta_{\Gamma,\varphi,x}$. Define the sequence $(a_n(x))_{n\in\mathbb{N}}$ as

$$a_n(x) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma,\, n-1\leq d(x,\gamma x)<n} e^{\int_x^{\gamma x}\widetilde{\varphi}}\,.$$

An elementary computation shows that the series $Z_{\Gamma,\varphi,x,s}$ converges (diverges) if, and only if, the series $\sum_{n\in\mathbb{N}} a_n(x)e^{-sn}$ converges (diverges), so that

$$\delta_{\Gamma,\varphi,x} = \limsup_{n\to+\infty}\frac{1}{n}\log\sum_{\gamma\in\Gamma,\, n-1\leq d(x,\gamma x)<n} e^{\int_x^{\gamma x}\widetilde{\varphi}}\,.$$

A simple computation shows that when $\delta_{\Gamma,\varphi,x} > 0$, it also satisfies

(19) $$\delta_{\Gamma,\varphi,x} = \limsup_{n\to+\infty}\frac{1}{n}\log\sum_{\gamma\in\Gamma, d(x,\gamma x)\leq n} e^{\int_x^{\gamma x}\widetilde{\varphi}}\,.$$

Remark 1. Contrarily to the negative curvature case, the critical exponent $\delta_{\Gamma,\varphi,x}$ does depend on the point x, and it does not seem possible to remove this dependance by an elementary reasoning. Observe however the following facts:

- For all $x \in \widetilde{M}$ we have $|\delta_{\Gamma,\varphi,x} - \delta_{\Gamma,0,x}| \leq \|\varphi\|_\infty$.
- $\delta_{\Gamma,0,x}$ is independent of x and coincides with the topological entropy of the geodesic flow, which is finite.
- The map $x \mapsto \delta_{\Gamma,\varphi,x}$ is Γ-invariant. When φ is continuous, the map $x \mapsto \delta_{\Gamma,\varphi,x}$ is moreover continuous on $\widetilde{M}$. It induces therefore a continuous, and therefore uniformly continuous, map on the quotient manifold M.

The last point is elementary to check, thanks to the uniform continuity of the lift $\widetilde{\varphi}$ of φ to $T^1\widetilde{M}$, and to the following geometric fact: in nonpositive curvature, given $\eta > 0$ and x, y satisfying $d(y,x) < \eta$, the geodesic segments $[x,\gamma x]$ and $[y,\gamma y]$ stay at distance less than η.

By the above remark, the following quantity is well defined and finite

$$\delta_{\Gamma,\varphi} \stackrel{\text{def}}{=} \max_{x\in\widetilde{M}} \delta_{\Gamma,\varphi,x}\,.$$

3.5. **Examples of compact rank one manifolds.** We want to mention some examples and discuss the hypotheses in our main results.

Example 1. A compact connected nonpositively curved manifold on which every geodesic eventually crosses the negatively curved part of the manifold is a compact rank one manifold where $\mathcal{H} = \emptyset$. By a result of Eberlein [10], its geodesic flow is Anosov (and our results are already well-known in this case). The typical example is a manifold where the curvature is negative everywhere except in a sufficiently small disk where it is equal to zero (Figure 1).

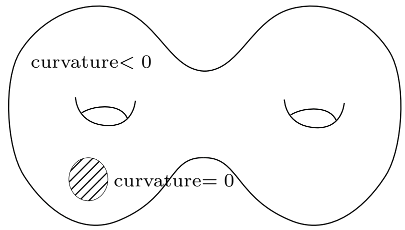

Figure 1. Nonpositively curved surface with Anosov geodesic flow

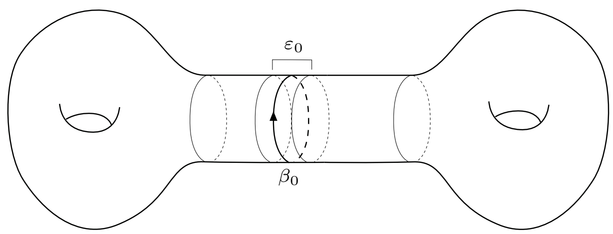

Figure 2. Surface with periodic euclidean cylinder

Example 2. Second, the simplest example of a compact rank one surface is a compact connected surface S of genus $g \geq 2$ which has a periodic euclidean cylinder of positive width somewhere and has negative curvature elsewhere. The set $\mathcal{H}$ of higher rank vectors is the set of vectors whose geodesic stays all the time in the flat cylinder. In the 'degenerate' case where the flat cylinder has width equal to zero, the set $\mathcal{H}$ is reduced to a single periodic orbit (Figure 2).

