The Determination of Price Discount in Pricing and Inventory Control of Perishable Goods: The Role of Time and Price Demand

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Abstract

Determining the appropriate inventory control policies and product prices is considered as an important dimension in the competitive markets of perishable products. The customers' willingness to pay for perishable product declines as the product's expiry date is approaching. In this paper, we considered the price discount in pricing model as an alternative approach to influence the consumers' purchase decision. The model simultaneously determines the optimal values of selling price, discount time, and replenishment schedule so that the total profit is maximized. However, since the demand increases during the discount interval process, different demand rate functions, i.e. the functions of price and time are used in the model. At first, we model the problem without discount and its solution shows an impossible result in reality because the replenishment time is very short. But then, by including discount in the model, more products are sold and thus the profit increases. Finally, we solved two numerical examples used an iterative algorithm by performing a sensitivity analysis of the model parameters. We discussed the specific managerial insights of the study as well.

Keywords: Pricing; Price discount; Inventory control; Demand rate function.

1. Introduction

Commodities, which lose their value over time, are called "perishable products". In this sense, medicines, fruits and vegetables, seasonal and fashion goods, electronics, etc. are considered as perishable products. Because of technological advances, competitive markets, and the significance of providing fresh products for customers, the sales management and pricing of fresh goods are important. The lifetime of goods and their expiry dates are considerable for the customers, so marking down the price in this period is an incentive approach for selling more goods.

In many businesses, dynamic pricing is considered as a mechanism to attract more customers. Diaz (2006) argued that the effect of price on consumer decision depends on how he evaluate the products. Especially in case of perishable food products, many consumers believe that fresh products have a higher value than the expired ones. When the prices are the same, they prefer the fresher products. To encourage the customers to buy perishable products approaching their expiry dates, using price discount is an effective approach. Tajbakhsh, Lee, and Zolfaghari (2011) designed an inventory model with stochastic price discount and numerical analysis to show the cost savings by using discount offer.

Inventory control and pricing of perishable goods have widely been considered by researchers (Nahmias, 1982; Raafat, 1991; Goyal & Giri, 2001; Karaesmen, Scheller–Wolf & Deniz, 2011). Ghare and Schrader (1963) were the first researchers who considered optimal replenishment policy and used Economic order quantity (EOQ) model for perishable goods. Eilon and Mallaya (1966) considered an inventory model with the demand function depending on the price. Wee (1995) addressed replenishment policy and pricing by considering price-dependent demand in which the price declines over time.

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In the aforementioned pricing and inventory control literature, the price discount model that directly affects demand rate function was not considered. In the literature of the available models, price markdown or discount is considered as an increasing and constant coefficient in the demand rate function in sale period. It means the demand rate function in these models is modified by constant coefficient and the models were analyzed by increased demand rate function (Maihami & Karimi, 2014). In this paper, due to the discount offering of the perishable goods like fashion ones in retail markets, two distinct prices are considered in initial of period and discount interval. Also, the demand rate has two functions that are explain in section 2.3.

The rest of this paper is organized as follows. In section 2, the model assumptions and notations are defined and the demand rate function during different intervals is introduced. Sections 3 and 4 include the mathematical models with and without discount and total profit functions. Afterwards, in section 5, two numerical examples are presented, and numerical values of the optimal price, optimal discount time, optimal replenishment time, and optimal order quantity are obtained. Also in this section, sensitivity analysis is done for example 1 and the results are extracted. Finally, conclusions and suggestions for future research are presented in section 6.

2. Model description
2.1. Assumptions
In this study, we assumed that the demand rate involves the two functions of time and price, where the demand rate has a jump in the discount time. Also because of the deterministic demand rate, the surplus and shortage of good are not considered at the end of period. Due to the high cost of transportation and supplying which are considered in purchase costs, determining the optimal frequency of order or replenishment schedule is important.

According to the assumption of surplus and shortage of goods, the order quantity of initial period is equal to the whole period. Other assumptions of the problem are considered as follows:

   i.   A single perishable item is assumed and there is no constraint in order quantity.
   ii.   The lead time is zero.
   iii.   All of the parameters are deterministic.
   iv.   The lead time is zero.
   v.   Time horizon is infinite.

The following notations are used throughout the paper:

2.2. Notation
$D_1(p,t)$ demand rate function before discount time, which depends on time and selling price.
$D_2(p,t)$ demand rate function after discount time, which depends on time and selling price.
c constant purchase cost per unit
A order cost
h holding cost per unit per unit time
p selling price per unit, where $p > c$
t_1 discount time
T length of replenishment cycle time
Q order quantity
α percentage of discount
$I(t)$ inventory level at time $t \in (0,T)$ for model without discount
$I_1(t)$ inventory level at time $t \in (0,t_1)$
$I_2(t)$ inventory level at time $t \in (t_1,T)$
p^* optimal selling price per unit
t_1^* optimal discount time
T^* optimal length of the replenishment cycle time
TP total profit per unit time of the inventory system
TP^* optimal total profit per unit time of the inventory system

