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Abstract

The spokes model of nonlocalized spatial competition provides a new analytical tool for differentiated oligopoly and a representation of spatial monopolistic competition. At the unique symmetric equilibrium of the spokes model, an increase in the number of firms leads to lower prices when consumers have relatively high product valuations, but, surprisingly, to higher prices for lower consumer valuations. New entry alters consumer and social welfare through price, market expansion, and matching effects. With free entry, there can be multiple equilibria in the number of firms, the market may provide too many or too few varieties from a social welfare perspective, and the equilibrium price remains above marginal cost even when the number of firms is arbitrarily large.

JEL Classification Number: D4, L1

Key Words: product differentiation, spatial oligopoly, nonlocalized competition, monopolistic competition

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1. INTRODUCTION

The study of markets with differentiated products is essential to understanding modern economies. The economics literature on product differentiation originates from the seminal paper of Hotelling (1929). The Hotelling model considers a market with two stores located symmetrically on a line, called the Main Street. Consumers are uniformly distributed on the line and incur transportation costs to purchase from either store. Even though the two firms' products are physically identical, they are differentiated to consumers at different locations due to the transportation costs. The Hotelling model has become a standard tool in oligopoly analysis.

To understand oligopoly interactions under product differentiation, it is important to develop tractable models with more than two firms. The circle model (Salop, 1979) extends the Hotelling model to allow an arbitrary number of differentiated oligopoly firms, and has proven to be an important tool for analyzing oligopoly markets. In symmetric equilibria of the circle model, price decreases in the number of firms, approaching the marginal cost as the number of firms gets large, and there is over-provision of varieties with free entry. Same as the Hotelling formulation, the circle model follows a spatial approach where consumer preferences (or product characteristics) are represented by addresses in a geographical (or characteristic) space.[1] A distinguishing feature of the circle model is that competition is localized, in that a small change in a firm's price only affects its two neighbors, not the rest of the firms. A drawback of the circle model is that symmetry requires incumbents to relocate in product space when new firms enter the market.

Parallel to the development of models of localized competition are models of nonlocalized competition, in the tradition of Chamberlin (1933). Under nonlocalized competition, each firm competes against the market, and a price change by one firm affects all other firms (more or less) equally. Nonlocalized competition is clearly important for many industries, and is becoming perhaps even more so with the developments of new trading institutions

[1] Lancaster (1966) pioneered the characteristics approach where goods are represented by points on some characteristic space.

such as the Internet. Studies of product differentiation in the Chamberlinian tradition include the representative consumer model pioneered by Spence (1996) and Dixit and Stiglitz (1977), and the random utility model exemplified by Perloff and Salop (1985). Following a nonspatial approach, these studies have offered new insights about prices and product varieties in differentiated-product industries. In particular, a market with nonlocalized competition can provide either too few or too many varieties.

The literature on differentiated product competition has focused on formalizing Chamberlin's concept of monopolistic competition by examining free entry equilibria in the limit as the number of competitors becomes arbitrarily large. In monopolistic competition, firms exercise market power, i.e. set price above marginal cost, while earning zero profits. With free entry into the market, the number of competitors increases either as the size of the market grows larger or as the fixed cost of market competition becomes smaller. Monopolistic competition holds in the limit if, as in Hart (1985a, 1985b), consumers care about only a limited number of product varieties,[2] or if the product space is unbounded and available product varieties are never close substitutes. These conditions ensure that, when each firm is negligibly small in the limiting market, the demand for a firm's product is not infinitely elastic.[3]

Despite the many important developments in the economics literature on product differentiation,[4] oligopoly competition with product differentiation has not been studied in a spatial model with nonlocalized competition. The spatial approach is attractive for oligopoly analysis, because it is based on a definite and easy to visualize physical foundation. In the present paper, we introduce the "spokes model" of non-localized spatial competition as

[2] This assumption has been justified by Wolinsky (1986) as arising from consumers' imperfect information about different brands.

[3] In the circle model, when the number of firms approaches infinite, the distance between any two firms approaches zero. In the random utility model of Perloff and Salop, when the random utility of each consumer is bounded, the difference between a consumer's utilities from her first and second most preferred brands approaches zero when the number of brands approaches infinite. In both cases the demand elasticity for each firm approaches infinite at the limit.

[4] See Eaton and Lipsey (1989) and Anderson et al (1992) for excellent reviews of the literature.

a tool for oligopoly analysis. The spokes model extends the classical Hotelling duopoly model to allow for arbitrary numbers of possible product varieties and of firms, and has the following structure. Starting at the midpoint (center) of a line of unit length, add lines of one-half length to form a radial network of N (≥ 2) lines (spokes). Each spoke (denoted as l_i) terminates at the center and originates at the other end. There are $i = 1, 2, ..., N$ distinct possible varieties of a product, with variety i located at the origin of spoke i. There are n ($\leq N$) firms, each producing a single variety (or brand). The brands are physically identical but are differentiated by their different locations. Consumers are uniformly distributed on the network of spokes. A consumer travels to a firm in order to purchase the firm's brand, and incurs transportation costs (or, alternatively, utility losses due to imperfect preference matching). For a consumer located on l_i, brand i is her first preferred brand (or local brand), and each of the other $N - 1$ brands is equally likely to be her second preferred brand. The consumer has value v for one unit of either her first or second preferred brands, and zero value for additional units or for other brands. The Hotelling model is a special case with $N = n = 2$.[5]

The spokes model is a special case of Hart's (1985a) general model of monopolistic competition, and inherits several attractive features that distinguish it from the circle model of spatial competition (Salop, 1979). First, the model maintains symmetry between all brands and between all firms without the need to change the locations of incumbents as new firms enter the market. Second, each firm is in direct competition with all other firms, even though each consumer is only interested in a fixed number of possible varieties.[6] Third, total output in the market is not fixed but depends on equilibrium prices and the num-

[5] A variant of the spokes model was initially suggested in Chen and Riordan (2003), in order to study how downstream market structure mattered for the competitive consequences of vertical integration and exclusive contracts. In that model, firms observe consumers' locations and deliver goods to consumers at individualized delivery prices; and it is thus not an extension of the standard Hotelling model where consumers' locations are not known.

[6] Our assumption that each consumer is only interested in two brands is obviously restrictive and is made mainly for tractability. We shall later discuss a possible motivation for this assumption based on consumers' imperfect information, as well as possible ways to relax this assumption.

ber of firms. Consequently, the new entry in the spokes model has a market expansion effect. Fourth, the model approximates monopolistic competition in the limit as $N \to \infty$ and $n = kN$ for some fixed $0 < k \leq 1$. Hart (1985a, 1985b) focuses on the limiting case of monopolistic competition. The additional structure of the spokes model allows a more detailed analysis of the effect of new entry on market conduct and performance away from the limit when N is finite. While other analyses of differentiated oligopoly do likewise (see, for example, Sattinger, 1984; and Anderson, dePalma, and Thisse, 1992), our spatial approach is novel and offers interesting, and at times surprising, new insights.

We use the model to reexamine core economic questions about differentiated product markets: How does price competition depend on market structure? What are the effects of new product entry on competition and welfare? Does the market provide too few or too many product varieties compared to the social optimum? And what properties hold when firms are small relative to the size of the market? Our analysis reveals intriguing relationships between market structure and equilibrium price: an increase in the number of firms reduce price if consumers value products highly, but raises price if consumer value is in an intermediate range. Consequently, firm profit can be non-monotonic in the number of competitors. New entry alters consumer and social welfare through price, market expansion, and matching effects. As with other models on nonlocalized competition, the number of product varieties can be either socially excessive or deficient. In the spokes model entry tends to be excessive (deficient) when entry cost is relatively low (high), and excessive or deficient entry can arise for the same set of parameter values due to multiple free-entry equilibria. Finally, the spokes model with free entry provides a representation of spatial monopolistic competition as the number of competitors becomes arbitrarily large, with interesting welfare properties of equilibrium in the limit.

Our result that equilibrium price can increase with entry is unusual,[7] and it has the

[7] Perloff, Suslow, and Sequin (2005) demonstrate a similar result in a spatial model comparing monopoly and duopoly. Other oligopoly models in which price rises with more firms are based on imperfect consumer information (e.g., Satterthwaite, 1979; Schulz and Stahl, 1996; and Stiglitz, 1987), or mixed-strategy pricing (e.g., Rosenthal, 1980). Our result is obtained under perfect information and with pure strategies.

following intuition. In equilibrium, each symmetrically positioned firm views itself as competing in a number of submarkets. The submarkets are distinguished by whether consumers' first and second preferred brands are available. In some submarkets, consumers lack an alternatively available brand and the firm is effectively a monopolist. In other submarkets, the firm is a duopolist competing with an alternative brand. A key property of the spokes model is that the price elasticity of demand can be lower in the monopoly submarkets than in duopoly submarkets.[8] Therefore, firms prefer a lower price in the monopoly submarkets, but, unable to discriminate, settle on a compromise price. The effect of new entry is to convert some monopoly submarkets into duopolies. This changes the compromise, and gives firms an incentive to raise price. This intuition shows that price-increasing entry depends on a particular ranking of elasticities across market structures.

The rest of the paper is organized as follows. Section 2 describes the basic model and derives each firm's demand function. Section 3 characterizes the unique (symmetric) equilibrium of the model for a given number of firms. The equilibrium price exhibits different properties corresponding to four mutually exclusive and connected regions of parameter values. Comparative static analysis of equilibrium shows how a change in the number of firms affects price, profits, and consumer welfare. Section 4 endogenizes the number of firms in a free-entry equilibrium, and shows that the free entry number of firms may exceed or fall short of the socially optimal number depending on parameter values, and, in some cases, on equilibrium selection. Section 5 studies monopolistic competition by examining the properties of the model when there is an arbitrarily large number of possible varieties and proportionally large number of firms. Section 6 concludes.