Example 3. Finally, let us mention examples introduced by Gromov [15] and further studied by Knieper [21] where $\#\Pi_{\mathcal{H}}(T)$ can have exponential growth in T. First, consider a hyperbolic punctured torus, and modify the neighbourhood of the puncture so that it becomes isometric to a flat cylinder $I \times C_1$, where I is an interval and C_1 a circle. Call this punctured and flattened torus T_1. Consider the three dimensional manifold $M_1 = T_1 \times C_2$, where C_2 is another circle. The boundary is the product (flat) torus $C_1 \times C_2$ of two circles. Consider another such manifold $M_2 = C_1' \times T_2$, where T_2 is also a flattened punctured torus, C_1' is isometric to C_1 and $C_2' = \partial T_2$ to C_2. Now glue M_1 and M_2 along their boundaries by identifying $C_1 \times C_2$ with $C_1' \times C_2'$. The resulting manifold (compare Figure 3) is a compact connected three-dimensional rank one manifold, whose singular periodic orbits have exponential growth [21]. Indeed, one observes that the product of a periodic orbit in T_1 and a point in C_2 is a periodic orbit of M_1 of rank two. Therefore, there is an injective map from the set of periodic orbits of T_1 into the set of singular periodic orbits of M_1. Moreover, the length of a periodic orbit in T_1 is at most the length of the periodic geodesic in the same homotopy class in the hyperbolic punctured torus. The exponential growth rate of periodic

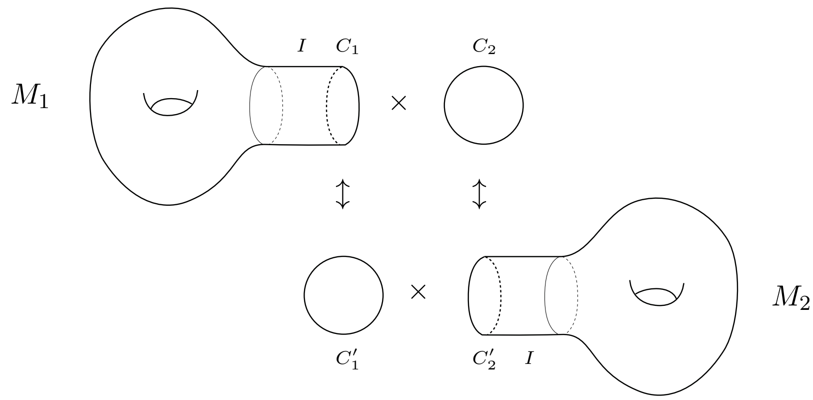

FIGURE 3. Gromov's example

orbits on the hyperbolic punctured torus implies therefore the exponential growth rate of singular periodic orbits of M_1.

Focusing on the Example 2, we would like to discuss the hypotheses of our main results. Let us develop a little bit more the case where the euclidean cylinder is of positive width, say of width L. Denote by h the topological entropy of the flow. Choose a (higher rank) vector v_0 tangent to a periodic geodesic inside the cylinder. We will define a potential $\varphi\colon T^1S \to \mathbb{R}$ that will not satisfy hypotheses and conclusions of Theorem 1.1.

Choose φ to be constant along the orbit β_0 of v_0 and satisfying $\varphi(v_0) = (1+\eta)\,h > h$ for some $\eta > 0$. Choose some $\varepsilon > 0$ such that $2\varepsilon/L < \eta/(1+\eta)$. Let $V_\varepsilon(v_0)$ be the set of unit vectors in T^1S such that the distance from their basepoint to the closed geodesic associated to v_0 is less than $\varepsilon/2$, and whose angle with the orbit of v_0 is less than $\varepsilon/2$. By an elementary euclidean argument already used in [8], if v is any unit tangent vector whose basepoint is outside the cylinder, the proportion of time spent by the piece of orbit $(g^t(v))_{0\le t\le T}$ inside $V_\varepsilon(v_0)$ is at most ε/L. Now extend the potential φ continuously to T^1S in such a way that $\varphi \ge 0$ and $\varphi \equiv 0$ outside $V_\varepsilon(v_0)$.

Observe that this potential φ does not satisfy assumption (4) of Theorem 1.1. If β is a rank one periodic orbit then the above remark shows that

$$\int \varphi\, d\nu_\beta \le \frac{\varepsilon}{L}\|\varphi\|_\infty\, \ell(\beta) = \frac{\varepsilon}{L}\,(1+\eta)h\,\ell(\beta)\,.$$

We deduce easily that

$$P_{Gur,\mathcal{R}}(\varphi) \le \frac{\varepsilon}{L}\|\varphi\|_\infty + h < \frac{\eta}{2(1+\eta)}\|\varphi\|_\infty + h = h(\frac{\eta}{2}+1)\,.$$

On the other hand, the topological pressure of φ certainly satisfies

$$P_{top}(\varphi) = P_{\mathcal{M}}(\varphi) \ge \int \varphi\, d\widehat{\nu}_{\beta_0} = h(1+\eta) > P_{Gur,\mathcal{R}}(\varphi)\,,$$

where $\widehat{\nu}_{\beta_0}$ is the normalized periodic measure supported by β_0. In particular, it proves that equalities (6) in Theorem 1.1 do not hold.

