Prethermalization and thermalization in models with weak integrability breaking

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Beyond Integrability: the mathematics and physics of integrability and its breaking in low-dimensional strongly correlated phenomena
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F.H.L. Essler, S. Kehrein, S. R. Manmana and N. J. Robinson, Phys. Rev. B **89**, 165104 (2014).
B. Bertini, F.H.L. Essler, S. Groha and N. J. Robinson, arXiv:1506.03502 (2015) & *in preparation*.

Outline

Introduction

- What's understood?

Weakly non-integrable quenches

The prethermalization plateau

- Our definition of prethermalization
- Deformed GGE description of the prethermalization plateau

The approach to thermalization

- A quantum Boltzmann equation

Global quantum quenches

General problem: how does a generic initial state time-evolve?

General procedure:

1. Consider a short-ranged Hamiltonian $H(U)$ isolated from environment.
2. Prepare system in ground state $|\Psi_0\rangle$ of Hamiltonian $H(U_0)$
3. At time $t = 0$ change the Hamiltonian $H(U_0) \to H(U)$
4. Time-evolve the initial state $|\Psi(t)\rangle = \exp[-iH(U)t]|\Psi_0\rangle$

General goal:

1. Study time-evolution of observables:

$$\langle\Psi(t)|\mathcal{O}(x)|\Psi(t)\rangle, \quad \langle\Psi(t)|\mathcal{O}_1(x)\mathcal{O}_2(y)|\Psi(t)\rangle$$

Time-evolution of observables – what's understood?

1. Closed (isolated) quantum systems do not relax globally
   - Initial state $|\Psi_0\rangle$ is pure
   - Time evolution $|\Psi(t)\rangle = \sum_a e^{-iE_a t}|E_a\rangle\langle E_a|\psi_0\rangle$
   - Can construct observables that never relax $|E_a\rangle\langle E_b| + \mathrm{H.c.} \to e^{-i(E_a-E_b)t}|E_a\rangle\langle E_b| + \mathrm{H.c}$
2. It can relax locally. Picture: the rest of the system acts like a bath to a subsystem.

Time-evolution of observables – what's understood?

Two paradigms for the long-time behavior of observables:

Non-integrable Hamiltonian

- "Generic" system
- Behaves thermally

Deutsch '91, Srednicki '94

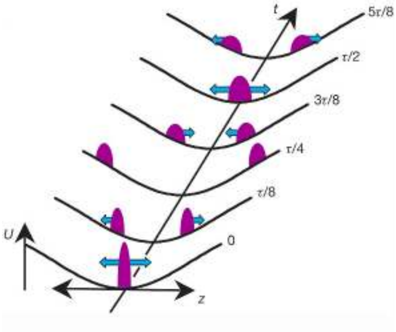
Integrable Hamiltonian

- More complicated
- Doesn't 'thermalize'

Rigol, Dunjko, Yurosvki & Olshanii '07

Illustration: the quantum Newton's cradle

Kinoshita, Wenger, Weiss '06

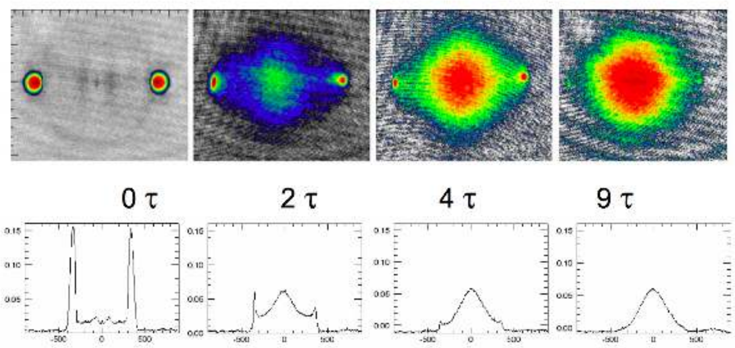


1. Separate two bunches of bosons in harmonic trap and release.
2. Essentially unitary time-evolution.
3. Approach steady state.
4. Big difference between 1D and 3D confinement.

