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Preview tracking control for a class of fractional-order linear systems

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Abstract

This paper studies the preview tracking control of a class of fractional-order linear systems. Firstly, we use the fractional derivative property to take the fractional derivative of both sides of the state equation several times, and we obtain a formal ordinary linear system. An augmented error system is constructed for the transformed ordinary linear system, the appropriate performance index function is introduced and relevant results of the optimal preview control are applied to design the optimal preview controller for the augmented error system when the reference signal is previewable. Based on the relationship between the original system and the augmented error system, the preview tracking controller of the original system can be obtained. It can guarantee the asymptotic tracking of the output of the original closed-loop system to the reference signal. The validity of the theoretical results is verified by numerical simulation.

Keywords: Fractional-order linear systems; Augmented error system; Tracking control; Preview control

1 Introduction

The fractional-order system refers to the control system described by a fractional differential equation. In traditional control theory, the derivatives in the control system described by ordinary differential equations are all of integer order. However, it was later discovered that many physical systems exhibit fractional kinetic behavior because of their specific material and chemical properties [1–4]. If the equations containing fractional derivatives are used to describe such a system, the essential properties of the object of study can be better revealed. Therefore, the fractional-order system theory has been propounded and applied in many fields of engineering science [5–8]. For example, in [7], by employing the spectral theorem, Duhamel's formula is proved for the time-fractional-order Schrödinger equations, and properties of solution operators are given. Reference [8] investigates the existence, uniqueness and Hölder continuity of solutions to the time-fractional Navier–Stokes equations. In recent years, the study of fractional-order systems has been extended to the field of robust control, optimal control, sliding mode control, fault-tolerant control, iterative learning control and other advanced control strategies [9–18]. Reference [9] studies the static output feedback control problem of fractional uncertain systems by using the linear matrix inequality method. In Refs. [10–12], the variational method of classical optimal control is extended to fractional-order systems, and the Euler–Lagrange equation of fractional-order variational problems is obtained. The variational method and Lagrange multiplier method are combined to solve the equations numerically. In [13], a sliding mode controller is designed for fractional-order linear systems, a new fractional-order switching surface is proposed, and the case with input delay and state delay is considered. In [14], the problem of robust fault-tolerant control for continuous-time fractional-order systems with interval parameters and sensor faults is studied by establishing sensor fault model and state observer. Based on the properties of fractional derivative and generalized Gronwall inequality, a P-type iterative learning control updating laws for a class of delay fractional-order systems is given in [15]. In [16], a PD^{α}-type distributed iterative learning control laws is proposed for consensus tracking of nonlinear fractional multi-agent systems. Reference [17] discusses the problem of complete tracking for a class of fractional-order systems in a finite-time interval, and gives fractional-order iterative learning control laws involving a local average operator associated with probability. In addition, the fractional-order system has been successfully applied to the design of the fractional-order damper, antilock braking systems, and other practical engineering problems [19–21].

In many practical cases, the future reference signal or the future disturbance signal of the control system is partly or completely known, such as the flight path of aircraft, the processing path of numerically-controlled machine tools, the driving path of vehicles, and so on. The future information can be used to design the controller to improve the control quality of closed-loop systems. Such problems are problems of preview control. Since Sheridan put forward the concept of preview control in the 1960s, it has received extensive attention and formed a relatively complete set of theories and methods. At present, preview control has made progress in the theoretical studies of integer-order systems such as continuous-time linear systems, random systems and multi-agent systems. In [22], the problem of optimal preview control for continuous-time systems is studied by using the augmented system method. On the basis of [22], [23] studies the situation in which both reference signal and disturbance signal can be previewed at the same time. Reference [24] studies the coordinated optimal preview tracking control problem for continuous-time multi-agent systems on directed graphs. In [25], the optimal preview control problem for a class of continuous-time stochastic systems is studied by constructing an auxiliary system. At the same time, preview control is being applied successfully to many engineering control problems such as vehicle active suspension systems, electromechanical servo systems, robots, and aircraft [26–28].

To date, there are no published research results on preview control of fractional-order systems. The combination of preview control and fractional-order systems has important theoretical and practical significance. For a class of fractional-order linear systems, the design method for a preview tracking controller is given in this paper. The main contributions of this paper can be summarized as follows. (1) Combining fractional-order system with preview control for the first time, the problem of preview control for fractional-order system is studied, and the theory of preview control is extended. (2) By using the properties of fractional calculus, the fractional derivatives on both sides of the equation of state are calculated many times to obtain a formal ordinary linear system. The error system is constructed for the obtained integer-order system, and the controller with preview effect is designed by introducing appropriate performance indicators. (3) Research shows that the preview control theory of ordinary systems is a special case of the preview control theory of fractional-order systems. The conclusions and methods of fractional-order systems can be directly applied to ordinary control systems.

The research contents are arranged as follows. Section 1 is the introduction. Section 2 provides a few basic concepts, for completeness. Section 3 presents the problem of preview control for fractional-order linear systems. The problem of preview controller design for such systems is discussed in Sect. 4. Sections 5 and 6 discuss the conditions under which the controller exists. Section 7 is for a numerical simulation. Finally, Sect. 8 is for a brief conclusion.

2 Preliminaries

In its theoretical development, many definitions of fractional-order calculus have emerged because of different angles of study. The scientific rationality of the definitions has been convincingly tested in practice. The definitions used in this article are described below and related properties are given. Additional definitions and properties of fractional-order calculus can be found in [29–31].

Definition 1 (Fractional-order integral [29]) For any $\alpha \in C$, $\mathrm{Re}(\alpha) > 0$, the order α integral of the function $f(t)$ is defined as

$$ {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \tag{1}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\,dt$.

Remark 1 ${}_{t_0}D_t^{\alpha}$ denotes the fractional integral operator, t is the independent variable, t_0 is the lower boundary of the variable.