2.3. Demand function
The basic demand rate is a function of time and price during the selling period. Due to the importance of time in buying and selling perishable products, demand change over time is considered exponential. Price is a necessary factor in buying perishable products and demand function is considered as the linear function of price (Yu-Chung & Ji, 2008). In the proposed model, price is considered high in the initial of period and then after a while, the price is reduced significantly for encouraging customers to buy those more. According to the field study done, these goods have the highest levels of sale in discount intervals. Pursuant to the evidence obtained by interviewing ten salesmen, the demand rate function is considered with significant jump after discount offering. In surveys conducted in almost? Cases, immediately after discount offering (the first day off), sales have had more than double growth in comparison to the day before offering

discount. So according to literature (e.g.Yu-Chung & Ji, 2008) and the simulation of assumed sales quantity obtained by interviewing, the demand rate function is considered as below:

$$D_i(p,t) = \begin{cases} D_1(p,t) = (a-bp)e^{-\lambda t}, & b>0, a>0, \lambda>0 \\ D_2(p,t) = (a-bp(1-\alpha))t^{\beta}e^{-\lambda t}, & b>0, a>0, \lambda>0, \beta>1 \end{cases} \tag{1}$$

The firm offers a lower price in a markdown interval to attract customers to purchase products approaching their expiry date. As a result, two different prices are set in the selling period:

$$P(t) = \begin{cases} P & [0, t_1] \\ P(1-\alpha) & [t_1, T] \end{cases} \tag{2}$$

3. Simple model without discount

There is no shortage and surplus at the end of period T, so maximum inventory level is equal to the demand quantity in this period, and is expressed as:

$$I(0) = \int_0^T D_1(p,t)dt = \int_0^T (a-bp)e^{-\lambda t}dt = \frac{e^{-T\lambda}(-1+e^{T\lambda})(a-bp)}{\lambda} \tag{3}$$

Inventory level during the time interval $[0, T]$ is calculated by:

$$I(t) = I(0) - \int_0^t D_1(p,t)dt = -\frac{e^{-t\lambda}(-1+e^{t\lambda})(a-bp)}{\lambda} + \frac{e^{-T\lambda}(-1+e^{T\lambda})(a-bp)}{\lambda}, \ 0 \le t \le T \tag{4}$$

3.1. Total profit function

To obtain profit function, we can obtain the costs and the sales revenue per cycle, which consist of the following elements:

A: the ordering cost

HC: the holding cost: with respect to I(t) can be calculated as:

$$HC = h\left[\int_0^T I(t)dt\right] = \frac{e^{-T\lambda}h(a-bp)(-1+e^{T\lambda}-T\lambda)}{\lambda^2} \tag{5}$$

PC: the purchasing cost: regarding to Q, purchasing cost is calculated according to:

$$PC = c*Q = c*\frac{e^{-T\lambda}(-1+e^{T\lambda})(a-bp)}{\lambda} \tag{6}$$

SR: the sales revenue: the revenue of selling the product with price p in interval [0,T] can be expressed as:

$$SR = p\frac{e^{-T\lambda}(-1+e^{T\lambda})(a-bp)}{\lambda} \tag{7}$$

As a result, total profit per unit time can be calculated with following equation:

$$TP = \frac{SR-A-HC-PC}{T} = \frac{-A+\frac{e^{-T\lambda}(-1+e^{T\lambda})(-c+p)(a-bp)}{\lambda} - \frac{e^{-T\lambda}h(a-bp)(-1+e^{T\lambda}-T\lambda)}{\lambda^2}}{T} \tag{8}$$

3.2. Optimal solution

TP (p, T) is a function of p and T. Thus, the necessary conditions for the total profit per unit time to be maximized are $\partial TP(p,T)/\partial p = 0$ and $\partial TP(p,T)/\partial T = 0$, and the equations have simultaneously solution. Hessian matrix is calculated and the determinant of it in our example was positive. So the profit function was concave.

$$\frac{\partial TP}{\partial p} = \frac{-\frac{be^{-T\lambda}(-1+e^{T\lambda})(-c+p)}{\lambda} + \frac{e^{-T\lambda}(-1+e^{T\lambda})(a-bp)}{\lambda} + \frac{be^{-T\lambda}h(-1+e^{T\lambda}-T\lambda)}{\lambda^2}}{T} = 0$$

$$\frac{\partial TP}{\partial T} = -\frac{-A+\frac{e^{-T\lambda}(-1+e^{T\lambda})(-c+p)(a-bp)}{\lambda} - \frac{e^{-T\lambda}h(a-bp)(-1+e^{T\lambda}-T\lambda)}{\lambda^2}}{T^2} + \frac{1}{T}\left((-c+p)(a-bp) - e^{-T\lambda}(-1+e^{T\lambda})(-c+p)(a-\right.$$

$$\left. bp) - \frac{e^{-T\lambda}h(a-bp)(-\lambda+e^{T\lambda}\lambda)}{\lambda^2} + \frac{e^{-T\lambda}h(a-bp)(-1+e^{T\lambda}-T\lambda)}{\lambda}\right) = 0$$

4. Modeling perishable products with discount

In the initial of the period, we assumed that the order quantity is Q and the lead time is zero. Maximum inventory in the first stage of the period is equal to order quantity because there is no inventory from the previous period. On the other hand, there is no surplus and shortage in the end of the period due to the deterministic demand and no shortage of perishable good is permitted. Note that in this paper, perishable means no inventory will be destroyed, but its value reduces in customers' view and the demand rate is decreased over time. So for encouraging the customers to buy more

and increase the demand rate, the price discount is taken into account. As mentioned before, the demand rate functions during [0, t_1] and [t_1, T] intervals are different. During the interval [0, t_1], reduction of inventory descends over time. The product is sold in price p and the inventory level in this interval at time t is shown by I_1 (t).

