

# PRICING AND INVENTORY MODELS FOR A RETAILER

By

Chirag Surti, B.Eng., M.Sc.

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DOCTOR OF PHILOSOPHY (2009)	McMaster University Hamilton, Ontario
MASTER OF SCIENCE (2003)	State University of New York Buffalo, New York
BACHELOR OF ENGINEERING (2001)	Mumbai University Mumbai, Maharashtra
TITLE:	Pricing and Inventory Models for a Retailer
AUTHOR:	Chirag Surti
SUPERVISORS:	Dr. Prakash Abad Dr. Elkafi Hassini
COMMITTEE:	Dr. Sourav Ray
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## Abstract

In this thesis we study three problems of joint pricing and inventory in a retail setting.

The first problem deals with pricing and ordering for a retailer facing uncertain supply as well as price-sensitive uncertain demand. We first formulate the problem as two cases of pricing: a simultaneous pricing strategy where the price and the order quantity are simultaneously determined and a postponed pricing strategy where the price and the order quantity are sequentially determined. We provide a solution procedure to find the optimal price and order quantity that maximizes the retailer's profit. By conducting sensitivity analysis, we find that if the supplier is very unreliable, then the retailer is better off postponing the pricing decision in order to maximize profit. Reducing supply variability does not have the same impact on retailer's profit as much as increasing the expected supply amount. Most importantly we find that the difference between the expected profits in the two cases is not due to higher expected revenue, but due to lower expected salvage and shortage losses when the pricing decision is postponed.

Next, we study a price setting retailer selling two substitutable goods to consumers. The retailer must decide on the optimal price and inventory that maximize the expected profit. Aside from making these decisions under demand uncertainty, the retailer must also account for the substitution that occurs upon stock out of one of the two products. Furthermore, we also take into account the related cannibalization of the available stock due to customers substituting. We formulate the problem and find the optimal prices analytically as well as conduct sensitivity analysis. We compare our findings to a model that does not consider substitution and the resultant cannibalization of inventory and find that the model that does not consider substitution tends to overestimate the expected profit for low degrees of substitution and tends to underestimate the expected profit for high degrees of substitution. Furthermore, the prices charged and the inventory held at the retailer for each product, tend to be suboptimal. The total quantity stocked in general, for both products, is lower when we account for substitution and cannibalization.

Lastly, we study the problem of finding optimal order quantities and prices for the bundle (a collection of two or more goods sold jointly at one price) and individual items as well as how a supplier can use bundles to achieve coordination with its retailer. In a decentralized supply chain, we show that bundling is not always a feasible or a very profitable strategy. This is especially true if the products or the bundle are discounted beyond a certain point, because it may make the supplier worse off while making the retailer better off. This reduces the effectiveness of the bundling strategy in a supply chain setting. We find that the supplier, retailer and the supply chain can simultaneously improve their profits by offering bundled goods to the consumers and achieve performance of a coordinated supply chain when the supplier charges the retailer a bundling fee upfront and in exchange offering a bundling discount to the retailer.

In the last chapter, we summarize our findings as well as provide direction for future research.

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## Dedication

To my mother and father. You have always been a source of comfort, motivation and inspiration to me. You have had confidence in me, even when I did not. I will always be grateful for sacrifices you have made to give me a better life. Thank you!

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# Chapter 1

## Introduction

### 1.1 Overview

Consumers have a wide variety of product choices as well as the choice of where to purchase from and from which channel (online, bricks and mortar stores). This has led to manufacturers and retailers revamping their supply chain to be more customer centric. This has in turn, led to a better integration of customer information in the decision making process and better customer service. One area of research that has recently gained attention, from academics and practitioners alike, is the role of price in an operations management context. In most operations literature demand was considered to be an exogenous factor outside the decision making framework. But with more information, better understanding of the consumer, and the ability to capture and incorporate the effect of pricing into the operations modeling, demand can be controlled as part of the decision making process in a supply chain jointly with the inventory level for the products. This important factor, when properly incorporated in the decision making process, will lead to higher profits and better operational efficiency for the supplier and the retailer.

An important aspect of a successful supply chain management practice is the efficient and correct matching of supply with demand. This includes stocking the right product at the right place with the right quantity and most importantly at the right price. Recent studies in the supply chain management and operations management literature have focused on these aspects by considering price-sensitive consumers as part of the operational decision making framework. In this environment, the manager uses price as a lever to manage demand such that it matches the supply. At the retail level, the operations manager is concerned with making the right stocking decision, at the right price, such that the supply meets demand while at the same time accounting for the demand related risks. These include the risk of stocking too much at high prices resulting in low demand and excess inventory leading to salvaging on one hand and stocking too little at low prices resulting in excess demand leading to costly shortages on the other hand. Operational decisions when taken independent

of the demand side considerations, although have been prevalent in the operations literature until recently, have limited the ability of managers to extract maximum profits out of the supply chain.

In this thesis we use the single period price-setting newsvendor framework to formulate, analyze and solve the problem of pricing and stocking of retail goods under various scenarios. Specifically, the models studied in this thesis are

1. A price-setting retailer making ordering decisions facing not only uncertain demand from customers but also uncertain supply from its vendor. The retailer would like to find an optimal price and order quantity such that the risk of overstocking and understocking are minimized while the expected profit is maximized.
2. A price-setting retailer making ordering decisions while facing uncertain demand with the possibility that the customers substitute a different product, if their preferred product is stocked out. The retailer would like to find an optimal price and order quantity for the two substitutes such that the risk of overstocking and understocking are minimized and the expected profit is maximized.
3. A price-setting retailer making ordering decisions for two products that can be sold either separately or jointly in a bundle, supplied by a single vendor. The retailer would also like to achieve a price based coordination such that the overall supply chain profits are maximized for the retailer and the vendor.

In the next section we introduce each problem in more detail as well as provide further motivation behind studying them.

## 1.2 Uncertain Supply

Supply chains are exposed to a variety of risks one of which is having too much or too little stock to sell. This risk is usually due to uncertainty in consumer demand and is amplified when the supply is uncertain. The uncertainty in supply can be due to many reasons including loss or damage in transit. Furthermore, increased focus on quality, safety and regulatory issues at the point of consumption, have resulted in added risk of some of the goods being rejected by customs or the regulators at the point of consumption or preventing retailers from selling all the goods once received<sup>1</sup>. Another source of uncertainty, that is widely cited and well researched, is that of manufacturing and process unreliability. Although total quality management (TQM) and statistical quality control tools, such as six sigma process control, are widely used to reduce manufacturing and shipment of defective items by the supplier,

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<sup>1</sup>As per a personal communication with an operations manager from a global pharmaceutical company.

they may not always prevent the shipment of items that do not meet all the quality standards set by the retailer, consumer or the government agency. Another source of unreliable supply can be capacity related, e.g., farm goods, where the amount of land (capacity) under cultivation for a crop (product) has an uncertain yield. Supply may be uncertain in many different environments such as electronic fabrication and assembly, chemical manufacturing and processes, discrete parts manufacturing, food and agricultural products, pharmaceutical products and medical supplies. For the most recent industry specific studies that have looked at the issue of supply uncertainty, we refer the readers to Jones et al. (2001) and Kazaz (2004) (agriculture applications), Bakal and Akcali (2006) (automotive parts resellers), Tang and Yin (2007b) (retail goods) and Hsieh and Wu (2008) (manufacturing).

We consider a retailer facing price-sensitive demand with additive errors. The supply is uncertain with lead times that may make a mid-season recourse impossible, if the demand outstrips the supply resulting in costly shortages mid-season. However, overstocking is not always desirable as it has its own downside. They include higher ordering, shipping and holding costs, salvage losses at the end of a selling season and limited storage space. An order quantity and a price that maximizes revenue and minimizes salvage and shortage losses needs to be determined such that these risks are balanced. The added uncertainty in supply makes the analytical determination of the optimal price and quantity much more complex. We consider the supply to be *stochastically proportional*, i.e. the proportion of salable units does not depend on the order size. It is an exogenous variable outside the control of the retailer and the supplier. In our study we develop models that support uncertain supply, with demand either being stochastic or deterministic, to yield results that explain the relationship between optimal price and order quantity and the uncertain supply. The existing body of knowledge on this issue does not always include price as one of the levers to reduce the additional risk from supply uncertainty.

### 1.3 Product Substitution

We consider a single period joint pricing and ordering problem for a retailer stocking two substitutable products. In case of a stockout of one product the customer may substitute with an alternative, if available, with some known probability. In some cases, when the stock of the other product is limited, this substitution may lead to a stockout of the other product as well, known as inventory cannibalization. We provide a model by which the retailer may optimize the expected profit by jointly determining the optimal prices and quantities of the two products when their demand is linear in price and uncertain.

According to a study by Gruen et al. (2002) conducted on behalf of Grocery Manufacturers of America (GMA), grocery retailers face a significant loss of sales due to stockouts in the fast moving consumer goods segment. The study focused on 11 product categories from around the world. According to the study, the average

rate of an item being stocked out worldwide is 8.3% and 72% of the stockouts are due to retailer’s in-store practices such as store ordering and stocking policies and incorrect forecasting. When a consumer faces a stockout situation for their preferred item, on average 45% of them choose to substitute with a different item, 46% choose to go to another store or delay purchase and the remainder choose not to buy the item at all. This results in an average annual loss of revenue of 4% for a typical grocery retailer. According to the study, 80% of the time retailers take more than 8 hours to respond to a stockout. Another interesting finding of this study was that higher safety stocks do not necessarily translate into lower occurrences of stockouts. In fact, excessive levels of inventory seems to impede the retailer’s response to stockouts and thus is a symptom of poor inventory management and ordering system at the store level. Another finding of this study was that the retailer risks losing the customer’s future business if the item is repeatedly stocked out. van Woensel et al. (2007) study consumer response to stockouts of perishable bakery products in Europe. One of their findings, that differs from Gruen et al. (2002), is that consumers have a higher willingness to substitute perishable products (on average 90% of the customers substitute) when their preferred product is out of stock compared to non-perishable items as indicated by Gruen et al. (2002). In Table 1.1, we reproduce the average worldwide consumer response to stockouts across 11 product categories (Gruen et al. (2002)).

Table 1.1: Average Worldwide Response to Stockouts

Do not Purchase Item	9%
Substitute-Different Brand	26%
Buy Item at Another Store	31%
Substitute-Same Brand (Different Size or Type)	19%
Delay Purchase	15%

In Table 1.2, we reproduce the average Canada-wide consumer response to stockouts across 8 product categories (Gruen et al. (2002)).

Table 1.2: Average Canada-wide Response to Stockouts

Do not Purchase Item	12%
Substitute-Different Brand	21%
Buy Item at Another Store	34%
Substitute-Same Brand (Different Size or Type)	15%
Delay Purchase	18%

Although studies such as Gruen et al. (2002); van Woensel et al. (2007) are applicable to most low margin retailers including grocery stores and department stores, the applicability of these studies to higher margin or premium retailers is not

clear. Given the possibility of high substitution rates and the positive relationship between safety stock and stockout occurrences instead of a negative one (Gruen et al. (2002)), there is a need for a more robust model incorporating both price and inventory so as to reduce stockout occurrences when possible or provide a substitute to the consumer. Most price-setting newsvendor type models do not incorporate product substitution and inventory cannibalization into the framework of joint pricing and ordering.

## 1.4 Product Bundling

An increasing number of products are being sold in bundles by retailers. Consumers have a choice to purchase these products as a part of a bundle from the retailer at a price that is usually lower than the price of the individual components purchased separately. Examples can be found across the product and price spectra. As an illustrative example, in year 2006, Microsoft released the new generation of game console called Xbox 360. The console is being sold in two package formats, the Xbox 360 Core, a no-frills game console and Xbox 360, a bundle that includes a certain number of accessories. The Core bundle sells for \$299.00 and the Xbox 360 bundle is being sold at \$399.00. The Xbox 360 bundle comes with additional accessories, which if purchased separately, would cost \$189.96. Microsoft, with its 360 Bundle, aims to provide a one stop shopping experience for its customers, not to mention a \$89.97 savings. Not to be outdone, the market leader at the time and its number one competitor, Sony, has similar retail options for its PlayStation 3 video game console.

Bundling has been found to be a useful marketing tool to boost sales and profit. The bundling literature usually focuses on studying the feasibility of the bundle in a retail setting. However, these studies fail to account for the fact that the product being bundled and sold can be manufactured, supplied and retailed by different entities. These studies usually consider a retailer as the sole decision maker, which is analogous to a centralized supply chain and thus fail to account for the fact that in a decentralized supply chain, the manufacturing, distribution and the retailing are usually undertaken by different entities trying to maximize their own profits without focusing on the profitability of the overall supply chain. This myopic behavior leads to double marginalization that prevents different parties from maximizing the overall profit. Thus, in a multi-entity or decentralized supply chain, bundling may not always be a feasible or a very profitable strategy. This is especially true if the products or the bundle are discounted beyond a certain point in an effort to boost sales, because it may make one or more parties worse off while making the other better off. This reduces the effectiveness of the bundling strategy in a supply chain setting.

There are several reasons why retailers would like to bundle products and consumers would like to consume bundles. For the retailer, it reduces logistics costs, inventory holding costs and increases revenue, leading to higher profits. For consumers, it may lead to greater consumer surplus, reduced decision making and provides the

ease and convenience of one stop shopping. Another rationale for bundling is that it serves as a tool for sorting consumers based on their reservation prices and hence extracts maximum profits for the retailer.

We analyze the case of an independent retailer selling two goods individually as well as in a bundle to consumers in a supply chain where the supplier, who is also independent, supplies the two goods to the retailer. Studies so far, have not analyzed bundling in a supply chain context. We find that selling bundles not only leads to higher sales and improves the profits for both, but also reduces the effect of double marginalization as bundling can serve as a coordination mechanism between the supplier and the retailer.

## 1.5 Organization of the Thesis

The remainder of the thesis consists of five chapters. Chapter 2 presents a survey of the literature on the topics introduced so far. Chapters 3 to 5 constitute the main body of this thesis. Each of the chapters is focused on a particular type of problem that the retailer might face. Each problem is first formulated by stating the underlying assumptions, then the necessary analytical modeling and analysis is performed to establish properties and procedures that help us find the optimal solution. Furthermore, we highlight our analytical finding with the help of numerical examples. We also conduct sensitivity analysis in order to study the impact of the assumptions and parameters on the optimal solution.

In Chapter 3 we study the problem of a retailer facing uncertain and price-sensitive consumer demand as well as uncertain supply. The retailer must arrive at an optimal price and order quantity such that its expected profit is maximized. The retailer, depending upon the situation, may either choose to simultaneously place an order and price the product or may do so sequentially such that the retail price is set after the order arrives. In the latter case, the retailer faces no further supply uncertainty. We call these two cases simultaneous pricing and postponed pricing, respectively. The problem is analytically complex and thus after presenting analytical results, we also present solution procedures required to optimally solve the problem. Next, we conduct sensitivity analysis based on parameters of the yield distribution as well as investigate and identify the factors that affect the final results.

In Chapter 4 we study the problem of single-period joint-pricing and ordering for two substitutable products. Demand for each product is price-sensitive and linear with additive errors following a general distribution. We incorporate the inventory cannibalization as a result of substitution. In case of cannibalization the shortage of one of the product results in demand spillover, due to substitution, which can result in a shortage for the other product. We perform analytical and numerical analysis to study the impact of these two phenomena and find that substitution and cannibalization can have a significant impact on prices, order quantities and the expected profit.



In Chapter 5 we study the problem of an independent retailer and an independent supplier selling two products in a supply chain. We propose that product bundling can act as a coordination mechanism, whereby these two independent entities can coordinate their decision making. We find that if the supplier offers a bundling discount to the retailer, thereby financially inducing the retailer to offer bundles to the consumers, the two can coordinate the decision making, leading the supply chain to produce more profits. Since the supplier and retailer are both trying to be better off, we ensure a Pareto-efficient result, i.e., both are better off, by having the supplier charge a bundling fee to the retailer to ensure that the supplier's profits do not suffer. Finally we identify the range of values for these fees for which bundling is an efficient response.

Finally in Chapter 6, we summarize our results and conclusions and identify future direction of research.

# Chapter 2

## Literature Review

The literature on the joint-pricing and inventory models can be broadly classified based on several factors as shown in the table below.

Factor	Level
1. Time Horizon	a. Single period b. Multi-period
2. Demand Function	a. Linear b. Non-linear
3. Demand Certainty	a. Deterministic b. Stochastic
4. Demand Errors	a. No error b. Additive error c. Multiplicative error
5. Nature of Supply	a. Certain b. Uncertain
6. Supply Chain Context	a. With coordination b. Without coordination
7. Number of Products	a. Single product b. Multi-product i. Substitutable products ii. Non-substitutable product A. Bundled products B. Non-bundled products

Our work in the following three chapters, is based upon the above seven factors and is classified in Table 2.2.

Table 2.2: Research Classification

Factor	1	2	3	4	5	6	7
Chapter 3	a	a	b	b	b	b	a
Chapter 4	a	a	b	b	a	b	b.i
Chapter 5	a	a	a	a	a	a	b.ii.A

Studies in the operations literature that investigate pricing problems have predominantly focused on linear demand with additive errors due to the analytical tractability it affords. As before, we focus on each topic individually to present a survey of existing literature as well as identify opportunities for further extensions and contributions to the field.

## 2.1 Uncertain Supply

The literature review is aimed at covering the most relevant literature to our problem. Although an extensive body of literature exists in the field of uncertain supply, most of the literature focuses on the problem where the demand faced by the retailer is either deterministic or price independent and stochastic.

The earliest model of a random supply and demand was explored by Karlin (1958) where the fraction received, i.e., the yield, is independent of the order quantity. This issue has been also explored by Silver (1976), Nahmias (1982), Noori and Keller (1984) and Shih (1980) where the yield is random and the demand is considered fixed. Silver (1976) explores the exact relationship between the optimal order quantity and the parameters of the yield distribution (he considers yield to be normally distributed) and finds that the optimal order quantity is a multiple of the economic order quantity (EOQ) and this multiple depends on the choice of mean and variance of the yield distribution. Noori and Keller (1984) provide closed form expressions for optimal order quantity for uniform and exponential demand distributions with yield having a general distribution.

The most comprehensive review of the literature on random yields is done by Yano and Lee (1995). They broadly classify yields into two classes: random process yields and variable capacity yields. Examples for random process yields include a discrete manufacturing system, producing one unit at a time. For a batch or a continuous type system, a stochastically proportional yield is used, where only a fraction of the whole batch is fit for consumption. For random capacity, where the actual process capacity is unknown, examples include land under agricultural cultivation or a scarce manufacturing resource such as a highly specialized machine that is not always available either due to breakdown, routine maintenance or because it is processing another job. In this paper, we use a stochastically proportional yield as the retailer usually does not have control over the supplier's manufacturing process

or their capacity.

Gerchak et al. (1988) study a periodic review inventory model with random yield and show that the reorder point is independent of the yield randomness. Bollapragada and Morton (1999) provide a myopic heuristic for the periodic review problem with stochastically proportional yield and stochastic demand. A more recent review of these problems can be found in Grosfeld-Nir and Gerchak (2004). Henig and Gerchak (1990) provide a comprehensive analysis of a periodic review inventory system with random yields. All of the studies mentioned here do not incorporate price as a decision variable, since they are focused on the production environment. Li and Zheng (2006) incorporate the simultaneous pricing decision in this model with full backlogging of unmet demand. They show, for a general stochastic demand function, that the objective function is jointly concave in price and order quantity. Any unmet demand at the end of the selling cycle is fully met with the help of a special order in which all the units are of good quality and surplus inventory is carried forward. Although such assumptions are normal in a production/manufacturing type setting, they are not realistic in a retail setting where the unmet demand at the end of the selling season is lost and surplus is usually salvaged.

Van Mieghem and Dada (1999) analyze the benefits of a price-postponement strategy under demand uncertainty. Although product postponement is well studied and understood as a means to reduce demand uncertainty, they propose a price-postponement strategy as a cost effective tool to reduce uncertainty in the supply chain when only demand is uncertain. Other work involving postponed pricing with uncertain yield and demand includes Jones et al. (2001), Kazaz (2004) and Bakal and Akcali (2006), where they take the problem and break it down into two stages such that stage one deals with the realization of random yield and stage two with a recourse action in terms of either placing a second order with no yield uncertainty (Kazaz (2004)) or placing a second order with yield uncertainty (Jones et al. (2001)) or setting a postponed price (Bakal and Akcali (2006)). Wang (2006) considers the problem of simultaneous pricing and ordering as well as postponed pricing in a decentralized supply chain setting with  $n$  manufacturers and a single retailer with certain yield for all products.