Definition 2 (Caputo fractional derivative [29]) Set n as a positive integer and $f(t)$ as a differentiable function of order n. When $n-1 < \alpha < n$, the Caputo derivative of order α in $f(t)$ is specified as

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} f(t) &= {}_{t_0}D_t^{-(n-\alpha)}\bigl(D^n f(t)\bigr) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau. \end{aligned} \tag{2}$$

Remark 2 (1) D^n denotes the derivative operator of integral order, i.e., $D^n f(t) = f^{(n)}(t)$; (2) from Definition 2, it can be seen that the order α Caputo derivative of $f(t)$ is to take the order n derivative of $f(t)$ first, then the order $n-\alpha$ integral; (3) similarly, when $n-1 < \alpha < n$, the function that has the derivative of α must be the order n derivative first; (4) $\lim\limits_{\alpha \to n^-} {}_{t_0}^{C}D_t^{\alpha} f(t) = f^{(n)}(t)$ [29].

Property 1 ([29]) The operation of the Caputo fractional derivative is linear, i.e., for arbitrary constants λ_1, λ_2,

$$ {}_{t_0}^{C}D_t^{\alpha}\bigl[\lambda_1 f_1(t) + \lambda_2 f_2(t)\bigr] = \lambda_1 {}_{t_0}^{C}D_t^{\alpha} f_1(t) + \lambda_2 {}_{t_0}^{C}D_t^{\alpha} f_2(t). \tag{3}$$

Property 2 ([29]) Let k, m, s all be positive integers, and $k + s \le m, f(t) \in C^1[0, T]$, where $T > 0$, then

$$ {}_0^C D_t^{\frac{k}{m}} \big({}_0^C D_t^{\frac{s}{m}} f(t) \big) = {}_0^C D_t^{\frac{k+s}{m}} f(t). \tag{4} $$

Remark 3 For simplicity, we write ${}_0^C D_t^{\frac{k}{m}} ({}_0^C D_t^{\frac{s}{m}} f(t))$ as ${}_0^C D_t^{\frac{k}{m}} {}_0^C D_t^{\frac{s}{m}} f(t)$ in this paper.

Property 3 ([29]) Let $\alpha > 0$, $n - 1 < \alpha < n, f(t)$ be continuously derivable on $[0, T]$. Then, the compound formula of fractional integration and equal-order Caputo fractional derivative calculation is

$$ {}_0D_t^{-\alpha} \big({}_0^C D_t^{\alpha} f(t) \big) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0), \tag{5} $$

where $f^{(0)}(t) = f(t)$.

Remark 4 When $f(t) = [f_1(t) \quad f_2(t) \quad \cdots \quad f_n(t)]^T$, let us say

$$ {}_{t_0}^C D_t^{\alpha} f(t) = \Big[{}_{t_0}^C D_t^{\alpha} f_1(t) \quad {}_{t_0}^C D_t^{\alpha} f_2(t) \quad \cdots \quad {}_{t_0}^C D_t^{\alpha} f_n(t) \Big]^T, $$

and all the above properties can be proved to be correct.

The following two lemmas need to be used in this article.

Lemma 1 ([32]) (A, B) *is stabilizable* (*controllable*) *if and only if the matrix* $[\lambda I - A \quad B]$ *has full row rank for all* $\lambda \in \bar{\mathbb{C}}^+$ (*for all* λ), $\bar{\mathbb{C}}^+$ *is the closed right-half complex plane*.

Lemma 2 ([32]) (A, B) *is controllable if and only if for any eigenvalue* λ *of* A *and the corresponding left eigenvector* x, *we have* $x^* B \neq 0$, *where* x^* *represents the conjugate transpose of vector* x.

3 Problem formulation

Consider a fractional-order linear system

$$ {}_0^C D_t^{\frac{1}{m}} x(t) = Ax(t) + Bu(t), \tag{6a} $$

$$ y(t) = Cx(t), \tag{6b} $$

where $m > 1$ is a given positive integer, $x(t) \in R^n$ is the state vector, $u(t) \in R^r$ is the control input and $y(t) \in R^p$ is the output vector; A, B and C are constant matrices of dimensions $n \times n$, $n \times r$, $p \times n$, respectively.

Let $y_d(t) \in R^p$ be the desired tracking, or reference signal. Define the difference between the reference signal and the output signal as the error signal $e(t)$, i.e.,

$$ e(t) = y(t) - y_d(t). \tag{7} $$

In addition, we make the following basic assumptions.

Assumption 1 (A^m, B) is stabilizable.

Assumption 2 The matrix $\left[\begin{smallmatrix} A^m & B \\ C & 0 \end{smallmatrix}\right]$ has full row rank.

Assumption 3 (C, A^m) is detectable.

Assumption 4 The reference signal $y_d(t)$ is a piecewise-continuously differentiable function satisfying

$$\lim_{t\to\infty} y_d(t) = \bar{y}_d, \qquad \lim_{t\to\infty} \dot{y}_d(t) = 0, \tag{8}$$

where $\bar{y}_d$ is constant vector. Furthermore, the reference signal is previewable, namely, the future value of $y_d(\tau)$ is available in $\{\tau | t \leq \tau \leq t + l_r\}$ at each instant of time t, where l_r is the preview length.

Remark 5 Consider an ordinary system $\left\{\begin{smallmatrix} \dot{x}(t) = A^m x(t) + Bu(t) \\ y(t) = Cx(t) \end{smallmatrix}\right.$: Assumption 1 is equivalent to saying that the system is stabilizable, and Assumption 3 is equivalent to saying that the system is detectable. This indicates that the fractional-order control system we consider has some internal relationship with the ordinary control system. In fact, based on this relationship, we can apply the preview control theory of ordinary systems to solve the design problem of the fractional-order system controller.

This paper aims at designing a controller to allow the output of System (6a)–(6b) to asymptotically track the reference signal, or to make the error signal asymptotically approach the zero vector. System (6a)–(6b) will be transformed into one that can design the controller through the optimal preview control method so as to achieve this goal.