2. SPOKES MODEL

There are $i = 1, 2, ..., N$ possible varieties of a differentiated product. Each variety (brand) is represented by a point that is the origin of a line with its length being $\frac{1}{2}$. The other end of the line is called its terminal. For variety i, its associated line is called l_i, and the terminals

[8] This generalizes a property of the Hotelling model (Chen and Riordan, 2004).

of all the lines meet at one point, called the center. This forms a radial network of lines (spokes network) connected at the center, and this network represents the preference space.

There are $j = 1, 2, ..., n$ firms in the market, $2 \leq n \leq N$. Firm j is located at the origin of l_j and produces variety j with constant marginal cost. For expositional simplicity, this variable production cost is normalized to zero; thus all values in the model are interpreted to be net of production costs. Each firm produces only one variety and posts a single price. Firms set prices simultaneously.

Consumers are uniformly distributed on the spokes network, and the total mass of consumers is normalized to unity. A consumer's location (ideal point) on the network is fully characterized by a vector (l_i, x_i), meaning that the consumer is on l_i at a distance x_i to variety i (the origin of l_i).[9] Since all the other varieties are symmetric, the distance from consumer (l_i, x_i) to any variety i', $i' \neq i$, is $\frac{1}{2} - x_i + \frac{1}{2} = 1 - x_i$. Any consumer must travel on the spokes to reach any firm (variety) where she wishes to purchase the product, incurring positive transportation costs. The unit transportation cost, t, is normalized to unity; thus all values in the model are expressed in transportation cost units. Variety i is consumer $(l_i, x_i)'s$ first preferred brand (or local brand), of which her valuation for one unit is v; she also has a second preferred brand, which is any $i' \neq i$ chosen by nature with probability $\frac{1}{N-1,}$ and of which her valuation for one unit is also v. The consumer places zero value on the brand that is not one of her two desired brands,[10] as well as on any additional units of any brand.

We notice immediately the following:

Remark 1 *The spokes model reduces to the Hotelling model when $N = n = 2$.*

We derive the demand for firm j for any given price profile $(p_1, p_2, ..., p_n)$. There are three relevant categories of consumers: consumers for whom brand j is preferred, and whose two preferred brands are both available; consumers for whom brand j is the first preferred

[9] We denote the consumer located at the center by $\left(l_1, \frac{1}{2}\right)$, and therefore every consumer's location representation is unique.

[10] We discuss later about the motivation for this assumption and how it can be relaxed without changing the results of our analysis.

brand, whose second preferred brand is not available; and consumers whose first brand is unavailable and for whom brand j is the second preferred brand.

For any consumer located on l_j or on l_k, denoted as (l_j, x_j) or (l_k, x_k), for $j, k \in \{1, ..., n\}$, both variety j and variety k are her desired brands with conditional probability $\frac{1}{N-1}$. Such a consumer is indifferent between variety j and k if $p_j + x_j = p_k + (1 - x_j)$ or $p_j + (1 - x_k) = p_k + x_k$. The marginal consumer between j and k is a distance

$$\hat{x} = \max\left\{\min\left\{\frac{1}{2} + \frac{p_k - p_j}{2}, 1\right\}, 0\right\}$$

from firm j. The number of such consumers served by firm j is

$$\frac{2}{N}\frac{1}{N-1}\sum_{k \neq j,\ k \in \{1,...,n\}} \max\left\{\min\left\{\frac{1}{2} + \frac{p_k - p_j}{2}, 1\right\}, 0\right\},$$

where $\frac{2}{N}$ is the density of consumers on l_j and on l_k.

For any consumer on l_j, with probability $\frac{1}{N-1}$ variety i is her second preferred brand where $i \notin \{1, ..., n\}$. Such a consumer prefers purchasing from firm j to no purchase if $p_j + x_j \leq v$. Firm $j's$ demand from this second category of consumers is

$$\frac{N-n}{N-1}\frac{2}{N}\min\{\max\{0, v - p_j\}, \frac{1}{2}\},$$

where $\frac{2}{N}$ is again the density of consumers on l_j, and $N - n$ varieties are unavailable.

Finally, for any consumer on l_i, $i \neq j$ and $i \notin \{1, ..., n\}$, variety j is her second preferred brand with probability $\frac{1}{N-1}$. Such a consumer prefers purchasing from firm j to not purchasing if $p_j + (1 - x_i) \leq v$. Firm $j's$ demand from this last consumer type is

$$\frac{N-n}{N-1}\frac{2}{N}\min\{\max\left\{0, v - p_j - \frac{1}{2}\right\}, \frac{1}{2}\}.$$

Summing up these three categories of consumers, and simplifying, we obtain firm $j's$ total demand as

$$q_j = \frac{1}{N-1}\frac{2}{N}\sum_{\substack{k \neq j,\\ k \in \{1,...,n\}}} \max\left\{\min\left\{\frac{1}{2} + \frac{p_k - p_j}{2}, 1\right\}, 0\right\} + \frac{N-n}{N-1}\frac{2}{N}\max\left\{\min\{v - p_j, 1\}, 0\right\},$$

which, provided $|p_k - p_j| \leq 1$ and $v - p_j > \frac{1}{2}$, can be re-written as

$$q_j = \begin{cases} \frac{1}{N-1}\frac{2}{N}\sum_{k\neq j,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2}+\frac{p_k-p_j}{2}\right)+\frac{N-n}{N-1}\frac{2}{N}(v-p_j) & if \quad 0 < v-p_j \leq 1 \\ \frac{1}{N-1}\frac{2}{N}\sum_{k\neq j,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2}+\frac{p_k-p_j}{2}\right)+\frac{N-n}{N-1}\frac{2}{N} & if \quad v-p_j > 1 \end{cases} . \quad (1)$$

Thus, firm j essentially sells to two consumer groups: consumers who have an alternative available, and those who do not. The firm, however, cannot price discriminate between the two consumer groups.

A restrictive assumption of the spokes model is that each consumer only cares about two possible brands, although the two desired brands differ for different consumers. This is a special case of Hart's (1985) restriction that each consumer cares only about a fixed finite number of possible varieties. It is a tractable way to introduce nonlocalized competition in a spatial setting. One possible motivation for the assumption, following Wolinsky (1986), is consumers' imperfect information. For instance, if the consumer has perfect information about her local brand but must search to find information about any other brand, and if she has zero cost for her first search but has a sufficiently high cost for any additional search, then she effectively will be interested only in her local brand and another randomly chosen brand even if other brands are also desirable.[11]

The purpose of the restriction is to assure the existence of a symmetric pure strategy equilibrium in prices with a minimum of fuss. For example, suppose alternatively that consumer (l_i, x_i) valued equally all varieties other than variety i. Then there would be a discontinuity in firm i's demand curve that would undermine a pure strategy equilibrium. There are various ways to extend the model to relax the assumption and still deal with the existence problem. For example, suppose that each consumer has a randomly selected third preferred brand valued at $v_3 < v - 1$, fourth preferred brand valued at $v_4 \leq v_3$, and so on. Consumer (l_i, x_i) travels distance $(1 - x_i)$ to purchase any of these lower-ranked brands, the same as if she purchases her second preferred brand. This formulation is similar to the model of Deneckere and Rothchild (1992), except that here the intensity of consumer

[11] In their symmetric random utility model, Perloff and Salop (1985) have also suggested that consumers may have imperfect information about the availability of competing brands, which can lead to a situation where every firm competes with every firm else but for different consumers, as in the spokes model here.

preferences over different brands is heterogeneous. If $v_3 < p^*$, where p^* is the equilibrium price, then the consumer only cares about two varieties in equilibrium, and all of our results remain true.

3. PRICE

Given the symmetry of the model, we focus on symmetric Bertrand-Nash (pure strategy) equilibria in which all firms set the same price p^*, serve an equal number of consumers q^*, and earn the same amount of profit $\pi^* = p^* q^*$ (recalling cost is normalized to zero). We assume:

$$1 \leq v \leq 2\frac{N-1}{n-1} + \frac{1}{2}\frac{2N-n-1}{N-n} \equiv \bar{v}(N,n). \tag{2}$$

If $v > \bar{v}(N,n)$, then a symmetric pure strategy equilibrium does not exist;[12] and, if $v < 1$, then firms effectively are independent monopolists. The equilibrium price is a continuous function of v, corresponding to four regions of the assumed parameter space. The regions are distinguished by the prevailing pattern of consumer demand, in particular, the extent to which consumers whose desired brands are available actually make a purchase and obtain a positive surplus in equilibrium. We have:

Proposition 1 *The spokes model has a unique symmetric equilibrium. The equilibrium price is*

$$p^* = \begin{cases} \frac{2N-n-1}{n-1} & if \quad 2\frac{N-1}{n-1} < v \leq \bar{v}(N,n) \ \textit{(Region I)} \\ v-1 & if \quad 2 \leq v \leq 2\frac{N-1}{n-1} \ \textit{(Region II)} \\ \frac{2(N-n)v+(n-1)}{4N-3n-1} & if \quad \frac{1}{2} + \frac{N-1}{2N-n-1} < v < 2 \ \textit{(Region III)} \\ v - \frac{1}{2} & if \quad 1 \leq v \leq \frac{1}{2} + \frac{N-1}{2N-n-1} \ \textit{(Region IV)} \end{cases}. \tag{3}$$

The proof of Proposition 1 is in the Appendix. Figure 1 illustrates how p^* depends on v over the four regions. Region I corresponds to "normal" oligopoly competition. All consumers whose desired brands are available purchase and enjoy a strictly positive surplus in equilibrium. Price is forced down by competition for consumers with a first and

[12] For any given $N \geq 3$, $\bar{v}(N,n)$ is a convex function of n and reaches its minimum at $n = \frac{2N+1}{3}$; and thus $\bar{v}(N,n) \geq \bar{v}(N, \frac{2N+1}{3}) = 5$. Notice also that $\bar{v}(N,n) = \infty$ when $n = N$.

second choice of available brands. Consequently, p^* depends on n and N, but not on v. In Region II, firms focus on monopolizing consumers who lack a second choice. All consumers whose desired brands are available again purchase, but the marginal consumer is indifferent between purchasing her second desired brand and purchasing nothing. Thus each firm's demand curve has a kink at $p^* = v - 1$, which fully extracts the surplus of the marginal consumer, and therefore rises linearly with v. In Region III, firms sell to both consumers who have a choice (the duopoly submarket) and those who do not (the monopoly submarket). The marginal consumer in the duopoly submarket is indifferent between two available varieties and gains a strictly positive surplus, while the marginal consumer in the monopoly submarket is indifferent between purchasing her second preferred variety and not purchasing at all. An increase in v motivates each firm to raise price in order to further exploit consumers in the monopoly submarket, and thus p^* rises with v. This region has the unusual property that equilibrium demand is *more* elastic in the monopoly submarket, implying that price increases with entry, as discussed further below. Finally, Region IV corresponds to a different kind of "kinked" equilibrium. All consumers whose first preferred variety is available, and only these consumers, purchase the product, with the marginal consumer indifferent between purchasing and not. Again p^* does not depend on n and N, and increases linearly with v.