During the interval [t_1, T], the product is sold in price discount p (1-α). Because of discount in this interval, first, a moderate growth accrues in demand; however, it reduces gradually. The inventory level in this interval at time t is shown by I_2 (t), Figure 1. As shown in the dashed line of the diagram, without discount, the level of inventory will continue.
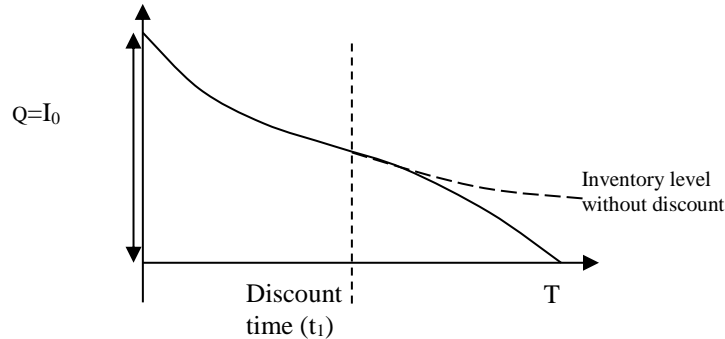

Figure 1. Inventory level in period T

There is neither shortage nor surplus at the end of period T, so maximum inventory level is equal to the demand quantity in this period, and is expressed as:

$$I_1(0) = \int_0^{t_1} D_1(p,t)dt + \int_{t_1}^T D_2(p,t)dt = \int_0^{t_1}(a-bp)e^{-\lambda t}dt + \int_{t_1}^T (a-bp(1-\alpha))t^3 e^{-\lambda t}dt$$

$$= \frac{e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)}{\lambda} + \frac{1}{\lambda^4}(a+bp(-1+\alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda))))$$

(9)

Inventory level during the time interval [0, t_1] is calculated by:

$$I_1(t) = I_1(0) - \int_0^t D_1(p,t)dt$$

$$= -\frac{e^{-t\lambda}(-1+e^{t\lambda})(a-bp)}{\lambda} + \frac{e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)}{\lambda} + \frac{1}{\lambda^4}(a+bp(-1+\alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))), \ 0 \le t \le t_1$$

(10)

Inventory level during the time interval [t_1, T] is calculated by:
$$I_2(t) = I_1(t) - \int_{t_1}^t D_2(p,t)dt$$

$$= -\frac{1}{\lambda^4}(a+bp(-1+\alpha))\left(e^{-t\lambda}\left(-6 - t\lambda(6 + t\lambda(3 + t\lambda))\right) + e^{-t_1\lambda}\left(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda))\right)\right) +$$
$$\frac{1}{\lambda^4}(a+bp(-1+\alpha))\left(e^{-T\lambda}\left(-6 - T\lambda(6 + T\lambda(3 + T\lambda))\right) + e^{-t_1\lambda}\left(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda))\right)\right), \ t_1 \le t \le T$$

(11)

4.1. Total profit function
Based on the obtained inventory levels, we can obtain the inventory costs and the sales revenue per cycle, which consist of the following elements:

A: the ordering cost

HC: the holding cost: with respect to I_1 (t) and I_2(t) can be calculated as:

$$HC = h[\int_0^{t_1} I_1(t)dt + \int_{t_1}^T I_2(t)dt]$$

$$= \frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)t_1\lambda^3 + (a-bp)\lambda^2(1-e^{-t_1\lambda}-t_1\lambda) + t_1(a+bp(-1+\alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda))))) + \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a +$$

(12)

$bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda))))))))$

PC: the purchasing cost: regarding Q, the purchase cost is calculated according to:

PC= $c * Q$

$$= c * \left[\frac{e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)}{\lambda} + \frac{1}{\lambda^4}(a + bp(-1 + \alpha))\left(e^{-T\lambda}\left(-6 - T\lambda\left(6 + T\lambda(3 + T\lambda)\right)\right) + e^{-t_1\lambda}\left(6 + t_1\lambda\left(6 + t_1\lambda(3 + t_1\lambda)\right)\right)\right)\right] \tag{13}$$

To obtain profit per unit time, initially total revenue and costs are calculated, and then, profit is obtained from their subtraction.

SR: The sales revenue: the revenue of selling the product with price p at interval [0, t_1] and the revenue of selling at discount price p (1-α) at interval [t_1,T] can be expressed as:

$$SR = p \int_0^{t_1} D_1(p,t) + p(1 - \alpha) \int_{t_1}^T D_2(p,t)$$

$$= p \frac{e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)}{\lambda} + (p(1 - \alpha)) \frac{1}{\lambda^4}(a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) \tag{14}$$

As a result, total profit per unit time can be calculated with following equation:

$$TP = \frac{SR - A - HC - PC}{T}$$

$$= \frac{1}{T}(-A + \frac{e^{-t_1\lambda}(-1 + e^{t_1\lambda})(-c + p)(a - bp)}{\lambda} + \frac{1}{\lambda^4}(-c + p(1 - \alpha))(a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)t_1\lambda^3 + (a - bp)\lambda^2(1 - e^{-t_1\lambda} - t_1\lambda) + t_1(a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda))))))))) \tag{15}$$

The main purpose of this paper is to determine an optimal ordering policy that corresponds to maximizing the total profit per unit time. First, we prove that the optimal solution of (t_1; T) exists for any given p. Then, a unique p exists that maximizes the total profit per unit time for any given value of t_1 and T.