Illustration: the quantum Newton's cradle

Non-integrable Hamiltonian

3D quantum Newton's cradle rapidly thermalizes ($\sim$ 3 collisions)

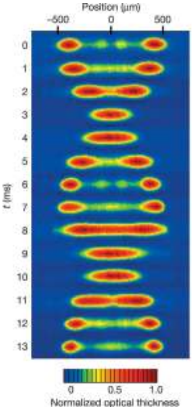


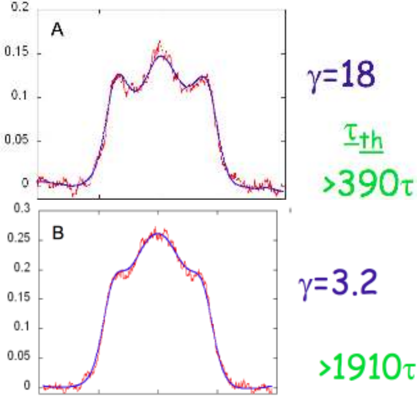
Kinoshita, Wenger, Weiss '06.

Illustration: the quantum Newton's cradle

Integrable Hamiltonian

1D quantum Newton's cradle slowly approaches non-thermal





Kinoshita, Wenger, Weiss '06.

Time-evolution of observables – what's understood?

Two paradigms for the long-time behavior of observables:

Non-integrable Hamiltonian

Thermalizes

$\langle \mathcal{O}(x) \rangle_{t\to\infty} = \mathrm{Tr}[\mathcal{O}(x)\rho_{\mathrm{th}}]$

$\rho_{\mathrm{th}} = \frac{1}{Z} e^{-\beta_{\mathrm{eff}} H}$

temperature $1/\beta_{\mathrm{eff}}$ fixed by

$\langle \Psi_0 | H | \Psi_0 \rangle = \mathrm{Tr}[H\rho_{\mathrm{th}}]$

Integrable Hamiltonian

Equilibrates

$\langle \mathcal{O}(x) \rangle_{t\to\infty} = \mathrm{Tr}[\mathcal{O}(x)\rho_{\mathrm{GGE}}]$

$\rho_{\mathrm{GGE}} = \frac{1}{Z} e^{-\sum_m \lambda_m I_m}$

where $[I_m, I_n] = 0$ are the conservation laws of H.

Lagrange multipliers fixed by

$\langle \Psi_0 | I_m | \Psi_0 \rangle = \mathrm{Tr}[I_m \rho_{\mathrm{GGE}}]$.

General Questions

Dichotomy in late-time behavior for integrable and non-integrable

Natural questions:
What happens when integrability is only "weakly" broken?
Is there memory of the integrable theory for some timescales?

What do we mean by weakly broken integrability?

- Consider two-parameter family of non-integrable Hamiltonians $H(g, U) = H_0(g) + UH_1(g)$ with $H_0(g)$ integrable.
- Quench $H(g_0, 0) \to H(g, U)$ to break integrability.
- We say integrability "weakly broken" when $U \ll$ all other energy scales $(g, g_0, |g - g_0|, \ldots)$.

Our quench protocol

To examine these questions, we want to study the influence of the integrability breaking term on the time-evolution.

$\rightarrow$ **We want $O(1)$ dynamics as well as $O(U)$!**

1. Start with density matrix ρ_0 which is not an eigenstate of $H(g, U)$ for any U (*including* $U = 0$).

   Example: ρ_0 ground state of $H(g_0, 0)$ with $g_0 \neq g$.
2. Time-evolve and compare expectation values for integrable $H(g, 0)$ and non-integrable $H(g, U \neq 0)$.

The model

$$H(\delta, J_2, U) = -J_1 \sum_j \left(1 + \delta(-1)^j\right)\left(c_j^\dagger c_{j+1} + \mathrm{H.c.}\right) - J_2 \sum_j \left(c_j^\dagger c_{j+2} + \mathrm{H.c.}\right) + U \sum_j n_j n_{j+1}$$

Integrable limits:

- $J_2 = 0$, $\delta = 0$: Anisotropic Heisenberg model
- $U = 0$: free fermions

For our problem, we will use the free theory

$$H(\delta, J_2, 0) = \sum_{\alpha=\pm} \sum_k \epsilon_\alpha(k, \delta, J_2) a_\alpha^\dagger(k) a_\alpha(k)$$