[Insert Figure 1 about here]

The effects of market structure on equilibrium prices follow easily from Proposition 1:

Corollary 1

$$\frac{dp^*}{dn} = \begin{cases} -2\frac{N-1}{(n-1)^2} < 0 & if \quad Region\ I \\ 0 & if \quad Region\ II \\ 2\frac{(N-1)(2-v)}{(3n-4N+1)^2} > 0 & if \quad Region\ III \\ 0 & if \quad Region\ IV \end{cases}. \tag{4}$$

A change in market concentration has a mixed effect on price across the regions of the parameter space: it is weakly decreasing in n for $v \geq 2$ but weakly increasing in n for

$v < 2$. In Region I, where v is high, an increase in n has the familiar effect of lowering equilibrium prices, due to increased competition. For Regions II and IV, where the demand is kinked, p^* is unaffected by the small changes in n, due to a discontinuity in the marginal profit function. What is most surprising is that p^* is strictly increasing in n in Region III. This is very different from the result in the circle model. While there are other oligopoly models in which price rises with more firms, these models rely either on imperfect consumer information (e.g., Satterthwaite, 1979; Schulz and Stahl, 1996; and Stiglitz, 1987) or on mixed strategy equilibrium in prices (e.g., Rosenthal, 1980). Our striking result is obtained under complete information and with pure strategies, and it has a novel economic intuition: In Region III of parameter values, each firm continues to sell to two segments of consumers, those it competes for against other firms (the competitive segment) and those for whom it provides the only desirable variety (the monopoly segment). It turns out, however, that demand is more elastic for the monopoly segment than for the competitive segment. This property is due to the fact that, as the firm lowers its price, the marginal consumer in the monopoly segment always has zero surplus from the alternative (not purchasing) while the marginal consumer in the competitive segment becomes increasingly attracted to the alternative (closer to the competing brands). As the number of firms becomes higher, the monopoly segment shrinks and the competitive segment expands, reducing the overall demand elasticity. This leads to a higher market price.

It is also interesting that changes in n can change equilibrium prices by changing the nature of the equilibrium, i.e. by shifting the equilibrium from one region to another. For instance, an increase in n can shift the equilibrium from Region II to Region I, decreasing the equilibrium price from $v-1$ to $2\frac{N-1}{n-1}-1$. On the other hand, an increase in n can shift the equilibrium from Region IV to Region III, resulting in a higher price.[13]

If an increase in n leads to lower prices, then it benefits consumers. On the other hand, an increase in n that leads to higher prices does not necessarily mean consumers are worse

[13] If unit transportation cost t were not normalized to 1, the equilibrium price would be $v-t$ in Region II and $v-\frac{t}{2}$ in Region IV, and an increase in t would lower price in these two regions. This "perverse" effect of t in kinked equilibria is similar to that of the circle model (Salop, 1979).

off, because it also increases the available product varieties, which has the positive market expansion and matching effects. Generally, an increase in n affects consumers in three ways:

- Market expansion effect: An increase in available varieties enables some consumers whose desired brands were previously unavailable to obtain a positive surplus.
- Price effect: Depending on the value of v, an increase in n can either reduce, increase or have no effect on equilibrium prices.
- Matching effect: Some consumers previously consuming their second choice, are able to consumer their first choice.

Equilibrium profit is calculated easily from Proposition 1 and equilibrium demand:

Corollary 2 *The profit of each firm at the unique symmetric equilibrium is:*

$$\pi^* = \begin{cases} \frac{(2N-n-1)^2}{(n-1)(N-1)N} & if \quad Region\ I \\ (v-1)\,\frac{2N-n-1}{(N-1)N} & if \quad Region\ II \\ \frac{(2(N-n)v+(n-1))^2(2N-n-1)}{(4N-3n-1)^2(N-1)N} & if \quad Region\ III \\ \left(v-\frac{1}{2}\right)\frac{1}{N} & if \quad Region\ IV \end{cases}. \tag{5}$$

Furthermore, π^* decreases in n for $v \geq 2$ (Regions I and II), but π^* may either decrease or increase in n if $\frac{1}{2}+\frac{N-1}{2N-n-1} < v < 2$ (Region III).

The unusual result that profits can be non-monotonic in the number of firms is a consequence of price-increasing entry. In Region III, an increase in n raises equilibrium price, but reduces each firm's output since some consumers switch to purchase from new entrants. If v is relatively large in Region III, then each firm sells to most consumers in its monopoly submarkets. Consequently, the firm experiences a large decrease in output when an increase in n converts some of these submarkets to duopolies, even though price increases, and π^* decreases in n. But if v is relatively small, the output effect dominates when n is small and the price effect dominates when n is large, resulting in a U-shaped curve, as demonstrated with the following example:

Example 1 *Assume* $N = 20$ *and* $v = \frac{3}{2}$. *Then, for* $n < N$, $\frac{1}{2} + \frac{N-1}{2N-n-1} < v < 2$, *and*

$$\pi^* = \frac{(2(N-n)v + (n-1))^2 (2N-n-1)}{(4N-3n-1)^2 (N-1) N} = \frac{(2n-59)^2 (39-n)}{380(3n-79)^2}.$$

Thus,

$$\frac{\partial \pi^*}{\partial n} = \frac{\left(6n^2 - 297n + 3179\right)(2n-59)}{380(79-3n)^3},$$

and $\pi^*(n)$ *is convex because*

$$\frac{\partial^2 \pi^*}{\partial n^2} = \frac{19(275-9n)}{10(3n-79)^4} > 0.$$

Since $\frac{\partial \pi^*}{\partial n} = 0$ *when* $n = 15.654$, *which is the solution to*

$$6n^2 - 297n + 3179 = 0,$$

π^* *decreases for* $n \leq 15$ *and increases for* $n \geq 16$.

4. VARIETY

The performance of markets under product differentiation depends not only on the equilibrium price, but also on the variety of products available in the market. The spokes model offers an interesting setting to investigate the issue of whether and how the variety provided by the market in equilibrium differs from the socially optimal level. Unlike the circle model, the spokes model has the desirable feature that as the number of firms increases, the symmetry of the model is maintained without the need to change the locations of the incumbent firms. In addition, there is a market expansion effect with the entry of new firms, namely some consumes who were not purchasers before will now consume the product, which is not present in the circle model or in the representative consumer model. Furthermore, the effect of entry or exit on market performance depends on the relationships between v, N, and n^*, the equilibrium number of firms (as determined by fixed cost).

Suppose that there are many identical potential firms who can enter to produce a brand by incurring a fixed entry cost $f > 0$. If n firms enter, then each earns profits $\pi^*(n)$ as characterized in Corollary 2. In a "free entry equilibrim", there are n^* active firms satisfying

$$\pi^*(n^*) \geq f \geq \pi^*(n^*+1)$$

if $n^* < N$, or $\pi^*(n^*) \geq f$ if $n^* = N$.

We separately consider two cases. Case A corresponds to combined Regions I and II of the parameter space, and Case B to Regions III and IV. For convenience we sometimes treat n as a continuous variable, in which case we use the notation $[n]^-$ to denote the largest integer smaller than n, and $[n]^+$ the smallest integer larger than n.

The reader can skip to Section 5 on monopolistic competition without much loss of continuity. Monopolistic competition provides a simpler framework for evaluating free-entry equilibria, with similar results.

4.1 Case A: $2 \leq v \leq \bar{v}(N, n^*)$.

We assume that

$$\frac{1}{2} \leq fN \leq \frac{2N-3}{N-1}(v-1). \tag{6}$$

The first inequality in this assumption ensures that the constraint $n \leq N$ is not binding for the socially optimal number of firms, and the second inequality ensures that a free-entry equilibrium can support at least two active firms (see Lemma 1 in the appendix).

By Corollary 2, π^* decreases in n for the relevant parameter space for Case A. Therefore if $\tilde{n} \geq 2$ satisfies

$$\pi^*(\tilde{n}) = f,$$

then the unique free-entry equilibrium has

$$n^* = [\tilde{n}]^-$$

firms.

In order to characterize the free entry equilibrium further, we need some additional notation. Define

$$\begin{aligned} \hat{n} &= 1 + 2\frac{N-1}{v}; \\ \hat{f} &= \frac{(2N - \hat{n} - 1)^2}{(\hat{n}-1)(N-1)N}. \end{aligned}$$

Then $\hat{n}$ is the critical value of n that divides the parameter space between Region I and II, and $\hat{f}$ is the corresponding value of f defined by the zero profit condition at this boundary point. Substituting $\hat{n}$ into $\hat{f}$, we obtain

$$\hat{f} = \frac{\left(2N - \left(1 + 2\frac{N-1}{v}\right) - 1\right)^2}{\left(1 + 2\frac{N-1}{v} - 1\right)(N-1)N} = 2\frac{(v-1)^2}{Nv}.$$

Thus $\hat{f}N > 1$ if $v > 2$. For $fN \leq \hat{f}N$, Region I is relevant, and for $\frac{1}{2} \leq fN < 1$ the constraint $n \leq N$ will be binding and hence $n^* = N$; for $1 \leq fN \leq \hat{f}N$ the zero profit condition is satisfied at

$$n_1 = 2N - 1 - \frac{N-1}{2}\left(\sqrt{fN(fN+8)} - fN\right). \tag{7}$$

Similarly, the zero profit condition in Region II is satisfied at

$$n_2 = 2N - 1 - \frac{fN(N-1)}{v-1}. \tag{8}$$

The next proposition, which provides a complete characterization of n^* and is proved formally in the appendix, establishes that Region I is the relevant region of the parameter space at a free-entry equilibrium for lower values of f, and Region II is relevant for higher values.