4 System transformation and its optimal tracking controller

Firstly, we use properties 1 and 2 of the Caputo fractional derivatives to transform System (6a)–(6b) and obtain a formal ordinary control system. Taking notice of ${}_0^C D_t^{\frac{k}{m}} {}_0^C D_t^{\frac{s}{m}} = {}_0^C D_t^{\frac{k+s}{m}}$, by applying ${}_0^C D_t^{\frac{1}{m}}$ to both sides of (6a), we get

$${}_0^C D_t^{\frac{2}{m}} x(t) = {}_0^C D_t^{\frac{1}{m}} \bigl(Ax(t) + Bu(t)\bigr) = A {}_0^C D_t^{\frac{1}{m}} x(t) + B {}_0^C D_t^{\frac{1}{m}} u(t), \tag{9}$$

then substitute (6a) into the right side of the above equation and get

$${}_0^C D_t^{\frac{2}{m}} x(t) = A\bigl(Ax(t) + Bu(t)\bigr) + B {}_0^C D_t^{\frac{1}{m}} u(t) = A^2 + ABu(t) + B {}_0^C D_t^{\frac{1}{m}} u(t). \tag{10}$$

We apply ${}_0^C D_t^{\frac{1}{m}}$ to both sides of (10) repeatedly, and substitute in (6a) until the left side becomes ${}_0^C D_t^{\frac{m}{m}} x(t)$, namely $\dot{x}(t)$, thus getting

$$\begin{aligned} \dot{x}(t) = {} & A^m x(t) + A^{m-1} Bu(t) + A^{m-2} B {}_0^C D_t^{\frac{1}{m}} u(t) + \cdots + AB {}_0^C D_t^{\frac{m-2}{m}} u(t) \\ & + B {}_0^C D_t^{\frac{m-1}{m}} u(t). \end{aligned} \tag{11}$$

Denote

$$\bar{A} = A^m \in R^{n\times n}, \qquad \bar{B} = \begin{bmatrix} A^{m-1}B & A^{m-2}B & \cdots & AB & B \end{bmatrix} \in R^{n\times mr},$$

$$v(t) = \begin{bmatrix} u(t) \\ {}_0^C D_t^{\frac{1}{m}} u(t) \\ {}_0^C D_t^{\frac{2}{m}} u(t) \\ \vdots \\ {}_0^C D_t^{\frac{m-1}{m}} u(t) \end{bmatrix} \in R^{mr}.$$

Equation (11) can be expressed as

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}v(t). \tag{12}$$

Obviously, the problem is transformed into one of designing the appropriate controller for System (12) so that the output $y(t) = Cx(t)$ of its closed-loop system can asymptotically track the reference signal $y_d(t)$. For this purpose, a quadratic performance index function is taken for System (12),

$$J = \frac{1}{2}\int_0^{\infty} \left[e^T(t)Q_e e(t) + \dot{v}^T(t)R\dot{v}(t)\right] dt, \tag{13}$$

where Q_e and R are $p \times p$ and $(mr) \times (mr)$ positive definite matrices, respectively. Note that the $\dot{v}(t)$ is introduced in the performance indicator function, which allows the integrator to be included in the controller, thus helping to eliminate static errors [23].

Then, adopting the methods of preview control theory, an augmented error system is constructed to transform the tracking problem into a regulation problem.

Differentiating both sides of (7), we have

$$\dot{e}(t) = \dot{y}(t) - \dot{y}_d(t) = C\dot{x}(t) - \dot{y}_d(t). \tag{14}$$

Differentiating both sides of (12), there is

$$\frac{d}{dt}\dot{x}(t) = \bar{A}\dot{x}(t) + \bar{B}\dot{v}(t). \tag{15}$$

Combining (14) and (15) we get

$$\dot{X}(t) = \tilde{A}X(t) + \tilde{B}\dot{v}(t) - \tilde{D}\dot{y}_d(t), \tag{16}$$

where $X(t) = \left[\begin{smallmatrix} e(t) \\ \dot{x}(t) \end{smallmatrix}\right] \in R^{p+n}$, $\tilde{A} = \left[\begin{smallmatrix} 0 & C \\ 0 & \bar{A} \end{smallmatrix}\right] \in R^{(p+n)\times(p+n)}$, $\tilde{B} = \left[\begin{smallmatrix} 0 \\ \bar{B} \end{smallmatrix}\right] \in R^{(p+n)\times mr}$, $\tilde{D} = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in R^{(p+n)\times p}$.

According to the output of the original system and the previewable characteristics of the reference signal, the error vector $e(t)$ should now be taken as the output, that is, the output equation should be taken as $e(t) = [I \quad 0]X(t)$, and finally we get

$$\begin{cases} \dot{X}(t) = \tilde{A}X(t) + \tilde{B}\dot{v}(t) - \tilde{D}\dot{y}_d(t), \\ e(t) = \tilde{C}X(t), \end{cases} \tag{17}$$

where $\tilde{C} = [I \quad 0] \in R^{p\times(p+n)}$.

Equation (17) is the augmented error system we need. The basic idea to solve the problem is to design the controller for System (17) through the method of optimal preview control, then obtain the controller of (12), which is the controller of System (6a)–(6b). It is not difficult to see that if we can design a state feedback to make part of component $e(t)$ of System (17) asymptotically stable to the zero vector, our goal is achieved. It is well known from optimal control theory that the controller which minimizes the performance index function given in Equation (13) allows the closed-loop system to have this property.

Using the relevant variable in (17), we can represent the quadratic index (13) as follows:

$$J = \frac{1}{2}\int_0^\infty \bigl[X^T(t)\tilde{Q}X(t) + \dot{v}^T(t)R\dot{v}(t)\bigr]\,dt, \tag{18}$$

where $\tilde{Q} = \tilde{C}^T Q_e \tilde{C} = \left[\begin{smallmatrix} Q_e & 0 \\ 0 & 0 \end{smallmatrix}\right] \in R^{(n+p)\times(n+p)}$.

It is noted that System (17) is similar in form to the system of [23], and the performance index function (18) is also similar in form to the literature [23]. Therefore, employing a similar derivation from the literature [23], the following theorem can be obtained.

