Prices and Deadweight Loss in Multi-Product Monopoly

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Abstract: The paper investigates prices and deadweight loss in multi-product monopoly (MPM) with linear interrelated demands and constant marginal costs. A prevailing view in the literature is that prices should be lower for complements and higher for substitutes, relative to independent goods. We argue that this view is invalid, and that the price for each good is independent of demand cross-effects, and of the characteristics and number of other goods. This conclusion is robust to various microeconomic foundations of linear demand, including those for Bowley or Shubik demands, and vertically or horizontally differentiated demands. Finally, we show that in all the underlying linear models the deadweight loss amounts to half the total monopoly profit.

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1. Introduction

Most firms do not only sell one, but many interrelated products. For example, supermarkets sell a multitude of substitute, complement or independent goods. Airlines and railway companies sell tickets with different conditions for the same route and oil corporations sell gasoline in petrol stations that differ by their locations.

This paper examines multi-product monopoly (MPM) facing linear interrelated demand and constant unit costs and shows that in this type of models optimal prices and welfare effects can be expressed in a very simple way: As in the textbook case of a single product monopoly, *the monopoly price of each good is the average of its own inverse demand intercept and its own marginal cost, and is thus independent of the characteristics of other products, the interactions between products and the number of products sold. We also show that the deadweight loss is half the monopoly profit.* In contrast, a common view in the literature is that monopoly prices should be lower for complements and higher for substitutes, relative to independent goods (see Section 2). Though intuitively plausible at first sight, this conclusion is actually invalid!

We obtain these elementary results for MPM facing three commonly used linear demand structures, corresponding to the three examples cited above: The standard models for demand with heterogeneous products, vertically (quality) differentiated products and horizontally (spatially) differentiated products. As seen below, this multitude of demand models is motivated by the diversity of economic settings where the issue of MPM pricing has been historically analyzed, often by founding fathers of modern industrial economics. Indeed, while product differentiation is nowadays more intimately connected to oligopolistic markets, several eminent neo-classical economists in the early 20th century have addressed the MPM problem. Some of these studies have arrived at conclusions stressing the relevance of cross-elasticities, which are thus not compatible with our findings. A key objective of this paper is to correct such prevalent oversights in the literature on the issue of MPM pricing. We also address the issue of MPM pricing under non-linear demand and show that our basic insights carry over.

Our results on the deadweight loss may appear surprising at first sight, since the welfare functions of the three models we consider are quite different. These results are a consequence of the fact that when a new product is added to a product line, existing prices do not change. It can be shown that this implies that monopoly profit and the deadweight loss always rise proportionally. Consequently the simple relation between the deadweight loss and profits in the single good case survives regardless of how many products or what types of products are added.

Based on an extensive literature search going back to the beginnings of modern economics, we believe that these simple properties of linear MPM have not been uncovered. An early reference to the problem of MPM is from Wicksell (1901) who argues that "*every retailer possesses, within his immediate circle, what we call an actual sales monopoly*" (1949 p. 87). Without solving for the monopoly prices, he recognizes that they are "*complex and … difficult to unravel*" (p.86). Similarly, Edgeworth (1925) analyzes railway fares of different classes but does not give an explicit solution. Hotelling (1932) provides a numerical example where a monopoly chooses the profit-maximizing prices for first and second-class railway tickets facing different consumer groups, but he does not solve for a general price rule since this is not the main focus of his paper. Robinson (1933) is the first to formally solve the problem of a monopolist selling the same product in different markets using third-degree price discrimination but she explicitly excludes price interdependence such as in "*the case of first- and third-class railway fares, analyzed by Edgeworth*" (Robinson, 1933, p. 181).

Coase (1946) goes beyond Robison's analysis and examines monopoly prices for two interrelated products using verbal and graphical arguments. He identifies two different effects of one good on the price of the other good and recognizes that the net effect could be positive or negative but does not provide a mathematical solution. Following a similar approach Holton (1957) considers the price discrimination by a supermarket selling interrelated products. He does not solve for optimal prices either, but argues that, *"supermarket operators do indeed establish prices with not only price elasticities but cross elasticities in mind"*, in contrast to our main conclusion.

While the complete solution of MPM pricing seems to have eluded economists' attention, some features of our results have emerged in the marketing literature. Shugan and Desiraju (2001) show that monopoly prices of two vertically differentiated products do not depend on each other's costs. Moorthy (2005) and Besanko et al (2005) find that with linear demand MPM prices do not respond to cost changes of other products. Neither of these studies derives the full scope of MPM pricing.

We think that our results can be useful in various contexts. A straightforward practical implication is that even in the presence of strong product interactions, neglecting such relations is part of good pricing practice for a monopolist. Besides its managerial relevance this insight also provides a theoretical justification for research in economics and marketing analyzing retail prices in a single product context. Another implication is that cross-subsidization cannot be part of optimal pricing for a non-regulated monopolist, at least under linear demand (Baumol, Panzar and Willig, 1982).

Our welfare results could serve as a simple and practical benchmark helping antitrust authorities estimate the social loss of MPM. If demand functions can be considered as approximately linear, it is not necessary to analyze in detail every product's price elasticity and the cross-elasticity with all other products. The social cost of MPM can simply be estimated by looking at the company's profit. For example this approach could help to evaluate the social cost of a local retail monopoly. Similarly the deadweight loss caused by a railway monopoly can be estimated from the company's profit without having to analyze the qualities and prices of the different tickets offered.

Finally our results have implications for joint profit maximization by oligopoly firms. Jointly maximizing the total profit is mathematically equivalent to the MPM price problem. Our findings indicate that with linear demand, even if products exhibit strong interdependencies, oligopoly firms do not need any information about their competitors' products and costs in order to set the prices that maximize joint profit.

The paper is organized as follows: Section 2 provides examples of misconceptions regarding monopoly prices. Section 3 derives the profit-maximizing prices for general linear MPM. In Section 4, we apply this result to three different models of interdependent products and obtain the MPM price in each case. Section 5 analyzes the relation between the deadweight loss and monopoly profits, and Section 6 extends our results to non-linear demand. Section 7 briefly concludes.

2. Illustrative Examples

An elementary fallacy in basic monopoly theory holds that a firm selling two complementary products will charge less for each than if it were selling each of them in an independent market. In its most succinct form, this conventional wisdom can be presented within the standard two-good paradigm.

Consider a representative consumer with utility function $U(x_1,x_2) = a(x_1 + x_2) - 0.5b(x_1^2 + x_2^2) + gx_1x_2 + y$, where y is income, and $|g| < b$. This gives rise to the standard symmetric inverse demand function $p_i = a - bx_i + gx_j$ (Bowley, 1924). The corresponding direct demand is then $x_i = a/(b - g) - (bp_i + gp_j)/(b^2 - g^2)$, $i, j = 1, 2$. While this can also be written as (Singh and Vives, 1984)

$$x_i = \alpha - \beta p_i + \gamma p_j, \text{ with } \alpha = a/(b - g), \ \beta = b/(b^2 - g^2) \text{ and } \gamma = -g/(b^2 - g^2),$$

it is important to observe that the constants α, β and γ are not autonomous here. In contrast, a, b and g are, except for the restriction that $|g| < b$.