More recent work on a single period random yield problem has been done by Rekik et al. (2007), Tang and Yin (2007b) and Hsieh and Wu (2008). Rekik et al. (2007) consider random demand for the case when yield and demand are either uniform or normal, obtain closed form expressions and provide some statistical insights for a choice of uniform distribution. Tang and Yin (2007b) develop a model with uncertain supply yield for the case when demand is *deterministic* and price-sensitive. They also develop a model for simultaneous pricing and postponed pricing. They do not incorporate shortage and salvage costs into their model both of which can arise due to the supply uncertainty even when demand is deterministic in the simultaneous pricing case. They model the problem where the yield distribution is discrete and uniform. Thus, the optimality conditions cannot be developed for a choice of

any general continuous distribution. Hsieh and Wu (2008) considers a decentralized supply chain consisting of a manufacturer, distributor and an original equipment manufacturer (OEM). The source of randomness in supply comes from capacity at the manufacturing end and random demand at the distributors end. Like Tang and Yin (2007b), they assume that the distributor of goods faces no shortage or salvage costs. They find that coordination, depending upon the scenario, may be beneficial to the manufacturer but not to the OEM. The distributor makes ordering and pricing decisions simultaneously.

We consider a retailer facing an uncertain supply and a price-sensitive demand with additive errors and no recourse in terms of placing a second order. Furthermore, it is not limited to a two stage problem as we also consider the case of simultaneous pricing and ordering policy, *i.e.*, *no price postponement* with lost unmet demand and surplus is salvaged at the end of the period.

## 2.2 Product Substitution

McGillivray and Silver (1978) study the effect of substitution from an inventory management perspective in a *single period* setting. Their model assumes the same unit variable and shortage costs for the two products with a fixed probability of substitution by the consumers. They develop a cost structure and a heuristic-based approach for an order-up-to periodic review policy. Ignall and Veinott (1969) and Deuermeyer (1980) study the *multi-period* case of this problem. Ignall and Veinott (1969) develop conditions under which the myopic single period solution is also the optimal long term solution. Deuermeyer (1980) develops a multi-period model for a class of products with interdependent demands and generalizes conclusions drawn by Ignall and Veinott (1969). Birge et al. (1998) consider the capacity problem of a firm producing two substitutable products where the firm is a price taker and determines the optimal capacity.

Parlar and Goyal (1984) develop a single period inventory problem for two substitutable products with no salvage and shortage costs. They show that the profit function is jointly concave in order quantities under some conditions on the parameter values. Parlar (1985) considers a two period Markov decision model to find an optimal inventory policy for two perishable and substitutable products. Pasternack and Drezner (1991) consider a single period stochastic demand model for two fully substitutable commodities, but at a different revenue level to account for substitution. They conclude that when either of the two products can be used as a substitute, inventory levels move in opposite directions with changes in revenue level. Total optimal stocking levels, however, may be more or less in this case than the case where there is no substitution.

Bassok et al. (1999) model multiple product classes with downward substitution and develop a greedy allocation policy for the case when demand is realized at the beginning of the period. Mahajan and van Ryzin (2001) consider an approximation

scheme to determine optimal order quantities for an assortment of products using an iterative stochastic optimization method since they show that the expected profit function is not even quasi-concave. They conclude that ignoring the impact of product substitution in a newsvendor setting leads to distortion in inventory decisions and thus suboptimal profits. Smith and Agrawal (2000) develop a probabilistic model to capture the effect of demand substitution and a methodology to determine profit maximizing assortment and order quantities. Rajaram and Tang (2001) develop a heuristic to study the impact of demand uncertainty, correlation and degree of substitution on the order quantity and expected profits by extending the basic newsvendor model for two products with same unit costs and prices. They conclude that product substitution leads to higher expected profits. Rather than the stocking levels, they find, that the degree of product substitutability plays a vital role in improving retailers profits.

All of the previous studies have assumed that the prices were exogenously determined parameters. There are a limited number of studies that have considered both pricing and order quantity as decision variables, albeit under some special conditions. Tang and Yin (2007a) develop a model with *deterministic* demand that investigates the optimal stocking policy for the retailer of two *perfectly* substitutable goods under fixed and variable pricing strategies. They extend the model to incorporate a capacity constraint and analyze the case of competition. One of their key findings is that as products become more substitutable, the optimal prices converge and the profit margins for the retailer diminish. Moinzadeh and Ingene (1993) consider a special case of joint pricing and ordering where the retailer stocks two substitutable products, one that is available for immediate delivery and the other that *requires waiting*, but sold at a lower price. The demand follows a Poisson distribution. The authors point at three limitations of their model (1) *only one of the prices is a decision variable* (2) *one of the two products is not available for immediate delivery* and (3) *they do not consider general demand distributions*. In our proposed model we avoid these limitations and deal with the case where both products can be available for immediate delivery. Karakul and Chan (2008) consider a similar problem where the firm introduces a new and innovative product that acts as a substitute for the existing product. They analyze the joint pricing and ordering decision made under decentralized and centralized scenarios, *when the price of the existing product is already set*. They show that the expected profit is unimodal and that higher profits can be derived by factoring in substitution.

Another important discussion that is usually missing from all studies stated above has been incorporating the impact of cannibalization into the newsvendor framework. When a customer, for whatever reason, chooses to substitute with another product, it has an effect of reducing the available supply of the substitute for customers whose first choice was the substitute itself. This cannibalization usually leads to further shortages at the retailer. Ernst and Kouvelis (1999), while incorporating this, model a newsvendor selling two non-substitutes separately as well as in

a *package*, which acts as a substitute in case of stockouts of either products. They conclude that using independent newsboy policy leads to suboptimal stocking levels. While the optimal policy results in a higher inventory cost for the retailer, it also results in higher profits. Shah and Avittathur (2007) considers cannibalization of inventory under one way substitution for highly customized products and using a heuristic determine the optimal portfolio of products to stock.

## 2.3 Product Bundling

We consider a price setting retailer selling product bundles. We define product bundles as a collection of two or more goods sold jointly by a retailer at one price. A bundle can be one physical entity (also called a package) or a mix and match type product where customers self select two or more goods that constitute a bundle and pay one price to the retailer. In this thesis, the term “bundle” and “bundling” refer to this phenomenon. Our literature review focuses on two main themes: bundling and supply chain coordination.

### 2.3.1 Bundling

There have been numerous studies on bundling from the economics literature. We refer to Adams and Yellen (1976) for a broad overview of bundling from an economics perspective. Extending their model, McAfee et al. (1989) provide a general and sufficient condition for optimality of mixed bundling in the two products case. Scott and Highfill (2001) considered the case of mixed bundling and its impact on profits and compared it to the case with no bundling. Stremersch and Tellis (2002) provide the most comprehensive review from a marketing and a legal standpoint. Schmalensee (1984) and Long (1984) study the problem of bundling using a Gaussian distribution to represent the customer valuations. Some of the conclusions reached by Schmalensee (1984) are that pure bundling has the ability to reduce customer heterogeneity whereas no bundling can extract higher prices from customers. Mixed bundling combines these two advantages by selling bundles to reduce customer heterogeneity for those buyers with high overall product valuations and make those who value only one product very highly, pay more for their preferred product. He finds that the solution to the mixed bundling problem is not easy to determine analytically nor is the profit function globally concave. Long (1984) focuses on the profitability of bundling in a slightly more general setting. He shows that mixed bundling is equivalent to a two part pricing with no bundles, where the components are offered in a bundle at a discount.

From the operations management literature, we have limited research that looks at some of the operations issues as well as the quantitative decision making problems about bundling. Hanson and Martin (1990) construct and solve a mixed integer programming problem to determine the optimal bundles and corresponding prices

with  $N$  customers and  $M$  products when customer reservation prices are known. Eppen et al. (1991) consider the strategic use of the bundles as a cost reduction device. They propose a managerial framework for successful implementation of bundling to increase demand. Bakos and Brynjolfsson (1999) focus on the bundling of information products. However, from an operations standpoint, distribution and marginal cost of production for such products tend to be negligible and hence the problem is not the most general case of bundles involving physical goods. Ernst and Kouvelis (1999) study a newsboy type model of a retailer selling items independently as well as in a bundle. Murthi and Sarkar (2003) study the role of management science in personalization, taking an interdisciplinary approach, to provide a framework for personalization of products and services. They propose bundling as one form of personalization albeit for Internet based content products with low marginal costs.

Hitt and Chen (2005) show that in a monopolistic setting bundling of low-marginal-cost goods, such as digital products, outperforms selling individual products to consumers with heterogeneous preference, when the customers self select the goods that go into making a bundle. Geng et al. (2005) provide a model for bundling information goods to consumers when the marginal utility of consuming an additional quantity of the same good decreases. Chen et al. (2005) characterize the joint optimal inventory control for a model where the retailer sells a physical good with optional value added service in a bundle. More recently, McCardle et al. (2007) have considered the case of pure bundling of retail products and its impact on operations decisions. Demand, in their case, is dependent on price and customer valuations are uniformly distributed. They establish the conditions under which such a strategy is profitable. Except for McCardle et al. (2007), who consider the case where two goods are sold as a pure bundle, all the other cited operations management studies have considered bundling of soft goods (e.g., software and digital content) or a bundle of a good with a service (Chen et al. (2005)).

### 2.3.2 Supply Chain Coordination

In the last decade there have been numerous studies on the importance of coordination in supply chains and on different coordination mechanisms. Spengler (1950) was among the first to identify the phenomenon of double marginalization which he characterized as inefficiency as a result of decentralized decision making. This effect leads to higher prices, lower sales and reduced profits in a supply chain. As vertical integration (centralized decision making) is often not possible, it is to the benefit of the supplier to seek ways to induce the retailer to order and price in a way that mimics the integrated supply chain and brings the sum of decentralized profits closer to the profit when the chain is operated by a single owner. Thus, most supply chain coordination literature focuses on devising remedies for the double marginalization effect by proposing a variety of coordination mechanisms. Jeuland and Shugan (1983) provides a comprehensive study of channel coordination.



A good starting source for the field of supply chain contract design can be found in the chapters by Anupindi and Bassok (1999) and Lariviere (1999). Among the most common mechanisms are quantity discounts (Weng (1995)), buy-back contracts (Emmons and Gilbert (1998)), quantity-flexibility contracts (Tsay (1999), Abad (1994)), consignment selling (Boyaci and Gallego (2002)), sales-rebate contracts (Taylor (2003)), order-delivery commitment (Schneeweiss et al. (2004)), and revenue sharing (Cachon and Lariviere (2005)). Recent studies that have looked at coordination include Li and Liu (2006) (quantity discounts), Chiou et al. (2007) (price discount), Lee (2007) (stocking and markdown pricing), Qin et al. (2007) (volume discounts) and Mathur and Shah (2008) (price compliance and quantity-based contracts).

Some operations management studies have discussed the benefits of bundling on operations efficiencies. For example Porteus (1985) has looked at how bundling products could save on setup costs and Kohli and Park (1994) have referred to the idea that bundling can serve as a mechanism for reducing buyer-seller transaction costs. Despite these indications, the issue of product bundling did not get enough attention from the operations management community and in particular, in the area of the ever growing field of supply chain coordination. To our knowledge, none of the published literature has looked at the impact of bundling on supply chain coordination. Motivated by the preceding discussion, our research therefore focuses on the overall operational aspect of bundling and the role it can play in supply chain coordination.

## Chapter 3

# Pricing and Ordering when Supply is Uncertain

We consider a price setting retailer facing uncertain supply as well as demand while making inventory decisions. The retailer must decide upon a profit maximizing price and quantity such that the risk of overstocking and understocking are minimized. In this paper we study two possible scenarios.

1. Profit maximizing price and order quantity are simultaneously decided by the retailer. The retailer places an order for the goods while simultaneously announcing the price of the product and before receiving the final shipment of goods from the supplier, i.e., before knowing how much of salable quantity will be received.
2. Profit maximizing price and order quantity are sequentially decided by the retailer. First the retailer places an order for the goods and waits for the supply to show up. Once the supply is inspected by the supplier, then the supplier announces the price of the product.

The demand form here and elsewhere in the thesis is considered to be linear and decreasing in price. This will enable us to achieve analytical tractability that will be otherwise lost. We develop the analytical results and conditions that guarantee optimality as well as provide simple yet effective solution procedures. The analytical results are sometimes intractable and closed form solutions are not always possible, especially when the yield and/or demand error distributions are complex such as the Beta and the Normal distributions. We provide numerical examples as well as conduct sensitivity analysis for the models using different yield scenarios to assess the impact of supply uncertainty on the expected profit for the retailer. We begin this chapter by first defining the relevant notation and then formulate the model. Next we analyze the model as well as provide a procedure to find the price and order quantity such that the profit is maximized. After that we provide a numerical

example to highlight the results as well as conduct sensitivity analysis. Finally we provide our conclusions.

### 3.1 Model with Stochastic Demand

#### 3.1.1 Notations and Assumptions

We define the notation used throughout this chapter in Table 3.1.1. In addition to

Table 3.1: Definition of terms

Symbol	Definitions
$c$	unit cost, $c \geq 0$ .
$p$	price per unit charged, a decision variable such that $p \geq c$ .
$Q$	lot size, a decision variable such that $Q > 0$ .
$h$	salvage price such that $0 \leq h \leq c$ .
$s$	shortage cost $s \geq c$ .
$r$	random variable representing the supply yield defined over the range $r \in [w, 1]$ , where $w \in (0, 1]$ , with mean $\mu_r$ and standard deviation $\sigma_r$ . Its probability density and cumulative functions are denoted by $f$ and $F$ , respectively.
$u$	realized value of yield $r$ .
$D(p, \varepsilon)$	is the demand function such that $D(p, \varepsilon) = y(p) + \varepsilon$ , where $y(p) = a - bp$ and $a > 0$ and $b > 0$ are known constants. $a$ is the market size parameter and $b$ is the price elasticity.
$\varepsilon$	random variable representing the uncertainty in demand defined over range $[A, B]$ , such that $A > -a$ , with mean $\mu_\varepsilon$ and standard deviation $\sigma_\varepsilon$ . Its probability density and cumulative functions are denoted by $g$ and $G$ , respectively.
$v$	realized value of the demand error $\varepsilon$ .
$D(p, v)$	is the realized demand $D(p, v) = y(p) + v$ .

the requirement that the unit price cannot be lower than the unit cost, i.e.,  $p \geq c$ , we also assume that the retailer only pays for the units that are *good* (salable). Such an assumption is consistent in a pure inventory setting such as ours, where costs are related to the salable quantity received by the retailer (Yano and Lee (1995)). Furthermore, in order to ensure that the demand is non-negative, we put an upper bound on the price  $p \leq \frac{a+A}{b}$ . This ensures that the demand  $D(p, \varepsilon) \geq 0$ .

### 3.1.2 Simultaneous Pricing and Ordering

In this case, the retailer decides about the order size  $Q$  and price  $p$  at the same time. The retailer has no recourse and thus must mitigate the risk of shortages with the risk of being left with excess inventory at the end of the selling period.

#### Expected Profit

The profit function is given as

$$\Pi(p, Q) = \begin{cases} pD(p, v) + h[uQ - D(p, v)] - cuQ, & \text{if } D(p, v) \leq uQ, \\ (p - c)uQ - s[D(p, v) - uQ], & \text{if } D(p, v) > uQ. \end{cases}$$

Upon substituting  $D(p, v) = y(p) + v$ , we get

$$\Pi(p, Q) = \begin{cases} p[y(p) + v] + h[uQ - y(p) - v] - cuQ, & \text{if } v \leq uQ - y(p), \\ (p - c)uQ - s[y(p) + v - uQ], & \text{if } v > uQ - y(p). \end{cases}$$

The expected profit function is given as

$$\begin{aligned} E[\Pi(p, Q)] &= \int_w^1 \int_A^{uQ-y(p)} \left\{ p[y(p) + v] + h[uQ - y(p) - v] - cuQ \right\} g(v)f(u)dvdu \\ &\quad + \int_w^1 \int_{uQ-y(p)}^B \left\{ (p - c)uQ - s[v - uQ + y(p)] \right\} g(v)f(u)dvdu. \end{aligned}$$

Upon simplification we get

$$\begin{aligned} E[\Pi(p, Q)] &= (p - h)[y(p) + \mu_\varepsilon] - (c - h)\mu_r Q \\ &\quad + (p + s - h) \int_w^1 \int_{uQ-y(p)}^B [uQ - y(p) - v]g(v)dvf(u)du. \end{aligned} \quad (3.1)$$

Note that the highest demand occurs when  $p = c$  and  $\varepsilon = B$  and the lowest yield  $u = w$ . Thus the upper bound on the order quantity  $Q$  can be set from  $wQ = D(c, B) = a - bc + B \Rightarrow Q = \frac{a-bc+B}{w}$ . Similarly the upper bound on  $p$  can be set by letting  $D(p, A) = 0 \Rightarrow p = \frac{a+A}{b}$ . The problem of maximizing the expected profit is thus formulated as

$$\begin{aligned} \max_{p, Q} \quad & E[\Pi(p, Q)] \\ \text{s.t.} \quad & 0 < Q \leq \frac{a-bc+B}{w}, \quad c \leq p \leq \frac{a+A}{b}. \end{aligned} \quad (P1)$$

The constraint on the price  $p$  in problem (P1) is slightly more restrictive than what is warranted. However as noted earlier, this ensures that the demand  $D(p, \varepsilon) \geq 0$  and we also avoid the added complexity of using the min/max function while evaluating the expected profit function in equation (3.1).

## Analysis

In this section, we analyze the expected profit function given by equation (3.1). First we show that the expected profit function is concave in each decision variable using the following proposition.

**Proposition 1** *For any given price  $p$ , (P1) is a convex program in  $Q$ . For any given order quantity  $Q$ , (P1) is a convex program in  $p$ .*

**Proof.** Consider

$$\frac{\partial^2 E(\Pi(p, Q))}{\partial Q^2} = -(p + s - h) \int_w^1 u^2 g(uQ - y(p)) f(u) du \leq 0, \quad (3.2)$$

and

$$\frac{\partial^2 E(\Pi(p, Q))}{\partial p^2} = -b \int_w^1 (2G(uQ - y(p)) + b(p + s - h) g(uQ - y(p))) f(u) du \leq 0.$$

From equation (3.2), for a given price  $p$ ,  $E(\Pi(p, Q))$  is concave and since the constraints are linear we conclude that (P1) is convex in  $Q$ . Similar arguments can be used for the case when  $Q$  is fixed. ■

Next we would like to analyze the hessian of the expected profit function to establish joint concavity analytically. However, the expression for the hessian is very complicated and proving joint concavity analytically is quite difficult. This is due to the fact that there exists a random variable ( $u$ ), as well as two decision variables ( $p$  and  $Q$ ) in the limits of the integration. This is in general true for all *price setting* newsvendor problems. Recent relevant examples include Petruzzi and Dada (1999) who propose complete enumeration, Agrawal and Nahmias (1997) who analyzed the problem for one decision variable, while fixing the other, thus guaranteeing concavity, as in this chapter and most recently this was shown by Karakul and Chan (2008). Furthermore, most previous studies have considered full backlogging of any unmet demand (Li and Zheng (2006)) as well as placing a special order with no supply yield uncertainty at the end of the season to meet any unsatisfied demand (Li and Zheng (2006), Kazaz (2004)), rendering the expected profit function analytically tractable. These assumptions are consistent in a manufacturing or production environment. However in retail operations, lost sales are the norm and thus makes the analysis of the expected profit function more complex as it is not possible to place a special order at the end of the season to satisfy unmet demand without supply uncertainty. However, we show the following result that helps us in finding the optimal solution to the problem (P1).