Theorem 1 *Suppose* $(\tilde{A},\tilde{B})$ *is stabilizable*, $(\tilde{Q}^{1/2},\tilde{A})$ *is detectable, and Assumption 4 holds. Let* $x(t) = 0$, $v(t) = 0$ *and* $y_d(t) = 0$ *for* $t < 0$. *Then, the optimal input of the system* (12) *with the minimum of the performance index function of* (13) *is*

$$v(t) = -K_e\int_0^t e(\sigma)\,d\sigma - K_x\bigl[x(t) - x(0)\bigr] + R^{-1}\tilde{B}^T\int_0^{l_r}\exp\bigl(\sigma\tilde{A}_c^T\bigr)P\tilde{D}y_d(t+\sigma)\,d\sigma, \tag{19}$$

where

$$\tilde{A}_c = \tilde{A} - \tilde{B}R^{-1}\tilde{B}^T P \tag{20}$$

is stable, P *is the unique semidefinite solution of the following algebraic Riccati equation*:

$$\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + \tilde{Q} = 0. \tag{21}$$

In addition, $K_e = R^{-1}\tilde{B}^T P_e$, $K_x = R^{-1}\tilde{B}^T P_x$, $P = [P_e \quad P_x]$.

5 The condition that the controller exists

In this section, we discuss the condition that the controller exists, that is, the condition of Theorem 1 is satisfied when the original system (6a)–(6b) satisfies the desired condition, rendering $(\tilde{A},\tilde{B})$ stabilizable and $(\tilde{Q}^{1/2},\tilde{A})$ detectable.

Lemma 3 ([23]) *The pair* $(\tilde{A},\tilde{B})$ *is stabilizable* (*controllable*) *if and only if* $(\bar{A},\bar{B})$ *is stabilizable* (*controllable*) *and* $\left[\begin{smallmatrix} \bar{A} & \bar{B} \\ C & 0 \end{smallmatrix}\right]$ *has full row rank.*

The proof of Lemma 3 is shown in the literature [23].

Theorem 2 *If* (A^m,B) *is stabilizable* (*controllable*), *then* $(\bar{A},\bar{B})$ (*that is*, $(A^m,\bar{B})$) *is stabilizable* (*controllable*).

Proof We know from the obvious inequality

$$\operatorname{rank}\begin{bmatrix}\lambda I - A^m & A^{m-1}B & A^{m-2}B & \cdots & AB & B\end{bmatrix} \geq \operatorname{rank}\begin{bmatrix}\lambda I - A^m & B\end{bmatrix}$$

that if $[\lambda I - A^m \quad B]$ has full row rank, then $[\lambda I - \bar{A} \quad \bar{B}]$, or

$$\begin{bmatrix}\lambda I - A^m & A^{m-1}B & A^{m-2}B & \cdots & AB & B\end{bmatrix}$$

is full row rank. The conclusion of the present theorem is obtained from Lemma 1. □

According to Lemma 3 and Theorem 2, the following theorem holds.

Theorem 3 *If Assumption 1 and Assumption 2 hold together, then $(\tilde{A}, \tilde{B})$ is stabilizable.*

Proof We know from the obvious inequality

$$\operatorname{rank}\begin{bmatrix}\bar{A} & \bar{B} \\ C & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}A^m & A^{m-1}B & A^{m-2}B & \cdots & AB & B \\ C & 0 & 0 & \cdots & 0 & 0\end{bmatrix} \geq \operatorname{rank}\begin{bmatrix}A^m & B \\ C & 0\end{bmatrix}$$

that when Assumption 2 is true, the matrix $[\begin{smallmatrix}\bar{A} & \bar{B} \\ C & 0\end{smallmatrix}]$ has full row rank. Besides, we know from Theorem 2 that when Assumption 1 holds, $(\bar{A}, \bar{B})$ is stabilizable. Then, according to Lemma 3, $(\tilde{A}, \tilde{B})$ is stabilizable when both Assumption 1 and Assumption 2 are true. Theorem 3 is proved. □

Theorem 4 *If Assumption 3 is true and Q_e is a positive definite matrix, then $(\tilde{Q}^{1/2}, \tilde{A})$ is detectable.*

Since $\bar{A} = A^m$, this is a result of the literature [23].

To sum up, the main theorem in this paper is obtained.

Theorem 5 *Suppose Assumption 1–Assumption 4 are all true and Q_e is a positive definite matrix. Let $x(t) = 0$, $u(t) = 0$ and $y_d(t) = 0$ for $t < 0$. Then, the input of System (6a)–(6b) with the minimum of the performance index function of (13) is*

$$u(t) = -K_{eu}\int_0^t e(\sigma)\,d\sigma - K_{xu}\big[x(t) - x(0)\big] + K_u\int_0^{l_r}\exp\big(\sigma\tilde{A}_c^T\big)P\tilde{D}y_d(t+\sigma)\,d\sigma, \tag{22}$$

where

$$K_e = \begin{bmatrix}K_{eu} \\ K_{e2}\end{bmatrix}, \qquad K_x = \begin{bmatrix}K_{xu} \\ K_{x2}\end{bmatrix},$$

$$R^{-1}\tilde{B}^T = \begin{bmatrix}K_u \\ K_{u2}\end{bmatrix}, \qquad K_{eu} \in R^{r\times p}, \qquad K_{xu} \in R^{r\times n}, \qquad K_u \in R^{r\times(p+n)}.$$

P is the unique semi-positive definite solution of the Riccati equation (21). In addition,

$$K_e = R^{-1}\tilde{B}^T P_e, \qquad K_x = R^{-1}\tilde{B}^T P_x, \qquad P = \begin{bmatrix}P_e & P_x\end{bmatrix}.$$

Proof According to Theorem 3 and Theorem 4, when Assumption 1, Assumption 2, and Assumption 3 are all true, $(\tilde{A},\tilde{B})$ is stabilizable and $(\tilde{Q}^{1/2},\tilde{A})$ is detectable. Therefore, when the assumptions of this theorem are satisfied, all of the conditions of Theorem 1 are satisfied. The optimal input of System (12) is obtained from Theorem 1 as shown in (19). Further, the first component vector of (19) is taken out and the input of System (6a)–(6b) is obtained.