Using the demand functions in the form $x_i = \alpha - \beta p_i + \gamma p_j$, $i, j = 1, 2$, and unit cost c for both products, one obtains both monopoly prices as $p^* = 0.5[c + \alpha/(\beta - \gamma)]$. Then, so goes the fallacy, this price is higher with substitute goods ($\gamma > 0$) and lower with complements ($\gamma < 0$), relative to the case of independent goods ($\gamma = 0$).[1] This would be correct if α and β remained constant when γ changes, which however is not the case. Indeed, using the relations between Greek and Roman letters, we obtain $\alpha/(\beta - \gamma) = a$, which is the intercept of inverse demand, or the consumer's willingness to pay at zero

[1] For example Motta (2004, p. 537) argues that, "*relative to the benchmark case where the two products are independent (...) the monopolist reduces the price of its products when they are complements (...) and it increases them when they are substitutes (...). The intuition for this result is straightforward. When the products are complements they exercise a positive externality on each other and the monopolist internalizes it by decreasing its prices*".

consumption i.e., $\partial U(0,0)/\partial x_1$, indeed a primitive constant.[2] Hence, $p^* = 0.5(c + a)$, which is also the optimal price for a monopolist selling only good i (and facing inverse demand $p_i = a - bx_i$ and unit cost c). Hence, with linear demand, a monopolist selling two goods prices each as if it were the only good sold. In other words, pricing is independent of the (substitute/complement) relations between the two goods.[3]

On the other hand, optimal outputs do depend on these relationships. Indeed, maximizing total profits with respect to outputs, one gets $x^* = (a - c)/2(b - g)$, which is higher for complements ($g > 0$) and lower for substitutes ($g < 0$), relative to independent products.[4] This is fully in line with basic economic intuition.

A key broader implication of this elementary example is that one should a priori view with suspicion any comparative statics with respect to changes in a parameter of the direct demand function. Such changes cannot be viewed in isolation, without due consideration of interdependencies with other demand parameters. This often appears to be overlooked, so various results in industrial economics might need some revisiting.

Another form this fallacy can commonly take is as follows. Considering MPM with general non-linear demand, one easily derives the optimal Lerner index, $(p_i - c_i)/p_i$, as $1/\varepsilon_{ii} - \sum_{j \neq i}(p_j - c_j)x_j\varepsilon_{ij}/(\varepsilon_{ii}p_i x_i)$, where ε_{ii} and ε_{ij} are the price elasticity and cross-elasticity. When every cross elasticity ε_{ij} is zero, we obtain the single-good monopoly condition $(p_i - c_i)/p_i = 1/\varepsilon_{ii}$. When the goods are substitutes, we have $\varepsilon_{ij} < 0$, and the price

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2 In contrast, α is the quantity demanded under zero prices. In the presence of substitutes (complements) one would expect it to be lower (higher), relative to the case of independent products.

3 This conclusion extends fully to linear demands for any number of goods, as well as to the alternative formulation of linear demand, due to Shubik (1959). The verification details are left to the reader.

4 Expressed in terms of the parameters of direct demand, $x^* = 0.5[\alpha - (\beta - \gamma)c]$, which would say that output is higher for "substitutes" ($\gamma > 0$) than for "complements" ($\gamma < 0$), in violation of standard intuition.

p_i would appear to be higher compared to the corresponding price in a separate (single-good) market.[5] However, again this argument would be correct only if elasticity ε_{ii} were to remain the same as we add new products. In the presence of substitute goods, the quantity demanded for a given product will fall and the value of ε_{ii} will rise. A higher ε_{ii} offsets the impact of ε_{ij}'s, and pushes p_i in an opposite direction. We will show that for linear demand, the two effects cancel out exactly and the prices are not affected.

We now move to a general investigation of MPM with linear demand.

3. The General Linear Model

We start by analyzing MPM pricing for a general linear demand model, which will be shown in Section 4 to encompass three commonly used but quite different linear demand structures. We refrain at this stage from specifying any microeconomic foundations, because we know of no single one that would cover all of our applications.

We consider a monopoly selling n products with constant marginal costs. Prices, quantities and marginal cost are denoted by p_i, x_i, and c_i respectively, $i = 1, \ldots, n$. The corresponding vectors for all n products are written as bold $\boldsymbol{p}$, $\boldsymbol{x}$ and $\boldsymbol{c}$. The linear demand function is specified by a constant Jacobian matrix $\partial \boldsymbol{x}/\partial \boldsymbol{p} = \boldsymbol{A}$, and a constant $n\times 1$ vector $\boldsymbol{\alpha}$, representing the vector of quantity demanded when all prices are zero, as

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p} \qquad (1)$$

When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{x}(\boldsymbol{c})$. We assume $\boldsymbol{A}$ is negative definite and symmetric, i.e., its elements $a_{ij} = a_{ji}$ for all i, j. The diagonal elements of $\boldsymbol{A}$ are all negative, i.e., $a_{ii} < 0$ for all i, as the demand for every good is downward sloping.

[5] See e.g., Tirole (1988, p. 70).

The off-diagonal elements, however, can be positive, negative or zero, according to products being substitutes, complements or independent. Given the demand function (1) and marginal costs vector $\boldsymbol{c} = (c_1, \ldots, c_n)$, we can write the monopoly profit as

$$\pi(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}) \qquad (2)$$

We will demonstrate that the monopoly prices can be expressed in a simple way using the vector of demand intercepts $\boldsymbol{p}^0$, which is the (minimal) price vector that exactly reduces demand for all products to zero. As matrix $\boldsymbol{A}$ is invertible, this vector is uniquely defined by $\boldsymbol{x}(\boldsymbol{p}^0) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}^0 = \boldsymbol{0}$, i.e., $\boldsymbol{p}^0 = -\boldsymbol{A}^{-1}\boldsymbol{\alpha}$.

PROPOSITION 1: The profit-maximizing prices are $\boldsymbol{p}^* = 0.5(\boldsymbol{p}^0 + \boldsymbol{c})$. *Under these prices only half of the socially optimal quantity of every good is sold.*

Proof: see Appendix A.

The impact of all the parameters of the demand function (1) on the monopoly price vector $\boldsymbol{p}^*$ is summarized in the demand intercept $\boldsymbol{p}^0$. This is similar to the single product case with linear demand, where the monopoly price does not depend on the slope. Hence, Proposition 1 can be interpreted as a generalized version of the solution to a single-good or a two-good monopoly (see Section 2). It implies that in general linear monopoly only 50% of a product's cost change is passed on to its price, and that a cost change for one product does not affect the prices of other products. This conclusion corroborates the findings by Moorthy (2005) and Besanko et al (2005).[6]

The optimal price vector is thus independent of inter-product relationships (substitutes/complements) if the vector $\boldsymbol{p}^0$ is. The latter in turn depends in general on the

[6] There are also some similarities between our setting and Ramsey's (1927) pioneering work on optimal taxation, although the motivation and the nature of the problem are quite different there.

microeconomic foundation invoked to derive demand. As seen in Section 2, for the standard (Bowley) quadratic utility, $\boldsymbol{p}^0$ indeed has the desired property (easily extended to n goods and to Shubik-type demands). In the next section, we show that this property also holds for three well-known distinct models of product differentiation in industrial organization, so that it actually holds for all widely used versions of linear demand.