Proposition 2 *The number of firms in a free-entry equilibrium is*

$$n^* = \begin{cases} N & if & \frac{1}{2} \leq fN < 1 \\ [n_1]^- & if & 1 \leq fN \leq \hat{f}N \\ [n_2]^- & if & \hat{f}N < fN \leq \frac{2N-3}{N-1}(v-1) \end{cases}, \tag{9}$$

assuming $2 \leq v \leq \bar{v}(N, n^*)$.

We next compare the free-entry number of firms with the number that maximizes social surplus (social welfare), n^o. Since $2 \leq v \leq \bar{v}(N, n^*)$, all consumers whose desired brands are available are served in a free-entry equilibrium. This means that social surplus cannot be increased by changing firms' prices, and any potential distortion in a market equilibrium comes from the possible distortion in the number of firms.

Accordingly, we compute the socially optimal number of firms under the assumption that available brands are allocated to consumers efficiently. With n firms, consumers on the n spokes receive their most preferred variety, and generate social surplus equal to

$$\frac{2n}{N}\int_0^{\frac{1}{2}}(v-x)\,dx=\frac{n}{N}\left(v-\frac{1}{4}\right),$$

where $\frac{2}{N}$ is the consumer density on each spoke. For consumers on the remaining $(N-n)$ spokes, whose first preferred variety is unavailable, each consumer is served by each of the n firms with probability $\frac{1}{N-1}$. Thus, the social surplus from serving all these consumers is

$$\frac{2n}{N}\frac{N-n}{N-1}\int_{\frac{1}{2}}^{1}(v-x)\,dx=\frac{n}{N}\frac{N-n}{N-1}\left(v-\frac{3}{4}\right).$$

Adding up, the social welfare with n firms is:

$$W(n)=\frac{\left(4\left(N-n\right)v+4\left(N-1\right)\left(v-1\right)+3\left(n-1\right)\right)n}{4\left(N-1\right)N}-fn.$$

We have:

$$W'(n)=-\frac{4v-1+4N-8Nv-6n+8nv}{4\left(N-1\right)N}-f$$

and

$$W''(n)=-\frac{(4v-3)}{2\left(N-1\right)N}<0.$$

Thus, ignoring integer constraints, the optimal n solves $W'(n)=0$, i.e.

$$n^w=\frac{2N-1}{2}-\frac{\left(2fN-1\right)\left(N-1\right)}{4v-3}. \tag{10}$$

We note that $n^w<N$ when $fN\geq\frac{1}{2}$. Furthermore, it is straightforward that $n^w>1$ if $v\geq 2$ and $N\geq 2$. If n^w happens to be integer, then $n^o=n^w$ is the socially optimal number of firms. Otherwise, either $n^o=[n^w]^-$ or $n^o=[n^w]^+$ by the concavity of $W(n)$.

The following table calculates pairs of (n^*,n^o) for various parameter configurations. An entry of "X" indicates that either n^* or n^o is less than 2.

TABLE 1: Equilibrium vs. socially optimal number of firms: (n^*,n^o)

	$fN =$	1	1.5	2	3	4	5	6	7	7.58
N	$v = 2$	10, 8	5, 6	X, 4	X, X	X, X	X, X	X, X	X, X	X, X
$=$	$v = 3$	10, 9	8, 8	7, 7	5, 5	X, 3	X, X	X, X	X, X	X, X
10	$v = 4$	10, 9	8, 8	7, 7	6, 6	5, 5	4, 3	X, 2	X, X	X, X
	$v = 5$	10, 9	8, 8	7, 8	6, 7	5, 6	5, 5	4, 4	3, 3	X, 2
N	$v = 2$	15, 12	8, 9	X, X	X, X	X, X	X, X	X, X	X, X	X, X
$=$	$v = 3$	15, 13	13, 11	11, 10	8, 7	X, 4	X, X	X, X	X, X	X, X
15	$v = 4$	15, 13	13, 12	11, 11	9, 9	8, 7	5, 5	X, 3	X, X	X, X
	$v = 5$	15, 14	13, 13	11, 12	9, 10	8, 9	7, 6	6, 5	4, 4	2, 3
N	$v = 2$	20, 16	10, 12	X, 8	X, X	X, X	X, X	X, X	X, X	X, X
$=$	$v = 3$	20, 17	17, 15	15, 13	10, 9	X, 5	X, X	X, X	X, X	X, X
20	$v = 4$	20, 18	17, 17	15, 15	12, 12	11, 9	7, 6	X, 3	X, X	X, X
	$v = 5$	20, 18	17, 17	15, 16	12, 14	11, 12	9, 9	6, 5	4, 4	2, 4

The table has several noteworthy features.

- First and foremost, the socially optimal number of firms can be greater than, equal to, or less than the equilibrium number depending on parameter values.

- Second, free entry tends to be excessive when fN is small[14], and deficient when fN is large.[15] The entry of an additional firm has the negative externality of reducing each incumbent firm's profit, but also has the positive externality of increasing consumer surplus through the market expansion and the matching effects. For given N, fN being small or large is the same as f being small or large. Thus, when f is small, the negative externality on profits is more likely to dominate; otherwise, the positive externality from the market expansion and matching effects tends to dominate.

- Third, the relationship between n^* and n^o is not monotonic in fN (or in f for fixed N). It can be readily verified that $n_1 - n^w$ is U-shaped as fN increases, and $n_2 - n^w$

[14] A sufficient condition for $n^* > n^o$ is $\frac{1}{2N} < f \leq 1$.

[15] When $fN = \frac{2N-3}{N-1}(v-1)$, $n_2 = 2$ but $n^w > 2$ if in addition $v < \frac{N}{2} + \frac{1}{8}$. Notice that in Table 1 $v < \frac{N}{2} + \frac{1}{8}$ since $N \geq 10$ and $v \leq 5$.

decreases as fN increases. Therefore, when fN is small and the equilibrium falls within Region I, $n^* > n^o$ initially but the opposite can be true for some intermediate values of fN.

- Fourth, for the same N and fN, as v increases, n^* remains the same if the nature of equilibrium does not change but can increase if the equilibrium switches from Region II to Region I; and n^o weakly increases. As a result, it is possible that the entry equilibrium changes from being excessive to being deficient, or vice versa, as v increases.

4.2 Case B: $1 \leq v < 2$

Case B combines Regions III and IV. Recall from Section 3 that

$$\pi^*(n) = \begin{cases} \frac{(2(N-n)v+(n-1))^2(2N-n-1)}{(4N-3n-1)^2(N-1)N} & if \quad \frac{1}{2} + \frac{N-1}{2N-n-1} < v < 2 \text{ (Region III)} \\ \left(v - \frac{1}{2}\right)\frac{1}{N} & if \quad 1 \leq v \leq \frac{1}{2} + \frac{N-1}{2N-n-1} \text{ (Region IV)} \end{cases},$$

and $\pi^*(n)$ is continuous. Notice that $\frac{1}{2} + \frac{N-1}{2N-n-1} \in (1, \frac{3}{2}]$ increases in n and is equal to $\frac{3}{2}$ when $n = N$. Thus, if $v \in [1, \frac{3}{2})$, as n increases from 2 to N, it is possible that the relevant region for π^* is first in Region III and then in Region IV. If $v \geq \frac{3}{2}$, the relevant region for π^* is always Region III.

The analysis of free-entry equilibria for this case is complicated by the possibility of multiple equilibria, due to the possibility that π^* is U-shaped in n in Region III. For given $f > 0$ that is not too large, n^* is a free-entry equilibrium if it satisfies one of the two conditions below:

1. $\pi^*(\cdot)$ is decreasing; and $\pi^*(n^*) \geq f \geq \pi^*(n^*+1)$ for $n^* < N$, or $\pi^*(n^*) \geq f$ for $n^* = N$.

2. $\pi^*(z) = f$ for some $z \leq N$ and $\pi^*(n)$ is weakly increasing for $n \geq z$ with

$$n^* = \arg\max\{\pi^*(n) - fn : z \leq n \leq N\}.$$

Obviously, if $\pi^*(n)$ is non-monotonic in n, both conditions can potentially be satisfied by different values of n^*.

Since in equilibrium not all consumers are served, the prices in the market equilibrium are not efficient. This complicates the determination of the socially optimal number of varieties, since one needs to consider whether prices are set efficiently (at marginal cost). Suppose that a social planner sets the price efficiently, then the socially optimal (the first-best) number of varieties is the same as before and is $n^o = [n^w]^-$ or $[n^w]^+$, where n^w is given by equation (10) earlier.

Suppose next that the social planner can regulate entry but not firm prices (i.e., the second-best solution). Then with n firms, consumers on the n spokes receive their most preferred variety, and generate social surplus equal to

$$\frac{2n}{N}\int_0^{\frac{1}{2}}(v-x)\,dx = \frac{n}{N}\left(v-\frac{1}{4}\right).$$

For consumers on the remaining $(N-n)$ spokes, whose first preferred variety is unavailable, if the parameter values are in Region III, then $(v-p^*)$ consumers on each of the $(N-n)$ spokes is served by each of the n firms with probability $\frac{1}{N-1}$, and the social surplus from serving all these consumers is

$$\begin{aligned}&\frac{2n}{N}\frac{N-n}{N-1}\int_{\frac{1}{2}}^{v-\frac{2(N-n)v+(n-1)}{4N-3n-1}}(v-x)\,dx\\ =\ &\frac{1}{4}\frac{(4N-n+2v-4Nv+2nv-3)(4N-5n+2v-12Nv+10nv+1)(N-n)n}{(3n-4N+1)^2(N-1)N}.\end{aligned}$$

If, on the other hand, the parameter values are in Region IV, then none of the consumers on the $(N-n)$ spokes is served.