4.2. Optimal solution

TP (p, t_1,T) is a function of p, t_1, T. Thus, the necessary conditions for the total profit per unit time (15) for any given p to be maximized are $\partial TP (p, t_1,T)/\partial t_1 = 0$ and $\partial TP(p, t_1,T)/\partial T = 0$, that is:

$$\frac{\partial TP}{\partial T} = -\frac{1}{T^2}(-A + \frac{e^{-t_1\lambda}(-1 + e^{t_1\lambda})(-c + p)(a - bp)}{\lambda} + \frac{1}{\lambda^4}(-c + p(1 - \alpha))(a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)t_1\lambda^3 + (a - bp)\lambda^2(1 - e^{-t_1\lambda} - t_1\lambda) + t_1(a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda))))))))) + \frac{1}{T}(-\frac{1}{\lambda^4}ht_1(a + bp(-1 + \alpha))(e^{-T\lambda}(-T\lambda(T\lambda^2 + \lambda(3 + T\lambda)) - \lambda(6 + T\lambda(3 + T\lambda))) - e^{-T\lambda}\lambda(-6 - T\lambda(6 + T\lambda(3 + T\lambda)))) + \frac{1}{\lambda^4}(-c + p(1 - \alpha))(a + bp(-1 + \alpha))(e^{-T\lambda}(-T\lambda(T\lambda^2 + \lambda(3 + T\lambda)) - \lambda(6 + T\lambda(3 + T\lambda))) - e^{-T\lambda}\lambda(-6 - T\lambda(6 + T\lambda(3 + T\lambda)))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}\lambda(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}\lambda(-24 + T\lambda(-12 + T\lambda(-4 - 2T\lambda + t_1\lambda) + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda))) + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda)))))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda)))))))))) = 0 \tag{16}$$

$$\frac{\partial TP}{\partial t_1} = \frac{1}{T}((-c + p)(a - bp) - e^{-t_1\lambda}(-1 + e^{t_1\lambda})(-c + p)(a - bp) + \frac{1}{\lambda^4}(-c + p(1 - \alpha))(a + bp(-1 + \alpha))(e^{-t_1\lambda}(t_1\lambda(t_1\lambda^2 + \lambda(3 + t_1\lambda)) + \lambda(6 + t_1\lambda(3 + t_1\lambda))) - e^{-t_1\lambda}\lambda(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)\lambda^3 + (a - bp)t_1\lambda^4 - e^{-t_1\lambda}(-1 + e^{t_1\lambda})(a - bp)t_1\lambda^4 + (a - bp)\lambda^2(-\lambda + e^{-t_1\lambda}\lambda) + (a + bp(-1 + \alpha))(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) + t_1(a + bp(-1 +$$

$\alpha))(e^{-t_1\lambda}(t_1\lambda(t_1\lambda^2 + \lambda(3 + t_1\lambda)) + \lambda(6 + t_1\lambda(3 + t_1\lambda))) - e^{-t_1\lambda}\lambda(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda))))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda)))))))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{t_1\lambda}\lambda(6 + T\lambda(6 + T\lambda(3 + T\lambda))) + e^{T\lambda}(t_1\lambda(t_1\lambda^2 + \lambda(6 + t_1\lambda)) + \lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}\lambda(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda))))))))))) = 0$ (17)

Theorem 1: For any given p, we have:
a) The system of (16) and (17) has a unique solution.
b) The solution in (a) satisfies the second order condition for maximum.

Proof: Please see appendix A

It is clear from the analysis so far that, for any given price, the points T* and t_1*, which maximize the total profit per unit time, not only exist, but also are unique. Next, the condition under which the optimal selling price also exists and is unique is considered. For any T* and t_1*, the first-order necessary condition for TP (p, t_1*, T*) to be maximized is ∂TP(p, t_1*, T*)/∂T=0, that is:

$\frac{\partial TP(p,t_1^*,T^*)}{\partial p} = \frac{1}{T}(-\frac{be^{-t_1\lambda}(-1+e^{t_1\lambda})(-c+p)}{\lambda} + \frac{e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)}{\lambda} + \frac{1}{\lambda^4}(a + bp(-1 + \alpha))(1 - \alpha)(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) + \frac{1}{\lambda^4}b(-c + p(1 - \alpha))(-1 + \alpha)(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^4}h(-be^{-t_1\lambda}(-1 + e^{t_1\lambda})t_1\lambda^3 - b\lambda^2(1 - e^{-t_1\lambda} - t_1\lambda) + bt_1(-1 + \alpha)(e^{-T\lambda}(-6 - T\lambda(6 + T\lambda(3 + T\lambda))) + e^{-t_1\lambda}(6 + t_1\lambda(6 + t_1\lambda(3 + t_1\lambda)))) - \frac{1}{\lambda^5}be^{-(T+t_1)\lambda}h(-1 + \alpha)(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4 - T\lambda + t_1\lambda)))))))))) = 0$ (18)

It is clear that equation (18) has a solution. In addition, the second-order deviation of TP (p, t_1*, T*) with respect to price is:

$\frac{\partial^2 TP(p,t_1^*,T^*)}{\partial p^2} = \frac{-\frac{2be^{-t_1\lambda}(-1+e^{t_1\lambda})}{\lambda} + \frac{2b(1-\alpha)(-1+\alpha)(e^{-T\lambda}(-6-T\lambda(6+T\lambda(3+T\lambda)))+e^{-t_1\lambda}(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda))))}{\lambda^4}}{T} < 0$

Consequently, TP (p, t_1*, T*) is a concave function of p for any given T* and t_1*. Also the value of price obtained from (18) is unique. So it is proved that the unique value of p obtained from (18) maximizes TP (p, t_1*, T*). For solving this problem, a heuristic algorithm is used.