The intuition of Proposition 1 can be explained more clearly when the problem is set up in quantities. A small change Δx_1 for good 1 (say), keeping the other quantities constant, has two effects on profits: a direct quantity effect $(p_1 - c_1)\Delta x_1$, and an indirect effect through price changes $\Delta x_1 \sum_{i=1}^{n} (\partial p_i / \partial x_1)\, x_i$. If profit is maximized, the two effects must exactly offset each other, i.e., $p_1 - c_1 + \sum_{i=1}^{n} (\partial p_i / \partial x_1)\, x_i = 0$. Given our definition of $\boldsymbol{p}^0$, good 1's inverse demand function is $p_1 = p_1^0 + \sum_{i=1}^{n} (\partial p_1 / \partial x_i) x_i$. The symmetry of $\partial p_1 / \partial x_i = \partial p_1 / \partial x_i$ implies $p_1 - p_1^0 = \sum_{i=1}^{n} (\partial p_i / \partial x_1)\, x_i$. Hence, we must have $p_1 - c_1 + p_1 - p_1^0 = 0$, which directly yields the optimal price $p_1^* = 0.5(p_1^0 + c_1)$.

With the two standard equivalent ways of solving the monopoly problem, this simple solution is not the easiest one to find. The first (second) approach is to use the direct (inverse) demand function and solve for monopoly prices (outputs) expressed in terms of the parameters of the direct (inverse) demand function. Only if one then expresses the optimal prices in terms of the parameters of the indirect demand function will the revealing expression for monopoly prices emerge (as done in Section 2).

4. Price Independence

In this section we show that the result derived in section 3 can be applied to MPM facing three commonly used models of linear demand for differentiated products, each having its own microeconomic foundation. In all three cases we obtain simple profit-maximizing prices, which are independent of other products and product relations. On the other hand, the latter do affect monopoly outputs (as seen in Section 2).

4.1 Heterogeneous products:

We first look at one of the standard models for heterogeneous products. There is a continuum of consumers indexed by a parameter λ, which can be interpreted as the marginal utility of income, with density function $f(\lambda)$. Each consumer λ has a utility function $h + (\boldsymbol{a}'\boldsymbol{y} - 0.5\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})/\lambda$, where h is the numeraire good whose price is 1, $\boldsymbol{y}$ is the consumption bundle of the monopoly products, $\boldsymbol{a}$ is an $n\times 1$ positive vector and $\boldsymbol{B}$ is an $n\times n$ matrix. Without loss of generality, let $\boldsymbol{B}$ be symmetric. We assume it to be positive definite so that the utility function is concave.

Each consumer chooses $\boldsymbol{y}$ to maximize utility subject to a budget constraint $h + \boldsymbol{p}'\boldsymbol{y} = m$. The first-order condition of utility maximization, $\boldsymbol{a} - \boldsymbol{B}\boldsymbol{y} - \lambda\boldsymbol{p} = \boldsymbol{0}$ yields an individual demand function $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{p})$. We denote the average λ, $\int \lambda f(\lambda) d\lambda$ by $\bar{\lambda}$. Integrating all $\boldsymbol{y}$ we get the aggregate demand function:

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{B}^{-1}(\boldsymbol{a} - \bar{\lambda}\,\boldsymbol{p}) \tag{3}$$

This demand function (3) follows our general version (1) with $\boldsymbol{B}^{-1}\boldsymbol{a} = \boldsymbol{\alpha}$, and $\bar{\lambda}\,\boldsymbol{B}^{-1} = -\boldsymbol{A}$. To ensure an interior solution we require the following condition.

ASSUMPTION 1: *For any* λ *,* $\boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{c}) > \boldsymbol{0}$.

Assumption 1 implies that when all prices are equal to marginal costs, every consumer has a positive demand for every product. This ensures that the demand function (3) is valid under the monopoly price. As $\boldsymbol{B}$ is symmetric and positive definite the demand function (3) satisfies the requirement of Proposition 1. The vector $\boldsymbol{p}^0$ of maximum prices is also easy to determine. As $\boldsymbol{B}^{-1}$ has full rank, $\boldsymbol{x}(\boldsymbol{p})$ is zero when $\boldsymbol{a} - \bar{\lambda}\boldsymbol{p} = \boldsymbol{0}$, so the maximum price vector $\boldsymbol{p}^0 = \boldsymbol{a}/\bar{\lambda}$. We obtain:

PROPOSITION 2: The MPM price for good i is $p_i^* = 0.5(c_i + a_i/\bar{\lambda})$.

As $a_i/\bar{\lambda}$ and c_i only depend on good *i*, product relations do not affect the optimal price. Note that $a_i/\bar{\lambda}$ can be interpreted as the marginal utility of product *i* to an average consumer when her consumption of all products is zero. This should not depend on product relations. If the monopolist can estimate this value, he can choose the optimal price easily, independently of how many goods he sells and how large their cross-elasticities are, in full contrast to the conclusions reached by Holton (1957).

If the monopoly sells each good in an independent market, good *i*'s demand function will reduce to $(a_i - \bar{\lambda}p_i)/b_i$ and the optimal price will be $0.5(c_i + a_i/\bar{\lambda})$, which is identical to the MPM price. So the MPM achieves optimal price coordination when it acts as if it were selling *n* products in *n* separate markets. This means again that product interdependence does not have any influence on the prices.

Obviously, this result can also be applied to multi-market rather than a multi-product setting. Consider for example a standard textbook example of third degree price discrimination, a cinema selling tickets to normal customers and students. The cinema faces demand functions $p_i = a_i - b_i x_i$ (i = 1, 2) for both types of customers. With marginal costs c, the prices for normal and student tickets are simply $p_i = 0.5(a_i + c)$, consistent with Robinson (1933). Plausibly, however, in reality some normal customers and students may prefer to see sets of movies. This leads to interdependent demand with product complementarities, i.e. an inverse demand function $p_i = a_i - b_i x_i + r x_j$, with $r > 0$. The conventional wisdom is to be that the price will depend on the demand interaction term r. As we have demonstrated, this is not true. If demand is linear, the price corresponding to Robinson' solution remains the same with interrelated products.[7]

Then what about "*first- and third-class railway fares*" that Robinson explicitly ruled out? Does the existence of third-class tickets affect the price of the first-class and vice versa? Unfortunately this question cannot be answered here because it requires a model of vertically differentiated products. Next we show that the optimal prices of "*first- and third-class railway fares*" are indeed independent.

4.2 Vertically differentiated products

We consider a model of n ($\geq$ 2) vertically differentiated products where product i has quality q_i (as in Mussa and Rosen, 1978). Without loss of generality, let $q_{i+1} > q_i$ for all i, so that q_n indicates the highest quality and q_1 the lowest. There is a continuum of consumers indexed by θ, which is uniformly distributed on [0,1]. Each consumer θ purchases at most one good. If he buys good i at price p_i, he obtains a surplus of $\theta q_i - p_i$.

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[7] Linearity is not always necessary for this result. For instance, if $p_i = a - bx_i^{\sigma} - rx_i^{0.5(\sigma - 1)} x_j^{0.5(\sigma + 1)}$, $\sigma >$ 0, the price is $(\sigma a + c)/(1 + \sigma)$, same as in a separate market ($p_i = a - bx_i^{\sigma}$).

The interpretation of $1/\theta$ is similar to the previous marginal utility of income λ. Each consumer chooses the product with the highest surplus, provided it is non-negative.