Thus, we can calculate the optimal  $Q^*(p)$  and  $p^*(Q)$  by solving the first order conditions given as

$$\begin{aligned}\frac{\partial E(\Pi(p, Q))}{\partial Q} &= -(c-h)\mu_r + (p+s-h) \int_w^1 \int_{uQ-y(p)}^B ug(v)f(u)dvdu = 0, \\ \frac{\partial E(\Pi(p, Q))}{\partial p} &= (y(p) + \mu_\varepsilon) - (p-h)b \\ &\quad + \int_w^1 \int_{uQ-y(p)}^B [(uQ-y(p)-v) + b(p+s-h)]g(v)f(u)dvdu = 0.\end{aligned}$$

### Solution Procedure

To find an optimal order quantity  $Q^*$ , price  $p^*$  and expected profit  $E[\Pi(p^*, Q^*)]$  we use the following search procedure:

1. Initialize  $i = 0$  and  $Q_i = 1$ .
2. Find  $\hat{p}_i$ , s.t.  $\frac{\partial E[\Pi(\hat{p}_i|Q_i)]}{\partial p} = 0$  and set  $p_i = \begin{cases} \hat{p}_i & \text{if } \hat{p}_i \geq c, \\ c & \text{if } \hat{p}_i < c. \end{cases}$
3. Calculate  $E[\Pi(p_i, Q_i)]$ . Store  $p_i$ ,  $Q_i$  and  $E[\Pi(p_i, Q_i)]$  in a table.
4. Let  $i = i + 1$  and  $Q_i = Q_{i-1} + \Delta$  where  $\Delta > 0$ . If  $Q_i \geq \frac{a-bc+B}{w}$  then end else goto Step 2.

The global maximum (or maxima) is identified by searching the table developed in Step 3.

### 3.1.3 Postponed Pricing

In this case, the retailer first determines the optimal order quantity  $Q$  in Stage I and after observing the value of supply yield, determines the optimal selling price  $p$  in Stage II. Thus the problem is formulated in stages with Stage II solved first where the retailer knows the value  $uQ \geq 0$ .

#### Expected Profit

The profit function for Stage II is

$$\Pi_{II}(p|uQ) = \begin{cases} pD(p, v) + h[uQ - D(p, v)] - cuQ, & \text{if } D(p, v) \leq uQ, \\ (p-c)uQ - s[D(p, v) - uQ], & \text{if } D(p, v) > uQ. \end{cases}$$

Note that in the above expression the value of the yield is already known and thus the notation  $uQ$  is used to indicate the fact that the retailer knows the available quantity for sale. Upon simplification of the above expression, we get

$$\Pi_{II}(p|uQ) = \begin{cases} p[y(p) + v] + h[uQ - y(p) - v] - cuQ, & \text{if } v \leq uQ - y(p), \\ (p - c)uQ - s[y(p) + v - uQ], & \text{if } v > uQ - y(p). \end{cases}$$

The expected profit function for Stage II is given as

$$\begin{aligned} E(\Pi_{II}(p|uQ)) &= (p - h)[y(p) + \mu_\varepsilon] + (h - c)uQ \\ &+ (p + s - h) \int_{uQ - y(p)}^B [uQ - y(p) - v] g(v) dv. \end{aligned}$$

The upper bound on  $p$  can be set by letting  $p = \frac{a+A}{b}$ . The problem of maximizing the expected profit in Stage II is

$$\max E[\Pi_{II}(p|uQ)], \text{ s.t. } c \leq p \leq \frac{a + A}{b}. \quad (\text{P2})$$

Let  $p^*(uQ)$  be the optimal price that maximizes the expected profit in Stage II for a realized value of supply yield  $u$  and order quantity  $Q$ . The expected profit function for Stage I is given as

$$E[\Pi_I(Q)] = \int_w^1 E(\Pi_{II}(p^*(uQ)|uQ)) f(u) du.$$

The highest demand occurs when  $p = c$  and  $\varepsilon = B$  and the lowest yield  $u = w$ . Thus the upper bound on the order quantity  $Q$  can be set from  $wQ = D(c, B) = a - bc + B \Rightarrow Q = \frac{a - bc + B}{w}$ . The problem of maximizing the expected profit in Stage I is

$$\max E[\Pi_I(Q)], \text{ s.t. } 0 < Q \leq \frac{a - bc + B}{w}. \quad (\text{P3})$$

## Analysis

Consider the second order condition for the expected profit in Stage II, given as

$$\frac{\partial^2 E(\Pi_{II}(p|uQ))}{\partial p^2} = -b[2G(uQ - y(p)) + b(p + s - h)g(uQ - y(p))] < 0.$$

Thus  $E(\Pi_{II}(p|uQ))$  is a concave function in  $p$ , for a given  $uQ$  value. Hence, there is a unique  $p$ , given as  $p^*(uQ)$ , that satisfies the first order condition

$$\begin{aligned} \frac{\partial E(\Pi_{II}(p|uQ))}{\partial p} &= (y(p) + \mu_\varepsilon) - (p - h)b \\ &+ \int_{uQ - y(p)}^B [(uQ - y(p) - v) + b(p + s - h)]g(v)dv = 0. \end{aligned}$$

### Solution Procedure

To find the order quantity  $Q$  that maximizes  $E(\Pi_I(Q))$ , we use the following solution procedure:

1. Using equation (3.3), we obtain a set of  $p^*(uQ)$  values for a set of  $(uQ)$  values,  $0 \leq uQ \leq \frac{a+B}{w}$ . Since in Stage II, the values of  $u$  and  $Q$  are numerically known, i.e., yield  $r$  takes a value  $u$  for a given order size  $Q$ , the problem simply reduces to finding a corresponding price  $p^*(uQ)$  that maximizes expected profit in (P2).
2. Next, we vary  $Q \in (0, \frac{a+B}{w}]$  by increment  $\Delta > 0$  to find the global optimum  $Q$ . For the problem in Stage II,  $p(uQ)$  can take values depending upon the value of  $Q$  and the realized value  $u$ . We calculate the expected profit for the value of  $Q$  using equation (3.1.3) as follows

$$\begin{aligned} E(\Pi_I(Q)) &= \int_w^1 E(\Pi_{II}(p, Q))|_{p=p(uQ)} f(u) du, \\ &\approx \sum_{u_j} E(\Pi_{II}(p, Q))|_{p=p(u_j Q)} f(u_j) \Delta u. \end{aligned}$$

We store the values of  $Q$  and the corresponding  $E[\Pi(p|Q)]$  in a table. The global maximum (or maxima) are then identified by searching the table. The average price, which we define as  $\bar{p} = \sum_{u_j} p(u_j Q)|_{Q=Q^*} f(u_j) \Delta u$ , can be calculated at this stage.

### 3.2 Model with Deterministic Demand

We now analyze the case when the demand is deterministic (i.e.,  $D(p) = y(p) = a - bp$ ) and the supply is unreliable. The reason for this analysis, as stated earlier in Section 2.1, are to provide a complete analysis of the pricing and ordering problem under supply uncertainty. Previous work including Tang and Yin (2007b) consider such model but make several restricting assumptions to derive an optimal solution. These include assuming the retailer will not face shortages or salvage losses and supply uncertainty following a discrete distribution, all of which we show limit the understanding of the problem. As before, we first develop and analyze the case when the price  $p$  and the order quantity  $Q$  are simultaneously determined and then develop and analyze the case when the price is set after realizing the supply yield  $r$ , for a given order quantity  $Q$ .



### 3.2.1 Simultaneous Pricing and Ordering

#### Expected Profit

The profit function is given as:

$$\Pi(p, Q) = \begin{cases} pD(p) - cuQ + h[uQ - D(p)], & uQ \geq D(p), \\ (p - c)uQ - s[D(p) - uQ], & uQ \leq D(p). \end{cases}$$

Upon simplification, we get

$$\Pi(p, Q) = \begin{cases} py(p) - cuQ + h[uQ - y(p)], & u \geq \frac{y(p)}{Q}, \\ (p - c)uQ - s[y(p) - uQ], & u \leq \frac{y(p)}{Q}. \end{cases}$$

The expected profit is given as

$$\begin{aligned} E[\Pi(p, Q)] &= \int_w^{y(p)/Q} (py(p) - cuQ + h[uQ - y(p)]) f(u) du \\ &\quad + \int_{y(p)/Q}^1 ((p - c)uQ - s[y(p) - uQ]) f(u) du \end{aligned}$$

Upon simplification, we get

$$\begin{aligned} E[\Pi(p, Q)] &= (p - h)y(p) - (c - h)\mu_r Q \\ &\quad + (p + s - h) \int_w^{y(p)/Q} [uQ - y(p)] f(u) du. \end{aligned}$$

For a given  $p$ , the upper bound on  $Q$  can be obtained by setting  $wQ = y(p)$  and the lower bound can be obtained by setting  $Q = y(p)$ . This implies that  $w \leq \frac{y(p)}{Q} \leq 1$ . The problem of maximizing the expected profit can now be stated as

$$\max E[\Pi(p, Q)], \text{ s.t. } w \leq \frac{y(p)}{Q} \leq 1, p \geq c. \quad (\text{P4})$$

#### Analysis

**Proposition 2** For a given price  $p$ , (P4) is a convex program in  $Q$ . For any given order quantity  $Q$ , (P4) is a convex program in  $p$ .

**Proof.** Consider

$$\frac{\partial^2 E(\Pi(p, Q))}{\partial Q^2} = -\frac{(p + s - h)y(p)^2}{Q^3} f\left(\frac{y(p)}{Q}\right) \leq 0 \quad (3.3)$$

From equation (3.3), for a given  $p$ ,  $E(\Pi(p, Q))$  is concave and since the constraints are linear we conclude that (P4) is a convex program in  $Q$ . Similar arguments can be used for the case when  $Q$  is fixed. ■

The first order conditions for the expected profit  $E[\Pi(p, Q)]$  are given as

$$\begin{aligned} \frac{\partial E(\Pi(p, Q))}{\partial Q} &= -(c-h)\mu_r + (p+s-h) \int_w^{y(p)/Q} uf(u)du, \\ \frac{\partial E(\Pi(p, Q))}{\partial p} &= y(p) - (p-h)b \\ &\quad + \int_w^{y(p)/Q} [(uQ - y(p)) + (p+s-h)b] f(u)du. \end{aligned} \quad (3.4)$$

By solving equation (3.4) for  $Q$ , we get the following implicit expression for  $Q^*$  as a function of  $p$ :

$$\int_w^{y(p)/Q} uf(u)du = \left[ \frac{c-h}{p+s-h} \right] \mu_r. \quad (3.5)$$

It is interesting to note that the condition (3.5) is similar to the fractile form in Whitin (1955). The solution procedure for the simultaneous pricing problem with deterministic demand is similar to the solution procedure in section 3.1.2, thus we do not repeat it.

### 3.2.2 Postponed Pricing

In this section we consider the scenario where the price will be set after the ordered quantity,  $Q$ , has been received and the outcome of the supply yield,  $r$ , is known.

#### Optimal Price

Given an order quantity  $Q$  and an observed value of supply yield  $u$ , the retailer can sell at most  $uQ$  units at price  $p$ . The postponed pricing problem can be formulated as

$$\max_p pD(p), \text{ s.t. } D(p) \leq uQ, \quad c \leq p \leq \frac{a}{b}. \quad (P5)$$

Here  $p = \frac{a}{b}$  is the price at which demand is equal to zero and let  $p^0 = \frac{a}{2b} = \arg \max_p p(a - bp)$ . Depending on the state of the first constraint in (P5), the optimal price can take two values:

1. If  $uQ > D(p^0)$  then  $p^0$ , the optimal price to the unconstrained version of (P5), also solves (P5). In this case we have ample supply and should charge  $p^0 = \frac{a}{2b}$  and salvage the remaining  $[uQ - D(p^0)]$  at a salvage price  $h$ .
2. If  $uQ \leq D(p^0)$  then the first constraint in (P5) will be binding at optimality, i.e., the price should be  $p^1$  such that  $D(p^1) = a - bp^1 = uQ$ , or  $p^1 = \frac{a-uQ}{b}$ . In this case we do not have enough supply to match the profit-maximizing demand and we should charge a price that will clear the supply with no quantity left for salvage.

Thus the price can take the following values depending upon the value of  $uQ$  relative to  $D(p^0)$ :

$$p^*(u, Q) = \begin{cases} p^0, & uQ \geq D(p^0), \\ p^1, & uQ < D(p^0). \end{cases}$$

Observe that  $c \leq p^0 \leq p^1 \leq \frac{a}{b}$  and so both  $p^0$  and  $p^1$  satisfy the second constraint in (P5).

### Optimal Order Quantity

Given the above discussion and noting that when  $uQ \leq D(p^0)$ , we can have either  $Q \geq D(p^0)$  or  $Q < D(p^0)$ , the profit, as a function of  $Q$ , is defined as

$$\Pi(Q) = \begin{cases} p^0 D(p^0) - cuQ + h[uQ - D(p^0)] & uQ \geq D(p^0), \\ (p^1 - c)uQ & uQ < D(p^0). \end{cases}$$

The largest value that  $Q$  can take is given as  $Q = \frac{D(p^0)}{w}$ . The optimization problem is stated as

$$\max E(\Pi(Q)), \quad 0 < Q \leq \frac{D(p^0)}{w}. \quad (\text{P6})$$

Note that in order to calculate the expected profit we will need to consider two cases.

**Case A:**  $Q < D(p^0)$ . Since  $Q < D(p^0)$  it follows that  $uQ < D(p^0)$  for all  $w < u \leq 1$ . From (3.6), the profit for Stage I in this case is given as

$$\Pi_{IA}(Q) = (p^1 - c)uQ.$$

and the expected profit is given as

$$\begin{aligned} E(\Pi_{IA}(Q)) &= \int_w^1 (p^1 - c)uQ f(u) du, \\ &= \int_w^1 \left( \frac{a - uQ}{b} - c \right) uQ f(u) du, \\ &= [a - bc] \frac{\mu_r Q}{b} - \frac{Q^2}{b} [\sigma_r^2 + \mu_r^2], \end{aligned} \quad (3.6)$$

The problem of maximizing the expected profit for Stage I can be formulated as

$$\max E(\Pi_{IA}(Q)), 0 < Q \leq D(p^0). \quad (P7)$$

From (3.6) we see that  $E(\Pi_{IA}(Q))$  is quadratic and concave in  $Q$  and it follows that the optimal unconstrained  $Q$  is the solution to the following first order necessary condition

$$\frac{dE(\Pi_{IA}(Q))}{dQ} = \frac{1}{b} [(a - bc)\mu_r - 2Q(\sigma_r^2 + \mu_r^2)] = 0.$$

Taking into account the constraint in (P7), it is easy to see that the solution to (P7) is as stated in the following proposition without proof.

**Proposition 3** *Given that  $Q \leq D(p^0) = \frac{a}{2}$ , the optimal order quantity that solves (P7) is*

$$Q_{IA}^* = \min \left[ \frac{(a - bc)\mu_r}{2(\sigma_r^2 + \mu_r^2)}, \frac{a}{2} \right]$$

**Case B:**  $Q \geq D(p^0)$ . Since  $Q \geq D(p^0) = \frac{a}{2}$ , we can either have  $uQ \geq D(p^0) = \frac{a}{2}$  or  $uQ < D(p^0) = \frac{a}{2}$ . From (3.6), the profit function in this case is

$$\Pi_{IB}(Q) = \begin{cases} p^0 D(p^0) - cuQ + h[uQ - D(p^0)], & uQ \geq D(p^0), \\ (p^1 - c)uQ, & uQ < D(p^0). \end{cases}$$

The expected profit is given as

$$\begin{aligned} E(\Pi_{IB}(Q)) &= \int_{D(p^0)/Q}^1 (p^0 D(p^0) - cuQ + h[uQ - D(p^0)]) f(u) du \\ &\quad + \int_w^{D(p^0)/Q} (p^1 - c) uQ f(u) du. \end{aligned}$$

After some algebra it can be shown that

$$\begin{aligned} E[\Pi_{IB}(Q)] &= (a - bc) \frac{\mu_r Q}{b} - \frac{Q^2}{b} (\sigma_r^2 + \mu_r^2) \\ &\quad + \int_{a/2Q}^1 \left[ a \left( \frac{a}{4b} - \frac{h}{2} \right) - (a - bh) \frac{uQ}{b} + \frac{Q^2}{b} u^2 \right] f(u) du. \end{aligned}$$

The problem of maximizing the expected profit in this case is

$$\max E(\Pi_{IB}(Q)), \frac{a}{2} \leq Q \leq \frac{a}{2w}. \quad (P8)$$

The first order necessary optimality conditions for the unconstrained version of (P8) is as stated below.

$$\frac{dE(\Pi_{IB}(Q))}{dQ} = (a - bc) \frac{\mu_r}{b} - \frac{2Q}{b} [\mu_r^2 + \sigma_r^2] - \int_{a/2Q}^1 \left[ (a - bh) \frac{u}{b} - \frac{2Q}{b} u^2 \right] f(u) du = 0. \quad (3.7)$$

As seen from (3.7), problem (P8) is more complex than problem (P7) and we will need to perform some analysis before we can establish the optimal order quantity. The second order condition for problem (P8) is

$$\frac{d^2 E(\Pi_{IB}(Q))}{dQ^2} = \frac{-2}{b} \int_w^{a/2Q} u^2 f(u) du + \frac{a^2 h}{4Q^3} f\left(\frac{a}{2Q}\right). \quad (3.8)$$

As one can see from equation (3.7), a closed form analytical solution is not possible in this case. However, we propose the following method to solve the optimization problem. We can see from equation (3.8) that the sign of the second derivative, for a given  $Q$ , is not clear. Thus, in order to find the profit maximizing quantity  $Q$ , we must further evaluate the first order condition given in (3.7). Let the quantities  $\hat{Q}_i$  where  $i \in \{1, 2, \dots, n\}$  and  $\hat{Q}_i \in \left[\frac{a}{2}, \frac{a}{2w}\right]$ , satisfy the first order condition (3.7). It is also possible that the solution to the problem (P8) may lie on the boundary of the feasible line, i.e., either  $Q = \frac{a}{2}$  or  $Q = \frac{a}{2w}$ . Thus for Case B, the optimal order quantity  $Q_B^*$  is the order quantity in the set  $\left[\frac{a}{2}, \hat{Q}_1, \dots, \hat{Q}_n, \frac{a}{2w}\right]$  that maximizes  $E(\Pi_{IB}(Q))$ . The optimal quantity  $Q^*$  that solves the original optimization problem (P5) is stated in the following proposition.

**Proposition 4** *The optimal order quantity that solves the optimization problem (P6) is:*

$$Q^* = \begin{cases} Q_A^* & \text{if } E(\Pi_{IA}(Q_A^*)) \geq E(\Pi_{IB}(Q_B^*)), \\ Q_B^* & \text{otherwise.} \end{cases}$$

### 3.3 Numerical Example and Sensitivity Analysis

In order to adequately represent all possible yield scenarios, we make use of the Beta distribution to represent stochastically proportional yield. Studies that make use of the Beta distribution, found in operations management and manufacturing literature include Stapper (1992) (VLSI yields), Seifi et al. (2000) (general manufacturing yield), Lin et al. (1997) (tolerance analysis), Kuo and Kim (1999) (semiconductor industry). Bollapragada and Morton (1999) also consider a periodic review problem where the supply yield has a Beta distribution. Although several recent studies on random yield in operations management, (e.g. Noori and Keller (1984), Rekik et al. (2007), Tang and Yin (2007b) etc.), make use of the uniform distribution to represent yield,

the choice is not ideal to model stochastically proportional yield in a supplier and a retailer setting. The stochastically proportional yield faced by a retailer can take different forms over the same support depending on the supplier's reliability. As the Uniform distribution exhibits no bias toward the mean, this can result in the retailer being even more risk averse than what is warranted. It is worthwhile to note that, for the Beta distribution, with  $\alpha = \beta = 1$ , the Beta distribution is uniform over its support  $[0, 1]$  and hence the Uniform distribution  $U[0, 1]$  is a special case of the Beta Distribution.