Now, let μ be a flow-invariant measure with support in the set $\mathcal{R}$ of regular vectors, and consider a generic recurrent vector v based outside the flat cylinder. We can find T large enough such that the ergodic average on $(g^t(v))_{0\le t\le T}$ is very close to μ, so that in particular,

$$\left|\int \varphi\, d\mu - \frac{1}{T}\int_0^T \varphi(g^t(v))\, dt\right| \le \varepsilon\,.$$

But the same argument as above shows that this piece of orbit $(g^t(v))_{0\le t\le T}$ cannot stay more than a proportion ε/L inside $V_\varepsilon(v_0)$. Therefore, $\int \varphi\, d\mu < \varepsilon + \|\varphi\|_\infty \varepsilon/L$, so that $h(\mu) + \int \varphi\, d\mu \le \varepsilon + h(1+\eta/2)$. If ε is small enough, this quantity is bounded from above by $h(1+\eta)$, so that

$$P_{top}(\varphi) > \sup_{\mu\in\mathcal{M}(\mathcal{R})}\left(h(\mu) + \int \varphi\, d\mu\right)\,.$$

In particular, it implies that $P_{top}(\varphi) = \|\varphi\|_\infty = h(1+\eta)$ and the periodic orbit measure $\widehat{\nu}_{\beta_0}$ is an equilibrium state for φ. This equilibrium state is unique when φ is chosen to be strictly decreasing in the neighborhood of v_0. (In the other case, it could happen that other periodic orbits in $V_\varepsilon(v_0)$ are also equilibrium states.)

As our definition of Gurevic pressures involves only primitive periodic orbits, and there is a unique primitive periodic orbit in $\Pi_{\mathcal{H}}(T)$, we see that $P_{Gur}(\varphi) = P_{Gur,\mathcal{R}}(\varphi)$ even though (4) is not satisfied. If our definitions of Gurevic pressures were modified to take into account all periodic orbits (not only primitive ones) then the above example would lead to a regular Gurevic pressure strictly less than the full Gurevic pressure. Moreover, the latter would be attained on singular periodic orbits and would be equal to the topological pressure.

To conclude on this example, observe that φ does not satisfy neither the assumption (7) of Theorem 1.2, nor its conclusions.

It would be interesting to adapt Example 2 with the cylinder to Gromov's Example 3 to provide an example for the strict inequalities $P_{Gur,\mathcal{R}}(\varphi) < P_{Gur}(\varphi) < P_{top}(\varphi)$.

3.6. **Proof of Theorem 1.1.** As each of the above quantities $\mathcal{P} = P_{\mathcal{M}}$, P_{top}, P_{Gur}, and $P_{Gur,\mathcal{R}}$ satisfy $\mathcal{P}(\varphi + c) = \mathcal{P}(\varphi) + c$ for every $c \in \mathbb{R}$, without loss of generality, in the following we can assume that all such pressures are positive.

By [32, Lemma 4], assumption $P_{top}(\varphi) > 0$, and (19), for all $x \in \widetilde{M}$ we have

$$\delta_{\Gamma,\varphi,x} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma, d(x,\gamma x)\le n} e^{\int_x^{\gamma x} \widetilde{\varphi}} \le P_{top}(\varphi)$$

In [32, Proof of Proposition 2, page 161], Pollicott proves (without stating explicitely) that $P_{Gur}(\varphi) \le \sup_{x\in\widetilde{M}} \delta_{\Gamma,\varphi,x} = \delta_{\Gamma,\varphi}$. His proof is written for φ Lipschitz, but his argument is valid for uniformly continuous potentials. As T^1M is compact, any continuous potential is uniformly continuous, so that its arguments apply. This together with the above shows the first claim (3) in Theorem 1.1.

Consider ε_0 as defined in (2). Let us prove that under the hypothesis (4) we have $P_{Gur,\mathcal{R}}(\varphi) \geq P_{Gur}(\varphi)$. Up to replacing φ by $\varphi - \inf_{\mu\in\mathcal{M}(\mathcal{R})}\int\varphi\,d\mu$, we can assume that $\inf_{\mu\in\mathcal{M}(\mathcal{R})}\int\varphi\,d\mu = 0$. The hypothesis then becomes $\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu < \varepsilon_0$. As $\int\varphi\,d\mu \geq 0$ for all $\mu\in\mathcal{M}(\mathcal{R})$, using (16) we get

$$P_{Gur}(\varphi) \geq P_{Gur,\mathcal{R}}(\varphi) \geq P_{Gur,\mathcal{R}}(0) = h > 0\,.$$

In particular, all pressures are positive, as required at the beginning of the proof. The fact that $P_{Gur,\mathcal{R}}(\varphi) > 0$ then implies that

(20)
$$P_{Gur,\mathcal{R}}(\varphi) = \limsup_{T\to+\infty}\frac{1}{T}\log\sum_{\beta\in\Pi_{\mathcal{R}}(T-1,T)} e^{\int\varphi\,d\nu_\beta} = \limsup_{T\to+\infty}\frac{1}{T}\log\sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta}\,.$$

As $P_{Gur}(\varphi) > 0$, the above equalities also hold for P_{Gur} with the sums taken over $\Pi(T-1,T)$ and $\Pi(T)$, respectively. In the case $\varphi = 0$, together with (9) it gives

$$\limsup_{T\to+\infty}\frac{1}{T}\log\#\Pi_{\mathcal{R}}(T-1,T) = \lim_{T\to+\infty}\frac{1}{T}\log\#\Pi_{\mathcal{R}}(T) = h > 0\,.$$

Observe that

$$\begin{aligned}\sum_{\beta\in\Pi(T)} e^{\int\varphi\,d\nu_\beta} &= \sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta} + \sum_{\beta\in\Pi_{\mathcal{H}}(T)} e^{\int\varphi\,d\nu_\beta}\\ &\leq \sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta} + e^{T\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu}\#\Pi_{\mathcal{H}}(T)\,.\end{aligned}$$