We need some assumptions to avoid technical difficulties. To ensure that every product has positive demand, we assume that the marginal cost of any product increases with its quality, while the consumer benefit increases more. Also the returns of quality are diminishing, i.e. marginal costs increase with quality at an increasing rate. If we write the marginal cost c_i of a product with quality q_i as $c(q_i)$, this translates into:

ASSUMPTION 2: *For any q, 0 < c'(q) < 1 and c"(q) > 0.*

We can determine the demand for a given good with quality q_i by identifying the highest and lowest type of consumers buying this good. The marginal consumer who is indifferent between buying product 1 and buying nothing gets a surplus $\theta_1 q_1 - p_1 = 0$, so all consumers with an index lower than $\theta_1 \equiv p_1/q_1$ will not buy any product. For consumer θ_i indifferent between buying products i and $i - 1$ we have $\theta_i q_{i-1} - p_{i-1} = \theta_i q_i - p_i$, so $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$. If $\theta_i < \theta_{i+1}$ for all $i < n$, and $\theta_n < 1$, we obtain positive demand for all goods as $x_i = \theta_{i+1} - \theta_i$ for $i < n$ and $x_n = 1 - \theta_n$. We will show that these conditions hold at the MPM prices. Substituting θ_is into these demand functions we get:

$$x_1 = \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1}, \qquad x_n = 1 - \frac{p_n - p_{n-1}}{q_n - q_{n-1}},$$

$$x_i = \frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}} \quad \text{for } 1 < i < n \tag{4}$$

It is easy to see that (4) is linear with a symmetric Jacobian matrix, so Proposition 1 applies. To find the MPM price the only information we need is the vector of demand intercepts $\boldsymbol{p}^0$. One can see that the demand for each product is zero when $p_i = q_i$ for all i. So $\boldsymbol{p}^0$ is equal to the vector of product qualities $\boldsymbol{q}$. In Appendix B we verify that the Jacobian matrix is negative definite and each good has a positive demand.

PROPOSITION 3: *The price for good i in MPM with vertically differentiated products is* $p_i^* = 0.5(c_i + q_i)$.

Proof: see Appendix B.

The monopoly price is simply the average of a product's quality and cost. It is again independent of other products' characteristics. Hence, the prices for "*first- and third-class railway fares*" only depend on the quality and cost of the service offered, not on those of other classes. In particular the prices are the same as the single good monopoly prices, i.e. the price if the monopoly only offers one class of tickets. In this case demand is $x_i = 1 - p_i/q_i$, and its optimal price is $0.5(c_i + q_i)$, which is identical to the MPM price.

According to Proposition 1 the monopoly only sells half the quantities sold in a competitive market. In a model of vertically differentiated products every consumer acquires at most one product. This means that compared to a competitive market, in monopoly some consumers switch to lower quality goods and in total fewer consumers will be served. While in the previous model each consumer buys half of the quantity of the social optimum, here the number of customers being served falls by half.

4.3 Horizontally differentiated products

We finally analyze a model of spatially (horizontally) differentiated products. The Hotelling model and its various extensions have been widely used to analyze oligopoly competition and location choices. Tirole (1988) discusses spatial discrimination by a monopolist selling one product (p. 140). However, little seems to be known about how a monopolist sets prices for a fixed number of horizontally differentiated products with predetermined locations. We will show that these prices are again independent of the features of other products.

We construct an extended version of the Hotelling model, which can be nested in our linear framework. Our model can be visualized as a star-shaped city with n (≥ 2) selling locations owned by a monopolist. The city has $n - 1$ roads radiating from the center and stretching indefinitely into suburbs. There is one shop at the city center and one branch shop along each road with one unit distance from the center. We do not address the question of how to choose locations but simply examine how a MPM sets profit-maximizing prices at these different shops. We assume that the central shop offers consumers a value v_1 at a price p_1, while branches offer v_i at p_i, for $i > 1$. Consumers reside along each road with uniform density. Each consumer incurs a unit travel cost τ, and maximizes his surplus $v_i - p_i - \tau s$, where s is distance.[8]

To ensure an interior solution where every shop has a positive demand under MPM prices, we need certain conditions. On one hand the shops' net values need to be sufficiently high relative to the travel cost so that all consumers between the centre and branch shops are covered. On the other hand, the differences between the net values of

[8] Chen and Riordan (2007) analyze an oligopoly version of this model with full symmetry across firms, and hence no firm at the center.

the centre and branch shops should be sufficiently small so that every shop can sell something. These requirements lead to the following conditions.

Assumption 3. $|v_1 - c_1 - v_i + c_i| < \tau < 0.2(v_1 - c_1 + v_i - c_i)$ *for all* $i > 1$.

In equilibrium no shop can charge a price higher than the value it offers, so we have $p_i < v_i$ for all i. If a branch shop can sell anything, we must have $v_i - p_i + \tau > v_1 - p_1$. Under these conditions we can derive the demand functions by identifying marginal consumers indifferent between buying at the center or a branch shop and those indifferent between a branch and buying nothing. For the former marginal consumers, we have $v_1 - p_1 - \tau y_i = v_i - p_i - \tau(1 - y_i)$, where y_i is the distance to the centre. Thus demand for the central shop $y_i = 0.5(v_1 - p_1 + p_i - v_i + \tau)/\tau$. Shop i serves the remaining $1 - y_i$ customers, but also attracts clients from the suburb up to a distance z_i, which is determined by $v_i - p_i - \tau z_i = 0$, so $z_i = (v_i - p_i)/\tau$. If $0 < y_i < 1$ for all $i > 1$, the demand function for the center $x_1 = \sum_{i=2}^{n} y_i$, and for branch shop i, $x_i = 1 - y_i + z_i$, i.e.,

$$x_1 = \frac{n-1}{2\tau}(\tau + v_1 - p_1) - \sum_{i=2}^{n} \frac{v_i - p_i}{2\tau},$$

$$x_i = \frac{\tau + 3v_i - 3p_i - v_1 + p_1}{2\tau} \qquad \text{for } i > 1. \tag{5}$$

Again (5) is linear in prices and the Jacobian matrix is symmetric. We also prove that this matrix is negative definite (Appendix C). One can verify that the demand for every good is zero when $p_1^0 = v_1 + 2\tau$ and $p_i^0 = v_i + \tau$ for any $i > 1$.[9] With these demand intercepts we can apply Proposition 1 and obtain the monopoly prices.

9 These hypothetical prices lie outside the permissible price range as demand should vanish when $p_i \geq v_i$.

Proposition 4: The MPM prices with horizontally differentiated products are p_1^* = $0.5(v_1 + c_1) + \tau$, and $p_i^* = 0.5(v_i + c_i + \tau)$ for $i > 1$.

Proof: see Appendix C.

The monopoly prices cannot be characterized by a single formula here, as the center shop differs from the others. Nonetheless, all prices again only depend on shop-specific parameters, not on other shops' values or costs. In fact, this property can be generalized to a model with different distances between the centre and branch shops.[10]

Similarly to the previous two cases, every shop only sells half of the socially optimal quantity. This is somehow surprising, because the market always covers all consumers between the center and branch shops. Only suburban residents stop buying any products due to monopoly pricing.

In the previous two models, every price is equal to the "naïve" monopoly prices, charged in independent markets or for a single product monopoly. In this case, if a branch shop is the only seller along its road, its price would be $0.5(v_i + c_i + \tau)$, which is again exactly the MPM price. However, if the central shop is the only seller, its price would be $0.5(v_1 + c_1)$, lower than the MPM price by τ. This result indicates that the MPM price is not always equal to separate monopoly prices. Nonetheless, for $n \geq 2$, the introduction of any new product/road will not affect the existing prices. In this sense we can still say that the MPM prices are independent of each other.

5. Welfare Loss

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[10] If we let s_i be the distance between the center and shop i, and normalize the average distance to 1, p_i^* will change slightly, with τ multiplied by $(s_i + 2)/3$, while p_1^* remains the same.