Adding up, the second-best social welfare with n firms is:

$$\tilde{W}(n) = \begin{cases} \begin{array}{c}\frac{n}{N}\left(v-\frac{1}{4}\right)-fn+\\ \frac{1}{4}\frac{(4N-n+2v-4Nv+2nv-3)(4N-5n+2v-12Nv+10nv+1)(N-n)n}{(3n-4N+1)^2(N-1)N}\end{array} & if \quad \frac{1}{2}+\frac{N-1}{2N-n-1}<v<2 \\ \frac{n}{N}\left(v-\frac{1}{4}\right)-fn & if \quad 1\le v\le \frac{1}{2}+\frac{N-1}{2N-n-1} \end{cases}. \tag{11}$$

The second-best number of varieties is integer $\tilde{n}^o$ that maximizes $\tilde{W}(n)$, and $\tilde{n}^o$ can be computed numerically.

In Table 2, we list for selected parameter values the equilibrium number of firms, the first-best number of firms, and the second-best number of firms, $(n^*, n^o, \tilde{n}^o)$. When n^* can take multiple values, the vector of n^* is entered. A number that is less than 2 is denoted with "X".

TABLE 2: Equilibrium vs. Socially Optimal Number of Firms: $(n^*, n^o, \tilde{n}^o)$

	$fN=$	0.74	0.76	0.96	1.1	1.2	1.4
N	$v=\frac{5}{4}$	$\begin{matrix}4\\10\end{matrix}$, 7, 10	2, 7, 10	X, 5, 4	X, 4, 2	X, 3, X	X, X, X
$=$	$v=\frac{6}{4}$	10, 8, 10	8, 8, 10	$\begin{matrix}5\\10\end{matrix}$, 7, 10	X, 6, 10	X, 5, 10	X, 4, X
10	$v=\frac{7}{4}$	10, 8, 10	10, 8, 10	10, 7, 10	7, 7, 10	5, 6, 10	2, 5, 10
N	$v=\frac{5}{4}$	$\begin{matrix}5\\15\end{matrix}$, 11, 15	$\begin{matrix}3\\15\end{matrix}$, 11, 15	X, 8, 6	X, 6, 2	X, 5, X	X, 2, X
$=$	$v=\frac{6}{4}$	15, 12, 15	15, 12, 15	$\begin{matrix}8\\15\end{matrix}$, 10, 15	X, 9, 9	X, 8, 7	X, 6, 4
15	$v=\frac{7}{4}$	15, 13, 15	15, 13, 15	15, 11, 12	10, 10, 11	8, 10, 10	3, 8, 8
N	$v=\frac{5}{4}$	$\begin{matrix}7\\20\end{matrix}$, 15, 20	3, 15, 20	X, 11, 8	X, 8, 3	X, 6, X	X, 2, X
$=$	$v=\frac{6}{4}$	20, 16, 20	20, 16, 20	$\begin{matrix}10\\20\end{matrix}$, 14, 20	2, 12, 12	X, 11, 10	X, 8, 6
20	$v=\frac{7}{4}$	20, 17, 20	20, 17, 20	20, 15, 17	14, 14, 15	10, 13, 13	4, 11, 11

Table 2 has the following notable features:

- The socially optimal number of firms, whether in the sense of first- or second-best, can be greater than, equal to, or less than the equilibrium number depending on parameter values. This can happen whether or not the free entry equilibrium is unique.

- Relative to the first best, free entry tends to be excessive when fN is small and deficient when fN is large. This is similar to the result in Case A. Relative to the second best, however, free entry tends to be deficient except possibly for some intermediate values of fN.

- For the same N and fN, as v increases, both n^* and n^o weakly increase, as in Case A; but the second best number $\tilde{n}^o$ can occasionally decrease, possibly due to the fact that price increases with n in Region III.

- If fN is relatively small and/or v is relatively large (close to 2), the second best number of firms tends to exceed the first best number; otherwise the opposite tends to be true. This may be due to the fact that under the second best price is too high and output is too low, which makes it more desirable to correct through more entry if fN is relatively small and/or v is relatively large.

4.3 Discussion

Deneckere and Rothschild (1985) suggest that markets tend to provide too many varieties under localized competition but not enough under nonlocalized competition. Our analysis indicates that the relationship between the nature of competition and entry is more complicated. In the spokes model, with nonlocalized competition, both under- and over-provision of product varieties are possible. This can happen for different parameter values, but sometimes also for the same parameter value due to the multiplicity of equilibria.

Our analysis further sheds light on when free entry is likely to be excessive or deficient. In both Case A and Case B, compared to the first best, free entry tends to be excessive when fN is small and deficient when fN is large. When entry cost is relatively low and/or post-entry profit is relatively high (fN is small), the business stealing effect tends to dominate the consumer surplus effect of entry associated with market expansion and improved product matching; and otherwise the business stealing effect tends to be dominated by the consumer surplus effect. Interestingly, deficient entry can also occur here because there are multiple equilibria and the market becomes "trapped" in a low-level equilibrium. If entry were sequential rather than simultaneous, then a "bandwagon" would eliminate such deficient entry equilibrium.

5. MONOPOLISTIC COMPETITION

We consider monopolistic competition with the spokes model by examining the limiting behavior of the market when the number of firms (n) is large. Since our model involves both N and n, we need to define what we mean by $n \to \infty$. Following Hart (1985), we assume $n = kN$, for a fixed parameter $k \in (0, 1]$, and let $N \to \infty$. We interpret this to mean that, as the number of possible varieties (N) increases, the fixed costs of market participation (f) decline appropriately to keep the free entry number of firms (n) in fixed proportion to N. In order to apply the results from Proposition 1 and Corollary 2 in this limit, we assume $\frac{v-c}{t} \in [1, \frac{2}{k} + \frac{1}{2}\frac{2-k}{1-k})$.

Hart (1985a, 1985b) argues that market power is key condition of true monopolistic competition. The following proposition establishes that, in the limit, as the market becomes unconcentrated, price in the spokes model remains bounded above zero, indicating that firms retain market power. Therefore, the spokes model provides a spatial representation of monopolistic competition. In fact, the spoke model of monopolistic competition is a special case of Hart's general model (Hart, 1985a).

Proposition 3 *If $n = kN$ and $N \to \infty$, then*

$$p^* \to \begin{cases} \frac{2-k}{k} & if \quad \frac{2}{k} < v \leq \frac{2}{k} + \frac{1}{2}\frac{2-k}{1-k} \text{ (Region I)} \\ v-1 & if \quad 2 \leq v \leq \frac{2}{k} \text{ (Region II)} \\ \frac{2(1-k)v+k}{4-3k} & if \quad \frac{1}{2} + \frac{1}{2-k} < v < 2 \text{ (Region III)} \\ v - \frac{1}{2} & if \quad 1 \leq v \leq \frac{1}{2} + \frac{1}{2-k} \text{ (Region IV)} \end{cases} . \tag{12}$$

As $N \to \infty$, $\pi^* \to 0$. But using Corollary 2, it is straightforward to show that $\pi^* N$ converges to a positive limit.

Corollary 3 *If $n = kN$ and $N \to \infty$, then*

$$\pi^* N \to R(v,k) \equiv \begin{cases} \frac{(2-k)^2}{k} & \text{if} \quad \text{Region I} \\ (v-1)(2-k) & \text{if} \quad \text{Region II} \\ \frac{(2(1-k)v+k)^2(2-k)}{(4-3k)^2} & \text{if} \quad \text{Region III} \\ \left(v-\frac{1}{2}\right) & \text{if} \quad \text{Region IV} \end{cases}. \tag{13}$$

In Regions I, II, and IV, the limiting value of $\pi^* N$ is decreasing in k. In Region III, however, the limiting value of $\pi^* N$ is a convex function of k with a minimum at

$$K(v) \equiv \frac{1}{6v-3}\left(9v - \sqrt{3}\sqrt{12v - 5v^2 - 4} - 6\right),$$

where $K(1) = 0$ and $K\left(\frac{7}{4}\right) = 1$.

In monopolistic competition the free entry zero-profit condition holds exactly at an interior equilibrium value of k, provided that in the limit $\pi^* N$ decreases in k. If $n = kN$ in equilibrium, then it must be that $\lim_{N\to\infty} f \to 0$, and

$$\lim_{N\to\infty} fN = R(v,k) \tag{14}$$

if $0 < k < 1$. This is a sufficient condition for a monopolistically competitive equilibrium in Regions I, II, and IV, while Region III requires the additional condition that $k < K(v)$ to insure that further entry decreases profits. The model also admits monopolistically competitive equilibria with $k = 1$ and $R(v,k) > \lim_{N\to\infty} fN$; in this case, even monopolistically competitive firms make positive profits, because there is no "room" in product space for further entry. Our analysis below focuses only on monopolistically competitive equilibria satisfying the zero profit condition (14).

We next use the zero profit condition to characterize the welfare properties of monopolistic competition. Recall from Section 4 that, ignoring integer constraints, welfare optimizing number of firms when available products are distributed efficiently to consumers is

$$n^w = \frac{2N-1}{2} - \frac{(2fN-1)(N-1)}{4v-3}. \tag{15}$$

Substituting

$$k^w = \frac{n^w}{N},$$

taking limits, and imposing the zero profit condition yields the following comparison of the socially optimal and equilibrium number of firms in monopolistic competition.