To do this, dividing the matrix K_e, K_x and $R^{-1}\tilde{B}^T$ into blocks, namely,

$$K_e = \begin{bmatrix} K_{eu} \\ K_{e2} \end{bmatrix}, \qquad K_x = \begin{bmatrix} K_{xu} \\ K_{x2} \end{bmatrix}, \qquad R^{-1}\tilde{B}^T = \begin{bmatrix} K_u \\ K_{u2} \end{bmatrix},$$
$$K_{eu} \in R^{r\times p}, \qquad K_{xu} \in R^{r\times n}, \qquad K_u \in R^{r\times(p+n)},$$

we substitute them into (19), take the first row on both sides of the equal sign, and get (22), so Theorem 5 can be proved. □

6 A little discussion

Remark 5 has indicated that the tracking problem studied in this paper is closely related to ordinary systems $\{\begin{smallmatrix} \dot{x}(t) = A^m x(t) + Bu(t) \\ y(t) = Cx(t) \end{smallmatrix}$. This section further discusses the relationship between the problem and system $\{\begin{smallmatrix} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{smallmatrix}$. One result follows.

Theorem 6 *A necessary condition for (A^m,B) to be able to control is that (A,B) is controllable.*

Proof Using contradiction, let (A^m,B) be controllable. According to Lemma 2, if (A,B) is uncontrollable, there must be a vector $w \neq 0$ and a complex number λ for

$$w^*A = \lambda w^*, \qquad w^*B = 0.$$

Repeating right multiplication A for $w^*A = \lambda w^*$ $m-1$ times to get

$$w^*A^m = \lambda^m w^*,$$

we have

$$w^*A^m = \lambda^m w^*, \qquad w^*B = 0.$$

This is in contradiction with (A^m,B) being controllable. Therefore, (A,B) is controllable.

Furthermore, is the inverse of Theorem 6 true? The answer is no. For example, take

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By utilizing Lemma 1, it is seen that (A,B) is controllable, but (A^2,B) is uncontrollable. □

According to the dual principle, Theorem 7 is obtained from Theorem 6.

Theorem 7 *The necessary condition for* (C, A^m) *to be observable is that* (C, A) *is observable.*

Theorem 6 and Theorem 7 are only necessary, but they are also meaningful. Because, if (A, B) cannot be controlled, (A^m, B) must not be controlled; if (C, A) is unobservable, then (C, A^m) must be unobservable. This provides some reference for us to judge the condition of Theorem 1.

7 Numerical simulation

7.1 Numerical simulation algorithm

In this section, the numerical solution of Eq. (6a) is discussed. The initial state $x(0)$ is known.

Firstly, the initial value problem of (6a) is transformed into the initial value problem of the corresponding integral equation. By applying ${}_0D_t^{-\frac{1}{m}}$ to both sides of (6a), we can obtain

$$ {}_0D_t^{-\frac{1}{m}}\left({}_0^CD_t^{\frac{1}{m}}x(t)\right) = {}_0D_t^{-\frac{1}{m}}\left(Ax(t) + Bu(t)\right). \tag{23}$$

Utilizing Property 3, the above formula is further reduced to

$$ x(t) = x(0) + \frac{1}{\Gamma(1/m)}\int_0^t (t-\tau)^{\frac{1}{m}-1}\left(Ax(\tau) + Bu(\tau)\right)d\tau. \tag{24}$$

Equation (24) is the integral equation corresponding to fractional differential (6a). It is well known that (6a) is the same as (24) in solution, so we only need to solve the integral equation (24) [31].

Taking the sampling interval as h, the interval $[0, kh]$ is divided into k equal points. We solve (24) on the interval $[0, kh]$. We obtain from Eq. (24)

$$\begin{aligned} x(kh) &= x(0) + \frac{1}{\Gamma(1/m)}\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1}\left(Ax(\tau) + Bu(\tau)\right)d\tau \\ &= x(0) + \frac{1}{\Gamma(1/m)}\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1}Ax(\tau)\,d\tau + \frac{1}{\Gamma(1/m)}\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1}Bu(\tau)\,d\tau \\ &= x(0) + \frac{1}{\Gamma(1/m)}\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1}Ax(\tau)\,d\tau \\ &\quad + \frac{1}{\Gamma(1/m)}\sum_{i=0}^{k-1}\int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1}Bu(\tau)\,d\tau. \end{aligned} \tag{25}$$

Adopting zero-order holder on $[ih, (i+1)h]$, thus we have

$$\begin{aligned} x(kh) &= x(0) + \frac{1}{\Gamma(1/m)}\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1}Ax(\tau)\,d\tau \\ &\quad + \frac{1}{\Gamma(1/m)}\sum_{i=0}^{k-1}\int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1}Bu(ih)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= x(0) + \frac{1}{\Gamma(1/m)} \int_0^{kh} (kh-\tau)^{\frac{1}{m}-1} Ax(\tau)\,d\tau \\
&\quad + \frac{1}{\Gamma(1/m)} \sum_{i=0}^{k-1} \left[\int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1}\,d\tau \right] Bu(ih).
\end{aligned}
\tag{26}
$$

Denote

$$
b_{ik} = \int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1}\,d\tau
$$

and let us take the integral

$$
b_{ik} = mh^{\frac{1}{m}} \bigl[(k-i)^{\frac{1}{m}} - \bigl(k-(i+1)\bigr)^{\frac{1}{m}} \bigr]. \tag{27}
$$

As we know, when the size of $k-i$ is close to that of $k-(i+1)$, directly calculating $(k-i)^{\frac{1}{m}} - (k-(i+1))^{\frac{1}{m}}$ will lose a lot of effective numbers and thus generate large errors. Therefore, we use the results of the following transformation:

$$
b_{ik} = \frac{m\sqrt[m]{h}}{(\sqrt[m]{k-i})^{m-1}(1+b_0+\cdots+b_0^{m-1})}, \qquad b_0 = \sqrt[m]{\frac{k-1-i}{k-i}}, \tag{28}
$$

and get

$$
x(kh) = x(0) + \frac{1}{\Gamma(1/m)} \int_0^{kh} (kh-\tau)^{\frac{1}{m}-1} Ax(\tau)\,d\tau + \frac{1}{\Gamma(1/m)} \sum_{i=0}^{k-1} b_{ik} Bu(ih). \tag{29}
$$