Estimating the deadweight loss in MPM with complex product relations might at first sight appear quite challenging. In this section we will show that the optimal prices determined in the previous section can be used to establish a simple relation between deadweight loss and monopoly profits. Since profits are usually observable, this relation provides an easy way to estimate the social loss caused by MPM.

Again our result can be understood as a generalization of a well-known property of the textbook example of a single product monopoly with linear demand: Deadweight loss equals half the monopoly profit. This relation remains valid in our three MPM models. This is unexpected because the welfare functions are fundamentally different across the three models and cannot be presented in a unified framework.

In the standard model with heterogeneous products every consumer's demand vector is $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{p})$. Substituting this into his utility function $(\boldsymbol{a}'\boldsymbol{y} - 0.5\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})/\lambda$ we obtain $0.5(\boldsymbol{a} + \lambda\boldsymbol{p})'\boldsymbol{B}^{-1}(\boldsymbol{a}/\lambda - \boldsymbol{p}) = 0.5\boldsymbol{a}'\boldsymbol{B}^{-1}\boldsymbol{a}/\lambda - 0.5\lambda\boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$. Since the first term is independent of prices, we only need to consider the second term. Integrating it for all λ the consumer total utility is determined as $-0.5\bar{\lambda}\,\boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$. Subtracting the total cost $\boldsymbol{c}'\boldsymbol{x}$ from the utility, we get the social welfare. The deadweight loss can be obtained by comparing the welfare under marginal cost pricing and MPM pricing.

The deadweight loss appears more complicated with vertically differentiated products. Recall that in this case we have $\theta_1 \equiv p_1/q_1$, and $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$ for $i > 1$. These θ_i's define the consumer demand for each product. Consumers purchasing product $i < n$ obtain utility $q_i\int_{\theta_i}^{\theta_{i+1}}\theta\, d\theta = 0.5q_i(\theta_{i+1}^2 - \theta_i^2)$. Those purchasing product n

obtain utility $q_n\int_{\theta_n}^{1}\theta\, d\theta = 0.5q_n(1-\theta_n^2)$. The total utility of all consumers is equal to 0.5 $\sum_1^{n-1} q_i\,(\theta_{i+1}^2 - \theta_i^2) + 0.5q_n(1-\theta_n^2)$. Subtracting from this function the total cost $\boldsymbol{c}'\boldsymbol{x}$, we obtain social welfare. With $\boldsymbol{p} = \boldsymbol{c}$ and the MPM price vector $\boldsymbol{p}^*$, we obtain the maximum welfare and its value under MPM. Their difference is the deadweight loss.

Finally in the case of horizontally differentiated products, the calculation of the deadweight loss involves transportation costs. We first consider the utility obtained by consumers residing along one road. The utility from the center is $\int_0^{y_i}(v_1 - \tau s)ds = v_1 y_i - 0.5\tau y_i^2$, where $y_i = 0.5(v_1 - p_1 - v_i + p_i + \tau)/\tau$, which is the position of marginal consumers who are indifferent between purchasing at the centre or branch shop *i*. Consumers who purchase from shop *i* obtain utility $\int_0^{1-y_i}(v_i - \tau s)ds + \int_0^{z_i}(v_i - \tau s)ds = v_i(1-y_i) - 0.5\tau(1-y_i)^2 + v_i z_i - 0.5\tau z_i^2$, where $z_i = (v_i - p_i)/\tau$. After adding the two utilities and subtracting the cost $c_1 y_i + c_i(1 - y_i + z_i)$, we obtain the social welfare along road *i*. Adding this welfare for all $i > 1$, we have the total social welfare. Given these fundamental structural differences the following result might be surprising.

PROPOSITION 5. *In all three MPM models, the deadweight loss is half the monopoly profit.*

Proof: see Appendix D.

The simple relationship known from the linear single product monopoly survives in all three MPM models. As long as demand and cost functions are linear, the relation between the deadweight loss and profits remains unchanged regardless of how many products or what kinds of goods are introduced.[11]

The intuition of this result can be best seen in the horizontal model. According to Proposition 4, adding a new road does not affect the prices along existing roads, and hence will not affect the relationship between welfare loss and monopoly profits there. But the additional deadweight loss along the new road is also half the additional profits, so the overall loss remains as half of the monopoly profits. For the other two models, these relations are more complex, as a new product affects the monopoly profits and deadweight loss from existing products. Nevertheless, the simple relation is always valid. If the linear model is a good approximation, this relation provides a good indication for the deadweight loss due to MPM pricing.

6. Non-linear Demand

Linear demand and cost functions are widely used in industrial economics and often a good approximation of real market conditions. However, even if demand is approximately linear the monopoly prices obtained for the linear model may not be approximately correct if they are very sensitive to non-linearity. In this section we show that this is not the case. Our results are still approximately true if the Hessian matrix of the consumer utility function does not vary significantly. We focus on the standard model for heterogeneous goods. We assume that the representative consumer's utility function is $h + u(\boldsymbol{x})$, where h is the numeraire good, $u(\boldsymbol{x})$ is continuously twice differentiable and strictly concave in $\boldsymbol{x}$, so that the Hessian matrix $u''(\boldsymbol{x})$ is negative

[11] Again the linearity is not always necessary. For example, given the inverse demand function in footnote 7, the deadweight loss is equal to the profit multiplied by $\sigma/(1 + \sigma)$, either one or two goods are sold.

definite and its determinant $|u''(\boldsymbol{x})| \neq 0$. The first-order condition for the utility maximization implies an inverse demand function $p(\boldsymbol{x}) = u'(\boldsymbol{x})$. As in our earlier result, the choke-off price vector $\boldsymbol{p}^0$ corresponds to zero demand, i.e., $\boldsymbol{p}^0 = u'(\boldsymbol{0})$. It equals the marginal utility at zero consumption, which is vector $\boldsymbol{a}$ in the linear case. If we used our simple rule we would obtain an estimated monopoly price as $\boldsymbol{p}^{\mathrm{m}} = 0.5[u'(\boldsymbol{0}) + \boldsymbol{c}]$. However, the true optimal price p^* should maximize the monopoly profit $[p(\boldsymbol{x}) - \boldsymbol{c}]'\boldsymbol{x}$. As $p'(\boldsymbol{x}) = u''(\boldsymbol{x})$, the first-order condition for the optimal $\boldsymbol{x^*}$ is as $p^* - \boldsymbol{c} + u''(\boldsymbol{x^*})\boldsymbol{x^*} = \boldsymbol{0}$.

How far will the estimated price $\boldsymbol{p}^{\mathrm{m}}$ be from the true optimum p^*? Given $\boldsymbol{x^*}$, we can always find a non-negative vector $\boldsymbol{w}_1 \leq \boldsymbol{x^*}$, such that $u'(\boldsymbol{x^*}) = u'(\boldsymbol{0}) + u''(\boldsymbol{w}_1)\boldsymbol{x^*}$. We can then write $u''(\boldsymbol{x^*})\boldsymbol{x^*}$ as $\boldsymbol{p^*} - u'(\boldsymbol{0}) + [u''(\boldsymbol{x^*}) - u''(\boldsymbol{w}_1)]\boldsymbol{x^*}$. Substituting this into the first-order condition for $\boldsymbol{x^*}$, we obtain:

PROPOSITION 6: The optimal price in non-linear MPM with heterogeneous products can be expressed as $\boldsymbol{p^*} = \boldsymbol{p}^m + 0.5[u''(\boldsymbol{w}_1) - u''(\boldsymbol{x^*})]\boldsymbol{x^*}$.