Proposition 4 *If $\pi^* = f$, $n = kN$, and $N \to \infty$, then*

$$k^w - k \to \begin{cases} 1 - k - \frac{2(2-k)^2 - k}{(4v-3)k} & in \quad Region\ I \\ 1 - k - \frac{(2(v-1)(2-k)-1)}{4v-3} & in \quad Region\ II \\ 1 - k - \frac{2[2v-k(2v-1)]^2(2-k)-(4-3k)^2}{(4v-3)(4-3k)^2} & in \quad Region\ III\ if\ k < K(v) \\ 1 - k - \frac{2(v-1)}{4v-3} & in \quad Region\ IV \end{cases}, \tag{16}$$

The proposition is summarized in Figure 2. The solid lines mark the boundaries of the four regions in (v, k), with "X" indicating regions in which a pure strategy equilibrium does not exist in the limit. The dashed lines divide the space in regions where entry is either deficient ($k^w > k$) or excessive ($k^w < k$). Clearly, entry can be excessive or deficient depending on (v, k). The value of k can be interpreted as an indicator of the degree of industry penetration in the market. The higher is k, the greater is product availability, and the greater the fraction of consumers who obtain the good in either monopolistic competition or a socially optimal allocation. In each of the four regions, entry has a business-stealing and a consumer surplus effect. The busines-stealing effect refers to the fact that a part of the profits of a new entrant is at the expense of incumbents, and therefore does not contribute to social welfare. The consumer surplus effect aries from market expansion and improved matching of consumers to possible varieties. Entry is excessive when the business stealing effect dominates the consumer surplus effect, and conversely (Mankiw and Whinston, 1986).

[Insert Figure 2 about here]

Figure 2 also shows that entry is deficient in Region I when v is sufficiently high and a pure strategy equilibrium exists. In this region, price is independent of v. Consequently, the consumer surplus effect dominates when v is large, and thus entry is deficient; the opposite is true when v is small. In the other regions, prices increase with v, eroding the consumer surplus effect. Consequently, when k is high in Regions II-IV, the business stealing effect dominates and entry is excessive; when k is small, the business stealing effect is small, and entry is deficient due to the dominating consumer surplus effect.

We next further compare the number of firms at the free entry equilibrium and the second-best outcome in the limit. In a second-best outcome, the social planner can choose entry but not prices. For Regions I and II, the number of firms is the same at the second-best and the first-best outcomes. For Regions III and IV, welfare under the second-best outcome is $\tilde{W}(n)$ defined by equation (11) in Section 4. Letting $n = kN$, $N \to \infty$, and assuming $fN \to F$, we have:

$$\tilde{W}(kN) \to \begin{cases} k\left[\left(v - \frac{1}{4}\right) + \frac{(4-k-4v+2kv)(4-5k-12v+10kv)(1-k)}{4(3k-4)^2} - F\right] & in \quad \text{Region III} \\ k\left[\left(v - \frac{1}{4}\right) - F\right] & in \quad \text{Region IV} \end{cases}.$$

Let k^s denote the second-best level of product variety that maximizes this function. Note, however that $F = R(v, k)$ in a zero-profit equilibrium. Therefore, for a given v, we can relate k^s to the equilibrium value of k, and compare second-best and equilibrium varieties.

[Insert Figure 3 about here]

A numerical comparison is summarized in Figure 3, which "blows up" Figure 2 on the restricted domain $v \in [1, 2]$. The diagram demarcates Regions IV and III and indicates where $k = k^w$ by the dashed line as before. The diagram yields several interesting observations. First, in Region IV, $k^s = 1$ and equilibrium product variety always is deficient relative to the second-best. Second, the area of Region III in which entry is excessive ($k^s < k$) is bounded by the dotted lines and $K(v)$ (which is the flatter upward-slopping solid curve marking the lower boundary of the X region); this region begins at about $v = \frac{5}{3}$ and occupies the upper corner of the diagram. Entry is deficient in the rest of Region III below $K(v)$. Finally, the second-best and the first-best varieties are not always directly comparable on the diagram, but the second-best variety clearly exceeds the first-best variety in Region IV and in part of Region III between the dashed and dotted lines where $k^w < k < k^s$.

6. CONCLUDING REMARKS

This paper has developed and analyzed a new spatial model of product differentiation. By extending the classical Hotelling duopoly to an oligopoly with nonlocalized competition and an arbitrary number of possible varieties and firms, the spokes model provides an attractive new tool for oligopoly analysis, as well as a representation of spatial monopolistic competition. In the spokes model, the symmetry between firms is maintained as new firms enter the market, without the need to relocate the incumbents in the preference space. Every brand (firm) competes directly with all other brands (firms), and both the number of buyers and industry output depend on prices and the number of firms. A unique symmetric (pure-strategy) equilibrium exists, and the nature of this equilibrium differs for different regions of the parameter space.

Analysis of the spokes model yields novel and interesting results on price and variety under product differentiation. In particular, an increase in the number of firms leads to lower prices when consumers have a relatively high willingness-to-pay for preferred varieties, but, surprisingly, to higher prices for lower consumer valuations. The entry of new firms affects consumers with the positive market expansion and matching effects, in addition to the possibly either positive or negative price effect; and each firm's profit can depend on the number of firms non-monotonically within some range, first decreasing and then increasing. With free entry, the market may provide either too many or too few varieties, and there can be multiple equilibria in the number of firms. Finally, when the number of firms and of potential product varieties both approach infinity, equilibrium price remains above marginal cost, and thus the spokes model provides a representation of spatial monopolistic competition.

As a tool for oligopoly analysis under nonlocalized competition, appropriate extensions of the spokes model of product differentiation have many possible applications in economics. For instance, the model provides an attractive framework to study firms' incentives to offer multiple products and the competition between multi-product firms. In particular, the model can address questions such as how market concentration affects the provision of product varieties by multiproduct firms, and how horizontal or vertical mergers affect competition and consumers. The spokes model can also be used to study how market structure affects firms' innovation incentives, for instance, whether a larger firm or firms in more concentrated markets have greater incentives to innovate. Furthermore, the spokes model is well suited for analyzing product choices by multiple firms, if firm locations on the network are determined endogenously. For some of these applications it would be necessary to modify the symmetric spokes model to introduce asymmetric firms.

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APPENDIX

The proofs for Proposition 1 and 2 follow.

Proof of Proposition 1. We consider in turn the four regions of parameter values. For each region, we construct a symmetric equilibrium where the equilibrium price satisfies a unique property that can hold only in the assumed region of parameter values, and any other price can be a symmetric equilibrium only in a different region of parameter values. The (symmetric) equilibrium is thus also unique.

Region I: Suppose that for this parameter region a symmetric equilibrium price satisfies

$$v > p^* + 1.$$

Then the demand facing firm j is

$$q_j = \frac{1}{N}\frac{n-1}{N-1}\left(1 + p^* - p_j\right) + \frac{N-n}{N-1}\frac{2}{N}$$

for prices p_j in the neighborhood of p^*. The corresponding profit of firm j is $\pi_j = p_j q_j$, and, firm j's first-order condition for profit maximization is

$$q_j - p_j \frac{1}{N}\frac{n-1}{N-1} = 0.$$

Therefore, at a symmetric equilibrium,

$$p^* = 1 + 2\frac{N-1}{n-1}.$$

It is straightforward that the second-order condition is satisfied and that p^* is a local maximum. Firm $j's$ output and profit at the proposed equilibrium are

$$q^* = \frac{2N-n-1}{\left(N-1\right)N}, \qquad \pi^* = \frac{\left(2N-n-1\right)^2}{\left(n-1\right)\left(N-1\right)N}.$$

The requirement that $v > p^* + 1$ is satisfied if and only if $v > \frac{N-1}{n-1}2$.

Finally, it is necessary to verify that a firm has no incentive to deviate globally. At the candidate equilibrium, the second-order condition is satisfied for $p_j \leq p^* + 1$. Furthermore, for $v - 1 \geq p_j > p^* + 1$, demand is perfectly inelastic and profit is increasing in p_j, and, for $p_j \geq v - 1$, firm j's profits are declining if $v \geq 2$. Therefore, the possibly most profitable deviation is $p_j = v - 1$, the profits from which are $\pi = (v-1)\frac{N-n}{N-1}\frac{2}{N}$. The deviation is not profitable if $\pi \leq \pi^*$, i.e.

$$(v-1)\frac{N-n}{N-1}\frac{2}{N} \leq \frac{(2N-n-1)^2}{(n-1)(N-1)N},$$

which holds if and only if

$$v \leq 1 + \frac{(2N-n-1)^2}{2(N-n)(n-1)} = 2\frac{N-1}{n-1} + \frac{1}{2}\frac{2N-n-1}{N-n} \equiv \bar{v}(N,n).$$

Thus $p^* = 1 + 2\frac{N-1}{n-1}$ is indeed a symmetric equilibrium in Region I, and it is the only symmetric equilibrium with the property that $v > p^* + 1$.

Region II: Suppose that for this parameter region a symmetric equilibrium price satisfies $p^* = v - 1$. Then

$$q_j = \begin{cases} \frac{2}{N}\left[\frac{1}{N-1}\sum_{k\neq j,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2}+\frac{p^*-p_j}{2}\right)+\frac{N-n}{N-1}(v-p_j)\right] & if \quad p_j \text{ is slightly above } v-1 \\ \frac{2}{N}\left[\frac{1}{N-1}\sum_{k\neq i,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2}+\frac{p^*-p_j}{2}\right)+\frac{N-n}{N-1}\right] & if \quad p_j \text{ is slightly below } v-1 \end{cases}.$$

In other words, the demand for firm j has a kink at $p_j = v - t$.

In order for $p^* = v - 1$ to be an equilibrium, a slight increase of p_j at p^* should not increase profit, i.e.

$$\begin{aligned} q_j + p_j\frac{\partial q_j}{\partial p_j} &= \frac{2}{N}\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right) - (v-1)\frac{2}{N}\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right) \\ &= -\frac{2}{N}\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right)(v-2) \leq 0, \end{aligned}$$

which holds if and only if $v \geq 2$. Also, a slight decrease of p_j at p^* should not increase profit, i.e.

$$\begin{aligned} q_j + p_j\frac{\partial q_j}{\partial p_j} &= \frac{2}{N}\left[\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right) - (v-1)\left(\frac{1}{2}\frac{n-1}{N-1}\right)\right] \\ &= \frac{2}{N}\left(\frac{n-1}{N-1}\right)\left[\frac{N-n}{n-1} - \frac{1}{2}(v-2)\right] \geq 0, \end{aligned}$$

which holds if and only if $v \leq 2\frac{N-1}{n-1}$.Therefore p^* is a local maximum. To show that p^* is also globally optimal, it suffices if firm j cannot benefit from any deviation to $p_j < v-1$. But since the second-order condition is satisfied for $p_j < v-1$ (any kink of the profit function makes it more concave), no global deviation can be profitable.