Further, by the properties of the integral, we have

$$
\int_0^{kh} (kh-\tau)^{\frac{1}{m}-1} Ax(\tau)\,d\tau = \sum_{i=0}^{k-1} \int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1} Ax(\tau)\,d\tau. \tag{30}
$$

Since the function $(kh-\tau)^{\frac{1}{m}-1}$ of τ is monotonous on the interval $[ih,(i+1)h]$, the integral mean value theorem can be applied to the integral $\int_{ih}^{(i+1)h}(kh-\tau)^{\frac{1}{m}-1}Ax(\tau)\,d\tau$. Because we can get the value of $x(ih)$,

$$
\int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1} Ax(\tau)\,d\tau \approx \left[\int_{ih}^{(i+1)h} (kh-\tau)^{\frac{1}{m}-1}\,d\tau \right] Ax(ih) = b_{ik}Ax(ih). \tag{31}
$$

Substituting (31) into (29),

$$
x(kh) = x(0) + \frac{1}{\Gamma(1/m)} \sum_{i=0}^{k-1} b_{ik} \bigl[Ax(ih) + Bu(ih) \bigr]. \tag{32}
$$

Let $k = 1, 2, \ldots$, we can obtain a numerical solution. Note that the input is (22) and the final iteration format is obtained. We have

$$\begin{cases} x((k+1)h) = x(0) + \frac{1}{\Gamma(1/m)} \sum_{i=0}^{k} b_{i,k+1}[Ax(ih) + Bu(ih)], \\ y(kh) = Cx(kh), \\ e(kh) = y(kh) - y_d(kh), \\ u(kh) = -K_{eu} \int_0^{kh} e(\sigma)\, d\sigma - K_{xu}[x(kh) - x(0)] \\ \qquad + K_u \int_0^{l_r} \exp(\sigma \tilde{A}_c^T) P \tilde{D} y_d(kh + \sigma)\, d\sigma. \end{cases} \tag{33}$$

Remark 6 Let us show that the results of this paper take ordinary control system ($m = 1$) as a special case. When $m = 1$, System (6a)–(6b) becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t). \end{cases} \tag{34}$$

At this time, whatever the value of i and k, always $b_{ik} = h$, Therefore, the first formula of iteration format (33) is

$$x\bigl((k+1)h\bigr) = x(0) + \sum_{i=0}^{k} hAx(ih) + \sum_{i=0}^{k} hBu(ih),$$

which is

$$x\bigl((k+1)h\bigr) = x(kh) + hAx(kh) + hBu(kh).$$

This is precisely the result of the discretization of System (34) by the Euler method.

7.2 Simulation case

In this section, the effectiveness of the designed controller is verified by numerical simulation. Two examples are given here.

Example 1 According to Ref. [1], a class of viscoelastic systems can be represented by the following fractional differential equations:

$$\begin{cases} M\ddot{x}(t) + \eta {}_0^C D_t^{\frac{1}{2}} x(t) + kx(t) = u(t), \\ x(0) = a_1, \quad \dot{x}(0) = a_2, \end{cases} \tag{35}$$

where M, η and k represent mass, damping coefficient, and elastic coefficient, respectively, $x(t)$ is the displacement function, and $u(t)$ is the input quantity.

Selecting a set of state variables

$$x_1(t) = x(t), \qquad x_2(t) = {}_0^C D_t^{\frac{1}{2}} x(t), \qquad x_3(t) = \dot{x}(t), \qquad x_4(t) = {}_0^C D_t^{\frac{3}{2}} x(t),$$

we have

$${}_0^C D_t^{\frac{1}{2}} x_1(t) = x_2(t), \qquad {}_0^C D_t^{\frac{1}{2}} x_2(t) = x_3(t), \qquad {}_0^C D_t^{\frac{1}{2}} x_3(t) = x_4(t) \tag{36}$$

and we use Eq. (35) to get

$$ {}_0^C D_t^{\frac{1}{2}} x_4(t) = \ddot{x}(t) = \frac{1}{M}\bigl(-\eta {}_0^C D_t^{\frac{1}{2}} x(t) - kx(t) + u(t)\bigr) = \frac{1}{M}\bigl(-kx_1(t) - \eta x_2(t) + u(t)\bigr). \tag{37} $$

The three expressions of (36) and (37) are written as matrix vectors, namely

$$ {}_0^C D_t^{\frac{1}{2}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{M} & -\frac{\eta}{M} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix} u(t). $$

The output of System (35) is $x(t)$. Let $M = 1$, $\eta = 1.5$, $k = 1$, $a_1 = 0.1$, $a_2 = 0$, We have

$$ \begin{cases} {}_0^C D_t^{\frac{1}{2}} X(t) = AX(t) + Bu(t), \\ y(t) = CX(t), \end{cases} \tag{38} $$

where

$$ X(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ {}_0^C D_t^{\frac{1}{2}} x(t) \\ \dot{x}_3(t) \\ {}_0^C D_t^{\frac{3}{2}} x_4(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1.5 & 0 & 0 \end{bmatrix}, $$

$$ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. $$

The initial state is $X(0) = [0.1 \quad 0 \quad 0 \quad 0]^T$.

For this example, it is verified that (A^2, B) can be controlled, (C, A^2) can be observed, and the matrix $\bigl[\begin{smallmatrix} A^2 & B \\ C & 0 \end{smallmatrix}\bigr]$ has full row rank. Let

$$ Q_e = 1, \qquad R = \begin{bmatrix} 15 & 14 \\ 14 & 15 \end{bmatrix}. $$

The reference signal is set as

$$ y_d(t) = \begin{cases} 0, & 0 \le t \le 10, \\ 0.2(t-10), & 10 < t \le 15, \\ 1, & t > 15. \end{cases} \tag{39} $$

Since the reference signal is piecewise-continuously differentiable, all the conditions of Theorem 5 are satisfied.