We provide a numerical example to highlight the analytical results for the simultaneous and the postponed pricing problems proposed in Section 3.1 and 3.2. For the distribution of the supply yield  $r$ , we consider the Beta distribution  $r \sim \text{Beta}[\alpha, \beta]$ , where  $\alpha \geq 1$  and  $\beta \geq 1$  are the shape parameters. The pdf of the Beta distribution is given as

$$f(u) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{(-1+\alpha)} (1-u)^{(-1+\beta)} & \text{if } 0 \leq u \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

and the mean and the variance are given as  $\mu_r = \frac{\alpha}{\alpha+\beta}$  and  $\sigma_r^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$  respectively. By varying the choice of  $\alpha$  and  $\beta$ , the supply yield distribution can take a variety of shapes such that the supplier, as perceived by the retailer, can be reliable if  $\beta > \alpha$  and unreliable when  $\beta < \alpha$ . For a complete discussion on the Beta distribution, we refer the readers to Chapter 25 in Johnson et al. (1995). For our numerical example, we consider the following parameter values  $a = 500$ ,  $b = 20$ ,  $c = 5$ ,  $h = 2$ ,  $s = 10$ ,  $A = -50$ ,  $B = 50$ ,  $w = 0$ ,  $\Delta u = 0.001$  and unless otherwise stated, the same values are used for all subsequent numerical analysis. For the supply yield we assume  $\alpha = 7$ ,  $\beta = 7$ ,  $\mu_r = 0.5$  and  $\sigma_r = 0.129$  and for the demand error we have  $\mu_\varepsilon = 0$ ,  $\sigma_\varepsilon = 16.67$ . To accurately represent the demand error  $\varepsilon$ , we use a Normal distribution that is truncated over the interval  $[A, B]$ , i.e.,

$$g(v) = \begin{cases} \frac{g_1(v)}{G(B)-G(A)} & \text{if } A \leq v \leq B, \\ 0 & \text{otherwise.} \end{cases}$$

where the pdf of the Normal distribution is given as:

$$g_1(v) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(v-\mu_\varepsilon)^2}{2\sigma^2}\right) & \text{if } -\infty \leq v \leq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

For a more complete discussion on the truncated Normal distribution, we refer the readers to Chapter 13 in Johnson et al. (1994). The optimal quantity and expected profits are included in Table 3.2 with corresponding prices. Note that the price in the case of the postponed pricing problem cannot be determined until the value of the supply yield is observed. Thus, an average price  $\bar{p}$  is calculated for a given value of  $Q$ , where  $\bar{p} = \sum_{u_j} p(u_j, Q) f(u_j) \Delta u$ ,  $j \in \{1, 2, \dots, 1000\}$ .

## Stochastic Demand

Table 3.2: Stochastic Demand.

	Simultaneous Pricing	Postponed Pricing
Price	15.59	15.46
Quantity	498.3	431.5
Expected Revenue	2832.35	2850.4
Expected Salvage	134.94	53.85
Expected Shortage	65.22	20.41
Expected Profit	1656.32	1805.1

From Table 3.2, we see that for the postponed pricing, the price and the quantity are lower whereas the expected profit is higher. This is to be expected as there is a greater uncertainty faced by the retailer, when the pricing and ordering decisions are made simultaneously. Once the supply yield uncertainty is resolved, the subsequent pricing decision is with regards to mitigating demand uncertainty alone. We also note that the increase in the expected profit is due to savings in inventory costs rather than an increase in sales revenue.

## Deterministic Demand

All parameter values are as before. For the postponed pricing problem, we make use of the relationship between  $\frac{\mu_r}{\sigma_r^2 + \mu_r^2}$  and  $\frac{a}{a-bc}$  to identify if Case A or Case B apply for calculating the optimal order quantity. Note that in the case of postponing the pricing decision cannot be determined till the supply yield  $u$  is observed. Thus, an average price  $\bar{p}$  is calculated for a given value of  $Q$ , where  $\bar{p} = \sum_{u_j} p(u_j, Q) f(u_j) \Delta u$ .

Table 3.3: Deterministic Demand.

	Simultaneous Pricing	Postponed Pricing
Price	15.69	14.90
Quantity	481.58	393.88
Expected Revenue	2818.40	2862.77
Expected Salvage	122.40	7.74
Expected Shortage	65.41	0.00
Expected Profit	1671.42	1885.80

Results are similar to the ones obtained when the demand was stochastic. Price and order quantity are lower and profits are higher for postponed pricing. The expected profit is higher for the deterministic demand case as compared to the stochas-

tic demand case. This is to be expected as there is a greater uncertainty in the previous setting, where the retailer in addition to uncertain supply also faces stochastic demand. This indicates that ignoring the demand uncertainty, as has been the case for most of the literature on pricing with uncertain supply, could lead to an overestimation of the expected profits. Note that for the case with postponed pricing, the retailer does not incur a shortage cost.

### 3.3.1 Sensitivity Analysis

We conduct sensitivity analysis by varying the shape parameters of the Beta distribution,  $\alpha$  and  $\beta$  such that the mean  $\mu_r$  varies from 0.2 to 0.9, holding  $\sigma_r = 0.125$  and the standard deviation  $\sigma_r$  varies from 0.078 to 0.2886, holding mean constant at  $\mu_r = 0.5$ . Our aim is to study the impact of supply uncertainty on the expected profit, order quantity and selling price. We hold all other parameters constant for our analysis.

#### Stochastic Demand

From Figure 3.1, we see that the expected profit in the case when the pricing decision is postponed is always higher than the case of simultaneous pricing. The order quantity  $Q$  for the simultaneous pricing problem is always higher than the postponed pricing case as there is greater uncertainty faced by the retailer. The retailer, being able to postpone the pricing decision, orders less from the supplier, as the retailer can set the price after observing the supply yield and thus, is in a better position to maximize the expected profit with lower uncertainty. As the supply uncertainty reduces, i.e. as  $\mu_r$  increases, the marginal increase in the expected profit diminishes. The expected price charged in the postponed pricing problem is always lower than the simultaneous pricing problem. The ability of the retailer to postpone the pricing decision, yields superior performance, as measured by the expected profit. However, as  $\mu_r$  increases, the expected profits for the postponed pricing and simultaneous pricing tend to converge along with the order quantities and prices charged. Thus, for a supplier who is “reliable”, the difference in the expected profit for the simultaneous and postponed pricing problem faced by the retailer is only marginally lower, whereas for a very “unreliable” supplier, the retailer’s profits are significantly greater for the postponed pricing case. As variability in supply decreases, i.e., as  $\sigma_r$  decreases, the expected profits and the prices charged converge, however the reduction in variability does not seem to have a significant impact on the order quantities for the postponed pricing problem as the retailer is able to first observe the supply yield. Similar results hold when the demand error distribution is assumed to be uniform, i.e.,  $\varepsilon \sim U[A, B]$ . To better understand the cause of the superior performance for the postponed pricing model we next investigate the expected revenue and expected salvage and shortage costs.



Figure 3.1: Sensitivity analysis for stochastic demand.

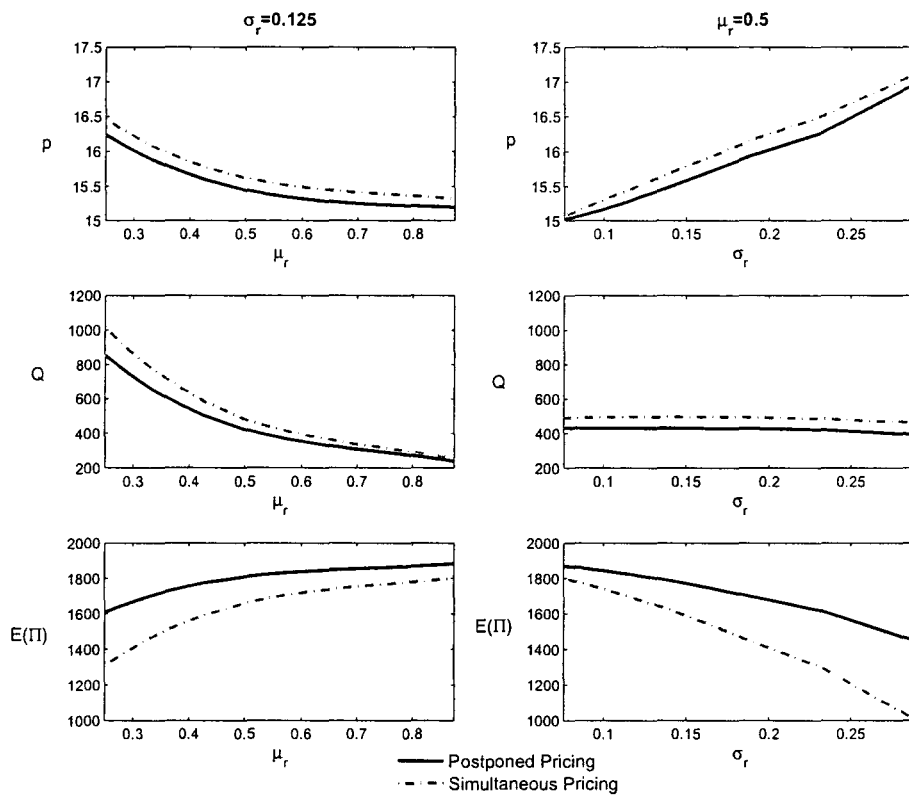
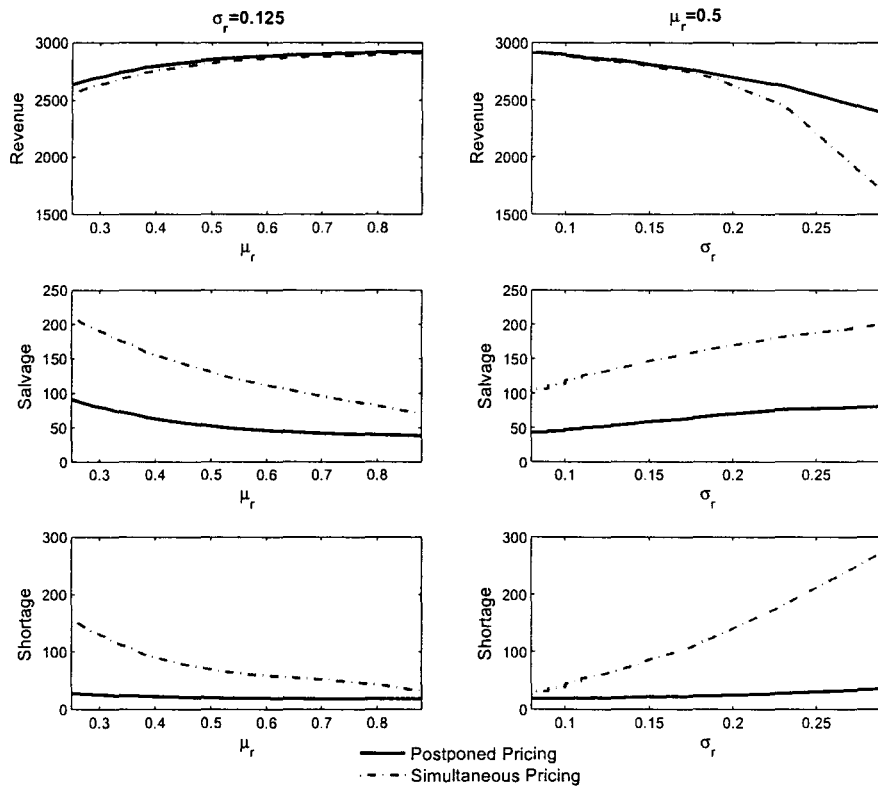


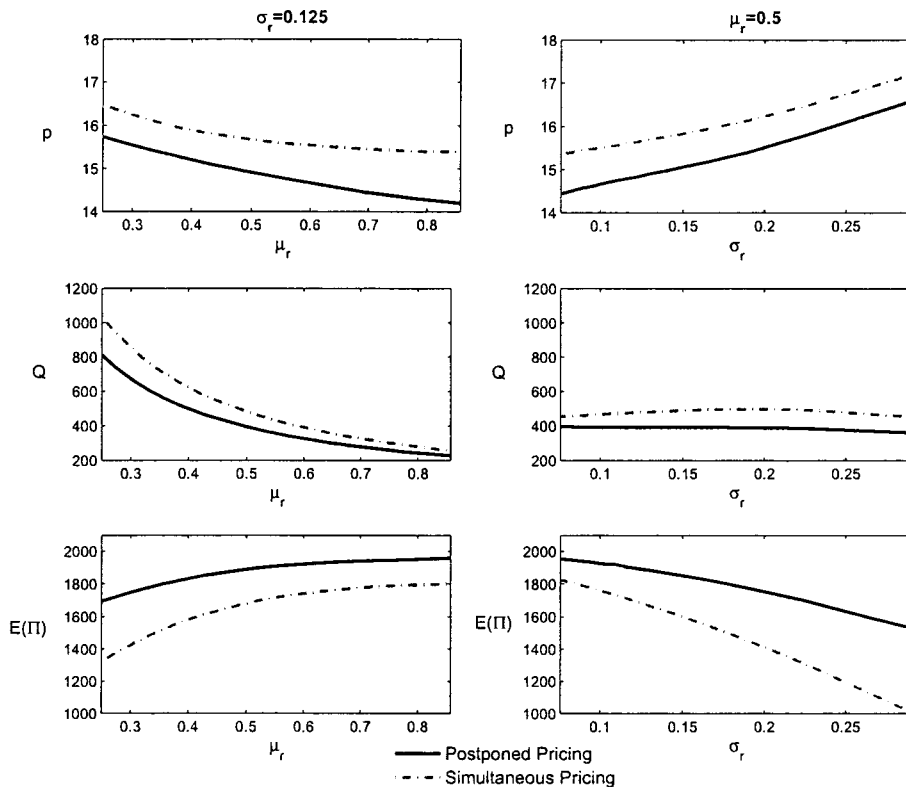
Figure 3.2: Performance analysis for stochastic demand.



In Figure 3.2, we note that the expected revenues have similar marginal increase when the supply yield becomes more certain. However, the marginal decrease in the expected inventory costs are not similar, in particular for the expected shortage costs under postponement. For the postponed pricing problem, the shortage costs seem to be almost insensitive to supply uncertainty.

### Deterministic Demand

Figure 3.3: Sensitivity analysis for deterministic demand.



From Figure 3.3, we see that the results for the sensitivity analysis are similar to those for stochastic demand. However, the prices charged do not converge as  $\mu_r$  increases and/or  $\sigma_r$  decreases. The problem of simultaneous pricing and ordering has a greater level of uncertainty than the postponed pricing case resulting in the retailer being more risk averse and thus charging a higher price even with demand being deterministic. For the postponed pricing case, the inventory costs are minimal, given that supply uncertainty is resolved before setting the price and there is no demand uncertainty in the problem.

### 3.4 Conclusions

We have studied a single period model with supply and demand uncertainty faced by a price-setting retailer. The work is aimed at developing a pricing strategy that can help mitigate the risk of supply and demand uncertainty faced by a retailer in a supply chain. Our model for both the stochastic and deterministic demand cases are more general than the ones previously developed in the literature. We also provide solution procedures for finding the optimal price  $p$  and order quantity  $Q$  for all scenarios under general distributions. We unify the results for the deterministic demand scenario and in addition, like Silver (1976), we show that there is a relationship between the optimal order quantity  $Q$  and the parameters of the yield distribution  $\mu_r$  and  $\sigma_r$  which can be explicitly observed in the postponed pricing case.

From our sensitivity analysis, we see that postponing the pricing decision can help produce more profits for the retailer even when the demand is price-sensitive and stochastic. This is due to greater uncertainty in simultaneous pricing that leads the retailer to be more risk averse in its pricing and ordering decisions. Thus, from a supply chain management standpoint, it is in the interest of the retailer facing unreliable supply to postpone the pricing decision until the supply uncertainty is resolved, whenever possible.

Under the deterministic framework, we can see that the postponing pricing decision can also lead to higher profits with a lower order quantity. This is due to the fact that once the supply uncertainty is resolved, the retailer faces no added risk and thus is free to price the product such that the profits are maximized. Further, we can see that in general, the expected profit when the demand is assumed to be deterministic is higher than the expected profit when it is stochastic. This means that a model that does not incorporate the stochastic demand tends to produce higher projected profits, results that may not very well represent reality where a retailer faces stochastic demand from consumers.

The sensitivity analysis also helped us gain insights into the role of price and order quantity decisions and their relationship to the supply uncertainty. As the supply uncertainty decreases in terms of a higher expected yield ( $\mu_r$ ), the expected profits for both simultaneous and postponed pricing scenarios tend to converge and the marginal difference between them decreases. This implies that for a supplier who has a stable production process overall or when the amount lost during transit is usually low, the added benefit of postponing the pricing decision is only incremental as measured by the expected profit. However, when the supplier is very unreliable, the exact opposite is true. Reducing the supply variability ( $\sigma_r$ ), does not seem to have a significant impact on the order quantity  $Q$  for both the simultaneous and postponed pricing case. However as variability reduces, the expected profits for both simultaneous and postponed pricing tend to converge and the marginal difference between them decreases. Most importantly, we find that the difference in the expected profits is not due to higher expected revenue, but due to lower expected

salvage and shortage losses when the pricing decision is postponed.

The implications, of unreliable supply, to the supplier are higher production and shipping costs as well as the cost of having unsalable goods left over at the end of a selling period when the uncertainty is high. For the retailer, it means higher shortage or salvage losses. Furthermore, postponement may not always be a feasible response to the supply uncertainty problem as the advertising and marketing effort required by the retailer may have to begin well in advance of the receipt of goods by the retailer and thus price postponement although, more profitable, may not be the feasible response. In such case, the retailer can optimally select a price and inventory such that the expected profit is maximized.

## Chapter 4

# Pricing and Ordering for Two Substitutable Products with Cannibalization

In this chapter we focus on a retailer selling two substitutable products to customers. The retailer faces price-sensitive linear demand with additive errors. In case of a stockout of one product, the customer may substitute with an alternative, if available, with some known probability called the probability of substitution. In some cases, when the stock of the other product is limited, this substitution may lead to a stockout of the other product as well, known as inventory cannibalization. The retailer must determine the prices and order quantities of these products such that the expected profit is maximized. Our model builds along the lines of Pasternack and Drezner (1991) and Parlar and Goyal (1984). We extend the earlier work by considering the most general case of a single period joint-pricing and ordering problem for a retailer selling two products. When the retailer stocks out of one of the two products, the customers may substitute with the next available one with some probability of substitution. We incorporate the effect of inventory cannibalization explicitly in our model. We perform analytical and numerical analysis to study the impact of substitution on the prices, order quantity and the expected profits. We begin the chapter by defining the notation and formulating the model and calculating the expected profit. Next, we provide the profit maximizing price as well as sensitivity analysis. Last, we highlight our findings with the help of a numerical example and provide our conclusions.

## 4.1 Model

### 4.1.1 Notation and Assumptions

The demand for product  $i$  is given as  $D_i = y_i + \varepsilon_i$  where  $y_i = a_i - b_i p_i$  and  $\varepsilon_i$  represents the random forecast error with mean  $\mu_i$  and variance  $\sigma_i^2$  defined over the support  $[A_i, B_i]$ . The parameters  $a_i > b_i$  and  $b_i > 0$  represent the market size and price sensitivity respectively and  $p_i$  is the retail price for product  $i$ , a decision variable. Further,  $c_i$ ,  $h_i$  and  $s_i$  are the corresponding unit cost, salvage value and shortage cost for the product such that  $h_i < c_i < s_i$ . Table 4.1 provides a summary of our notation.

Table 4.1: Summary of key notations.

Symbol	Explanation
Variables	
$p_i$	Price of product $i$ where $i = 1, 2$
$Q_i$	Order quantity.
Parameters	
$a_i$	Market size parameter.
$b_i$	Price sensitivity.
$c_i$	Unit purchase cost.
$h_i$	Salvage value, ( $h_i \geq -c_i$ ).
$s_i$	Shortage cost, ( $s_i \geq c_i$ ).
$\alpha_{ij}$	Probability that customer will substitute product $j$ , when $i$ is stocked out, ( $\alpha_{ij} \in (0, 1]$ ), $i \neq j$ .
$\theta_{ij}$	Rate of stock $j$ being cannibalized by customers looking for product $i$ , ( $\theta_{ij} \in [0, 1]$ ).
$y_i(p_i)$	Deterministic price sensitive demand defined as $y_i(p_i) = a_i - b_i p_i$ .
$D_i$	Stochastic demand $D_i = y_i + \varepsilon_i$ .
$\varepsilon_i$	Demand error defined over the support $[A_i, B_i]$ , where $A_i \geq -B_i$ .
$\mu_i, \sigma_i^2$	Mean and variance of the random error $\varepsilon_i$ .
$u, v$	Realized values of $\varepsilon_1$ and $\varepsilon_2$ , respectively.
$f(\cdot), g(\cdot)$	Corresponding pdfs of $\varepsilon_1$ and $\varepsilon_2$ , respectively.
$F(\cdot), G(\cdot)$	Corresponding cdfs of $\varepsilon_1$ and $\varepsilon_2$ , respectively.