Thus $p^* = v - 1$ is indeed a symmetric equilibrium in Region II.

Region III: Suppose that the symmetric equilibrium price p^* is such that

$$\frac{1}{2} < v - p^* < 1,$$

so that

$$q_j = \frac{2}{N}\left[\frac{1}{N-1}\sum_{k\neq j,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2}+\frac{p^*-p_j}{2}\right)+\frac{N-n}{N-1}(v-p_j)\right],$$

$$\frac{\partial q_j}{\partial p_j} = -\frac{2}{N}\left[\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right].$$

The first-order condition for firm j is

$$q_j - p_j\frac{2}{N}\left[\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right] = 0.$$

At a symmetric equilibrium

$$\frac{2}{N}\left[\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}(v-p^*)-\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right)p^*\right]=0,$$

which simplifies to

$$\begin{aligned}&\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}(v-p^*)-\left(\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}\right)p^*\\ =\ &\frac{1}{2}\frac{n-1}{N-1}+\frac{N-n}{N-1}v-\frac{4N-3n-1}{2(N-1)}p^*=0,\end{aligned}$$

or

$$p^* = \frac{2(N-n)v+(n-1)}{4N-3n-1},$$

and p^* is a local maximum since the second order condition is satisfied at p^*. Furthermore, since

$$\begin{aligned}v-p^* &= v-\frac{n-1}{4N-3n-1}-\frac{2(N-n)}{4N-3n-1}v\\ &= \left[1-\frac{2(N-n)}{4N-3n-1}\right]v-\frac{n-1}{4N-3n-1}=\left[\frac{2N-n-1}{4N-3n-1}\right]\left[v-\frac{n-1}{2N-n-1}\right]\\ &> \left[\frac{2N-n-1}{4N-3n-1}\right]\left[\frac{1}{2}+\frac{N-1}{2N-n-1}-\frac{n-1}{2N-n-1}\right]=\frac{1}{2},\end{aligned}$$

and

$$\begin{aligned} v - p^* &= v - \frac{2(N-n)v + (n-1)}{4N-3n-1} \\ &= \frac{v(2N-n-1)-(n-1)}{4N-3n-1} < \frac{2(2N-n-1)-(n-1)}{4N-3n-1} = 1, \end{aligned}$$

p^* indeed satisfies $\frac{1}{2} < v - p^* < 1$. Finally, to verify that p^* is also globally optimal, notice that, since $v - 1 < p^* < v - \frac{1}{2}$, it suffices if any deviation to any $p < p^*$ cannot be profitable. But since the second-order condition is satisfied for $p < p^*$, no global deviation can be profitable. Thus p^* is indeed a symmetric equilibrium in Region III.

Region IV: Suppose $p^* = v - \frac{t}{2}$. Then

$$q_j = \begin{cases} \frac{2}{N}(v - p_j) & if \quad p \text{ is slightly above } p^* \\ \frac{2}{N}\frac{1}{N-1}\sum_{k\neq i,\ k\in\{1,\ldots,n\}}\left(\frac{1}{2} + \frac{p^*-p_j}{2}\right) + \frac{2}{N}\frac{N-n}{N-1}(v-p_j) & if \quad p \text{ is slightly below } p^* \end{cases}.$$

In order for $p^*$to be an equilibrium, a slight increase of p_j at p^* should not increase profit, i.e.

$$q_j - (p_j - c)\frac{2}{N} = \frac{1}{N} - \left(v - \frac{1}{2}\right)\frac{2}{N} \leq 0,$$

which holds if and only if $v \geq 1$.Also, a slight decrease of p_j at p^* should not increase profit, i.e.

$$\begin{aligned} q_j - (p_j - c)\frac{2}{N}\left(\frac{1}{2}\frac{n-1}{N-1} + \frac{N-n}{N-1}\right) &= \frac{2}{N}\left[\frac{1}{2} - \left(v - \frac{1}{2}\right)\left(\frac{1}{2}\frac{n-1}{N-1} + \frac{N-n}{N-1}\right)\right] \\ = \frac{2}{N}\left[\frac{1}{2} - \left(v - \frac{1}{2}\right)\left(1 - \frac{1}{2}\frac{n-1}{N-1}\right)\right] &\geq 0, \end{aligned}$$

which holds if and only if $v \leq \frac{1}{2} + \frac{N-1}{2N-n-1}$.Therefore, p^* is a local optimum for firm j, and, since the second-order condition is satisfied for both $p < v - \frac{1}{2}$ and $p > v - \frac{1}{2}$, it is also a global optimum.

To prove that the equilibrium is unique, suppose that there is another symmetric equilibrium, $\tilde{p}$, in some region, say Region I. Then $v \leq \tilde{p} + 1$. If $v - \tilde{p} = 1$, then $\tilde{p}$ can be a symmetric equilibrium only in Region II; if $\frac{1}{2} < v - \tilde{p} < 1$, then $\tilde{p}$ can be a symmetric equilibrium only in Region III; and if $\frac{1}{2} = v - \tilde{p}$, then $\tilde{p}$ can be a symmetric equilibrium only in Region IV. This is a contradiction. Thus there is no other symmetric equilibrium in Region I. The arguments for the other regions are similar. **Q.E.D.**

Proof of Proposition 2. We first establish the following lemma:

Lemma 1 *Assume* $2 \leq v \leq \bar{v}(N, n^*)$. *Then (i)* $n_1 \geq 2$ *and* $n_2 \geq 2$; *(ii) if* $fN \leq 1$, *then* $n_1 \geq N$ *and* $n_2 \geq N$; *and (iii) if* $1 < fN$, *then* $n_1 < N$.

Proof. (i) First, since $\frac{1}{2} < fN \leq \frac{2N-3}{N-1}(v-1)$, we have

$$n_2 - 2 = 2N - 3 - \frac{fN(N-1)}{v-1} \geq 2N - 3 - (2N-3) = 0.$$

Thus if fN is sufficiently large, the equilibrium will be in Region II. Hence

$$\hat{f}N = \frac{2(v-1)^2}{v} \leq \frac{2N-3}{N-1}(v-1),$$

and it follows that $v \leq 2(N-1)$.

Next, since $\sqrt{fN(fN+8)} - fN$ increases in fN, and $fN \leq \hat{f}N = \frac{2(v-1)^2}{v}$ in Region I, we have

$$\begin{aligned} n_1 - 2 &= 2N - 3 - \frac{N-1}{2}\left(\sqrt{fN(fN+8)} - fN\right) \\ &\geq 2N - 3 - \frac{N-1}{2}\left(\sqrt{\frac{2(v-1)^2}{v}\left(\frac{2(v-1)^2}{v} + 8\right)} - \frac{2(v-1)^2}{v}\right) \\ &= 2N - 3 - \frac{N-1}{2}\left(4\frac{v-1}{v}\right) = \frac{2(N-1) - v}{v} \geq 0. \end{aligned}$$

(ii) If $fN \leq 1$,

$$\begin{aligned} n_1 &= 2N - 1 - \frac{N-1}{2}\left(\sqrt{fN(fN+8)} - fN\right) \\ &\geq 2N - 1 - \frac{N-1}{2}\left(\sqrt{(1+8)} - 1\right) = 2N - 1 - (N-1) = N, \end{aligned}$$

and

$$n_2 = 2N - 1 - \frac{fN(N-1)}{v-1} \geq 2N - 1 - \frac{(N-1)}{v-1} \geq 2N - 1 - (N-1) = N.$$

(iii) If $1 < fN$,

$$n_1 = 2N - 1 - \frac{N-1}{2}\left(\sqrt{fN(fN+8)} - fN\right) < 2N - 1 - \frac{N-1}{2}\left(\sqrt{1(1+8)} - 1\right) = N.$$

Q.E.D.

Note that $\hat{f}N = \frac{2(v-1)^2}{v} \geq 1$. We can now provide a complete characterization of the equilibrium number(s) of firms. First, if $\frac{1}{2} < fN \leq 1$, then $n_1 = n_2 = N$ from Part (ii) of Lemma 1. Thus $n^* = N$ firms will enter the market and earn non-negative profits. This is the only equilibrium since no additional firm can enter due to the constraint that $n \leq N$. Next, if $\frac{1}{N} < f \leq \hat{f}$, we have

$$n_1 \geq \hat{n} \text{ and thus } 2\frac{N-1}{n_1-1} \leq v.$$

Therefore $n^* = [n_1]^-$, and, $n_1 < N$ from part (iii) of Lemma 1. Finally, if $f > \hat{f}$, we have

$$n_2 < \hat{n} \text{ and thus } 2\frac{N-1}{n_2-1} > v.$$

Therefore $n^* = [n_2]^-$, and $n_2 < N$ since otherwise we would have $2 > v$, a contradiction.

Q.E.D.