4.3. Algorithm

We used a simple heuristic algorithm (Maihami & Kamalabadi, 2012) to obtain the optimal solution (p*, t_1*, T*).
Step 1: start with the initial value of p_j=p_1.
Step 2: find the optimal values of t_1*, T* by solving (16) and (17) for any given p_j.
Step 3: from the result of the last Step, determine the optimal p_{j+1} by (18).
Step 4: if the difference between p_j, p_{j+1} is small, set p*= p_{j+1}. Then (p*, t_1*, T*) is the optimal solution, and stop. Otherwise, go back to Step 2.
By using the above algorithm, the optimal solution (p*, t_1*, T*) is obtained. Since the order quantity is equal to the first inventory, Q* can be obtained by (9) and TP* is obtained by (15).

5. Numerical examples

The above algorithm is applied to solve the following numerical examples and illustrate the solution process and results. Mathematica 9 was applied for solving the examples.

Example1. The following parameters and functions are used for the example.
$D_1(p,t) = (500 - 0.5p)e^{-0.98t}$, $D_2(p,t) = (500 - 0.5p)t^3e^{-0.98t}$, c=200/ per unit, α=0.3, h=40 /per unit /per unit time, A=250/ per order
Without discount,t we just use $D_1(p,t)$ for all cycle. After computation, the results are:

P*=600.77, T*= 0.078, TP*=73517.45, Q*=14.98.

With discount, we use both demand functions. First, we use p_1=600. After five iteration, we have P*=854.79 , t_1*=0.85, T*=4.10, TP*=58105.40, Q*=772.96. The computational results are shown in Table 1.
As the results show, without discount, the replenishment time that is obtained is so small. It increases the order cost and very high order cost in reality is impossible. So use discount mechanism is a good method for increasing demand and

profit. Also in T*=4.10, if the retailer sells the product without discount, the profit becomes $TP = 17646.46$. We can understand passing (passage) of time has a negative effect on profit so with discount we can increase the profit.

Table 1. Computational results of Example 1 with discount.

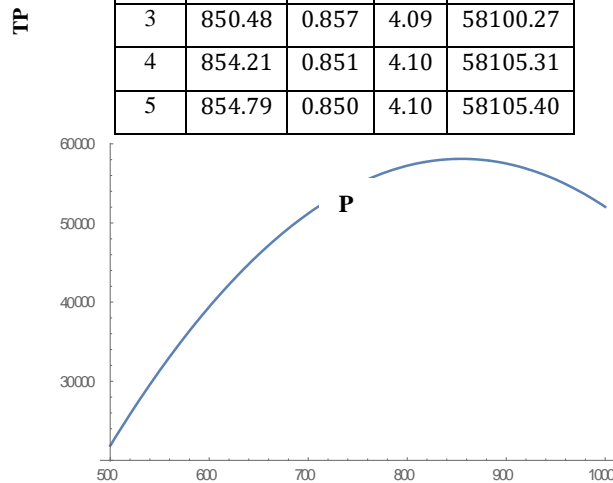| Step	p_j	t_1^*	T^*	TP*
1	600	1.11	3.62	41963.52
2	826.61	0.891	4.02	57908.03
3	850.48	0.857	4.09	58100.27
4	854.21	0.851	4.10	58105.31
5	854.79	0.850	4.10	58105.40

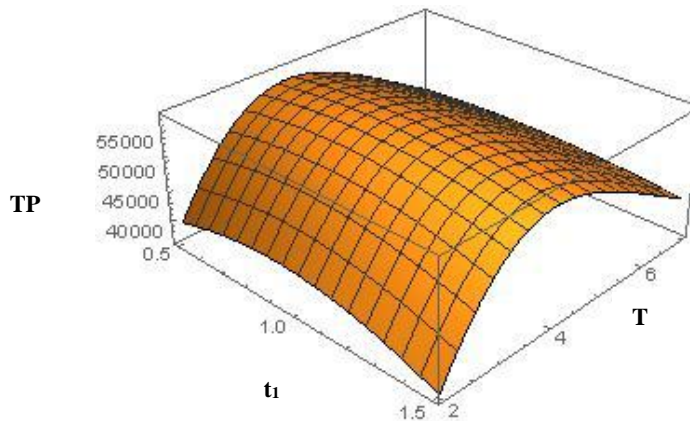**Figure 2.** Graphical representation of TP(p| t_1^*, T^*)

Figure 3. Graphical representation of TP (p^*, t_1, T)

Note that we run the numerical results at the price interval [400,100]. The numerical results reveal that TP^* is strictly concave in p, (see Figure 2). Also as shown in figure 3, TP is concave of t_1 and T. In this figure, the interval of t_1 is [0.5, 1.5] and the interval of T is [0.5, 1.5] and the surface has been run by optimal price. As a result, the local maximum obtained here from the heuristic algorithm is indeed the global maximum solution.

Example 2. The following data are used for the example:

$D_1(p,t) = (200 - 3p)e^{-0.98t}$, $D_2(p,t) = (200 - 3p)t^3 e^{-0.98t}$, c=20/ per unit, α=0.5, h=1 /per unit/per unit time, A=250/ per order.

First we use p_1=40. After five iterations, we have P^*=59.67 , t_1^*=0.80, T^*=4.6, TP^*=1339.23, Q^*=327.738. The results are shown in Table 2.