We conducted numerical simulation for $l_r = 0$ (without reference signal preview), $l_r = 1.5$ and $l_r = 3.0$, respectively. According to Theorem 5, the solution of the Riccati equation and the feedback gain matrix of the controller are obtained:

$$P = \begin{bmatrix} 2.715493353897455 & 3.035936029440240 & -1.242510196107315 & 1.608803950946188 & 0.236743393846281 \\ 3.035936029440240 & 5.685568170113166 & -2.392087410321241 & 2.407705257751323 & 0.950278182238873 \\ -1.242510196107315 & -2.392087410321241 & 3.136796987702080 & -2.938618940017380 & 0.826127889263837 \\ 1.608803950946188 & 2.407705257751323 & -2.938618940017380 & 2.969986982371575 & -0.838897582427755 \\ 0.236743393846281 & 0.950278182238873 & 0.826127889263837 & -0.838897582427755 & 0.978748459579978 \end{bmatrix},$$

$$K_{eu} = 0.717850060356720,$$

$$K_{xu} = \begin{bmatrix} 0.786609803962952 & -1.918795674136359 & 1.941185203088351 & -0.906411798984001 \end{bmatrix},$$

$$K_u = \begin{bmatrix} 0 & 0 & 0 & 0.517241379310345 & -0.482758620689655 \end{bmatrix}.$$

The step length $h = 0.01$ is selected and the tracking effect is shown in Fig. 1. Figure 2 shows the tracking error of the system output to the reference signal under different preview lengths.

It can be seen from Fig. 1 that the output of the closed-loop system can track the reference signal by taking different preview lengths. In fact, the adjustment time is 20.24 s, 18.82 s, and 17.90 s, respectively. As can be seen from Fig. 2, the tracking error decreases with the increase of the preview length.

Example 2 Consider the fractional-order system (6a)–(6b), where

$$m = 3, \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

After verification, (A^3, B) can be stabilized, (C, A^3) can be observed, and the matrix $\left[\begin{smallmatrix} A^3 & B \\ C & 0 \end{smallmatrix}\right]$ has full row rank. Let

$$Q_e = 1, \qquad R = 10I_3.$$

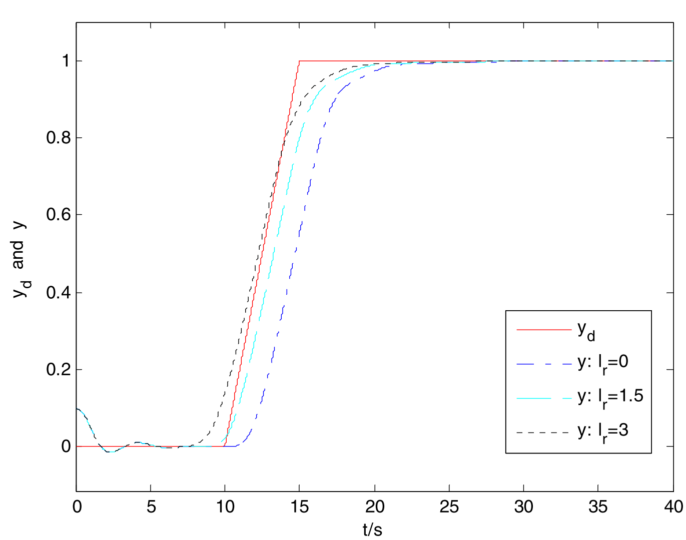

Figure 1 The output response of System (38)

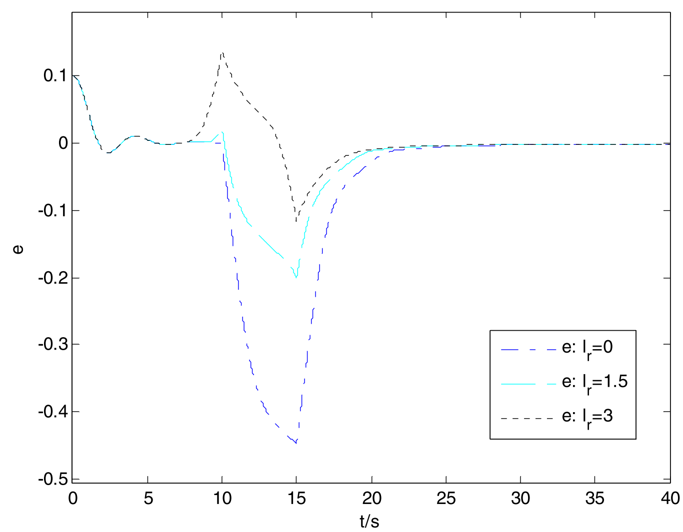

Figure 2 The tracking error of System (38)

The reference signal is taken as a step function

$$y_d(t) = \begin{cases} 0, & t < 10, \\ 1, & t \geq 10. \end{cases} \tag{40}$$

Notice that $y_d(t)$ is piecewise-continuously differentiable, so all of the conditions of Theorem 5 are satisfied.

We also carried out numerical simulations for three cases $l_r = 0$, $l_r = 1.5$ and $l_r = 3.5$. Respectively, the solutions of the Riccati equation and the feedback gain matrix of the controller are

$$\begin{aligned} P &= \begin{bmatrix} 1.630667100202825 & 0.912870929175287 & 0.202762262890926 \\ 0.912870929175287 & 2.321921924271059 & 0.475792303348216 \\ 0.202762262890926 & 0.475792303348216 & 0.097691175263477 \end{bmatrix}, \\ K_{eu} &= 0.182574185835057, \\ K_{xu} &= \begin{bmatrix} 0.464384384854212 & 0.095158460669643 \end{bmatrix}, \\ K_u &= \begin{bmatrix} 0 & 0.2000000000000000 & 0 \end{bmatrix}. \end{aligned}$$

Selecting the constant step length $h = 0.01$, the initial value is set as $x(0) = [0 \quad 0.05]^{\mathrm{T}}$. The tracking effect is shown in Fig. 3. It can be seen from Fig. 3 that the overshoot and the adjustment time decrease as the preview length increases. It is known that the adjustment time of the three cases is 22.75 s, 21.63 s and 21.36 s, respectively. Similarly, as the preview length increases, the overall tracking error gradually decreases.