Choose $0 < \delta < \varepsilon_0 - \sup_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu$. By Theorem 2.1, for T sufficiently large we have $\#\Pi_{\mathcal{H}}(T) \leq e^{-T(\varepsilon_0-\delta)}\#\Pi_{\mathcal{R}}(T)$. As $\inf_{\mu\in\mathcal{M}(\mathcal{R})}\int\varphi\,d\mu = 0$, the above terms can be estimated further by

$$\begin{aligned}&\leq \sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta} + e^{T\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu}e^{-T(\varepsilon_0-\delta)}\#\Pi_{\mathcal{R}}(T)\\ &\leq \sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta} + e^{T\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu}e^{-T(\varepsilon_0-\delta)}\sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta}\\ &= \sum_{\beta\in\Pi_{\mathcal{R}}(T)} e^{\int\varphi\,d\nu_\beta}\left(1 + e^{T(\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu-\varepsilon_0+\delta)}\right)\,.\end{aligned}$$

As $\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu < \varepsilon_0 - \delta$, considering the limsup of $\frac{1}{T}\log$ of the above quantities leads to $P_{Gur}(\varphi) \leq P_{Gur,\mathcal{R}}(\varphi)$. This proves (5).

Consider now the case where M is a rank one surface, where we can apply results of [7]. Assume in addition that φ is Hölder and that $\varphi|_{\mathcal{H}}$ is constant. In restriction to any basic set Λ_k, (17) holds. Obviously, periodic orbits of the geodesic flow restricted to Λ_k are in $\Pi_{\mathcal{R}} \subset \Pi$. Thus, naturally we have $P(\varphi,\Lambda_k) \leq P_{Gur,\mathcal{R}}(\varphi) \leq P_{Gur}(\varphi) \leq \sup_{x\in\widetilde{M}}\delta_{\Gamma,\varphi,x} = \delta_{\Gamma,\varphi} \leq P_{top}(\varphi) =$

$P_{\mathcal{M}}(\varphi)$ by what precedes. Further, by (18) this lower bound converges to $P_{top}(\varphi)$. This proves (6) and finishes the proof of Theorem 1.1. □

4. Equilibrium states

Given a potential, it is interesting to identify (ergodic) equilibrium measures, as they reflect the dynamics weighted by the potential.

In our situation, the existence of equilibrium states follows immediately from the upper semi-continuity [35] of the entropy map $\mu \mapsto h(\mu)$ on the set of flow-invariant probability measures since the geodesic flow is smooth [23]. By Bowen [4, Theorem 3.5], this upper semi-continuity can be deduced from the fact that the geodesic flow is h-expansive [21, Proposition 3.3].

In general, the uniqueness of equilibrium states can be deduced from the existence of a *Gibbs measure* (see [18, Section 20.3]). On the other hand, when the pressure map $\varphi \mapsto P_{\mathcal{M}}(\varphi)$ is differentiable at φ in every direction or in a set of directions that is dense in the weak topology [33, Corollary 3.6.14] then there exists also a unique equilibrium state. But we are still not able to apply one of these two strategies to rank one geodesic flows.

Classical arguments now lead us to the following results. We will always assume that φ is Hölder continuous and $\varphi|_{\mathcal{H}}$ is constant.

Remark 2. Let M be a smooth compact rank one surface and consider an increasing family of basic sets $(\Lambda_k)_{k\in\mathbb{N}}$ as provided in [7]. With respect to the restriction of the geodesic flow to Λ_k, the potential φ admits a unique equilibrium state that we denote by $\mu_{\varphi,k}$. Then any accumulation point (with respect to the weak$*$ topology) of the sequence of measures $(\mu_{\varphi,k})_{k\in\mathbb{N}}$ is an equilibrium state for the potential φ (with respect to the flow on T^1M).

Remark 3. Given $\beta \in \Pi(T)$, denote by ν_β is the Lebesgue measure on the periodic orbit β and $\widehat{\nu}_\beta = \ell(\beta)^{-1}\nu_\beta$ the normalized (probability) measure. In [35, Theorem 9.10], Walters shows that the accumulation points of weighted averages of Dirac measures on (ε, T)-separated sets that approximate well the topological pressure are equilibrium measures for φ. By Lemma 1, for every sufficiently small $\varepsilon > 0$ and for all $T > 0$, the set $\Pi(T-1, T)$ of periodic orbits of length approximately T is (ε, T)-separated. Thus, by Theorem 1.1 when M is a smooth rank one *surface*, these sets $\Pi(T-1, T)$ allow to approximate the topological pressure. In this situation, any accumulation point of the following weighted averages on periodic orbits

$$\frac{\sum_{\beta\in\Pi(T-1,T)} e^{\int \varphi\, d\nu_\beta}\, \widehat{\nu}_\beta}{\sum_{\beta\in\Pi(T-1,T)} e^{\int \varphi\, d\nu_\beta}},$$

is an equilibrium measure of φ (with respect to the flow on T^1M).

5. Hyperbolic Potentials

In this section we discuss assumptions (4) and (7). For $t > 0$ define $\varphi^t\colon v \mapsto \int_0^t \varphi(g^\tau(v))\,d\tau$. We first show some preliminary result based on classical arguments (see for example [16]) that we repeat for completeness.