Let the probability of product substitution in case of a stockout be given as  $\alpha_{ij}$ , i.e., if a customer looking to purchase product  $i$  does not find it, she will substitute with product  $j$ , if available, with a probability  $\alpha_{ij}$ ,  $i \neq j$  and  $\alpha_{ij} \in (0, 1]$ . Anupindi

et al. (1998) present methods for estimating these substitution probabilities. When a customer, looking to buy product  $i$ , does not find it and substitutes with product  $j \neq i$ , with probability  $\alpha_{ij}$ , it may cannibalize the available stock of product  $j$  for customers looking to purchase product  $j$  in the first place. Because product  $i$  is already stocked out, customers looking for product  $j$  will not be able to substitute with product  $i$ , if the available stock of product  $j$  gets depleted by customer substitution, leaving their demand unsatisfied and resulting in a shortage cost being assessed to the retailer for stockout of product  $j$  as well. Although the retailer is able to make a sale to customers looking for one product and upon not finding it, substituting with another product, who would have otherwise left the store without making a purchase (when at least one of the two products is available), it nonetheless is of interest to the retailer to know how many customers looking for product  $j$  were left unsatisfied due to customers looking for product  $i$  substituting with product  $j \neq i$ . We use  $\theta_{ij} \in [0, 1]$  to measure this cannibalization. It represents the rate at which product  $j$  is cannibalized by the product  $i$  customers. The retailer assesses a shortage cost  $s_i$  for the product  $i$  when the customer either substitutes with product  $j \neq i$  or leaves the store without making a purchase. The parameter  $\theta_{ij}$  assigns a fraction of sales of product  $j \neq i$ , in case of stockout of product  $i$ , to customers who were looking for product  $i$  in the first place. In doing so, we penalizes the retailer who still makes a sale of product  $j$  to the customer looking for product  $i$ , who would be otherwise left unsatisfied, as it depletes the stock of product  $j$  for customers looking for product  $j$  in the first place, which may lead to further shortages.

Knowing the individual customer arrival rates of  $i$  and  $j$  type of customers into the store, their substitution probabilities  $\alpha_{ij}$  and stocks of products  $i$  and  $j$ , the retailer may be able to estimate cannibalization of the two products. In our model, we consider this parameter to be exogenous. Though somewhat restrictive, this assumption allows us to provide a tractable solution to the pricing and ordering problem studied here. We also believe this assumption may not be overly restrictive because in most practical settings, the retailer may actually have to estimate the cannibalization rates from past observations.

#### 4.1.2 Model Development

For brevity, we will refer to the demand  $y_i(p_i)$  simply as  $y_i$ . For analytical and notational convenience, we will use  $z_i = Q_i - y_i$ ,  $A_i \leq z_i \leq B_i$  similar to Petruzzi and Dada (1999), where  $z_i$  represents the stocking factor. Thus,  $z_i$  acts as a surrogate variable for  $Q_i$ , i.e., the profit will be a function of  $z_i$  and  $p_i$ ,  $i = 1, 2$ . Depending upon the relative magnitudes of the demand and to facilitate comprehensive exposition, we decompose the problem into four scenarios. We state the profit function  $\Pi(p_1, p_2, z_1, z_2)$  and then derive the expected profit  $E[\Pi(p_1, p_2, z_1, z_2)]$ . We will also refer to the profit as  $\Pi$  and the expected profit as  $E(\Pi)$ , henceforth.



**Case I:**  $u \leq z_1$  and  $v \leq z_2$ .

In this case, there are no shortages for either product and the customers buy the type of product they desire. The inventory left over at the end of the selling season is disposed of at a salvage cost  $h_i$ . The profit function is given as

$$\begin{aligned} \Pi_1 = & (p_1 - h_1)(y_1 + u) + (p_2 - h_2)(y_2 + v) \\ & - (c_1 - h_1)(y_1 + z_1) - (c_2 - h_2)(y_2 + z_2), \quad \text{if } u \in I_1 \text{ and } v \in I_2, \end{aligned}$$

where  $I_1 = [A_1, z_1]$  and  $I_2 = [A_2, z_2]$ .

**Case II:**  $u \leq z_1$  and  $v \geq z_2$ .

In this case, there are no shortages for product 1, but there are shortages for product 2. Customers looking to buy product 2 and not finding it will substitute with product 1 with a known probability  $\alpha_{21}$ . Furthermore, if the substitution demand is high enough, i.e.,  $u + \alpha_{21}(v - z_2) \geq z_1$ , then demand cannibalization occurs at the rate of  $\theta_{21} \in [0, 1]$ , i.e., the rate at which customers looking for product 2 cannibalize the available stock of product 1, resulting in a shortage of product 1 for customers looking to purchase product 1 in the first place. Thus, the retailer will not only experience a shortage cost for stocking out of product 2, but may also experience a shortage cost for product 1 due to cannibalization. The profit function in this case is given as

$$\begin{aligned} \Pi_2 = & p_1 \min[y_1 + u + \alpha_{21}(v - z_2), (y_1 + z_1)] + p_2(y_2 + z_2) - s_2(v - z_2) \\ & + h_1 \max[z_1 - u - \alpha_{21}(v - z_2), 0] - s_1 \theta_{21} \max[0, u + \alpha_{21}(v - z_2) - z_1] \\ & - c_1(y_1 + z_1) - c_2(y_2 + z_2). \end{aligned}$$

Upon simplification we get

$$\Pi_2 = \begin{cases} (p_1 - h_1)[y_1 + u + \alpha_{21}(v - z_2)] + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_2(v - z_2) - (c_1 - h_1)(y_1 + z_1), & \text{if } u \in I_1 \text{ and } v \in I_3, \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_2(v - z_2) - \theta_{21} s_1 [u - z_1 + \alpha_{21}(v - z_2)], & \text{if } u \in I_1 \text{ and } v \in I_4, \end{cases}$$

where  $I_3 = [z_2, \Psi_2]$ ,  $I_4 = [\Psi_2, B_2]$  and  $z_2 \leq \Psi_2 = \frac{1}{\alpha_{21}}(z_1 - u) + z_2 \leq B_2$ .

**Case III:**  $u \geq z_1$  and  $v \leq z_2$ .

This case is similar to Case II, except that product 1 has stocked out instead of product 2. The profit function in this case is given as

$$\begin{aligned} \Pi_3 = & p_1(y_1 + z_1) + p_2 \min[y_2 + v + \alpha_{12}(u - z_1), (y_2 + z_2)] - s_1(u - z_1) \\ & + h_2 \max[z_2 - v - \alpha_{12}(u - z_1), 0] - s_2 \theta_{12} \max[0, z_2 - v - \alpha_{12}(u - z_1)] \\ & - c_1(y_1 + z_1) - c_2(y_2 + z_2). \end{aligned}$$

Upon simplification we get

$$\Pi_3 = \begin{cases} (p_1 - c_1)(y_1 + z_1) + (p_2 - h_2)[y_2 + v + \alpha_{12}(u - z_1)] \\ \quad - s_1(u - z_1) - (c_2 - h_2)(y_2 + z_2), & \text{if } u \in I_5 \text{ and } v \in I_2, \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_1(u - z_1) - \theta_{12}s_2[v - z_2 + \alpha_{12}(u - z_1)], & \text{if } u \in I_6 \text{ and } v \in I_2, \end{cases}$$

where  $I_5 = [z_1, \Psi_1]$ ,  $I_6 = [\Psi_1, B_1]$  and  $z_1 \leq \Psi_1 = \frac{1}{\alpha_{12}}(z_2 - v) + z_1 \leq B_1$ .

**Case IV:**  $u \geq z_1$  and  $v \geq z_2$ .

In this case, the original demand for the two products exceeds the quantity stocked. Unlike the previous two cases however, the substitution pattern is not very clear. Since one of the two products will stockout earlier than the other one, there will be some cannibalization of the inventory of the remaining product, but we do not know which one. Therefore, modeling this cannibalization will not be as straight forward as before. Since the substitution occurs when one of the two products will be stocked out and we assume the cannibalization will occur with a probability of  $(1 - \gamma)$  and  $\gamma$  respectively for product 1 and product 2 in general, where  $\gamma \in [0, 1]$ .

$$\Pi_4 = (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) - (s_1 + \gamma s_2 \theta_{12} \alpha_{12})(u - z_1) \\ - (s_2 + (1 - \gamma) s_1 \theta_{21} \alpha_{21})(v - z_2), \text{ if } u \in I_4 \text{ and } v \in I_8,$$

where  $I_7 = [z_1, B_1]$  and  $I_8 = [z_2, B_2]$ . The above four cases are summarized below.

$$\Pi = \begin{cases} (p_1 - h_1)(y_1 + u) + (p_2 - h_2)(y_2 + v) \\ \quad - (c_1 - h_1)(y_1 + z_1) - (c_2 - h_2)(y_2 + z_2) & \text{if } u \in I_1 \text{ and } v \in I_2 \\ (p_1 - h_1)[y_1 + u + \alpha_{21}(v - z_2)] + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_2(v - z_2) - (c_1 - h_1)(y_1 + z_1) & \text{if } u \in I_1 \text{ and } v \in I_3 \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_2(v - z_2) - \theta_{21}s_1[u - z_1 + \alpha_{21}(v - z_2)] & \text{if } u \in I_1 \text{ and } v \in I_4 \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - h_2)[y_2 + v + \alpha_{12}(u - z_1)] \\ \quad - s_1(u - z_1) - (c_2 - h_2)(y_2 + z_2) & \text{if } u \in I_5 \text{ and } v \in I_2 \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) \\ \quad - s_1(u - z_1) - \theta_{12}s_2[v - z_2 + \alpha_{12}(u - z_1)] & \text{if } u \in I_6 \text{ and } v \in I_2 \\ (p_1 - c_1)(y_1 + z_1) + (p_2 - c_2)(y_2 + z_2) \\ \quad - (s_1 + \gamma s_2 \theta_{12} \alpha_{12})(u - z_1) \\ \quad - (s_2 + (1 - \gamma) s_1 \theta_{21} \alpha_{21})(v - z_2) & \text{if } u \in I_4 \text{ and } v \in I_8 \end{cases} \quad (4.1)$$

The expected profit, after some algebraic manipulations and rearranging the

limits of the integration, is given as

$$\begin{aligned}
E(\Pi) &= (p_1 - h_1)(y_1 + \mu_1) - (c_1 - h_1)(y_1 + z_1) + (p_2 - h_2)(y_2 + \mu_2) - (c_2 - h_2)(y_2 + z_2) \\
&\quad - (p_1 - h_1 + s_1) \int_{z_1}^{B_1} (u - z_1) f(u) du - (p_2 - h_2 + s_2) \int_{z_2}^{B_2} (v - z_2) g(v) dv \\
&\quad + \alpha_{21} (p_1 - h_1) \int_{A_1}^{z_1} \int_{z_2}^{\Psi_2} (v - z_2) g(v) dv f(u) du \\
&\quad + \alpha_{12} (p_2 - h_2) \int_{A_2}^{z_2} \int_{z_1}^{\Psi_1} (u - z_1) f(u) du g(v) dv \\
&\quad - \int_{A_1}^{z_1} \int_{\Psi_2}^{B_2} ((p_1 - h_1 + \theta_{21}s_1)(u - z_1) + \theta_{21}s_1\alpha_{21}(v - z_2)) g(v) dv f(u) du \\
&\quad - \int_{A_2}^{z_2} \int_{\Psi_1}^{B_1} ((p_2 - h_2 + \theta_{12}s_2)(v - z_2) + \theta_{12}s_2\alpha_{12}(u - z_1)) f(u) du g(v) dv \\
&\quad - \int_{z_1}^{B_1} \int_{z_2}^{B_2} (\gamma\theta_{12}\alpha_{12}s_2(u - z_1) + (1 - \gamma)\theta_{21}\alpha_{21}s_1(v - z_2)) g(v) dv f(u) du \quad (4.2)
\end{aligned}$$

where lines 1–2 in the above expression are the expected standard newsvendor profit from selling products 1 and 2. Lines 3–4 of the expected profit function are the additional profits attained due to substitution. Lines 5–7 are the additional losses suffered by the retailer due to substitution. Our aim is to jointly optimize the expected profit function  $E(\Pi)$  given as

$$\max E(\Pi), \text{ s.t. } p_i \geq c_i, \text{ and } A_i \leq z_i \leq B_i, \quad i = 1, 2. \quad (\text{P})$$

### 4.1.3 Analytical Properties

The analysis of this problem is complex due to the presence of four decision variables as well as the fact that  $z_1$  and  $z_2$  are inter-related due to substitution and cannibalization. Karakul and Chan (2008) pointed out that the analysis of price-setting newsvendor problems by regular analysis methods is quite complex and concavity is not guaranteed. Furthermore, Mahajan and van Ryzin (2001), through counter-examples show that even for a problem with only order quantities as decision variables, the expected profit function is not even quasi-concave. As in the previous studies, joint concavity w.r.t. all four decision variables is not obvious for the problem at hand. Below we provide some structural properties for the problem that will help us in determining the optimal prices and order quantities.

**Theorem 1** *For a given set of  $z_i$  values, where  $i = 1, 2$ , the expected profit function is jointly concave in prices  $p_i$ .*

**Proof.** Let  $E(\hat{\Pi}) = E(\Pi(p_1, p_2 | z_1, z_2))$ . The Hessian of the expected profit  $E(\hat{\Pi})$  is given as

$$H \left[ E(\hat{\Pi}) \right] = \begin{bmatrix} \frac{\partial^2 E(\hat{\Pi})}{\partial p_1^2} & \frac{\partial^2 E(\hat{\Pi})}{\partial p_1 \partial p_2} \\ \frac{\partial^2 E(\hat{\Pi})}{\partial p_2 \partial p_1} & \frac{\partial^2 E(\hat{\Pi})}{\partial p_2^2} \end{bmatrix} = \begin{bmatrix} -2b_1 & 0 \\ 0 & -2b_2 \end{bmatrix}.$$

The Hessian is negative definite. Thus for a given set of  $z_i$  values, the expected profit  $E(\hat{\Pi})$  is jointly concave in  $p_i$ . ■

The following proposition gives us the optimal price  $p_i^*$  for a given set of  $z_i$  values.

**Proposition 5** *The optimal prices  $p_i^*(z_i)$  are given as*

$$p_1^*(z_1, z_2) = \frac{a_1 + b_1 c_1}{2b_1} + \frac{z_1 + \int_{A_1}^{z_1} [(u - z_1) G(\Psi_2) + \alpha_{21} \int_{z_2}^{\Psi_2} (v - z_2) g(v)] f(u) du}{2b_1}, \quad (4.3)$$

$$p_2^*(z_1, z_2) = \frac{a_2 + b_2 c_2}{2b_2} + \frac{z_2 + \int_{A_2}^{z_2} [(v - z_2) F(\Psi_1) + \alpha_{12} \int_{z_1}^{\Psi_1} (u - z_1) f(u)] g(v) dv}{2b_2}. \quad (4.4)$$

**Proof.** The first order conditions w.r.t.  $p_i$  are given as

$$\begin{aligned} \frac{\partial E(\hat{\Pi})}{\partial p_1} &= a_1 - b_1 p_1 + \mu_1 - (p_1 - h_1) b_1 + (c_1 - h_1) b_1 - \int_{z_1}^{B_1} (u - z_1) f(u) du \\ &\quad + \alpha_{21} \int_{A_1}^{z_1} \int_{z_2}^{\Psi_2} (v - z_2) g(v) dv f(u) du - \int_{A_1}^{z_1} \int_{\Psi_2}^{B_2} (u - z_1) g(v) dv f(u) du, \\ \frac{\partial E(\hat{\Pi})}{\partial p_2} &= a_2 - b_2 p_2 + \mu_2 - (p_2 - h_2) b_2 + (c_2 - h_2) b_2 - \int_{z_2}^{B_2} (v - z_2) g(v) dv \\ &\quad + \alpha_{12} \int_{A_2}^{z_2} \int_{z_1}^{\Psi_1} (u - z_1) f(u) du g(v) dv - \int_{A_2}^{z_2} \int_{\Psi_1}^{B_1} (v - z_2) f(u) du g(v) dv. \end{aligned}$$

Solving them yields equations (4.3–4.4). Since  $E(\Pi)$  is jointly concave in  $p_i(z_i)$ , from Theorem 1, equations (4.3–4.4) provide the optimal prices. ■

Given results in Theorem 1 and Proposition 5, we substitute equations (4.3) and (4.4) in equation (4.2). The expected profit now becomes a function of  $z_i$ . Thus, the optimization problem (P) can be rewritten as

$$\max E(\Pi(z_i | p_i^*(z_i))), \text{ s.t. } A_i \leq z_i \leq B_i, i = 1, 2. \quad (P1)$$

The optimization problem can now be solved for  $z_i$  using a grid search on the  $z_i$  plane.

#### 4.1.4 Special Cases

Under some restrictions on the parameters to the above problem, we have some observations that allow us to view three previous studies as special cases of this problem,

**Observation 1** *For a given pricing policy  $p_i$ , when  $s_i = 0$ ,  $h_i = 0$ ,  $b_i = 0$ ,  $\theta_{ij} = \theta_{ji} = 0$  the problem reduces to that of Parlar and Goyal (1984). Thus, the expected profit  $E(\Pi)$  is strictly concave in  $Q_i$  for a wide range of problem parameter values.*

**Observation 2** For a given pricing policy  $p_i$ , when  $b_i = 0$ ,  $\alpha_{ij} = \alpha_{ji} = 1$  and  $\theta_{ij} = \theta_{ji} = 0$  the problem reduces to that of Pasternack and Drezner (1991). Thus, the expected profit  $E(\Pi)$  is jointly concave in  $Q_i$ .

**Observation 3** Substituting price  $p_1$  (a known value),  $\alpha_{ij} \in [0, 1]$ ,  $\alpha_{ji} = 0$ ,  $\theta_{ij} = \theta_{ji} = 0$  in equation (4.1), the joint pricing and ordering problem defined by Karakul and Chan (2008) becomes a special case of the problem proposed in this chapter.

## 4.2 Sensitivity Analysis

In this section, we analyze the impact of substitution  $\alpha_{ij}$ , stocking factor  $z_i$  and cannibalization  $\theta_{ij}$  on the expected profit, optimal prices and order quantities. The analysis will be analytical, when possible, and numerical otherwise.

### 4.2.1 Analytical Results

#### Impact of $\alpha_{ij}$ and $\theta_{ij}$ on Expected Profit

The following proposition analyzes the impact of the substitution parameter  $\alpha_{ij}$  and cannibalization parameter  $\theta_{ij}$  on the expected profit  $E(\Pi)$ . In order to isolate the effect of substitution, we ignore the cannibalization (i.e.,  $\theta_{ij} = 0$ ) while differentiating the expected profit w.r.t.  $\alpha_{ij}$ . We could still make similar conclusions with  $\theta_{ij} \neq 0$ , however the analysis gets more complicated with no significant additional insights.

**Proposition 6** We state the following

1. The expected profit  $E(\Pi)$  is increasing in  $\alpha_{ij}$ , when  $\theta_{ij} = 0$ .
2. The expected profit  $E(\Pi)$  is decreasing in  $\theta_{ij}$ .

**Proof.** Differentiating  $E(\Pi)$  w.r.t.  $\alpha_{ij}$ , when  $\theta_{ij} = 0$ , we get

$$\frac{\partial E(\Pi)}{\partial \alpha_{12}} = (p_2 - h_2) \int_{A_2}^{z_2} \int_{z_1}^{\Psi_1} (u - z_1) f(u) du g(v) dv \geq 0,$$

$$\frac{\partial E(\Pi)}{\partial \alpha_{21}} = (p_1 - h_1) \int_{A_1}^{z_1} \int_{z_2}^{\Psi_2} (v - z_2) g(v) dv f(u) du \geq 0.$$

Differentiating  $E(\Pi)$  w.r.t.  $\theta_{ij}$  we get

$$\begin{aligned}\frac{\partial E(\Pi)}{\partial \theta_{12}} &= - \int_{A_1}^{z_1} \int_{\Psi_2}^{B_2} s_1 \alpha_{21} (v - \Psi_2) g(v) dv f(u) du \\ &\quad - (1 - \gamma) s_1 \alpha_{21} \int_{z_2}^{B_2} \int_{z_1}^{B_1} (v - z_2) f(u) du g(v) dv \leq 0, \\ \frac{\partial E(\Pi)}{\partial \theta_{21}} &= - \int_{A_2}^{z_2} \int_{\Psi_1}^{B_1} s_2 \alpha_{12} (u - \Psi_1) f(u) du g(v) dv \\ &\quad - \gamma s_2 \alpha_{12} \int_{z_2}^{B_2} \int_{z_1}^{B_1} (u - z_1) f(u) du g(v) dv \leq 0.\end{aligned}$$

■

From Proposition 6, we see that the expected profit for the retailer is increasing as the probability of substitution increases and is decreasing as the cannibalization rate for the inventory increases.