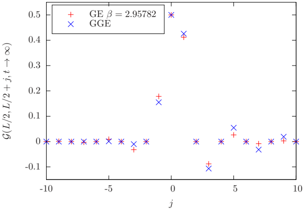
$$c_j = \frac{1}{\sqrt{L}} \sum_{k>0} \sum_{\alpha=\pm} \gamma_\alpha(j, k|\delta) a_\alpha(k)$$

Quenches in the free theory

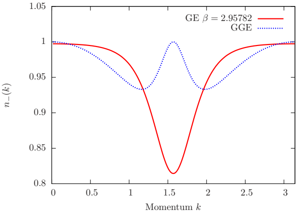
Integrable: $H(\delta, 0, 0)$

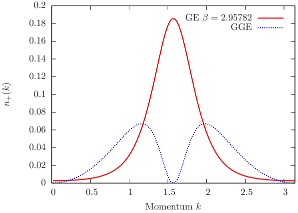
- Prepare system in ground state of $H(\delta_i, 0, 0)$
- Time-evolve according to $H(\delta_f, 0, 0)$

Green's function $\langle c^\dagger_{\frac{L}{2}} c_{\frac{L}{2}+j} \rangle_{t\to\infty}$



Very non-thermal mode occupation numbers!





Computing the time-evolution: the equations of motion

Heisenberg Equations of motion (EoM)

$$\frac{\mathrm{d}}{\mathrm{d}t} a^\dagger_\alpha(k) a_\beta(k) = i[H(\delta, J_2, U), a^\dagger_\alpha(k) a_\beta(k)]$$

Keep terms to second order. Apply Wick's theorem (assume 4+ particle cumulants are negligible).

$$\begin{aligned}
\dot{n}_{\alpha\beta}(k,t) &= i\epsilon_{\alpha\beta}(k) n_{\alpha\beta}(k,t) + 4iUe^{it\epsilon_{\alpha\beta}(k)} \sum_{\gamma_1} K_{\gamma_1\alpha}(k;t) n_{\gamma_1\beta}(k,0) - K_{\beta\gamma_1}(k;t) n_{\alpha\gamma_1}(k,0) \\
&- U^2 \int_0^t \mathrm{d}t' \sum_{\gamma} \sum_{k_1,k_2} L^{\gamma}_{\alpha\beta}(k_1,k_2|k|t-t') n_{\gamma_1\gamma_2}(k_1,t') n_{\gamma_3\gamma_4}(k_2,t') \\
&- U^2 \int_0^t \mathrm{d}t' \sum_{\vec{\gamma}} \sum_{k_1,k_2,k_3} M^{\vec{\gamma}}_{\alpha\beta}(k_1,k_2,k_3|k|t-t') n_{\gamma_1\gamma_2}(k_1,t') n_{\gamma_3\gamma_4}(k_2,t') n_{\gamma_5\gamma_6}(k_3,t')
\end{aligned}$$

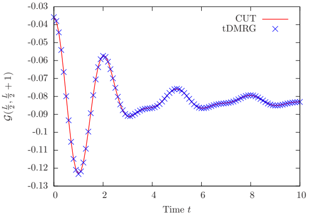
$n_{\alpha\beta}(k,t) = \langle\Psi(t)|a^\dagger_\alpha(k) a_\beta(k)|\Psi(t)\rangle$, $\epsilon_{\alpha\beta}(k) = \epsilon_\alpha(k) - \epsilon_\beta(k)$.

See also: Nessi & Iucci '14 '15

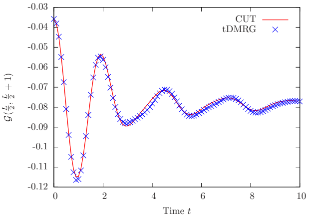
Non-integrable quenches comparison with TDMRG