Lemma 2. $\displaystyle\max_{\mu\in\mathcal{M}(\mathcal{H})}\int \varphi\,d\mu = \lim_{t\to\infty}\max_{v\in\mathcal{H}}\frac{1}{t}\varphi^t(v) = \inf_{t>0}\max_{v\in\mathcal{H}}\frac{1}{t}\varphi^t(v).$

Proof. Given $\mu\in\mathcal{M}$, by flow invariance, for $t>0$ we have

$$\frac{1}{t}\int_{T^1M}\varphi^t\,d\mu = \frac{1}{t}\int_{T^1M}\int_0^t \varphi\circ g^s\,ds\,d\mu = \frac{1}{t}\int_0^t\int_{T^1M}\varphi\circ g^s\,d\mu\,ds = \int_{T^1M}\varphi\,d\mu\,.$$

Thus

$$\int_{T^1M}\varphi\,d\mu \le \max_{\mathcal{H}}\frac{1}{t}\varphi^t\,.$$

Taking the supremum over $\mu\in\mathcal{M}(\mathcal{H})$, and then the infimum over $t>0$ gives

$$\max_{\mu\in\mathcal{M}(\mathcal{H})}\int\varphi\,d\mu \le \inf_{t>0}\max_{\mathcal{H}}\frac{1}{t}\varphi^t\,.$$

It remains to show the opposite inequality. Given $n\ge 1$ choose v_n in the compact invariant set $\mathcal{H}$ such that the function $\frac{1}{n}\varphi^n$ attains its maximum in v_n. Consider the probability measure ν_n defined by

$$\int\psi\,d\nu_n \stackrel{\text{def}}{=} \frac{1}{n}\int_0^n \psi(g^s(v_n))\,ds \quad \text{for every } \psi\in C^0(T^1M,\mathbb{R})\,.$$

Choose a subsequence $(n_k)_{k\ge1}$ of positive integers such that

$$\lim_{k\to\infty}\frac{1}{n_k}\varphi^{n_k}(v_{n_k}) = \limsup_{n\to\infty}\frac{1}{n}\varphi^n(v_n) = \limsup_{n\to\infty}\max_{\mathcal{H}}\frac{1}{n}\varphi^n\,.$$

Possibly taking a subsequence, the sequence of measures $(\nu_{n_k})_{k\ge1}$ converges in the weak$*$ topology to an invariant probability measure μ supported in $\mathcal{H}$. We obtain

$$\int\varphi\,d\mu = \lim_{k\to\infty}\int\varphi\,d\nu_{n_k} = \lim_{k\to\infty}\frac{1}{n_k}\varphi^{n_k}(v_{n_k}) = \limsup_{n\to\infty}\max_{\mathcal{H}}\frac{1}{n}\varphi^n\,.$$

This together with the above proves the claim. □

The following provides an immediate stronger version of condition (4).

Corollary 1. *Any continuous potential $\varphi\colon T^1M\to\mathbb{R}$ satisfying*

$$\max_{\mathcal{H}}\frac{1}{t}\varphi^t < \inf_{\mathcal{R}}\frac{1}{t}\varphi^t + \varepsilon_0$$

for some $t>0$ also satisfies condition (4).

We now study condition (7). Given continuous potentials $\varphi\colon T^1M\to\mathbb{R}$, consider

$$\alpha(\varphi) \stackrel{\text{def}}{=} P(\varphi) - \max\varphi^1.$$

The potential φ is said to be *hyperbolic* if there exists $t > 0$ such that

$$t\,P(\varphi) - \max \varphi^t > 0\,.$$

Two continuous potentials $\varphi, \psi\colon T^1M \to \mathbb{R}$ are said to be *co-homologous* (with respect to the flow) if there exists a continuous function $\eta\colon T^1M \to \mathbb{R}$ such that $\varphi^t - \psi^t = \eta \circ g^t - \eta$ for every t. This is equivalent to the fact that $\varphi - \psi = \lim_{t\to 0}(\eta \circ g^t - \eta)/t$.

In the following, we require that $\alpha(\varphi) > 0$. This seems a very restrictive hypothesis. However, following [16, Proposition 3.1] verbatim, in the case of a flow, we get the following equivalences.

Lemma 3. *The following facts are equivalent:*

- $\alpha(\varphi) > 0$.
- *The potential φ is hyperbolic.*
- *The metric entropy of each equilibrium state of φ is strictly positive.*
- *There exists a continuous potential ψ co-homologous to φ such that $\alpha(\psi) > 0$.*
- *Every continuous potential co-homologous to φ is hyperbolic.*

As one of our main interests is the unstable Jacobian $\varphi^{(u)}$ defined in (12), we observe the following.

Lemma 4. *For all $t < 1$ we have $\alpha(t\varphi^{(u)}) > 0$.*

Proof. Recall first that $t \mapsto P(t\varphi^{(u)})$ is convex and hence continuous. For $t = 0$, we have $P(0) = h > 0$. For $t = 1$, Ruelle's inequality implies $P(\varphi^{(u)}) \le 0$, Pesin's formula implies $P(\varphi^{(u)}) = 0$, and the restricted Liouville measure $\widetilde{m}$ is an equilibrium state for $\varphi^{(u)}$, recall (13). In particular, we have $P(t\varphi^{(u)}) \ge t\int \varphi^{(u)}\,d\widetilde{m} > 0$ for all $t < 1$.