### Impact of $\alpha_{ij}$ and $\theta_{ij}$ on Optimal Prices

The following proposition analyzes the impact of the substitution parameter  $\alpha_{ij}$  and cannibalization parameter  $\theta_{ij}$  on the optimal prices  $p_i^*(z_i)$ .

**Proposition 7** *The optimal price  $p_i^*(z_i)$  is*

1. *Independent of  $\alpha_{ij}$  and increasing in  $\alpha_{ji}$ ,*
2. *Decreasing in  $z_i$  and increasing in  $z_j$ ,*
3. *Independent of  $\theta_{ij}$  and  $\theta_{ji}$ ,*

where  $i = 1, 2$  and  $i \neq j$ .

**Proof.** We show the analysis for  $p_1^*(z_1, z_2)$ . Differentiating  $p_1^*(z_1, z_2)$  w.r.t.  $\alpha_{12}$  and  $\alpha_{21}$  yields

$$\frac{\partial p_1^*(z_1, z_2)}{\partial \alpha_{12}} = 0 \text{ and } \frac{\partial p_1^*(z_1, z_2)}{\partial \alpha_{21}} = \frac{1}{2b_1} \int_{A_1}^{z_1} \int_{z_2}^{\psi_2} (v - z_2) g(v) dv f(u) du \geq 0.$$

Differentiating  $p_1^*(z_1, z_2)$  w.r.t.  $z_1$  and  $z_2$  yields

$$\begin{aligned}\frac{\partial p_1^*(z_1, z_2)}{\partial z_1} &= -\frac{1}{2b_1} \alpha_2 \int_{A_1}^{z_1} \int_{z_2}^{\Psi_2} g(v) dv f(u) du \leq 0 \text{ and} \\ \frac{\partial p_1^*(z_1, z_2)}{\partial z_2} &= \frac{1}{2b_1} \left( \int_{z_1}^{B_1} f(u) du + \int_{A_1}^{z_1} \int_{\Psi_2}^{B_2} g(v) dv f(u) du \right) \geq 0\end{aligned}$$

Differentiating  $p_1^*(z_1, z_2)$  w.r.t.  $\theta_{ij}$  and  $\theta_{ji}$  yields

$$\frac{\partial p_1^*(z_1, z_2)}{\partial \theta_{12}} = 0 \text{ and } \frac{\partial p_1^*(z_1, z_2)}{\partial \theta_{21}} = 0.$$

Similar analysis can be done for  $p_2^*(z_1, z_2)$ . ■

From Proposition 7 we see that the optimal price charged for product  $i$  increases as the willingness of the customers to substitute from product  $j$  to product  $i$  increases but is independent of the willingness of customers looking for product  $i$  to switch to product  $j$ . The optimal price charged for product  $i$  decreases as the stocking factor  $z_i$  increases and increases as the stocking factor  $z_j$  increases. Finally, inventory cannibalization does not have an impact on the product prices.

## 4.2.2 Numerical Study

In addition to confirming the analytical results of the previous section, the numerical study will also be used to investigate the effect of substitution and cannibalization on the order quantities and the consequence of ignoring these effects. To achieve the last objective, we compare our results to the case where the two products are managed using two independent joint pricing and ordering newsvendor models such as the one proposed by Petruzzi and Dada (1999). The optimal prices, order quantities and the sum of the expected profits for the two independent newsvendor problems (2INV) are given as  $p_1^{NV}$ ,  $p_2^{NV}$ ,  $Q_1^{NV}$ ,  $Q_2^{NV}$  and  $E(\Pi^{NV}) = (E(\Pi_1^{NV}) + E(\Pi_2^{NV}))$  respectively. First, we vary the substitution rate  $\alpha_{ij}$ , while holding all other parameters constant and then vary the cannibalization rate  $\theta_{ij}$  to study their impact on the optimal prices, inventory and the expected profits. Furthermore, we define the total quantity held by the retailer under substitution as  $Q = Q_1 + Q_2$  and under 2INV as  $Q^{NV} = Q_1^{NV} + Q_2^{NV}$ . This analysis will enable us to demonstrate the need to incorporate product substitution and demand cannibalization in the joint pricing and ordering framework, as well as provide insights into the role of substitution and cannibalization and its impact on the decision variables and the expected profit. Unless stated otherwise, we assume the following parameters values:  $a_1 = 800$ ,  $a_2 = 750$ ,  $b_1 = 20$ ,  $b_2 = 30$ ,  $c_1 = 10$ ,  $c_2 = 5$ ,  $h_1 = 5$ ,  $h_2 = 3$ ,  $s_1 = 15$ ,  $s_2 = 10$ ,  $A_1 = -100$ ,  $B_1 = 100$ ,  $A_2 = -150$  and  $B_2 = 150$ .

So far, our analysis has assumed a general distribution for the demand errors  $\varepsilon_i$ . Although we are able to derive many insights with this assumption, for our computation we will assume the distribution of the error terms  $\varepsilon_i$  to be uniformly distributed over the support  $[A_i, B_i]$ . This reduces the computational effort required to optimally solve the problem. For the first set of sensitivity analysis, we hold  $\alpha_{21} = 0.67$  and vary  $\alpha_{12}$  from 0.1 to 1.0 while fixing the values of  $\theta_{12} = \theta_{21} = 0.50$ . Next we repeat this procedure by holding  $\alpha_{12} = 0.67$  constant and varying  $\alpha_{21}$  from 0.1 to 1.0. For the second set of sensitivity analysis, we hold  $\theta_{21} = 0.50$  constant and vary  $\theta_{12}$  from 0.0 to 1.0 while holding  $\alpha_{12} = \alpha_{21} = 0.67$  constant. Next we repeat this procedure by holding  $\theta_{12} = 0.50$  constant and varying  $\theta_{21}$  from 0.0 to 1.0.

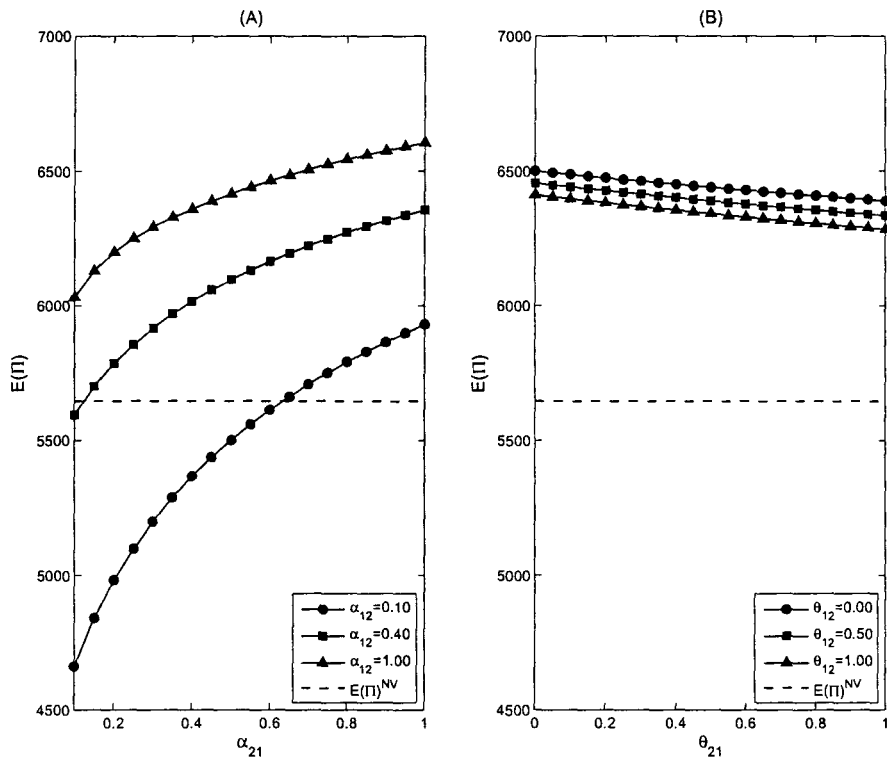


Figure 4.1: Impact of  $\alpha_{ij}$  and  $\theta_{ij}$  on the expected profit



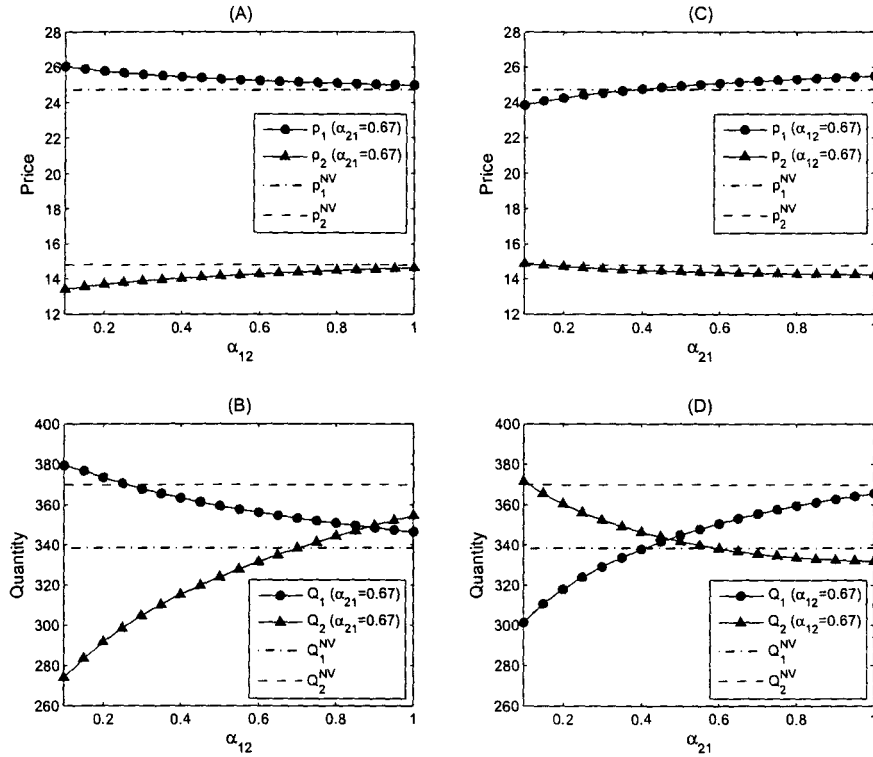


Figure 4.2: Impact of  $\alpha_{ij}$  on optimal prices and order quantity

In Figure 4.1(A), we show how the expected profit varies with the substitution parameter  $\alpha_{21}$  for different values of  $\alpha_{12}$ . In Figure 4.1(B), we show how the expected profit varies with the cannibalization parameter  $\theta_{21}$  at different values of  $\theta_{12}$ . The expected profit  $E(\Pi)$  is increasing as the substitutability of the two products increase. Comparing it to 2INV, we see that the 2INV tends to over estimate the expected profit for the retailer for low degrees of product substitution, i.e.,  $\alpha_{12}, \alpha_{21} \leq 0.40$  and under estimate the expected profit when the degree of substitution is high, i.e.,  $\alpha_{12}, \alpha_{21} \geq 0.40$ . The expected profit is decreasing in  $\theta_{ij}$  for all values of  $\theta_{ij}$ . Given that empirically the substitution rates have been found to be generally higher than 40% (e.g., see Gruen et al. (2002), van Woensel et al. (2007) and Ketzenberg et al. (2000)), we can say that ignoring the substitution and cannibalization effects will lead to underestimating the expected profits.

In Figure 4.2(A), we analyze the optimal prices by varying the substitution parameter  $\alpha_{12}$  holding  $\alpha_{21}$  constant. In Figure 4.2(B), we analyze the order quantity by varying the substitution parameter  $\alpha_{12}$  holding  $\alpha_{21}$  constant. Similar analysis

is done in 4.2(C) and 4.2(D), by varying  $\alpha_{21}$  while holding  $\alpha_{12}$  constant. From Proposition 7, Figure 4.2(A) and 4.2(C) we see that the prices  $p_1$  and  $p_2$  may increase or decrease as the degree of substitution increases. Furthermore, we see that the 2INV model tends to under price product 1 and over price product 2, when  $\alpha_{12}$  is varied from  $(0, 1]$  and  $\alpha_{21}$  is held constant at 0.67. Similarly, when  $\alpha_{21} \in (0, 1]$  and  $\alpha_{12} = 0.67$ , the retailer tends to over price product 2 and under price product 1. We see that the stocking quantities  $Q_1$  and  $Q_2$  change depending upon the substitution and cannibalization parameters. However, in Figure 4.2 we see that the order quantities tend to move in the same direction as the prices, i.e., as the price  $p_1$  in Figure 4.2 (A–B) decreases, the order quantity decreases as well. Similarly, when  $p_2$  increases,  $Q_2$  increases as well. Likewise, in Figure 4.2(C–D) we see that as price  $p_1$  increases,  $Q_1$  increases and as price  $p_2$  decreases,  $Q_2$  decreases as well. This appears to be counterintuitive at first, since, as prices increase, demand should decrease and vice versa. However, in Figure 4.2 (A–B), the prices and order quantities move in response to the degree of substitution  $\alpha_{12}$ . When the willingness of customers to substitute product 2 is high upon not finding product 1, the retailer tends to price product 1 lower and stock fewer units knowing that customers will substitute. Simultaneously, pricing product 2 higher and stocking more units. Similar analysis holds when varying  $\alpha_{21}$ .

In Figure 4.3(A), we analyze the optimal prices by varying the cannibalization parameter  $\theta_{12}$  holding  $\theta_{21}$  constant. In Figure 4.3(B), we analyze the order quantity by varying the cannibalization parameter  $\theta_{12}$  holding  $\theta_{21}$  constant. Similar analysis is done in 4.3(C) and 4.3(D), by varying  $\theta_{21}$  while holding  $\theta_{12}$  constant. From Proposition 7, Figure 4.3(A) and Figure 4.3(C) we see that the optimal prices  $p_1^*$  and  $p_2^*$  are unaffected by the cannibalization. However, the order quantities  $Q_1$  and  $Q_2$  change depending upon the values of  $\theta_{12}$  and  $\theta_{21}$ . As  $\theta_{ij}$  increases,  $Q_j$  increases and  $Q_i$  decreases; to account for cannibalization the retailer orders more of product  $j$ . Compared to 2INV, we observe that ignoring the effects of substitution and cannibalization will always result in overstocking at least one of the two products. This may partly explain the findings of Gruen et al. (2002) that although there is ample safety stock (of at least one product in store), the retailer will still have stockout situations under substitution.

In Figure 4.4(A), we analyze  $Q$  vs.  $Q^{NV}$  while varying  $\alpha_{ij}$ , holding  $\alpha_{ji}$  constant. Similarly in Figure 4.4(B), we analyze  $Q$  vs.  $Q^{NV}$  while varying  $\theta_{ij}$ , holding  $\theta_{ji}$  constant. From Figure 4.4(A), we see that the total quantity  $Q$  stocked by the retailer increases as  $\alpha_{12}$  and/or  $\alpha_{21}$  increase. However, the total quantity stocked is still less than the quantity stocked under 2INV. In fact, 2INV tends to significantly overstock for low degrees of substitution. This may again explain the empirical findings of Gruen et al. (2002). From Figure 4.4(B), we see that the impact of cannibalization is not as straightforward as that of substitution. The total quantity stocked overall is still less than 2INV, however, as  $\theta_{21}$  increases, the retailer stocks more and as  $\theta_{12}$  increases, the retailer tends to stock less. Although this may appear

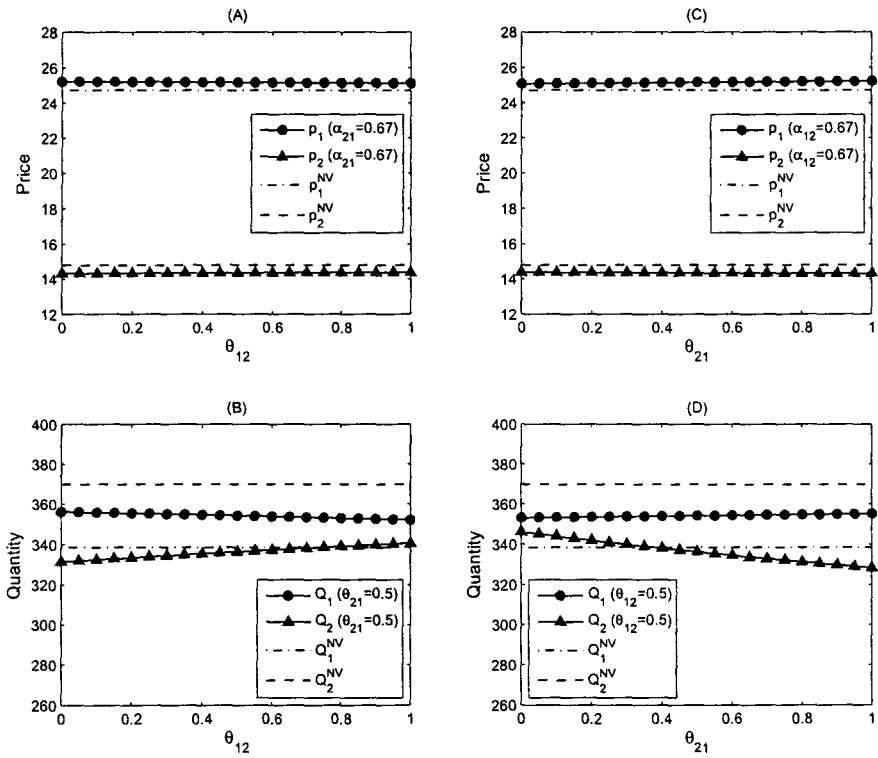


Figure 4.3: Impact of  $\theta_{ij}$  on optimal prices and order quantity

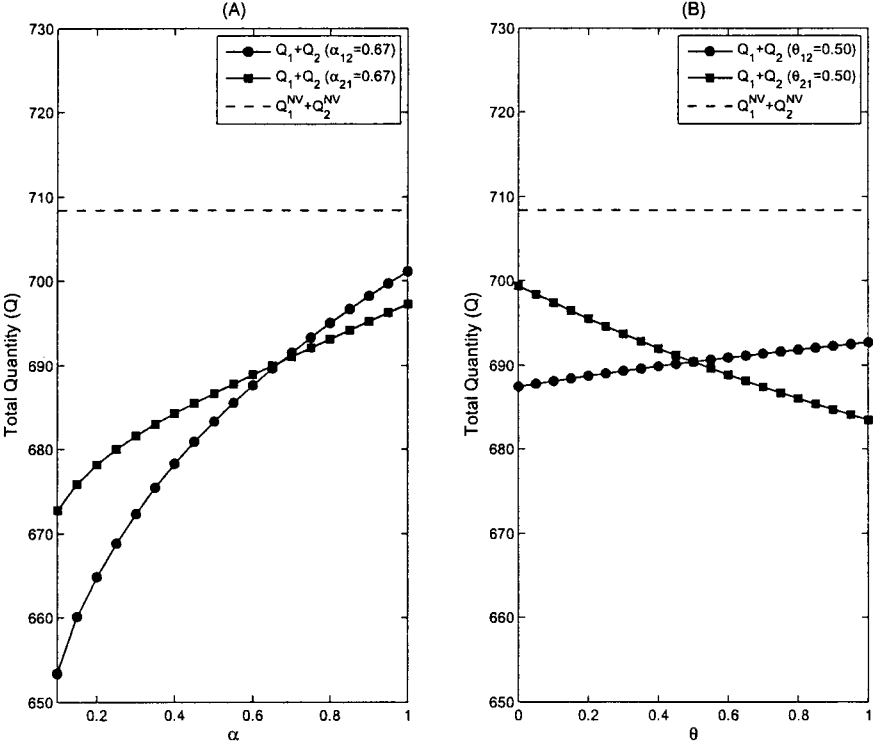


Figure 4.4: Impact of  $\alpha_{ij}$  and  $\theta_{ij}$  on total order quantity

to be counter intuitive at first, we must look at Figure 4.4(B) and Figure 4.3(B–D) jointly to understand the reasoning. As the cannibalization of product 2 increases, as product 1 becomes unavailable, the retailer stocks more of product 2. When the cannibalization of product 1 increases, as product 2 becomes unavailable, the retailer stocks more of product 1. Since product 1’s profit margin is greater than product 2’s profit margin, i.e.,  $(p_1^* - c_1) \geq (p_2^* - c_2)$ , the retailer is willing to under stock product 2 relative to product 1.