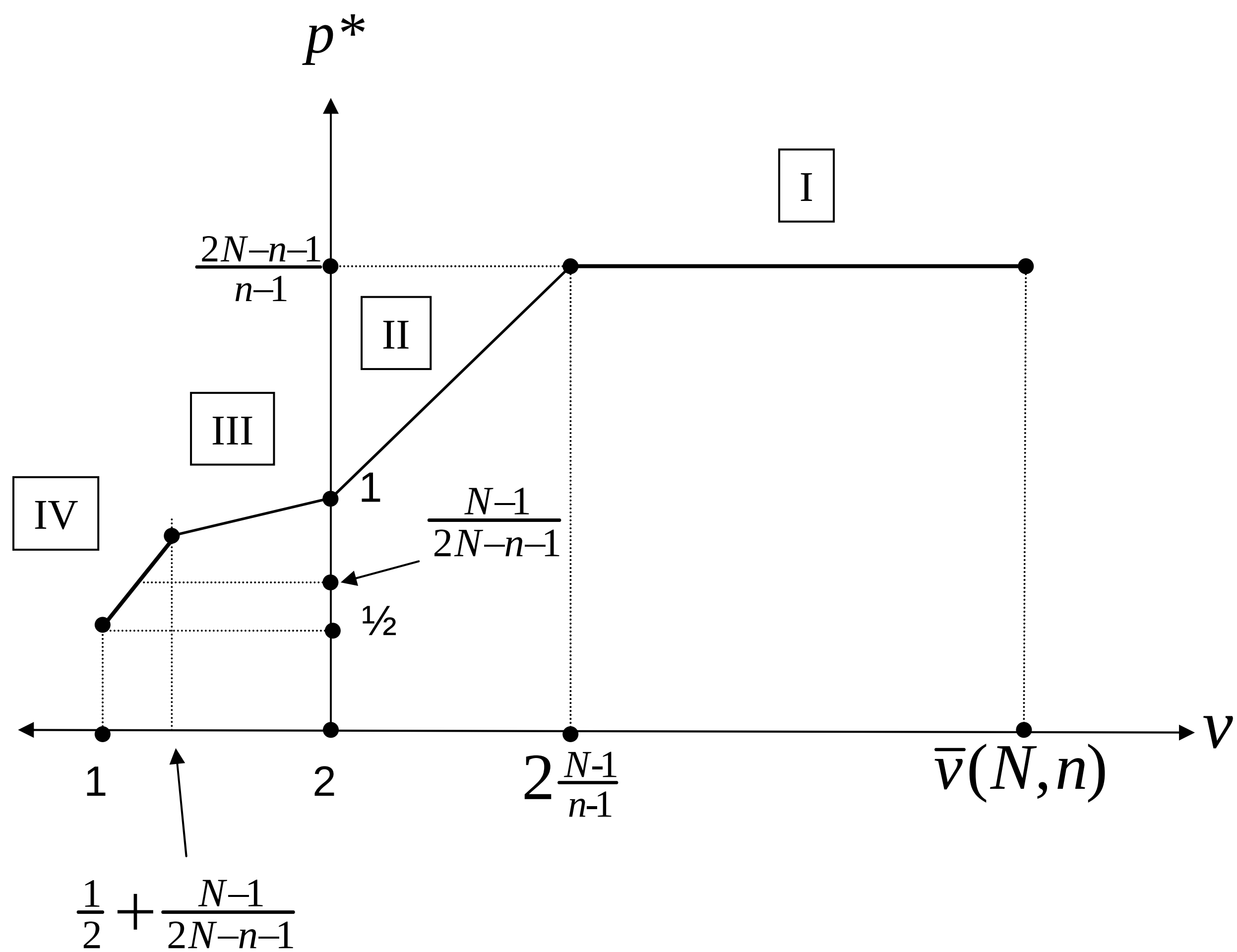

Figure 1
Price as a function of value

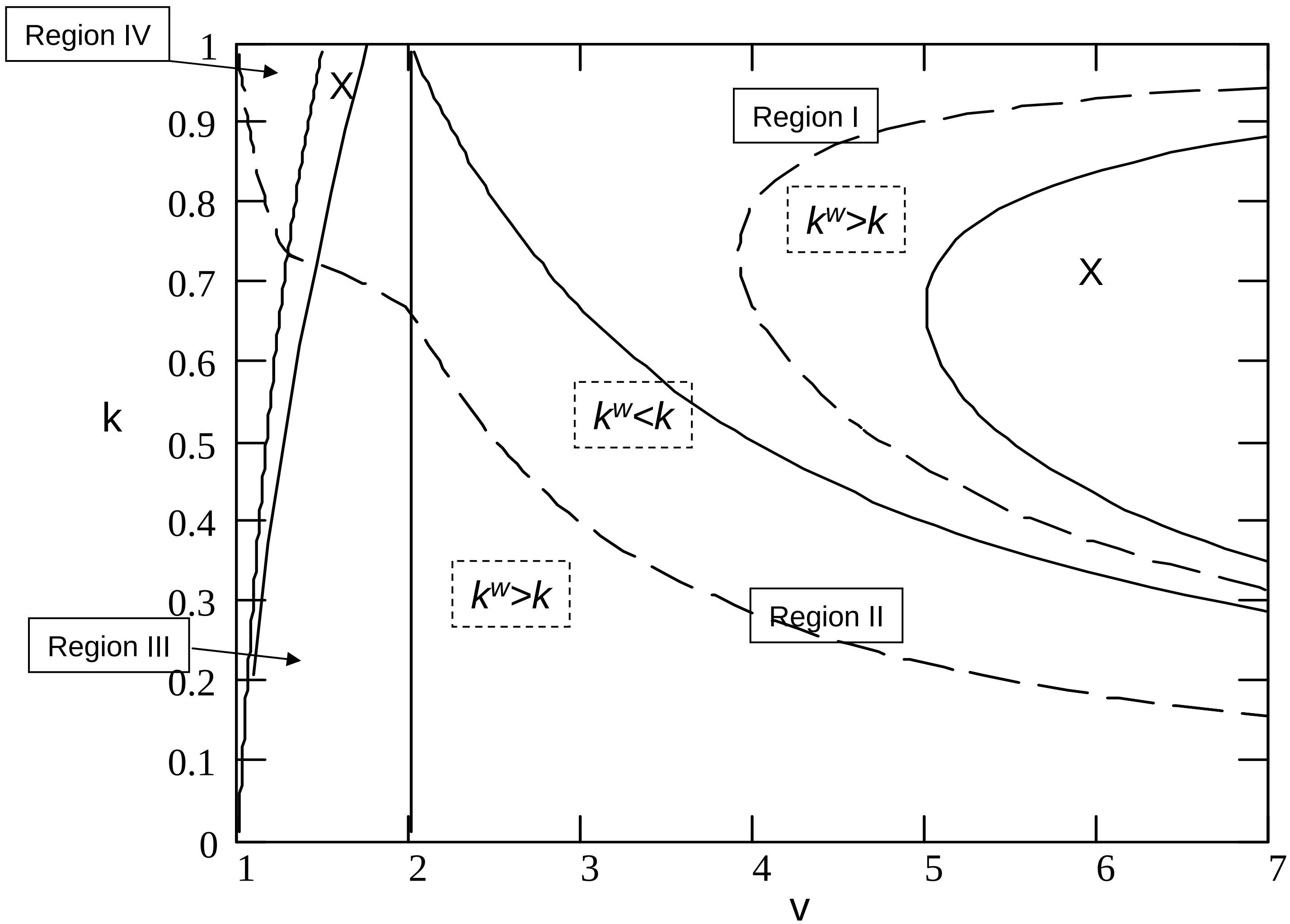

Figure 2
Equilibrium and optimal variety

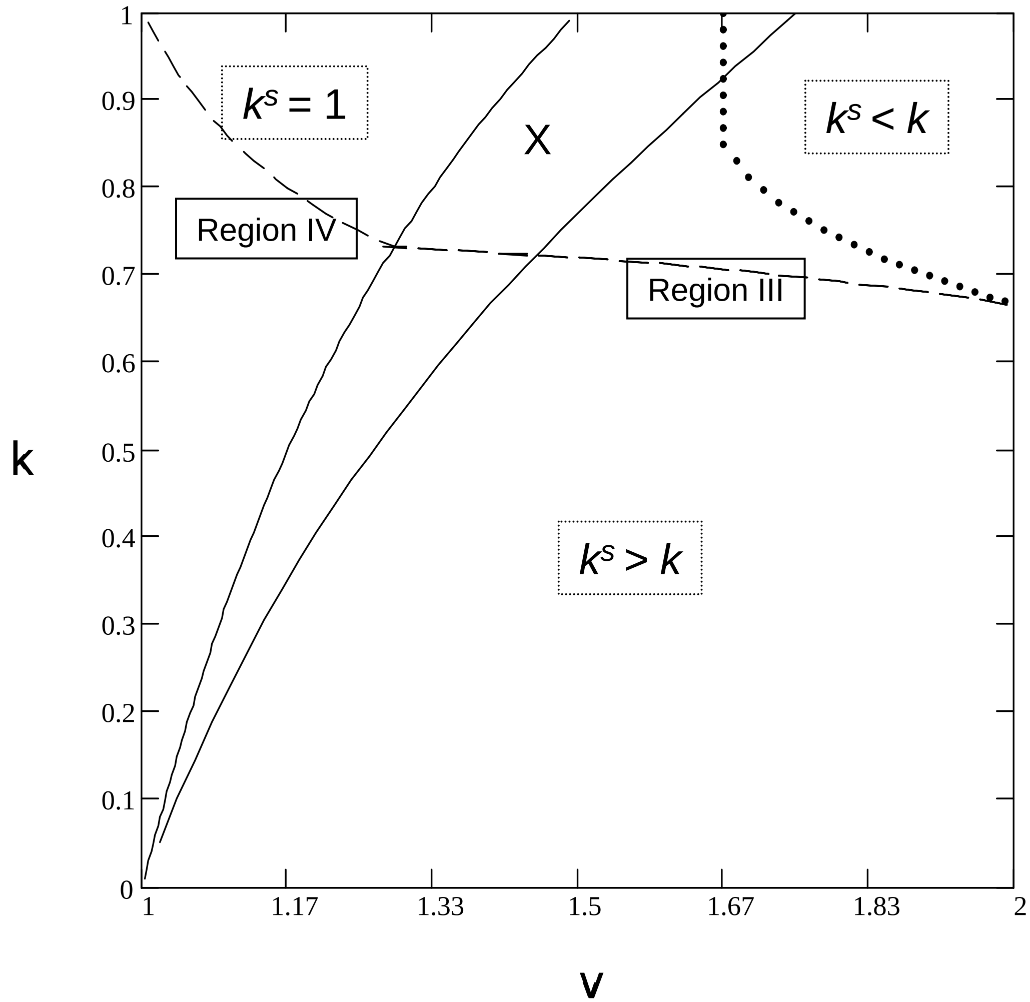

Figure 3
Equilibrium and second-best variety