Proposition 6 shows that our simple rule yields a price that differs from the true optimum only by the last error term. If the Hessian matrix of the utility function, $u''(\boldsymbol{x})$, does not vary significantly, this error term will be close to zero and our simple pricing rule will yield prices that are close to the true optimal prices. A small change in $u''(\boldsymbol{x})$ cannot lead to a significant gap between these two prices. Interestingly, the error term is not clearly linked to products being substitutes or complements, which implies that even in the non-linear case there is no clear-cut relationship between product relations and monopoly prices. This is indeed confirmed by an example below.

A similar approach can be used to show that the monopoly output $\boldsymbol{x}^{\mathrm{m}}$ under the estimated price $\boldsymbol{p}^{\mathrm{m}}$ is roughly half of the socially optimal level $\boldsymbol{x}^{\mathrm{c}}$ when $\boldsymbol{p} = \boldsymbol{c}$. We can always find a vector $\boldsymbol{w}_2$ ($\boldsymbol{x}^{\mathrm{m}} \leq \boldsymbol{w}_2 \leq \boldsymbol{x}^{\mathrm{c}}$) such that $u'(\boldsymbol{x}^{\mathrm{c}}) = u'(\boldsymbol{x}^{\mathrm{m}}) + u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. On the other hand, we can find a non-negative vector $\boldsymbol{w}_3 \leq \boldsymbol{x}^{\mathrm{m}}$, such that $u'(\boldsymbol{x}^{\mathrm{m}}) = u'(\boldsymbol{0}) + u''(\boldsymbol{w}_3)\boldsymbol{x}^{\mathrm{m}}$. Since $u'(\boldsymbol{x}^{\mathrm{c}}) = \boldsymbol{c}$ and $u'(\boldsymbol{x}^{\mathrm{m}}) = 0.5[u'(\boldsymbol{0}) + \boldsymbol{c}]$, these two equations imply $u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) = u''(\boldsymbol{w}_3)\boldsymbol{x}^{\mathrm{m}}$. Then we have

Proposition 7: The relation between the socially optimal output and the monopoly output under $\boldsymbol{p}^{m}$ *is:* $\boldsymbol{x}^{c} = 2\boldsymbol{x}^{m} + [u''(\boldsymbol{w}_2)]^{-1}[u''(\boldsymbol{w}_3) - u''(\boldsymbol{w}_2)]\boldsymbol{x}^{m}$.

If $u''(\boldsymbol{w}_3) - u''(\boldsymbol{w}_2)$ is very small and the elements of $[u''(\boldsymbol{w}_2)]^{-1}$ are finite, $\boldsymbol{x}^{\mathrm{c}}$ must be close to $2\boldsymbol{x}^{\mathrm{m}}$. These conditions hold if the Hessian matrix does not change significantly and $|u''(\boldsymbol{x})|$ is not close to zero. The latter is generally true when the demand for each product is significantly downward sloping, rather than horizontal. This does not seem to be a very restrictive requirement for a monopoly[12]. Again a smooth variation in $u''(\boldsymbol{x})$ should not radically change the simple relation $\boldsymbol{x}^{\mathrm{c}} = 2\boldsymbol{x}^{\mathrm{m}}$, which we find in the linear case.

Finally, the deadweight loss (*DL*) caused by the estimated monopoly price $\boldsymbol{p}^{\mathrm{m}}$ remains close to half of the monopoly profit. Given the definitions of $\boldsymbol{x}^{\mathrm{c}}$ and $\boldsymbol{x}^{\mathrm{m}}$, $DL = u(\boldsymbol{x}^{\mathrm{c}}) - u(\boldsymbol{x}^{\mathrm{m}}) - \boldsymbol{c}'\boldsymbol{x}^{\mathrm{c}} + \boldsymbol{c}'\boldsymbol{x}^{\mathrm{m}}$. We can find a vector $\boldsymbol{w}_4$, $\boldsymbol{x}^{\mathrm{m}} \leq \boldsymbol{w}_4 \leq \boldsymbol{x}^{\mathrm{c}}$, and write $u(\boldsymbol{x}^{\mathrm{c}})$ as $u(\boldsymbol{x}^{\mathrm{m}}) + u'(\boldsymbol{x}^{\mathrm{m}})(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) + 0.5(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})'u''(\boldsymbol{w}_4)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. Substitute this into *DL* we get $(\boldsymbol{p}^{\mathrm{m}} - \boldsymbol{c})'(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) + 0.5(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})'u''(\boldsymbol{w}_4)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. Substitute $\boldsymbol{p}^{\mathrm{m}} - \boldsymbol{c} = -u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$ into this function and rearrange its terms, we find the following result.

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[12] This requirement may not be necessary. In the case of footnote 7, the determinant $|u''(\boldsymbol{x})|$ can approach zero when r is close to 1. But $\boldsymbol{x}^{\mathrm{c}}$ is still nearly equal to $2\boldsymbol{x}^{\mathrm{m}}$ so long as σ is not far from 1.

PROPOSITION 8: *The deadweight loss due to the simple monopoly price* $\boldsymbol{p}^m$ *is* $0.5\pi^m + 0.5(\boldsymbol{x}^c - \boldsymbol{x}^m)'\{[u''(\boldsymbol{w}_4) - u''(\boldsymbol{w}_2)](\boldsymbol{x}^c - \boldsymbol{x}^m) - u''(\boldsymbol{w}_2)(\boldsymbol{x}^c - 2\boldsymbol{x}^m)\}$.

In the last term, we know that both $u''(\boldsymbol{w}_4) - u''(\boldsymbol{w}_2)$ and $\boldsymbol{x}^c - 2\boldsymbol{x}^m$ are small when the Hessian matrix $u''(\boldsymbol{x})$ does not vary significantly. Hence, the deadweight loss is close to half the monopoly profit, even if there are minor variations of $u''(\boldsymbol{x})$.

Our analysis shows that in each case the error term introduced by our simple estimation based on the linear model is limited by the variation of the Hessian matrix. If we can estimate the consumer utility function, we should be able to roughly estimate whether our simple results offer reasonable solutions, without having to precisely solve the non-linear problem. Similar approaches can be used to demonstrate that our findings for vertically and horizontally differentiated products hold approximately with nonlinear demand, if the Jacobian matrix does not vary significantly.

We close this section with a closed-form example showing that the conventional intuition of Section 2 need not apply even in markets with non-linear demand.

Example:

Consider a two-good monopolist with zero costs facing a symmetric demand function

$p_i = a - bx_i^\sigma - rx_j$, $i, j = 1, 2$, $\sigma > 0$. The profit function is then

$$\pi = (a - bx_1^\sigma - rx_2)x_1 + (a - bx_2^\sigma - rx_1)x_2.$$

The first-order condition for x_i implies: $a - b(\sigma + 1)(x^*)^\sigma - 2rx^* = 0$. (i)

The monopoly price is $p^* = a - b(x^*)^\sigma - rx^* = \sigma b(x^*)^\sigma + rx^*$. (ii)

In two independent markets, demand and profit are: $p_i = a - bx_i^\sigma$ and $\pi = (a - bx_i^\sigma)x_i$.