### 4.3 Conclusions

Retailers operate on very low margins and can lose as much as 4% (Gruen et al. (2002)) of their annual revenues due to stockouts, thus stockout based substitution and resultant cannibalization can have a significant impact on the retailer’s bottom line and should be incorporated in the overall pricing and ordering decisions. In this chapter, we study a general pricing and ordering problem for two products with stockout based substitution and cannibalization of inventory.

The analytical model helped us find the optimal prices to charge as well as the effect of substitution and cannibalization on the expected profit. Furthermore, we were able to show that the problem is concave in prices for a given inventory policy. The prices for the two products tend to move in opposite directions as the rate of substitution changes. We compare our findings to a model that does not consider substitution and the resultant cannibalization of inventory and find that the model that does not consider substitution tends to overestimate the expected profit for low degrees of substitution and tends to underestimate the expected profit for high degree of substitution. Furthermore, the prices charged and the inventory held at the retailer for each product, tend to be suboptimal. The total quantity stocked in general, for both products, is lower when we account for substitution and cannibalization. Although cannibalization plays an important role in the overall problem as it tends to reduce the retailers expected profit, this effect is not as significant as that of substitution. However, since the retailers operate on low margins (Gruen et al. (2002)), it is in the interest of the retailer to incorporate this effect in the overall decision making. Overall, with prudent management of the pricing and ordering process, the retailer can boost the expected profit by reducing the amount of inventory that the retailer holds as well as charging an optimal price for the goods.

# Chapter 5

## Supply Chain Coordination through Product Bundling

Bundling has been found to be an useful marketing tool to boost sales and profit in literature. Bundling literature that focuses on studying the feasibility of the bundling strategy, tends to focus on studying the problem with the assumption that there is only a single entity involved in the manufacturing, supplying, distributing, marketing and retailing of the products to consumers. Thus the analysis fails to consider a more general and realistic supply chain setting where the products being sold are usually manufactured, supplied and retailed by different entities, acting independently of each other. When the studies look at single entity, in supply chain parlance, that is referred to as a centralized supply chain and a multi-entity as a decentralized supply chain. In this chapter we focus on a two tier supply chain consisting of a supplier and a retailer selling two products either separately or in a bundle, in a monopolistic setting. We consider three bundling scenarios

1. Retailer sells the two products separately, we call it *No Bundle*,
2. Retailer sells the two products jointly but not separately, we call it *Pure Bundle*,
3. Retailer sells the two products jointly as well as separately, we call it *Mixed Bundle*

and two supply chain scenarios

1. Centralized supply chain,
2. Decentralized supply chain.

Our aim is to find the optimal price and discount that the retailer must offer consumers on the products as well as the optimal order quantity that the retailer must stock such that the profit is maximized. The supplier and the retailer may choose to either act in a centralized fashion, making decisions jointly or in a decentralized

fashion where the retailer and the supplier act independently and set the prices such that their individual profits are maximized. This leads to what is known as double marginalization and reduces the overall profit of the supply chain. In this chapter we would like to investigate if there exists a *price based* bundling mechanism whereby the retailer and the supplier can coordinate while maintaining their independence and at the same time eliminating the effect of double marginalization. In the following sections, we define the notation used in this chapter and formulate models for three bundling scenarios and two supply chain scenarios, to find the profit maximizing price and quantity. Then, we analyze and compare the performance of the decentralized supply chain and quantify double marginalization and suggest a remedy. We highlight our findings with the help of a numerical example and finally provide our conclusions. Our contribution is twofold. First, we provide a unified framework for studying bundling of two goods and present closed form solutions for optimal pricing and ordering under different scenarios. Second, we look at the role bundling can play within a supply chain coordination context. In addition to increasing sales and profits in a monopolistic centralized setting, we show that bundling may not be as beneficial to all parties in a supply chain and that the allocation of benefits depends on the bundling discount offered by the supplier to the retailer and the relative bargaining power of each party in the supply chain.

## 5.1 Supply Chain Model

We consider a supplier making two products  $A$  and  $B$  at costs of  $c_A$  and  $c_B$ , respectively, and sells them to a retailer, separately or in bundles, who would then sell them to consumers separately as well as in a bundle  $AB$  at prices  $P_A$ ,  $P_B$  and  $P_{AB}$ , respectively (we assume that the retailer will not unbundle). We assume that the retailer and the supplier know the distribution of the valuation that potential customers attach to each product  $j$ ,  $j = \{A, B\}$  and denote them as  $V_j$  where  $V_j \in [L_j, H_j]$ . The retailer buys these products from the supplier at a wholesale price  $w_j, j = \{A, B, AB\}$ , such that  $c_j \leq w_j \leq P_j \leq H_j$ . Consumers will not buy product  $j = \{A, B, AB\}$ , if the price of product  $P_j$  exceeds their valuation  $V_j$  and they will not buy the bundle if the bundle price exceeds the sum of their valuations for the components, *i.e.*,  $P_{AB} \geq V_A + V_B$ . This assumption is illustrated in Figure 5.1 for the case where the two products are offered separately. The horizontal axis measures the value of Product  $A$  and the vertical axis measures the value of Product  $B$ . The value rectangle, thus defined, represents all four possible combinations of values that consumers attach to  $A$  and  $B$ . For example, the region marked by “Purchase  $A$  only” in Figure 5.1 represents those consumers with product  $A$  valuations that exceed its price  $P_A$  and with product  $B$  valuations that are lower than its price  $P_B$ .

We also assume that the marginal utility of consuming a second unit of either product, together or separate, is zero and that the total market size  $M$  is a fixed known parameter. The joint distribution of customer valuations for products  $A$  and

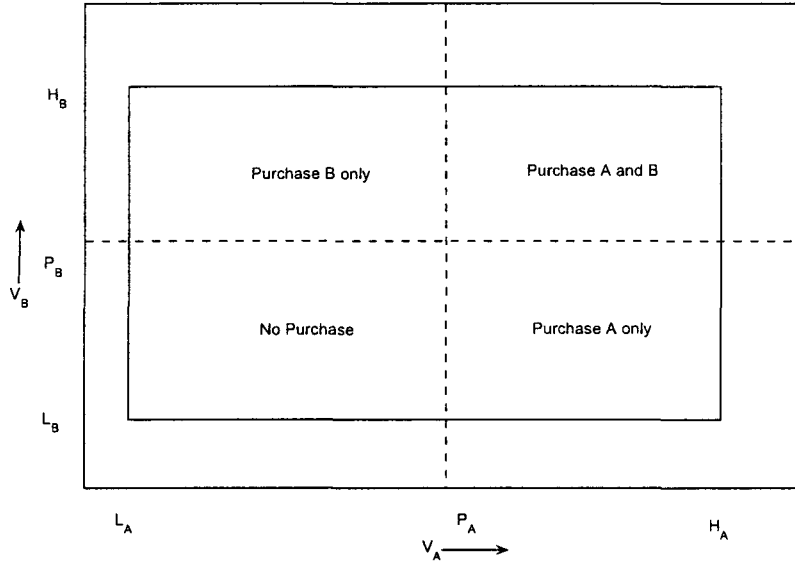


Figure 5.1: Customer valuations for No Bundles.

$B$  is represented by  $f(V_A, V_B)$  and is assumed to be uniformly distributed. It is worth mentioning here that uniformly distributed customer valuations correspond to a downward sloping linear demand function used throughout this thesis (e.g., see Phillips (2005) p. 47). Furthermore, Schmalensee (1984) and Long (1984) have made use of the Gaussian distribution to show that bundling properties derived using the uniform distribution are still valid in general. In a decentralized supply chain, the supplier is the leader and charges the retailer a wholesale price  $w_j$ . The goal of the supplier is to charge the right wholesale price to maximize its profit of  $\Pi_S(w_A, w_B, w_{AB})$ . Likewise, the retailer needs to decide on prices  $P_A$ ,  $P_B$  and  $P_{AB}$  with the goal of maximizing its profit  $\Pi_R(P_A, P_B, P_{AB})$ .

## 5.2 Model Analysis

As in previous coordination studies (e.g., Weng (1995); Lee (2007)), in this section we analyze the supply chain profits under two cases: (i) a centralized system where the supplier and the retailer act as a single business entity with the aim of maximizing the overall supply chain profits and (ii) a decentralized system where the supplier and retailer act independently to maximize their individual profits.



### 5.2.1 Centralized Supply Chain

Here we look at the case where the supplier and retailer function as one entity and offer the goods to the consumer. We will consider three scenarios: (1) No bundling, where the products are offered separately, (2) Pure bundling where the products are offered only in bundles and (3) Mixed bundling where the products are offered separately as components as well as jointly in a bundle.

#### No Bundling

In this section we focus on the case where the products are offered only as separate units. We will use this case as a benchmark to study the effect of bundling later. The fraction of the market that will purchase each good  $j \in \{A, B\}$  is given as

$$Q_A = M \int_{L_B}^{H_B} \int_{P_A}^{H_A} f(V_A, V_B) dV_A dV_B, \quad (5.1)$$

$$Q_B = M \int_{P_B}^{H_B} \int_{L_A}^{H_A} f(V_A, V_B) dV_A dV_B. \quad (5.2)$$

The profit is

$$\Pi_C^{NB} = (P_A - c_A)Q_A + (P_B - c_B)Q_B. \quad (5.3)$$

The supply chain's profit optimization problem is to

$$\max \Pi_C^{NB} \text{ s.t. } c_j \leq P_j \leq H_j, \quad j \in \{A, B\}.$$

The optimal prices and quantities stocked are as stated in Theorem 2.

**Theorem 2** *The optimal prices and corresponding quantities are as follows:*

$$P_j^{CNB} = \frac{H_j + c_j}{2}, \quad j \in \{A, B\}, \quad (5.4)$$

and

$$Q_j^{CNB} = \frac{M}{2} \cdot \frac{H_j - c_j}{H_j - L_j}, \quad j \in \{A, B\}. \quad (5.5)$$

The corresponding optimal profit is

$$\Pi_C^{NB} = \frac{M}{4} \sum_{j=A,B} \frac{(H_j - c_j)^2}{H_j - L_j}. \quad (5.6)$$

**Proof.** The proof is in the Appendix. ■

### Pure Bundling

In this section we look at the case where the retailer decides to offer a bundle of goods  $A$  and  $B$  instead of offering them separately. The bundle in this case has a unit cost of  $c_{AB} = c_A + c_B - c$  and is being sold at a price  $P_{AB}$ , where  $c$  is the “cost” to bundle. If  $c > 0$  then there is a cost savings realized by the supplier from bundling and if  $c < 0$  then there is a cost of bundling for the supplier. If  $c = 0$ , then there is no cost to bundle. Customers whose sum of valuations exceeds the bundle price, i.e.,  $P_{AB} \leq V_A + V_B$ , will buy the bundle. Everyone else is priced out of the market.

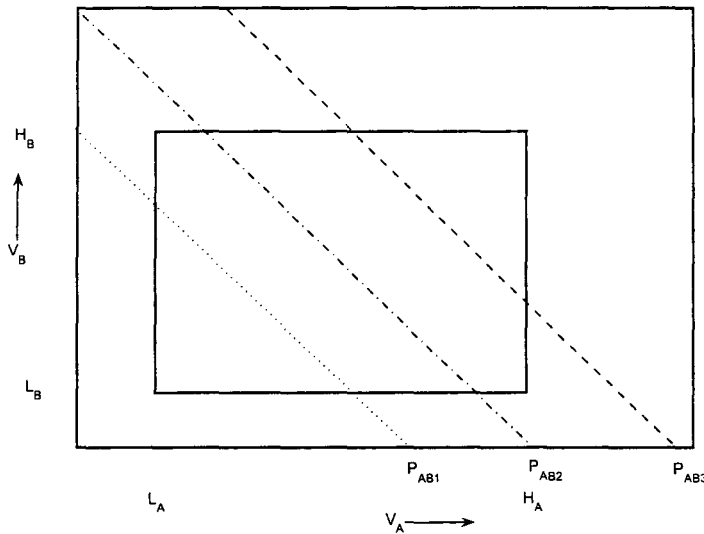


Figure 5.2: Customer valuations for Pure Bundle

As illustrated in Figure 5.2, depending on  $P_{AB}$ , the order quantity is defined as follows:

$$Q_{AB} = \begin{cases} M \left( 1 - \int_{L_B}^{P_{AB}-L_A} \int_{L_A}^{P_{AB}-V_B} f(V_A, V_B) dV_A dV_B \right), & \text{if } P_{AB} \in I_1 \\ M \left( \int_{L_B}^{H_B} \int_{P_{AB}-V_B}^{P_{AB}-L_B} f(V_A, V_B) dV_A dV_B + \int_{L_B}^{H_B} \int_{P_{AB}-L_B}^{H_A} f(V_A, V_B) dV_A dV_B \right), & \text{if } P_{AB} \in I_2 \\ M \left( \int_{P_{AB}-H_A}^{H_B} \int_{P_{AB}-V_B}^{H_A} f(V_A, V_B) dV_A dV_B \right), & \text{if } P_{AB} \in I_3. \end{cases}$$

where

$$\begin{aligned} I_1 &= [L_A + L_B, \min(H_A + L_B, H_B + L_A)], \\ I_2 &= [\min(H_A + L_B, H_B + L_A), \max(H_A + L_B, H_B + L_A)] \text{ and} \\ I_3 &= [\max(H_A + L_B, H_B + L_A), H_A + H_B]. \end{aligned}$$

When the reservation prices are independent and uniformly distributed for products, the order quantity is given as

$$Q_{AB}^{CPB} = \begin{cases} M \left( 1 - \frac{(P_{AB} - L_A - L_B)^2}{2(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_1, \\ \frac{M}{2} \left( \frac{2H_A + H_B + L_B - 2P_{AB}}{(H_A - L_A)} \right), & \text{if } P_{AB} \in I_2, \\ \frac{M}{2} \left( \frac{(H_A + H_B - P_{AB})^2}{(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_3. \end{cases}$$

The profit is given as

$$\Pi_C^{PB} = \begin{cases} M(P_{AB} - c_{AB}) \left( 1 - \frac{(P_{AB} - L_A - L_B)^2}{2(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_1, \\ \frac{M(P_{AB} - c_{AB})}{2} \left( \frac{2H_A + H_B + L_B - 2P_{AB}}{(H_A - L_A)} \right), & \text{if } P_{AB} \in I_2, \\ \frac{M(P_{AB} - c_{AB})}{2} \left( \frac{(H_A + H_B - P_{AB})^2}{(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_3. \end{cases}$$

The supply chain's profit optimization problem is to

$$\max \Pi_C^{PB} \text{ s.t. } c_{AB} \leq P_{AB} \leq H_A + H_B.$$

The optimal bundle price is given by Theorem 3.

**Theorem 3** *In a centralized supply chain selling bundles, the optimal bundle price is given as*

$$P_{AB}^{CPB} = \begin{cases} P_{AB1} = \frac{c_{AB} + 2(L_A + L_B) + \sqrt{[c_{AB} - (L_A + L_B)]^2 + 6(H_A - L_A)(H_B - L_B)}}{3}, & \text{if } c_{AB} \in I_4, \\ P_{AB2} = \frac{2H_A + H_B + L_B + 2c_{AB}}{4}, & \text{if } c_{AB} \in I_5, \\ P_{AB3} = \frac{H_A + H_B + 2c_{AB}}{3}, & \text{if } c_{AB} \in I_6. \end{cases}$$

where

$$\begin{aligned} I_4 &= \left[ L_A + L_B, \frac{3H_B + 4L_A - 2H_A - L_B}{2} \right], \\ I_5 &= \left[ \frac{3H_B + 4L_A - 2H_A - L_B}{2}, \frac{2H_A + 3L_B - H_B}{2} \right] \text{ and} \\ I_6 &= \left[ \frac{2H_A + 3L_B - H_B}{2}, H_A + H_B \right], \end{aligned}$$

and without loss of generality, we assume  $H_B - L_B \leq H_A - L_A$ .

**Proof.** The proof is in the Appendix. ■

Two interesting questions to look at here are how do bundle prices compare to the individual product prices and is bundling always profitable? We provide answers to these questions in Propositions 8.

**Proposition 8** Given that  $0 \leq c_{AB} = c_A + c_B - c$ , where  $c \leq c_A + c_B$  and assuming, without loss of generality, that  $H_B - L_B \leq H_A - L_A$ , we have

(a)  $P_{AB}^{CPB} \leq P_A^{CNB} + P_B^{CNB}$ .

(b) If  $c_{AB} \in I_5 \cup I_6$  then  $\Pi_C^{PB} \geq \Pi_C^{NB}$  if and only if  $c \geq \bar{c}$ .

**Proof.** The Proof is in the Appendix. ■

where  $\bar{c} = \frac{2}{\sqrt{M}} \sqrt{(H_A - L_A) \Pi_C^{NB}} + c_A + c_B - H_A - \frac{1}{2}(H_B + L_B) \geq 0$ . Proposition 8 suggests that as long as the cost savings from bundling is larger than a threshold,  $\bar{c}$ , then it would be profitable to sell in pure bundles. We also infer that pure bundles will always be sold at a discount, i.e.,  $P_{AB}^{CPB} \leq P_A^{CNB} + P_B^{CNB}$ . Note that  $\bar{c}$  can be negative in which case it would be still beneficial to bundle even when the cost of bundling is higher than that of the sum of the component costs.

### Mixed Bundling

Here we look at the case where the retailer offers the consumer the option of either buying the components separately or buying a bundle at prices  $P_j$  where  $j \in \{A, B, AB\}$  respectively, where  $\max(P_A, P_B) \leq P_{AB} \leq P_A + P_B$ . The market segmentation is shown in Figure 3.

Let  $P_{AB} = P_A + P_B - \alpha$ , where  $\alpha \in [0, \min(P_A - L_A, P_B - L_B)]$  is a decision variable representing the discount that the consumer gets when she chooses to buy a bundle instead of the two products separately. Similarly, let  $c_{AB} = c_A + c_B - c$ , where  $0 \leq c \leq c_A + c_B$  is the savings obtained by the supplier selling bundles to the retailer, a known parameter. The quantities stocked are given as follows:

$$\begin{aligned} Q_A &= M \int_{L_B}^{P_B - \alpha} \int_{P_A}^{H_A} f(V_A, V_B) dV_A dV_B, \\ Q_B &= M \int_{P_B}^{H_B} \int_{L_A}^{P_A - \alpha} f(V_A, V_B) dV_A dV_B, \\ Q_{AB} &= M \left( \int_{P_B - \alpha}^{P_B} \int_{P_A + P_B - \alpha - V_B}^{P_A} f(V_A, V_B) dV_A dV_B + \int_{P_B}^{H_B} \int_{P_A - \alpha}^{P_A} f(V_A, V_B) dV_A dV_B \right. \\ &\quad \left. + \int_{P_B - \alpha}^{H_B} \int_{P_A}^{H_A} f(V_A, V_B) dV_A dV_B \right). \end{aligned}$$

For the case where the reservation prices are uniformly distributed for products A and B such that  $V_A \sim U[L_A, H_A]$  and  $V_B \sim U[L_B, H_B]$  the quantities stocked are

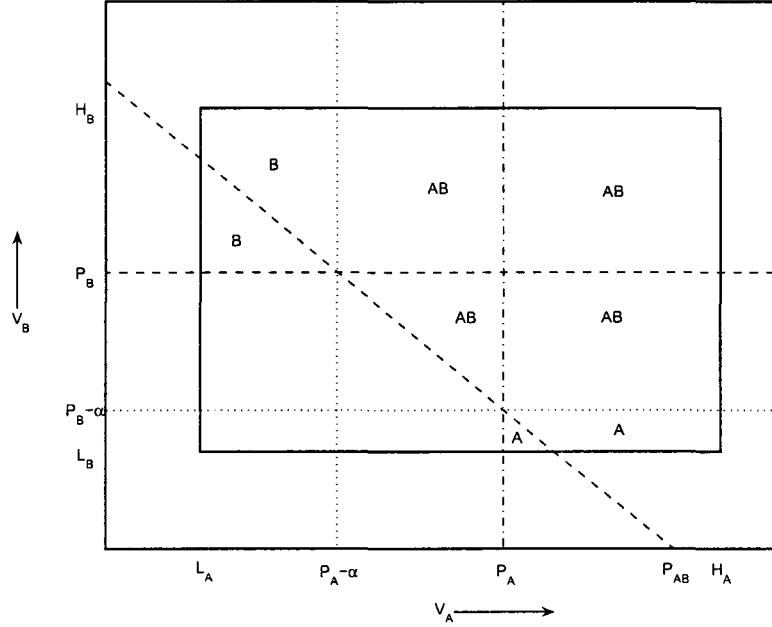


Figure 5.3: Customer valuations for Mixed Bundle

$$Q_A^{CMB} = M \left[ \frac{(H_A - P_A)(P_B - L_B - \alpha)}{(H_A - L_A)(H_B - L_B)} \right]$$

$$Q_B^{CMB} = M \left[ \frac{(H_B - P_B)(P_A - L_A - \alpha)}{(H_A - L_A)(H_B - L_B)} \right]$$

$$Q_{AB}^{CMB} = M \left[ \frac{(H_A - P_A)(H_B - P_B) + \alpha [(H_A - P_A) + (H_B - P_B)] + \frac{1}{2}\alpha^2}{(H_A - L_A)(H_B - L_B)} \right]$$

The profit for the centralized supply chain is

$$\Pi_C^{MB} = \sum_{j=A,B,AB} (P_j - c_j) Q_j. \quad (5.7)$$

The supply chain's profit optimization problem is to

$$\max \Pi_C^{MB} \text{ s.t. } c_j \leq P_j \leq H_j, \quad j \in \{A, B\} \text{ and } 0 \leq \alpha \leq \min(H_A - L_A, H_B - L_B).$$

Note that we have relaxed the upper bound on  $\alpha$ . This relaxation will be helpful in identifying a simple solution procedure. As noted in previous studies, in general the

above problem is not convex and numerical simulation would be required to solve it. However, in Theorem 4 we show that under certain conditions on consumer valuations and production costs, it is possible to reduce the problem to a one-dimensional optimization problem that can be solved by a simple line search for the bundle discount  $\alpha$ .