It is not hard to check that $\varphi^{(u)} \le 0$ (see, for example [7, Lemma 2.4]). Further, $\min \varphi^{(u)} = \min_{\mathcal{R}} \varphi^{(u)} < 0$. Thus, $\max(t\varphi^{(u)}) = -|t| \min \varphi^{(u)} > 0$ for $t < 0$, and $\max(t\varphi^{(u)}) = 0$ for $t \ge 0$. In particular, $t \mapsto P(t\varphi^{(u)})$ is non-increasing.

We deduce that $P(t\varphi^{(u)}) - \max(t\varphi^{(u)}) = P(t\varphi^{(u)}) > 0$ for $0 \le t < 1$ and $P(t\varphi^{(u)}) - \max(t\varphi^{(u)}) \ge P(0) + |t| \min \varphi^{(u)} > P(0) > 0$ for $t < 0$. □

When $\alpha(\varphi) > 0$, we will see in the next section that the contribution of periodic orbits with small positive Lyapunov exponent in the growth rate of the definition (14) of Gurevich pressure is negligible.

6. Level-2 Large Deviation Principle

In this section, we assume that M is a surface, and φ is a continuous potential such that $\varphi_{|\mathcal{H}}$ is constant and $\alpha(\varphi) = P_{\mathcal{M}}(\varphi) - \max_{T^1M} \varphi > 0$. Adding a constant to φ, we will assume that $\varphi_{|\mathcal{H}} = 0$. We denote by $P(\varphi)$ the topological pressure, which coincides with all other pressures by Theorem 1.1. The assumption $\alpha(\varphi) > 0$ with $\varphi_{|\mathcal{H}} = 0$ implies in particular $P(\varphi) > 0$.

In order to formalize our level-2 large deviation results [1], let us first introduce a rate function. Closely related approaches can be found, for example, in [31]. Let $\mathcal{P}$ be the space of all (not necessarily invariant) Borel probability measures on T^1M endowed with the topology of weak$*$ convergence, and $\mathcal{M}$ is the subspace of invariant measures under the geodesic flow. Given $\varphi \in C^0(T^1M,\mathbb{R})$, we define $Q_\varphi\colon C^0(T^1M,\mathbb{R}) \to \mathbb{R}$ by

$$Q_\varphi(\psi) \stackrel{\text{def}}{=} P(\varphi+\psi) - P(\varphi).$$

Note that Q_φ is a continuous and convex functional. Within the framework of the theory of conjugating functions (see for example [2]), Q_φ can be characterized by

$$Q_\varphi(\psi) = \sup_{\nu\in\mathcal{P}} \left(\int \psi\, d\nu - I_\varphi(\nu) \right),$$

where I_φ is the convex conjugate of Q_φ defined by

$$I_\varphi(\mu) \stackrel{\text{def}}{=} \sup_{\psi\in C^0(T^1M,\mathbb{R})} \left(\int \psi\, d\mu - Q_\varphi(\psi) \right) \tag{21}$$

for all $\mu \in \mathcal{P}$ and $I_\varphi(\mu) = \infty$ for any other signed measure μ. Since $I_\varphi\colon \mathcal{P} \to \mathbb{R}$ is a pointwise supremum of continuous and affine functions, it is a lower semi-continuous and convex functional. Given $\nu \in \mathcal{P}$, we call

$$\widehat{h}(\nu) \stackrel{\text{def}}{=} \inf_{\psi\in C^0(T^1M,\mathbb{R})} \left(P(\psi) - \int \psi\, d\nu \right) \tag{22}$$

the *generalized entropy* of f with respect to ν. It follows from the definition that $h(\nu) \le \widehat{h}(\nu)$ for every $\nu \in \mathcal{M}$. A (not necessarily invariant) measure $\mu \in \mathcal{P}$ is called a *generalized equilibrium state* for φ if $P(\varphi) = \widehat{h}(\mu) + \int \varphi\, d\mu$. This terminology is justified by the *dual variational principle* $h(\nu) = \widehat{h}(\nu)$ [35, Chapter 9.4]. Observe that

$$\begin{aligned} P(\varphi) - \widehat{h}(\mu) - \int \varphi\, d\mu &= P(\varphi) - \int \varphi\, d\mu + \sup_\psi \left(\int (\psi+\varphi)\, d\mu - P(\psi+\varphi) \right) \\ &= \sup_\psi \left(\int \psi\, d\mu - P(\psi+\varphi) + P(\varphi) \right) = I_\varphi(\mu) \end{aligned}$$

Thus, for all $\mu \in \mathcal{M}$, the equality $h(\mu) = \widehat{h}(\mu)$ implies

$$I_\varphi(\mu) = P(\varphi) - \left(h(\mu) + \int \varphi\, d\mu \right) \ge 0. \tag{23}$$

Moreover, for $\mu \in \mathcal{M}$ we have $I_\varphi(\mu) = 0$ if, and only if, μ is an equilibrium state for φ. Therefore, on can think of the functional I_φ as a "distance" from μ to the set of all generalized equilibrium states of φ.

Following closely [31, Section 2], we obtain the following result.

[1] Level-2 deviation refers to deviations of empirical measures to distinguish from so-called level-1 deviations of empirical (Birkhoff) sums or integrals of an observable, see [12].