The first order condition for x_i is $a - b(\sigma + 1)(x^0)^\sigma = 0$, (iii)

The separate price satisfies $p^0 = a - b(x^0)^\sigma = \sigma b(x^0)^\sigma$. (iv)

Subtracting (iv) from (ii), we get: $p^* - p^0 = \sigma b[(x^*)^\sigma - (x^0)^\sigma] + rx^*$. (v)

Subtracting (iii) from (i), we get: $b(\sigma + 1)[(x^*)^\sigma - (x^0)^\sigma] = -2rx^*$. (vi)

Substituting (vi) into (v), we obtain:

$$p^* - p^0 = \frac{1-\sigma}{1+\sigma} rx^*. \tag{vii}$$

If $\sigma = 1$, (vii) confirms our result for the linear case. If $\sigma < 1$, (vii) implies that $p^* > p^0$ if and only if goods are substitutes ($r > 0$), in line with conventional wisdom, as reported e.g. in Motta (2004), see our Section 2. However, if $\sigma > 1$, (vii) implies that $p^* > p^0$ if and only if goods are complements ($r < 0$), in total violation of conventional wisdom. This example confirms that, while compelling at first sight, the conventional intuition about MPM pricing need not be valid in non-linear models either.

7. Concluding Remarks

The paper analyzes pricing and welfare effects of MPM with linear demand and cost functions. Our main result is that the MPM price for each good depends only on the marginal cost and the inverse demand intercept of that good, the nature of any number of other goods being immaterial. This conclusion is at odds with much literature, old and new in industrial organization, including basic textbooks, stressing the role of substitute/complement products and cross-elasticities in MPM pricing. The underlying oversight in the literature seems to originate from a general tendency to interpret the parameters of direct demand functions as being autonomous, which unfortunately leads to questionable comparative statics conclusions in other settings not covered here.

Another result is that the deadweight loss is half of the MPM profit. In other words, relations known from the simple one-product textbook linear model generalize verbatim to three workhorse linear models of interdependent products: heterogeneous (a la Bowley or Shubik), vertically and horizontally differentiated.

Due to their basic nature, the results presented here can be relevant to a wide range of contexts, covering theoretical and policy issues. Some examples are emerging areas in industrial organization such as bundling and tying. More broadly, the simple insights from this paper could contribute to fields as different as antitrust theory, regulation, urban and spatial economics and marketing. Our welfare results are potentially useful in regulatory design, as a way to estimate deadweight loss in complex situations.

While we limited our analysis mainly to linear demand, our results in Section 6 indicate that our main insight is robust, the linear demand being special only insofar as it leads to the two effects of adding a substitute or complement product to an existing product line being clearly identified and exactly canceling out. We hope that this paper might lead to renewed interest in the topic of monopoly pricing, which was addressed by economists in the early years, but seems to have been largely ignored recently.

Appendix A: We differentiate the profit function (2), $(\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{Ap})$ and get the first-order condition $d\pi/d\boldsymbol{p} = \boldsymbol{\alpha} + \boldsymbol{A}'(\boldsymbol{p} - \boldsymbol{c}) + \boldsymbol{Ap} = \boldsymbol{0}$. Since $\boldsymbol{A}$ is symmetric, we have $\boldsymbol{\alpha} + \boldsymbol{A}(2\boldsymbol{p} - \boldsymbol{c}) = \boldsymbol{0}$. The Hessian matrix of the profit function is equal to $2\boldsymbol{A}$. So the second-order condition holds since $\boldsymbol{A}$ is negative definite. Then the optimal price can be solved from the first-order condition, as $\boldsymbol{p}^* = 0.5(\boldsymbol{c} - \boldsymbol{A}^{-1}\boldsymbol{\alpha})$.

If we plug $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ into the demand function (1), we get $\boldsymbol{x} = \boldsymbol{0}$. So $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ is the demand intercept vector $\boldsymbol{p}^0$. The optimal price $\boldsymbol{p}^*$ can be written as $0.5(\boldsymbol{c} + \boldsymbol{p}^0)$.

Putting $\boldsymbol{p}^*$ into the demand function (1), we get $\boldsymbol{x}^* = 0.5(\boldsymbol{\alpha} + \boldsymbol{Ac})$. When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{\alpha} + \boldsymbol{Ac}$, which is twice of the monopoly output $\boldsymbol{x}^*$. ||

Appendix B: (i) The demand function (3) is clearly linear in prices. To apply Proposition 1, we need to show that Jacobian matrix $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_{i+1} = 1/(q_{i+1} - q_i) = \partial x_{i+1}/\partial p_i$ for all i, and $\partial x_i/\partial p_j = 0$ for any $j \neq i$ and $|j - i| > 1$, the matrix is indeed symmetric.

To show it is negative definite, we see the sum of the first row of $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is equal to $-1/q_1$, and the sum of every other row is zero. Hence the matrix has a quasi-dominant diagonal and must be negative definite (McKenzie 1960, Theorem 2).

(ii) We then need to find $\boldsymbol{p}^0$. Substituting $\boldsymbol{p} = \boldsymbol{q}$ into the demand function (3), we get $\boldsymbol{x} = \boldsymbol{0}$. So $\boldsymbol{p}^0 = \boldsymbol{q}$, and the MPM price $p_i^* = 0.5(c_i + q_i)$. We also have $\boldsymbol{x}(\boldsymbol{p}^*) = 0.5\boldsymbol{x}(\boldsymbol{c})$ from Proposition 1.

(iii) To complete the proof, we need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, we need to show $\frac{c_2 - c_1}{q_2 - q_1} \geq \frac{c_1}{q_1}$, or $\frac{c_2}{q_2} \geq \frac{c_1}{q_1}$. This holds since $c''(q) > 0$. For $x_n \geq 0$, we must have $c_n - c_{n-1} \leq q_n - q_{n-1}$. This is true given $c'(q) < 1$.

For $1 < i < n$, $x_i \geq 0$ holds if $\frac{c_{i+1} - c_i}{q_{i+1} - q_i} \geq \frac{c_i - c_{i-1}}{q_i - q_{i-1}}$. To prove this, we write $c_{i+1} - c_i$ as $(q_{i+1} - q_i)c'(\omega_i)$ and $c_i - c_{i-1} = (q_i - q_{i-1})c'(\omega_{i-1})$, where $q_{i-1} \leq \omega_{i-1} \leq q_i \leq \omega_i \leq q_{i+1}$. As $c''(q) > 0$, $\omega_{i-1} \leq \omega_i$, we get $c'(\omega_i) \geq c'(\omega_{i-1})$, so $x_i \geq 0$.

Finally, we show that no consumer receives a negative surplus under $\boldsymbol{p}^*$. The marginal consumer buying from good 1 receives a zero surplus. For $i > 1$, the marginal consumer $\theta_i = (p_i - p_{i-1})/(q_i - q_{i-1})$, receives a positive surplus if $\theta_i q_i \geq p_i$ or $p_i q_{i-1} \geq p_{i-1} q_i$. Using p_i^* and p_{i-1}^*, it becomes $c_i/q_i \geq c_{i-1}/q_{i-1}$, which holds given $c''(q) > 0$. ||

Appendix C: (i) To apply Proposition 1, again we need to show that the $n \times n$ Jacobian matrix $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_1 = \partial x_1/\partial p_i = 0.5/\tau$ for all $i > 1$, and $\partial x_i/\partial p_j = 0$ for i and $j \neq 1$, it is indeed symmetric.

Moreover since $\partial x_1/\partial p_1 = -0.5(n-1)\tau$, $\partial x_i/\partial p_i = -1.5/\tau$, the sum of the first row of $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is 0, and the sum of any other row is $-1/\tau < 0$. By McKenzie (1960) $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ must be negative definite.