First order EoMs

$U = 0 \to 0.15,\ \delta = 0.75 \to 0.25,\ J_2 = 0$

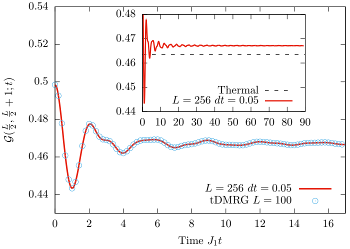


$U = 0 \to 0.5,\ \delta = 0.75 \to 0.25,\ J_2 = 0$



Second order EoMs

$U = 0 \to 0.4,\ J_2 = 0,\ \delta = 0.8 \to 0.4$

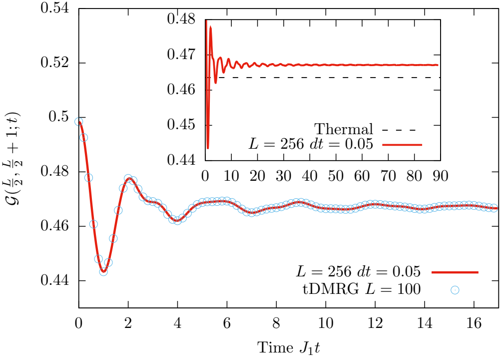


Robust prethermalization!

See also: Moekel & Kehrein '08, Kollar et al '11, Marcuzzi et al '13

Our definition of Prethermalization

On intermediate time scales correlation functions relax to a non-thermal plateau which retains information about the proximate integrable theory.



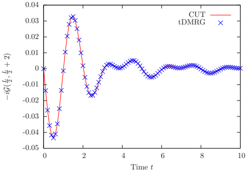
$$U = 0 \to 0.4,\ J_2 = 0,\ \delta = 0.8 \to 0.4$$

Non-integrable quenches comparison to TDMRG

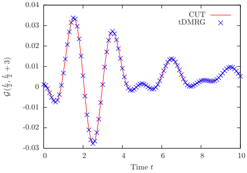
Equally good agreement for other separations of Green's function.

$U = 0 \to 0.15,\ \delta = 0.75 \to 0.25,\ J_2 = 0$

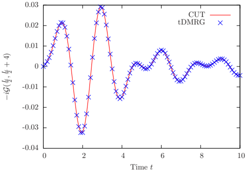
$\mathcal{G}(L/2, L/2+2)$



$\mathcal{G}(L/2, L/2+3)$



$\mathcal{G}(L/2, L/2+4)$



Non-integrable quenches: prethermalization and the dGGE

Truncating the EoM at first order, can construct operators conserved up to U^2 corrections.

$$Q_\alpha(q) \equiv n_{\alpha\alpha}(q) - U \sum_\gamma \sum_{\mathbf{k}} N_\alpha^\gamma(\mathbf{k}|q) a^\dagger_{\gamma_1}(k_1) a_{\gamma_2}(k_2) a^\dagger_{\gamma_3}(k_3) a_{\gamma_4}(k_4).$$

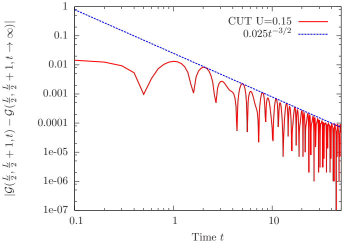
Can be used to construct "deformed GGE" with charges $O(U)$ *different* to integrable quench.

$$\rho_{dGGE} = \frac{1}{Z_{dGGE}} \exp\left[-\sum_{\alpha,q} \lambda_q^\alpha Q_\alpha(q)\right]$$

Non-integrable quenches: prethermalization and the dGGE

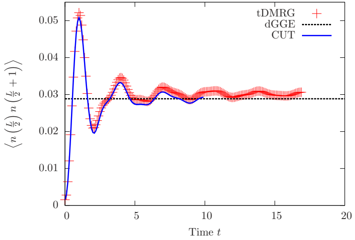
$$\rho_{dGGE} = \frac{1}{Z_{dGGE}} \exp\left[-\sum_{\alpha,q} \lambda_q^\alpha Q_\alpha(q) \right]$$

Approach dGGE as power law:



$U = 0 \to 0.15$, $\delta = 0.75 \to 0.25$, $J_2 = 0$

Also works for 4-point functions



$U = 0 \to 0.4$, $\delta = 0.8 \to 0.4$, $J_2 = 0$

Moving off the prethermalization plateau

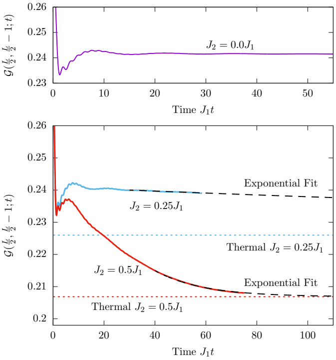
Prethermalization is not the full story!