Further, we continued to increase the preview length, and found that when it exceeds a certain value, the effect of the preview effect changes little. Figure 4 shows the step response of Example 2 when $l_r = 3$, $l_r = 5$ and $l_r = 20$. As can be seen from the figure, the

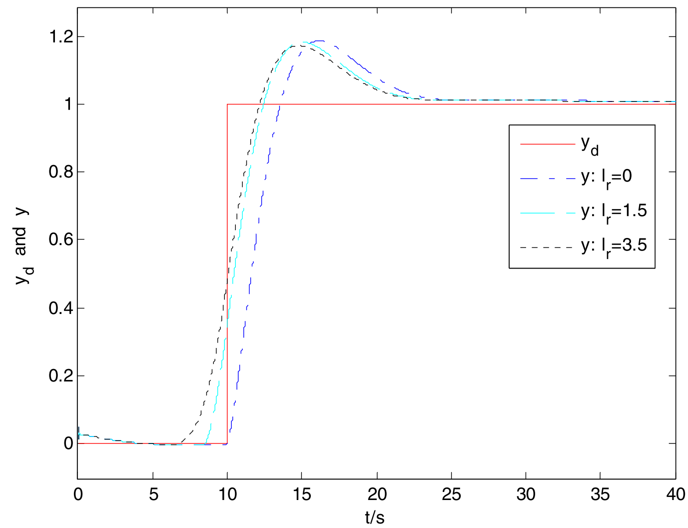

Figure 3 The step response of the system in Example 2

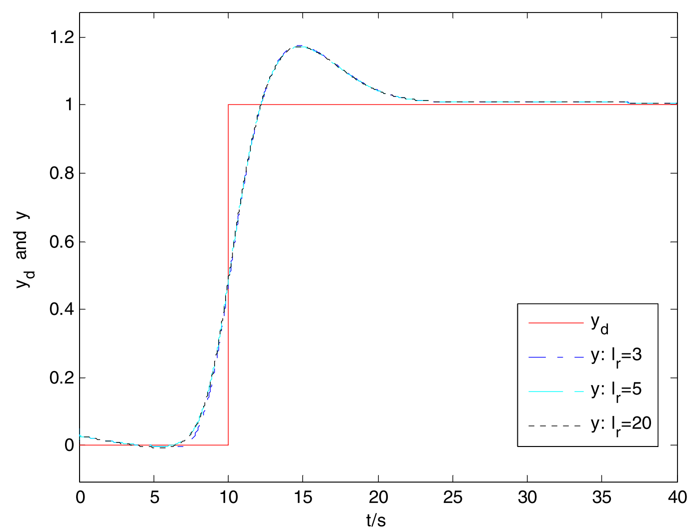

Figure 4 The step responses of the system in Example 2 for different preview lengths

output response curves of $l_r = 5$ and $l_r = 20$ are completely identical, and only the output response of $l_r = 3$ is slightly different. It can be seen from Figs. 3 and 4 that all features of the preview control theory of ordinary systems are completely retained in the preview control of fractional-order systems.

In Example 2, when $m = 1$ is the ordinary control system, let

$$Q_e = 1, \qquad R = 13.$$

The reference signal is still taken (40), so all the conditions of Theorem 4.3 in Ref. [11] are satisfied.

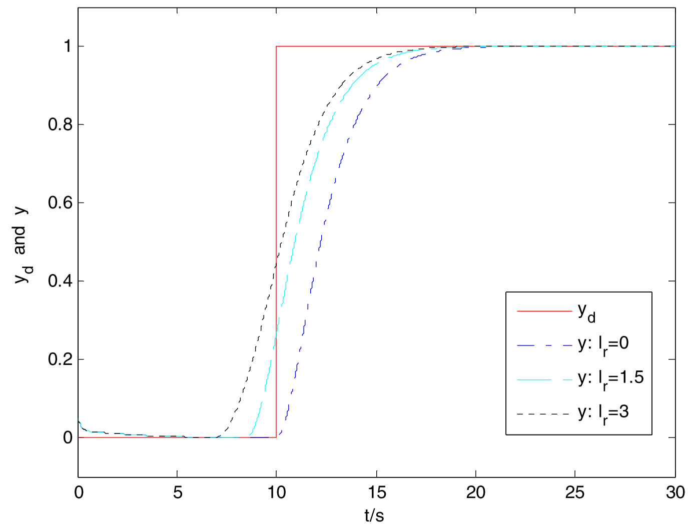

Figure 5 The step response of an ordinary system

The three cases of $l_r = 0$, $l_r = 1.5$ and $l_r = 3.0$ are numerically simulated. The solution of the Riccati equation and the feedback gain matrix of the controller are obtained:

$$P = \begin{bmatrix} 2.618310767549180 & 1.802775637731997 & 0.835457432601274 \\ 1.802775637731997 & 7.970226863749043 & 1.945657640326236 \\ 0.835457432601274 & 1.945657640326236 & 0.549679396948806 \end{bmatrix},$$

$$K_{eu} = 0.277350098112615,$$

$$K_{xu} = \begin{bmatrix} 1.226188748269084 & 0.299331944665575 \end{bmatrix},$$

$$K_u = \begin{bmatrix} 0 & 0.153846153846154 & 0 \end{bmatrix}.$$

Selecting the step length $h = 0.01$, the initial value is $x(0) = [0 \quad 0.05]^{\mathrm{T}}$. The tracking effect is shown in Fig. 5.

Figure 5 shows that increasing the preview length can shorten the adjustment time and reduce overshoot. The calculation shows that the adjustment time is 17.18 s, 15.92 s and 15.42 s, respectively. Thus, up to now, the ordinary system preview control theory is a special case of the fractional system preview control theory. The conclusion and method of fractional-order systems can be directly applied to ordinary control systems.

8 Conclusion

The basic theory of preview control is extended to the fractional control system. We study the preview tracking control problem for a class of fractional linear systems. By taking the fractional derivative of both sides of the state equation repeatedly until the left end of the equation becomes the first derivative of the state vector, the fractional linear system is transformed into a formal integer linear system. An error system is constructed for the ordinary integer-order system, and the appropriate performance index is introduced to design the preview controller. Further, the controller of the original fractional-order system is derived. The numerical simulation method of the fractional-order linear system is also given in the paper. It is shown that the preview controller of the ordinary integer-order system is a special case of the preview controller of fractional-order systems given in this paper. Numerical simulation shows that the designed controller is very effective.

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