Lemma 5. *Let M be a smooth compact rank one surface. Let $\varphi\colon T^1M\to\mathbb{R}$ be a continuous potential, and $\mathcal{K}\subset\mathcal{M}$ be a compact set. Then we have*

$$\limsup_{T\to\infty}\frac{1}{T}\log\sum_{\beta\in\Pi(T-1,T),\nu_\beta\in\mathcal{K}}e^{\int\varphi\,d\nu_\beta}\le P_{top}(\varphi)-\inf_{\nu\in\mathcal{K}}I_\varphi(\nu),$$

where ν_β denotes the invariant probability measure supported on the periodic orbit which projects to β.

Proof. Let $\rho\overset{\text{def}}{=}\inf_{\nu\in\mathcal{K}}I_\varphi(\nu)$. For all $\nu\in\mathcal{K}$, by (21) we have

$$\rho\le\sup_\psi\left(\int\psi\,d\nu-Q_\varphi(\psi)\right). \tag{24}$$

Hence, given $\nu\in\mathcal{K}$ and $\varepsilon>0$ there exists $\psi=\psi(\nu,\varepsilon)\in C^0(T^1M,\mathbb{R})$ such that $\rho-\varepsilon<\int\psi\,d\nu-Q_\varphi(\psi)$. Thus, we obtain that

$$\mathcal{K}\subset\bigcup_\psi\left\{\nu\in\mathcal{M}\colon\int\psi\,d\nu>Q_\varphi(\psi)+\rho-\varepsilon\right\}. \tag{25}$$

It is a covering of $\mathcal{K}$ by open sets. By compactness there exists a finite cover $\mathcal{U}_1,\ldots,\mathcal{U}_N$ of $\mathcal{K}$ determined by functions $\psi_1,\ldots,\psi_N$ through

$$\mathcal{U}_i\overset{\text{def}}{=}\left\{\nu\in\mathcal{M}\colon\int\psi_i\,d\nu-Q_\varphi(\psi_i)-\rho+\varepsilon>0\right\}. \tag{26}$$

We get

$$\begin{aligned}\sum_{\beta\in\Pi(T-1,T),\ \nu_\beta\in\mathcal{K}}e^{\int\varphi\,d\nu_\beta}&\le\sum_{i=1}^N\sum_{\beta\in\Pi(T-1,T),\ \nu_\beta\in\mathcal{U}_i}e^{\int\varphi\,d\nu_\beta}\\&<\sum_{i=1}^N\sum_{\beta\in\Pi(T-1,T),\ \nu_\beta\in\mathcal{U}_i}e^{\int\varphi\,d\nu_\beta}\cdot\exp\left[T\left(\int\psi_i\,d\nu_\beta-Q_\varphi(\psi_i)-\rho+\varepsilon\right)\right]\\&\le\sum_{i=1}^N\exp[-T(Q_\varphi(\psi_i)+\rho-\varepsilon)]\sum_{\beta\in\Pi(T-1,T)}e^{\int(\varphi+\psi_i)\,d\nu_\beta}.\end{aligned}$$

Now (3) in Theorem 1.1 applied to the pressure of $\varphi+\psi_i$ gives us

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log\sum_{\beta\in\Pi(T-1,T),\nu_\beta\in\mathcal{K}}e^{\int\varphi\,d\nu_\beta}\\&\le\max_{1\le i\le N}\{-Q_\varphi(\psi_i)-\rho+\varepsilon+P_{top}(\varphi+\psi_i)\}\\&=P_{top}(\varphi)-\rho+\varepsilon.\end{aligned} \tag{27}$$

Since $\varepsilon>0$ was arbitrary, this concludes the proof. □

We now are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first prove item 1. Consider the compact subset

$$\mathcal{K} \stackrel{\text{def}}{=} \{\mu \in \mathcal{M} \colon \chi(\mu) \le \alpha(\varphi) - \delta\}.$$

For $\nu \in \mathcal{K}$, using definitions of $\mathcal{K}$, I_φ and Ruelle's inequality, we get

$$I_\varphi(\nu) \ge P_{top}(\varphi) - \max \varphi - h(\nu) = \alpha(\varphi) - h(\nu) \ge \alpha(\varphi) - \chi(\nu) \ge \delta.$$

It follows that $\inf_{\nu \in \mathcal{K}} I_\varphi(\nu) \ge \delta$. Now Lemma 5 implies

$$\limsup_{T\to\infty} \frac{1}{T} \log \sum_{\beta \in \Pi(T-1,T), \chi(\beta) \le \alpha - \delta} e^{\int \varphi \, d\nu_\beta} \le P_{top}(\varphi) - \delta. \tag{28}$$

Theorem 1.1 (6) gives $P_{top}(\varphi) = P_{Gur}(\varphi)$. Hence there is a subsequence $T_k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{T_k} \log \sum_{\beta \in \Pi(T_k-1,T_k)} e^{\int \varphi \, d\nu_\beta} = P_{top}(\varphi). \tag{29}$$

So for any $\varepsilon > 0$ there exists $T_0 \ge 1$ so that for every $k \ge 1$ with $T_k \ge T_0$ we have

$$\sum_{\beta \in \Pi(T_k-1,T_k)} e^{\int \varphi \, d\nu_\beta} \ge e^{T_k(P_{top}(\varphi)-\varepsilon)}.$$