Table 2. Computational results of Example 1

Step	p_j	$t1*$	$T*$	TP*
1	40	1.24	3.44	738.21
2	56.43	0.89	4.5	1322.51
3	59.31	0.81	4.6	1339.03
4	59.64	0.80	4.6	1339.23
5	59.67	0.80	4.6	1339.23

5.1. Sensitivity analysis

The effects of changes in the system parameters on p^*, t_1^*, T^*, TP^* and Q^* are done, and based on this sensitivity analysis, some managerial implications are extended. The results of Example 1 are used for sensitivity analysis.

Table 3. Sensitivity analysis with respect to the model parameters

Parameter	Value	p^*	t_1^*	T^*	TP^*	Q^*
A	125	854.83	0.85	4.10	58135.9	772.89
	188	854.83	0.85	4.10	58120.5	772.89
	313	855.14	0.85	4.11	58090.4	774.89
	375	855.14	0.85	4.11	58075.3	774.89
C	100	788.07	0.9	4.15	78262.9	890.08
	150	821.35	0.88	4.13	67859.6	832.46
	250	889.72	0.8	4.09	49008.4	717.17
	300	926.36	0.73	4.07	40583.9	657.33
h	20	831.38	0.87	4.46	68548.5	898.04
	60	869.36	0.83	3.71	48976.3	651.75
	100	808.62	0.94	2.43	35220.1	345.11
	120	786.61	0.98	2.26	31212.0	311.15
α	0.1	705.19	0.98	4.02	63151.4	732.09
	0.2	773.12	0.94	4.05	61387.7	747.07
	0.4	924.22	0.71	4.1	52442.2	837.13
	0.45	991.41	0.35	4.18	49082.6	866.02

The effects of changes in the value of parameters A, c, h or α on the p^*, t_1^*, T^*, TP^* and Q^* based on Example 1 are shown in Table 3. The sensitivity analysis is performed by changing each value of the parameters by -50%, -25%, +25%, and +50%, taking one parameter at a time and keeping the remaining parameter values unchanged.

The following observations are indicated based on the sensitivity analysis shown in Table 3:

1. When the values of parameters A, c or α increase, the optimal selling price of p^* will increase. And when the value of parameter h increases, p^* will first increase and then decrease. Moreover, p^* is weakly positively sensitive to changes in parameter A, whereas p^* is highly positively sensitivity to changes in parameters c or α. It is reasonable that the purchase cost and discount have strong and positive effect on the optimal selling price.

2. When the value of parameter A increases, the optimal discount time t_1^* remains unchanged while it decreases as the values of parameters c or α increase. Also when h increases, the optimal discount time t_1^* will initially decrease and then increase. That is, the more increase in the cost and discount percentage is observed, the sooner the discount time will start.

3. When the values of parameters A or α increase, the optimal length of the replenishment cycle time T^* will increase. This shows that the higher the order cost and discount percentage are, the longer the length of the replenishment cycle will be. While the values of parameters c or h increase, the optimal length of the replenishment cycle time T^* will

decrease. It means that increasing the purchase cost and the holding cost makes short the length of the replenishment cycle to sell the products quickly.

4. When the values of parameters A or α increase, the optimal order quantity Q^* will increase. Due to discount, the increase rate is slow. In fact, without discount, the order quantity must decrease when the value of parameter A increases. The corresponding managerial insight is that as the purchase cost and the order cost increase, the order quantity decreases.

5. When the values of parameters A, c, h or α increase, the optimal total profit per unit time TP^* will decrease. This implies that increases in costs have a negative effect on the total profit per unit time.

6. Conclusion

In this paper, a model of inventory control and pricing for perishable goods in terms of discounts was offered. Also a simple model without discount is presented to show the efficiency of the proposed model. In the proposed model, it was proven that the objective function value obtained from the optimal values is unique and optimal. Finally, two numerical examples using the heuristic algorithm were described in the model and the results were compared. We solved the first example without discount and showed the result. As shown, the replenishment time acquired from simple model is very short and in real world, it is impossible. So discount mechanism could help retailer to save the costs of transportation. Sensitivity analysis was done to show the effects of changes in the value of parameters on the decision variable and objective function. The results demonstrate d that the objective function is concave, and the optimal value for profit is global. The model presented in this paper is comprehensive and flexible to different values of the parameters of the demand function. This paper can be extended in several ways. For instance, we can extend it by considering a variable percentage discount. Another aspect is advertising policies, delays in payment, and models of coordination in the system (supply chain). It is possible to consider multi product and their complementary and substitution effects and examine the results.

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Appendix A

A.1. proof of theorem 1

Because of very high complications in Eqs.(18) and (19), the straightforward proof does not exist. So, only the process of proof is explained. First, t_1 (or T) is obtained based on T (or t_1) from Eqs.(18) and (19) (call this function F(x)). For F(x), the first-order derivative with respect to x is taken and shown that F(x) is strictly a decreasing or increasing function. Next, the "Intermediate Value Theorem" is used and the proof is completed. The simple kind of this proof can be found in the literature (e.g. ; Chang, Teng, Ouyang, & Dye, 2006; Dye, 2007; Yang, Quyang & WU, 2009). Let (t_1^*, T^*) be the solution of Eqs.(18) and (19)which can be obtained as follows:

$$\frac{\partial^2 TP}{\partial T^2}\Big|_{(t_1^*,T^*)} = \frac{1}{T^3}2(-A + \frac{e^{-t_1\lambda}(-1+e^{t_1\lambda})(-c+p)(a-bp)}{\lambda} + \frac{1}{\lambda^4}(-c+p(1-\alpha))(a+bp(-1+\alpha))(e^{-T\lambda}(-6-T\lambda(6+T\lambda(3+T\lambda))) + e^{-t_1\lambda}(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda)))) - \frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)t_1\lambda^3 + (a-bp)\lambda^2(1-e^{-t_1\lambda}-t_1\lambda) + t_1(a+bp(-1+\alpha))(e^{-T\lambda}(-6-T\lambda(6+T\lambda(3+T\lambda))) + e^{-t_1\lambda}(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda))))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda))))))))) - \frac{1}{T^2}2(-\frac{1}{\lambda^4}ht_1(a+bp(-1+\alpha))(e^{-T\lambda}(-T\lambda(T\lambda^2+\lambda(3+T\lambda))-\lambda(6+T\lambda(3+T\lambda)))-e^{-T\lambda}\lambda(-6-T\lambda(6+T\lambda(3+T\lambda)))) + \frac{1}{\lambda^4}(-c+p(1-\alpha))(a+bp(-1+\alpha))(e^{-T\lambda}(-T\lambda(T\lambda^2+\lambda(3+T\lambda))-\lambda(6+T\lambda(3+T\lambda)))-e^{-T\lambda}\lambda(-6-T\lambda(6+T\lambda(3+T\lambda))))-\frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}\lambda(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda(-24+T\lambda(-12+T\lambda(-4-2T\lambda+t_1\lambda)+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))))))) + \frac{1}{T}(-\frac{1}{\lambda^4}ht_1(a+bp(-1+\alpha))(e^{-T\lambda}(-2T\lambda^3-2\lambda(T\lambda^2+\lambda(3+T\lambda)))-2e^{-T\lambda}(-T\lambda(T\lambda^2+\lambda(3+T\lambda))-\lambda(6+T\lambda(3+T\lambda))) + e^{-T\lambda}\lambda^2(-6-T\lambda(6+T\lambda(3+T\lambda)))) + \frac{1}{\lambda^4}(-c+p(1-\alpha))(a+bp(-1+\alpha))(e^{-T\lambda}(-2T\lambda^3-2\lambda(T\lambda^2+\lambda(3+T\lambda)))-2e^{-T\lambda}(-T\lambda(T\lambda^2+\lambda(3+T\lambda))-\lambda(6+T\lambda(3+T\lambda))) + e^{-T\lambda}\lambda^2(-6-T\lambda(6+T\lambda(3+T\lambda))))-\frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}\lambda^2(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda(T\lambda(-2T\lambda^2+2\lambda(-4-2T\lambda+t_1\lambda)) + 2\lambda(-12+T\lambda(-4-2T\lambda+t_1\lambda)+\lambda(3t_1+T(-4-T\lambda+t_1\lambda))))) + \frac{1}{\lambda^4}2e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}\lambda(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda(-24+T\lambda(-12+T\lambda(-4-2T\lambda+t_1\lambda))+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))-\frac{1}{\lambda^3}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda))))))))))$$

$$\frac{\partial^2 TP}{\partial t_1^2}\Big|_{(t_1^*,T^*)} = \frac{1}{T}(-(-c+p)(a-bp)\lambda + e^{-t_1\lambda}(-1+e^{t_1\lambda})(-c+p)(a-bp)\lambda + \frac{1}{\lambda^4}(-c+p(1-\alpha))(a+bp(-1+\alpha))(e^{-t_1\lambda}(2t_1\lambda^3+2\lambda(t_1\lambda^2+\lambda(3+t_1\lambda)))-2e^{-t_1\lambda}\lambda(t_1\lambda(t_1\lambda^2+\lambda(3+t_1\lambda))+\lambda(6+t_1\lambda(3+t_1\lambda))) + e^{-t_1\lambda}\lambda^2(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda))))-\frac{1}{\lambda^4}h(2(a-bp)\lambda^4-e^{-t_1\lambda}(a-bp)\lambda^4-2e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)\lambda^4-(a-bp)t_1\lambda^5 + e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)t_1\lambda^5 + 2(a+bp(-1+\alpha))(e^{-t_1\lambda}(t_1\lambda(t_1\lambda^2+\lambda(3+t_1\lambda))+\lambda(6+t_1\lambda(3+t_1\lambda)))-e^{-t_1\lambda}(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda)))) + t_1(a+bp(-1+\alpha))(e^{-t_1\lambda}(2t_1\lambda^3+2\lambda(t_1\lambda^2+\lambda(3+t_1\lambda)))-2e^{-t_1\lambda}\lambda(t_1\lambda(t_1\lambda^2+\lambda(3+t_1\lambda))+\lambda(6+t_1\lambda(3+t_1\lambda))) + e^{-t_1\lambda}\lambda^2(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda)))))-\frac{1}{\lambda^3}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))))) + \frac{1}{\lambda^4}2e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{t_1\lambda}\lambda(6+T\lambda(6+T\lambda(3+T\lambda))) + e^{T\lambda}(t_1\lambda(t_1\lambda^2+\lambda(6+t_1\lambda))+\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda))))))))-\frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(2e^{t_1\lambda}\lambda^2(6+T\lambda(6+T\lambda(3+T\lambda))) + e^{T\lambda}(2t_1\lambda^3+2\lambda(t_1\lambda^2+\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda^2(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda))))))))))$$