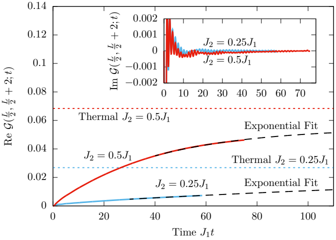
- $J_2 = 0$ has robust prethermalization plateau
  $\rightarrow$ no signs of drifting for times we can compute
- Introduce $J_2 \neq 0$ in attempt to tune thermalization timescale
  $\rightarrow$ breaks particle-hole symmetry
  $\rightarrow$ can increase number of $\Delta E = 0$ scattering channels

Moving off the prethermalization plateau

Relaxation compatible with:

$$G(i,j;t) \sim G(i,j)_{\mathrm{th}} + A_{ij}(J_2,\delta,U)e^{-t/\tau_{ij}(J_2,\delta,U)}$$





Thermal initial state with $\beta = 2,\ \delta = 0 \to 0.1,\ U = 0 \to 0.4,\ J_2 = 0 \to J_2$

A quantum Boltzmann equation

EoMs: $\delta_f \to 0$ then $n_{+-}(k, t \gg 1) \approx 0$

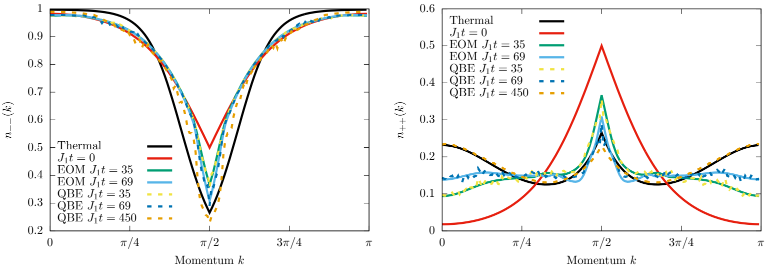
Derive QBE in limit $U \to 0$, $t \to \infty$ with $\tau = tU^2$ fixed.

$$\begin{aligned}\dot{n}_{\alpha\alpha}(k,\tau) = &- \sum_{\gamma,\delta} \sum_{p,q} \widetilde{K}^{\gamma\delta}_{\alpha\beta}(p,q;k) n_{\gamma\gamma}(p,\tau) n_{\delta\delta}(q,\tau) \\ &- \sum_{\gamma,\delta,\epsilon} \sum_{p,q,r} \widetilde{L}^{\gamma\delta\epsilon}_{\alpha\beta}(p,q,r;k) n_{\gamma\gamma}(p,\tau) n_{\delta\delta}(q,\tau) n_{\epsilon\epsilon}(r,\tau)\end{aligned}$$

See e.g. Erdös, Salmhofer, Yau '04; Lukkarinen, Spohn, '09.

A quantum Boltzmann equation

Mode occupation numbers approach thermal values (computed via perturbation theory) in the long-time limit for QBE



Green's functions:

$$G(i,j;t) \sim G(i,j)_{\mathrm{th}} + A_{ij}(J_2,\delta,U)e^{-t/\tau_{ij}(J_2,\delta,U)}$$

$$\rightarrow \text{QBE compatible with } \tau_{ij}(J_2,\delta,U) \propto U^{-2}$$

Conclusions

- Equations of motion useful for computing real-time dynamics
- Prethermalization plateau well-approximated by dGGE
- Introducing J_2 (next-nearest neighbor hopping) we see drifting from prethermalization plateau.
- Strength of drifting is very strongly dependent on J_2.
- Exponential approach to thermalization: fixed J_2, δ $\tau(J_2, \delta, U) \propto U^{-2}$.
- For $\delta_f = 0$ a QBE captures behavior well and mode occupation number approach thermal distribution.

See also the poster presented by **Stefan Groha**

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