On the other hand, if T_0 is large enough, then by (28) we also have

$$\sum_{\beta \in \Pi(T_k-1,T_k), \chi(\beta) \le \alpha-\delta} e^{\int \varphi \, d\nu_\beta} \le e^{T_k(P_{top}(\varphi)-\delta+\varepsilon)}.$$

Combining the two above inequalities, we obtain

$$\sum_{\beta \in \Pi(T_k-1,T_k), \chi(\beta) > \alpha-\delta} e^{\int \varphi \, d\nu_\beta} \ge e^{T_k(P_{top}(\varphi)-\varepsilon)} \left(1 - e^{T_k(-\delta+2\varepsilon)}\right).$$

For $\varepsilon < \delta/2$, this implies

$$\limsup_{T_k\to\infty} \frac{1}{T_k} \log \sum_{\beta \in \Pi(T_k-1,T_k), \chi(\beta) > \alpha-\delta} e^{\int \varphi \, d\nu_\beta} \ge P_{top}(\varphi) - \varepsilon\,.$$

The left hand side is clearly smaller than the Gurevic pressure $P_{Gur}(\varphi)$, which, by Theorem 1.1, coincides with $P_{top}(\varphi) = P_{\mathcal{M}}(\varphi)$. As ε was arbitrary, together with (29) this completes the proof of item 1.

Now we prove item 2. We refer to section 2.1 for geometric observations. Fix some $\eta > 0$. By uniform continuity of φ and of its lift $\widetilde{\varphi}$ to $T^1\widetilde{M}$, we can choose $r \in (0, \min\{1/4, \eta, \rho\})$, such that for any two vectors v, w in $T^1\widetilde{M}$, $d(v,w) \le r$ implies $|\widetilde{\varphi}(v) - \widetilde{\varphi}(w)| \le \eta$. Let $\{B(x_i, r)\}_{i=1}^N$ be a finite cover of M. By item 1 above, we have

$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\beta \in \Pi(T-1,T), \chi(\beta) > \alpha(\varphi)-\delta} e^{\int \varphi \, d\nu_\beta} = P_{top}(\varphi) \tag{30}$$

Let $T \ge T_0 + 1 + 2r$, and $\beta \in \Pi(T-1, T)$ a periodic orbit of the geodesic flow with $\chi(\beta) > \alpha - \delta$. The closed geodesic associated to β intersects some $B(x_i, r)$. Let $\gamma \in \Gamma$ be an isometry whose axis projects to M onto this closed geodesic, and whose translation length is $\ell(\beta)$. One can lift x_i and the closed geodesic associated to β in such a way that this lift intersects $B(\widetilde{x}_i, r)$ and $B(\gamma\widetilde{x}_i, r)$, where $\widetilde{x}_i$ is the lift of x_i. Therefore, the geodesic from $\widetilde{x}_i$ to $\gamma\widetilde{x}_i$ projects on M to a loop $(\beta_\gamma(t))_{0\le t\le T_\gamma}$ with $\beta_\gamma(0)=\beta_\gamma(T_\gamma)=x_i$. Moreover, a simple triangular inequality gives $|T_\gamma-\ell(\beta)|\le 2r$. By construction of β_γ, by uniform continuity of φ, and elementary considerations in nonpositive curvature (see Section 2.1), as $\ell(\beta)\in(T-1,T)$, we get

$$\left|\int_\beta \varphi\, d\nu_\beta - \int_0^{T_\gamma} \varphi(\beta_\gamma'(t))\, dt\right| \le \eta\, \ell(\beta) + 2r\, \|\varphi\|_\infty \le \eta T + 2\eta\, \|\varphi\|_\infty$$

A different closed orbit β may lead to a different point $\widetilde{x}_i$ from the cover. Summing over i, using the fact that $\int_0^{T_\gamma}\varphi(\beta_\gamma'(t))dt=\int_{\widetilde{x}_i}^{\gamma\widetilde{x}_i}\widetilde{\varphi}$, we obtain

$$\sum_{\beta\in\Pi(T-1,T),\ \chi(\beta)>\alpha(\varphi)-\delta} e^{\int \varphi\, d\nu_\beta} \le e^{\eta T+2\eta\|\varphi\|_\infty} \sum_{i=1}^N \sum_{\gamma\in\Gamma_{\alpha(\varphi)-\delta},\ d(\widetilde{x}_i,\gamma\widetilde{x}_i)\le T} e^{\int_{\widetilde{x}_i}^{\gamma\widetilde{x}_i}\widetilde{\varphi}}\,.$$

Now taking $\limsup_{T\to\infty}\frac{1}{T}\log$, with (30) we obtain

$$P_{top}(\varphi) \le \max_{1\le i\le N} \limsup_{T\to\infty} \frac{1}{T}\log \sum_{\gamma\in\Gamma_{\alpha(\varphi)-\delta}, d(\widetilde{x}_i,\gamma\widetilde{x}_i)\le T} e^{\int_{\widetilde{x}_i}^{\gamma\widetilde{x}_i}\widetilde{\varphi}} + \eta\,.$$

Thanks to property (3) in Theorem 1.1, the latter term is bounded from above by $\max_{i=1,\dots,N}\delta_{\Gamma,\varphi,x_i}+\eta\le\delta_{\Gamma,\varphi}+\eta$. As η can be taken arbitrarily small, this finishes the proof. □

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