(ii) One can verify that demand function (5) is zero when $p_1 = 2\tau + v_1$, and $p_i = \tau + v_i$. So the MPM prices $p_1^* = 0.5(v_1 + c_1 + 2\tau)$, and $p_i^* = 0.5(v_i + c_i + \tau)$ for $i > 1$.

We need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, it suffices to show $\tau + v_1 - c_1 \geq v_i - c_i$. For $x_i \geq 0$, we need $v_i - c_i + 3\tau \geq v_1 - c_1$. Assumption 3 guarantees both of them.

Finally, every marginal consumer must receive a non-negative surplus. For a consumer indifferent between the center and shop i, her surplus from the center is $v_1 - p_1 - \tau y_i = v_1 - p_1 - 0.5(v_1 - p_1 - v_i + p_i + \tau) = 0.25(v_1 - c_1 + v_i - c_i - 5\tau)$. It is positive given Assumption 3. A marginal consumer outside of shop i receives a zero surplus. ||

Appendix D: (i) Substituting the monopoly price $\boldsymbol{p}^*$ and marginal cost pricing $\boldsymbol{c}$ into function $-0.5\bar{\lambda}\, \boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$, and subtracting one from the other, we get the total utility loss as $0.5\bar{\lambda}\, (\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{B}^{-1}(\boldsymbol{p}^* + \boldsymbol{c})$. But we know $\bar{\lambda}\, (\boldsymbol{p}^* - \boldsymbol{c}) = 0.5(\boldsymbol{a} - \bar{\lambda}\, \boldsymbol{c})$, and $\boldsymbol{B}^{-1}(\boldsymbol{a} - \bar{\lambda}\, \boldsymbol{c}) = \boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, so the utility loss is equal to $0.5(\boldsymbol{p}^* + \boldsymbol{c})'\boldsymbol{x}^*$. On the other hand, since the monopoly pricing reduces the outputs by half, the total cost falls by $\boldsymbol{c}'\boldsymbol{x}^*$. Thus, the deadweight loss, the sum of the utility and cost changes, is equal to $0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^*$, which is half of the total monopoly profit.

(ii) Vertically differentiated products: We first write the twice of the total utility of all consumers as $2u = \sum_1^{n-1} q_i\,(\theta_{i+1}^2 - \theta_i^2) + q_n - q_n\theta_n^2$. Regrouping the summation items, it becomes $q_n - q_1\theta_1^2 - \sum_2^n \theta_i^2 (q_i - q_{i-1})$. Substituting $\theta_1 \equiv p_1/q_1$ and $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$, it becomes $q_n - \theta_1 p_1 - \sum_2^n \theta_i (p_i - p_{i-1})$. Regrouping the summation items again, it changes to $q_n + \sum_1^{n-1} p_i (\theta_{i+1} - \theta_i) - \theta_n p_n$. As $\theta_{i+1} - \theta_i = x_i$ and $1 - \theta_n = x_n$, we get $2u = \sum_1^n p_i x_i + q_n - p_n$.

Moreover, we write $q_n - p_n$ as $q_n - \sum_2^n (p_i - p_{i-1}) - p_1 = q_n - \sum_2^n \theta_i (q_i - q_{i-1}) - \theta_1 q_1$. Regrouping the summation items, we get $q_n - p_n = q_n + \sum_1^{n-1} q_i (\theta_{i+1} - \theta_i) - \theta_n q_n = \sum_1^n q_i x_i$. Substitute this into $2u$ expression, we get $u = 0.5\sum_1^n (p_i + q_i) x_i = 0.5(\boldsymbol{p} + \boldsymbol{q})'\boldsymbol{x}$.

Social welfare (SW) $u - \boldsymbol{c}'\boldsymbol{x} = 0.5(\boldsymbol{p} + \boldsymbol{q} - 2\boldsymbol{c})'\boldsymbol{x}$. When $\boldsymbol{p} = \boldsymbol{c}$, we get $SW(\boldsymbol{c}) = 0.5(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}(\boldsymbol{c})$; when $\boldsymbol{p} = \boldsymbol{p}^* = 0.5(\boldsymbol{q} + \boldsymbol{c})$, $SW^* = 0.75(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}^*$. Given $\boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, the deadweight loss $SW(\boldsymbol{c}) - SW^* = 0.25(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}^*$.

As $\boldsymbol{q} - \boldsymbol{c} = 2(\boldsymbol{p}^* - \boldsymbol{c})$, the deadweight loss is $0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 0.5\pi^*$.

(ii) Horizontally differentiated products: The utility obtained by consumers along one road is $u_i = [v_1 - 0.5\tau y_i]y_i + [v_i - 0.5\tau(1 - y_i)](1 - y_i) + [v_i - 0.5(v_i - p_i)z_i]z_i$, where $z_i = (v_i - p_i)/\tau$ and $y_i = 0.5(v_1 - p_1 - v_i + p_i + \tau)/\tau$. Substitute y_i and z_i in brackets [], $u_i = 0.25(3v_1 + p_1 + v_i - p_i - \tau)y_i + 0.25(3v_i + p_i + v_1 - p_1 - \tau)(1 - y_i) + 0.5(v_i + p_i)z_i$, which simplifies to $0.5(v_1 + p_1)y_i + 0.5(v_i + p_i)(1 - y_i + z_i) + 0.25(v_1 - p_1 + v_i - p_i - \tau)$.

Note that $v_1 - p_1 + v_i - p_i = \tau(2y_i + 2z_i - 1)$, and $1 - y_i + z_i = x_i$, we write this utility as $u_i = 0.5(v_1 + p_1)y_i + 0.5(v_i + p_i)x_i + 0.5\tau(y_i + z_i - 1)$. Replacing z_i by $y_i + x_i - 1$, we get $u_i = 0.5(v_1 + p_1 + 2\tau)y_i + 0.5(v_i + p_i + \tau)x_i - \tau$. One-road welfare is $u_i - c_1 y_i - c_i x_i$.

Recall that $\sum_2^n y_i = x_1$. The welfare from all roads is $0.5x_1(v_1 + p_1 + 2\tau - 2c_1) + 0.5\sum_2^n x_i(v_i + p_i + \tau - 2c_i) - n\tau$. When $\boldsymbol{p} = \boldsymbol{c}$, we get the maximum total welfare as $0.5x_1(\boldsymbol{c})(v_1 + 2\tau - c_1) + 0.5\sum_2^n x_i(c)(v_i + \tau - c_i) - n\tau$. Since $\boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, $v_1 + 2\tau - c_1 = 2(p_1^* - c_1)$, and $v_i + \tau - c_i = 2(p_i^* - c_i)$, we can write this welfare as $2x_1^*(p_1^* - c_1) + 2$

$\sum_{2}^{n} x_i^*(p_i^* - c_i) - n\tau = 2\pi^* - n\tau$. When $\boldsymbol{p} = \boldsymbol{p^*}$, the welfare is $0.5x_1^*(1.5v_1 + 3\tau - 1.5c_1) + 0.75\sum_{2}^{n} x_i^*(v_i + \tau - c_i) - n\tau$, which is equal to $1.5x_1^*(p_1^* - c_1) + 1.5\sum_{2}^{n} x_i^*(p_i^* - c_i) - n\tau = 1.5\pi^* - n\tau$. Subtracting two welfare values, we get the deadweight loss $0.5\pi^*$. ||

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