$$\frac{\partial^2 TP}{\partial t_1 \partial T}\Big|_{(t_1^*,T^*)} = -\frac{1}{T^2}((-c+p)(a-bp)-e^{-t_1\lambda}(-1+e^{t_1\lambda})(-c+p)(a-bp) + \frac{1}{\lambda^4}(-c+p(1-\alpha))(a+bp(-1+\alpha))(e^{-t_1\lambda}(t_1\lambda(t_1\lambda^2+\lambda(3+t_1\lambda))+\lambda(6+t_1\lambda(3+t_1\lambda)))-e^{-t_1\lambda}\lambda(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda))))-\frac{1}{\lambda^4}h(e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)\lambda^3 + (a-bp)t_1\lambda^4-e^{-t_1\lambda}(-1+e^{t_1\lambda})(a-bp)t_1\lambda^4 + (a-bp)\lambda^2(-\lambda+e^{-t_1\lambda}\lambda) + (a+bp(-1+\alpha))(e^{-T\lambda}(-6-T\lambda(6+T\lambda(3+T\lambda))) + e^{-t_1\lambda}(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda)))) + t_1(a+bp(-1+\alpha))(e^{-t_1\lambda}(t_1\lambda(t_1\lambda^2+\lambda(3+t_1\lambda))+\lambda(6+t_1\lambda(3+t_1\lambda)))-e^{-t_1\lambda}\lambda(6+t_1\lambda(6+t_1\lambda(3+t_1\lambda))))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{T\lambda}(24+t_1\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))))))-\frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a+bp(-1+\alpha))(e^{t_1\lambda}\lambda(6+T\lambda(6+T\lambda(3+T\lambda))) + e^{T\lambda}(t_1\lambda(t_1\lambda^2+\lambda(6+t_1\lambda))+\lambda(18+t_1\lambda(6+t_1\lambda))) + e^{t_1\lambda}\lambda(-24+\lambda(6t_1+T(-24+\lambda(6t_1+T(-12+\lambda(3t_1+T(-4-T\lambda+t_1\lambda)))))))))) +$$

$\frac{1}{T}(-\frac{1}{\lambda^4}h(a + bp(-1 + \alpha))(e^{-T\lambda}(-T\lambda(T\lambda^2 + \lambda(3 + T\lambda)) - \lambda(6 + T\lambda(3 + T\lambda))) - e^{-T\lambda}\lambda(-6 - T\lambda(6 + T\lambda(3 + T\lambda)))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}\lambda(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}\lambda(-24 + T\lambda(-12 + T\lambda(-4-2T\lambda + t_1\lambda) + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda))) + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda)))))) - \frac{1}{\lambda^5}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{t_1\lambda}\lambda(T\lambda(T\lambda^2 + \lambda(3 + T\lambda)) + \lambda(6 + T\lambda(3 + T\lambda))) + e^{T\lambda}\lambda(t_1\lambda(t_1\lambda^2 + \lambda(6 + t_1\lambda)) + \lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}\lambda^2(-24 + T\lambda(-12 + T\lambda(-4-2T\lambda + t_1\lambda) + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda))) + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda)))))) - \frac{1}{\lambda^3}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{T\lambda}(24 + t_1\lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda)))))))) + \frac{1}{\lambda^4}e^{-(T+t_1)\lambda}h(a + bp(-1 + \alpha))(e^{t_1\lambda}\lambda(6 + T\lambda(6 + T\lambda(3 + T\lambda))) + e^{T\lambda}(t_1\lambda(t_1\lambda^2 + \lambda(6 + t_1\lambda)) + \lambda(18 + t_1\lambda(6 + t_1\lambda))) + e^{t_1\lambda}\lambda(-24 + \lambda(6t_1 + T(-24 + \lambda(6t_1 + T(-12 + \lambda(3t_1 + T(-4-T\lambda + t_1\lambda))))))))))$

Thus, the determinant of Hessian matrix H at the stationary point (t_1^*, T^*) is:

$$H = \begin{bmatrix} \frac{\partial^2 TP}{\partial T^2} & \frac{\partial^2 TP}{\partial T \partial t_1} \\ \frac{\partial^2 TP}{\partial t_1 \partial T} & \frac{\partial^2 TP}{\partial t_1^2} \end{bmatrix} \rightarrow Det(H) = \left(\frac{\partial^2 TP}{\partial T^2}\Big|_{(t_1^*, T^*)}\right) * \left(\frac{\partial^2 TP}{\partial t_1^2}\Big|_{(t_1^*, T^*)}\right) - \left[\frac{\partial^2 TP}{\partial T \partial t_1}\Big|_{(t_1^*, T^*)}\right]^2 > 0$$

By assuming, $\lambda > 0$ we have:

$$\frac{\partial^2 TP}{\partial T^2}\Big|_{(t_1^*, T^*)} < 0, \frac{\partial^2 TP}{\partial t_1^2}\Big|_{(t_1^*, T^*)} < 0, \left|\frac{\partial^2 TP}{\partial T^2}\Big|_{(t_1^*, T^*)}\right| > \left|\frac{\partial^2 TP}{\partial T \partial t_1}\Big|_{(t_1^*, T^*)}\right|, \left|\frac{\partial^2 TP}{\partial t_1^2}\Big|_{(t_1^*, T^*)}\right| > \left|\frac{\partial^2 TP}{\partial T \partial t_1}\Big|_{(t_1^*, T^*)}\right|$$

$Det(H) > 0$

Hence, the Hessian matrix H at point (t_1^*, T^*) is negative definite. Consequently, we can conclude that the stationary point (t_1^*, T^*) is a global maximum for our optimization problem. This completes the proof.