**Theorem 4** *When*

$$\max \left( \frac{3}{2} \min(H_A - L_A, H_B - L_B) - \frac{c}{2}, \frac{c}{2} \right) \leq \sqrt{(H_A - L_A)(H_B - L_B)},$$

*then the optimal product prices can be found by solving the following system of linear equations for a given value of bundle discount  $\alpha$*

$$P_i^{CMB} = \max(\min(H_i, P_i^1), c_i), i \in \{A, B\}$$

where

$$P_i^1 = \frac{H_i + c_i}{2} + \frac{3\alpha^2 + (4H_j - 6P_j + 2c_j - 2c)\alpha + 2c(P_j - H_j)}{4(H_j - L_j)}, i, j \in \{A, B\}, i \neq j \quad (5.8)$$

and the optimal bundle discount  $\alpha^{CMB} \in [0, \min(H_A - L_A, H_B - L_B)]$ .

**Proof.** The proof is in the Appendix. ■

We note that when  $\alpha = 0$  and  $c = 0$ , the mixed bundling problem reduces to the no bundle problem. In Proposition 9 we show that in the case of a centralized supply chain mixed bundling will always lead to higher profits than when no bundles are offered.

**Proposition 9**  $\Pi_C^{MB} \geq \Pi_C^{NB}$ .

**Proof.** Note that

$$\frac{\partial \Pi_C^{MB}}{\partial \alpha} \Big|_{P_A = \frac{H_A + c_A}{2}, P_B = \frac{H_B + c_B}{2}, \alpha = 0} = \frac{M}{4} \left[ \frac{(H_A - c_A)(H_B - c_B) + 2c(H_A - c_A + H_B - c_B)}{(H_A - L_A)(H_B - L_B)} \right] > 0$$

where  $P_A = \frac{H_A + c_A}{2}$  and  $P_B = \frac{H_B + c_B}{2}$  are the optimal prices under no bundling. Therefore, increasing  $\alpha$ , i.e., selling mixed bundles, increases the profit  $\Pi_C^{MB}$ . ■

## 5.2.2 Decentralized Supply Chain

In this section we consider the case where the retailer and the supplier are independent entities such that the retailer has to pay a wholesale price  $w_j$  to the supplier, who will then sell to the consumer at retail price  $P_j$ . As in the previous section we also consider the three scenarios of no bundling, pure bundling and mixed bundling.

## No Bundling

The supplier has a cost of input  $c_j$ ,  $j \in \{A, B\}$  and charges the retailer a wholesale price  $w_j > c_j$  for product  $j$ . The retailer orders quantities  $Q_A$  and  $Q_B$ . The profits for the retailer and supplier are

$$\Pi_R^{NB} = (P_A - w_A) Q_A + (P_B - w_B) Q_B, \text{ and} \quad (5.9)$$

$$\Pi_S^{NB} = (w_A - c_A) Q_A + (w_B - c_B) Q_B, \quad (5.10)$$

respectively. The objective of the supplier is to

$$\max \Pi_S^{NB} \text{ s.t. } c_j \leq w_j \leq H_j, \quad j \in \{A, B\},$$

and that of the retailer is to

$$\max \Pi_R^{NB} \text{ s.t. } w_j \leq P_j \leq H_j, \quad j \in \{A, B\}.$$

The total supply chain profit is  $\Pi_{DC}^{NB} = \Pi_S^{NB} + \Pi_R^{NB}$ . Theorem 5 establishes the optimal wholesale prices for the supplier and the optimal prices and order quantities for the retailer.

**Theorem 5** *When the supplier and retailer are independent entities, the optimal prices and order quantities for the supplier and retailer are given as*

$$w_j^{DNB} = \frac{H_j + c_j}{2}, \quad (5.11)$$

$$P_j^{DNB} = \frac{3H_j + c_j}{4}, \quad (5.12)$$

$$Q_j^{DNB} = \frac{M}{4} \cdot \frac{H_j - c_j}{H_j - L_j}, \quad j = \{A, B\}. \quad (5.13)$$

The retailer and supplier profits are

$$\Pi_R^{NB} = \frac{M}{16} \sum_{j=A,B} \frac{(H_j - c_j)^2}{H_j - L_j}, \quad (5.14)$$

$$\Pi_S^{NB} = \frac{M}{8} \sum_{j=A,B} \frac{(H_j - c_j)^2}{H_j - L_j}. \quad (5.15)$$

**Proof.** Following the same logic as the one used to prove Theorem 2, we find that

$$P_j^{DNB} = \frac{H_j + w_j^*}{2}, \quad j = \{A, B\}, \quad (5.16)$$

$$Q_j^{DNB} = \frac{M}{4} \cdot \frac{H_j - w_j^*}{H_j - L_j}, \quad j = \{A, B\}. \quad (5.17)$$

After substituting these values in the supplier's profit function (5.10) and solving for  $w_j^{DNB}$  we obtain (5.11). Substituting (5.11) in (5.16) and (5.17) we get the desired results (5.12) and (5.13). Finally, substitute (5.11–5.13) in (5.9–5.10) to get (5.14–5.15). ■

Thus, the total profit for the decentralized supply chain is given as

$$\Pi_{DC}^{NB} = \Pi_R^{NB} + \Pi_S^{NB} = \frac{3M}{16} \sum_{j=A,B} \frac{(H_j - c_j)^2}{H_j - L_j}.$$

### Pure Bundling

In the situation where the supplier and retailer are independent but both sell only bundles  $AB$ , the retailer's and supplier's profit is given as

$$\Pi_R^{PB} = (P_{AB} - w_{AB})Q_{AB} \text{ and} \quad (5.18)$$

$$\Pi_S^{PB} = (w_{AB} - c_{AB})Q_{AB} \quad (5.19)$$

respectively, where  $c_j \leq w_j \leq P_j$ . The retailer will order

$$Q_{AB}^{DPB} = \begin{cases} M \left( 1 - \frac{(P_{AB} - L_A - L_B)^2}{2(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_1, \\ \frac{M}{2} \left( \frac{2H_A + H_B + L_B - 2P_{AB}}{(H_A - L_A)} \right), & \text{if } P_{AB} \in I_2, \\ \frac{M}{2} \left( \frac{(H_A + H_B - P_{AB})^2}{(H_A - L_A)(H_B - L_B)} \right), & \text{if } P_{AB} \in I_3. \end{cases}$$

The optimal retailer price and supplier wholesale price that maximize their profits are given in Theorem 6.

**Theorem 6** *When the supplier and retailer are independent entities, the optimal prices for the supplier and retailer are:*

$$w_{AB}^{DPB} = \begin{cases} w_{AB1} = \frac{c_{AB} + 2(L_A + L_B) + \sqrt{[c_{AB} - (L_A + L_B)]^2 + 6(H_A - L_A)(H_B - L_B)}}{3}, & \text{if } c_{AB} \in I_4, \\ w_{AB2} = \frac{2H_A + H_B + L_B + 2c_{AB}}{4}, & \text{if } c_{AB} \in I_5, \\ w_{AB3} = \frac{H_A + H_B + 2c_{AB}}{3}, & \text{if } c_{AB} \in I_6. \end{cases}$$

and

$$P_{AB}^{DPB} = \begin{cases} P_{AB1} = \frac{w_{AB}^* + 2(L_A + L_B) + \sqrt{[w_{AB}^* - (L_A + L_B)]^2 + 6(H_A - L_A)(H_B - L_B)}}{3}, & \text{if } w_{AB}^* \in I_4, \\ P_{AB2} = \frac{2H_A + H_B + L_B + 2w_{AB}^*}{4}, & \text{if } w_{AB}^* \in I_5, \\ P_{AB3} = \frac{H_A + H_B + 2w_{AB}^*}{3}, & \text{if } w_{AB}^* \in I_6. \end{cases}$$



**Proof.** The results follow directly from Theorem 3. ■

Note that depending on  $c_{AB}$  and  $w_{AB}$  and the condition  $c_{AB} \leq w_{AB} \leq P_{AB}$ , the optimal bundle price can take several different forms (six possible combinations). As an example, below we include three possible forms:

$$P_{AB}^{DPB} = \begin{cases} P_{AB1} = \frac{c_{AB} + 8(L_A + L_B) + 4\sqrt{[c_{AB} - (L_A + L_B)]^2 + 6(H_A - L_A)(H_B - L_B)}}{9}, & \text{if } w_{AB}^* \in I_4 \text{ and } c_{AB} \in I_4, \\ P_{AB2} = \frac{6H_A + 3H_B + 3L_B + 2c_{AB}}{8}, & \text{if } w_{AB}^* \in I_5 \text{ and } c_{AB} \in I_5, \\ P_{AB3} = \frac{5H_A + 5H_B + 4c_{AB}}{9}, & \text{if } w_{AB}^* \in I_6 \text{ and } c_{AB} \in I_6. \end{cases}$$

The total supply chain profit is given as  $\Pi_{DC}^{PB} = \Pi_R^{PB} + \Pi_S^{PB}$ .

### Mixed Bundling

In the situation when the retailer and the supplier are independent, the retailer buys the products at a wholesale price  $w_j$  from the supplier where  $j \in \{A, B, AB\}$ . The bundling discount offered by the supplier to the retailer is defined as  $\delta = w_A + w_B - w_{AB}$ , where  $\delta \in [0, \min(w_A - L_A, w_B - L_B)]$ . The retailer offers to sell the goods to consumers in a mixed bundle format at price  $P_j$  where  $j \in \{A, B\}$  and offer a bundling discount  $\alpha \in [0, \min(P_A - L_A, P_B - L_B)]$ . The profits for the retailer and the supplier are

$$\begin{aligned} \Pi_R^{MB} &= \sum_{j=A,B,AB} (P_j - w_j)Q_j \text{ and} \\ \Pi_S^{MB} &= \sum_{j=A,B,AB} (w_j - c_j)Q_j, \end{aligned}$$

respectively. The supplier's goal is to

$$\max \Pi_S^{MB} \text{ s.t. } c_j \leq w_j \leq H_j, j \in \{A, B\}, \text{ and } 0 \leq \delta \leq \min(w_A - L_A, w_B - L_B).$$

Similarly, the retailer's goal is to

$$\max \Pi_R^{MB} \text{ s.t. } w_j \leq P_j \leq H_j, j \in \{A, B\}, \text{ and } 0 \leq \alpha \leq \min(P_A - L_A, P_B - L_B).$$

The total supply chain profit is  $\Pi_{DC}^{MB} = \Pi_R^{MB} + \Pi_S^{MB}$ . Although some restricted version of the optimization problems in the decentralized mixed bundling case can be solved (e.g., using results found in Theorem 4), the general problems are non-convex and complex and thus we will use numerical methods to determine the optimal solutions as well as derive insights.

### 5.3 The Role of Bundling in Coordinating the Supply Chain

We note that under a decentralized no-bundling situation, the supplier gets  $\frac{2}{3}$  of the chain's profit and the remaining  $\frac{1}{3}$  goes to the retailer. When the supplier and retailer do not coordinate this leads to double marginalization in the supply chain which reduces the overall profit. This profit gap is

$$\Pi_{Gap}^{NB} = \Pi_C^{NB} - \Pi_{DC}^{NB} = \frac{M}{16} \sum_{j=A,B} \frac{(H_j - c_j)^2}{H_j - L_j} > 0. \quad (5.20)$$

Most supply chain coordination literature focuses on reducing this gap to zero by a variety of different mechanisms. In this section we investigate how bundling can help coordinate between the retailer and supplier to increase the supply chain profits and reduce this gap. In particular we show how the supplier and retailer profits depend on the bundling discount  $\delta$  and show that a bundling discount alone is not sufficient to achieve supply chain coordination. To guarantee joint profit maximization we propose a bundling fee to serve as a coordination mechanism.

#### 5.3.1 Pure Bundles

In this case the supplier does not offer to sell each components separately. The supplier can prohibit the individual selling of the products by setting  $w_A = H_A$  and  $w_B = H_B$ . Thus, to sell only in pure bundles the supplier sets the wholesale price for the bundle as  $w_{AB} = H_A + H_B - \delta$ ,  $\delta \in [0, H_A + H_B - c_{AB}]$ . By increasing the bundling discount  $\delta$  the supplier can make bundling more attractive for the retailer. In Proposition 10 we show how the different profits change with the bundle discount. Since in this section we are interested in the role of bundling in coordination we only consider cases where the bundle cost savings are sufficient to warrant  $\Pi_C^{NB} \leq \Pi_C^{PB}$ , i.e.,  $c \geq \bar{c}$  where  $\bar{c}$  is as defined in Proposition 8.

**Proposition 10** *Depending on the value of  $\delta$  we have four regions:*

- (a) *Region I:  $\delta \in [\delta_0, \delta_1]$  where  $\delta_0 = 0$  and  $\delta_1$  is such that  $\Pi_{DC}^{PB} = \Pi_{DC}^{NB}$ . Here we have  $\Pi_S^{PB}$ ,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing in  $\delta$  and  $\Pi_{DC}^{PB} = \Pi_R^{PB} + \Pi_S^{PB} \leq \Pi_{DC}^{NB} \leq \Pi_C^{NB} \leq \Pi_C^{PB}$ .*
- (b) *Region II:  $\delta \in [\delta_1, \delta_2]$ , where  $\delta_2 = \underset{\delta}{\operatorname{argmax}} \Pi_S$ . Here we have  $\Pi_S^{PB}$ ,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing in  $\delta$ ,  $\Pi_S^{PB}$  achieves its maximum  $\Pi_S^{PB}(\delta_2)$ , and  $\Pi_{DC}^{NB} \leq \Pi_{DC}^{PB} = \Pi_R^{PB} + \Pi_S^{PB} \leq \Pi_C^{NB} \leq \Pi_C^{PB}$ .*
- (c) *Region III:  $\delta \in [\delta_2, \delta_3]$  where  $\delta_3$  is such that  $\Pi_{DC}^{PB} = \Pi_C^{NB}$ . Here we have  $\Pi_S^{PB}$  is decreasing,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing in  $\delta$ , and  $\Pi_{DC}^{NB} \leq \Pi_{DC}^{PB} = \Pi_R^{PB} + \Pi_S^{PB} \leq \Pi_C^{NB} \leq \Pi_C^{PB}$ .*

(d) Region IV:  $\delta \in [\delta_3, \delta_4]$  where  $\delta_4 = H_A + H_B - c_{AB}$  is such that  $\Pi_{DC}^{PB} = \Pi_C^{PB}$ . Here we have  $\Pi_S^{PB}$  is decreasing,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing in  $\delta$ , and  $\Pi_{DC}^{NB} \leq \Pi_C^{NB} \leq \Pi_{DC}^{PB} = \Pi_R^{PB} + \Pi_S^{PB} \leq \Pi_C^{PB}$ .

**Proof.** The proof is in the Appendix. ■

In Figure 5.4 we include an illustration of the impact of the bundle discount on the overall profits of the supply chain as well as the supplier and retailer. In Region I the supplier and the retailer are both better off acting independently. Both of them improve their profits and the supply chain can achieve the same profits as when no bundles are sold in a decentralized setting, i.e.,  $\Pi_{DC}^{NB} = \Pi_{DC}^{PB}$ . In Region II the supplier and the retailer are both better off, the total supply chain profit is higher, i.e.,  $\Pi_{DC}^{NB} < \Pi_{DC}^{PB}$  and the supplier maximizes her profits. In Region III and IV the retailer continues to improve her profits, however the supplier is worse off than before as  $\Pi_S^{PB}$  begins to decrease.

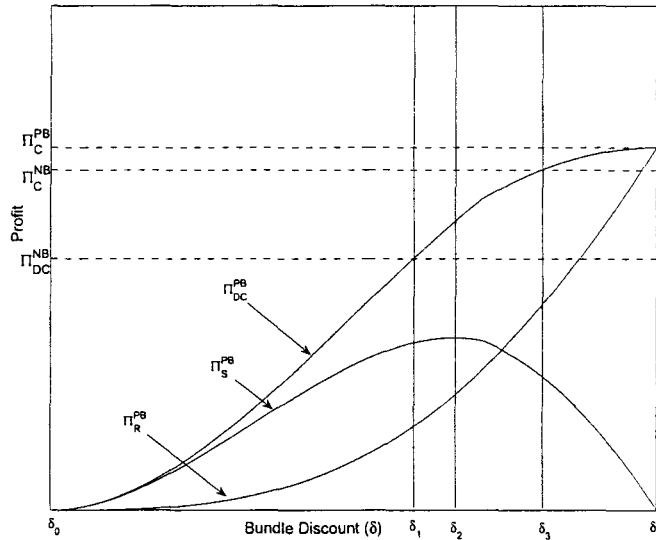


Figure 5.4: Role of Bundling Discount in the Pure Bundling Case.

### 5.3.2 Mixed Bundles

In case of mixed bundles, the supplier offers to sell the components and bundle to the retailer at wholesale prices  $w_A, w_B$  and  $w_{AB}$ . The supplier can offer the retailer a discount on each of the two products as well as a bundle to induce the retailer into buying more. Depending upon the magnitude of this discount, the retailer may increase her buying and thus can achieve higher profits for the whole supply chain.

Solving the decentralized mixed bundling problem is more complicated than the pure bundling case as we have six decision variables in this case, i.e.,  $w_j$  and  $P_j$  where  $j \in \{A, B, AB\}$ . Also, unlike pure bundling, the bundling discount  $\delta$  offered under mixed bundling is mathematically different. The wholesale price for the bundle can be written as

$$w_{AB} = w_A + w_B - \delta_{AB},$$

where  $w_j = w_j^* - \delta_j$ , where  $w_j$  is the wholesale price for component  $j \in A, B$  and  $w_j^* = \frac{H_j + c_j}{2}$  is defined as the optimal wholesale price charged by the supplier to the retailer in the decentralized supply chain selling no bundle and  $\delta_j, j \in \{A, B, AB\}$  is the individual product discount offered by the supplier to the retailer while engaging in mixed bundling. The constraints of which are given as  $\delta_j \in [0, H_j - c_j], j \in A, B$  and  $\delta_{AB} \in [0, \min(w_A - L_A, w_B - L_B)]$ . Thus, for mixed bundling we define the bundling discount offered by the supplier to the retailer as  $\delta = \sum \delta_j, j \in \{A, B, AB\}$ . Furthermore, through our analysis in Sections 5.2.1 and 5.2.2, we have constraints on the bundling discount  $\alpha$  offered by the retailer to the customer and is given as  $\alpha \in \{0, \min(P_A - L_A, P_B - L_B)\}$ . By increasing the bundling discount  $\delta \in [\delta_1, \delta_4]$ , the supplier makes bundling more attractive for the retailer. We study the effect of the bundle discount numerically and in Figure 5.5 we show the impact of this discount ( $\delta$ ) on the overall profits of the supply chain as well as each agent.

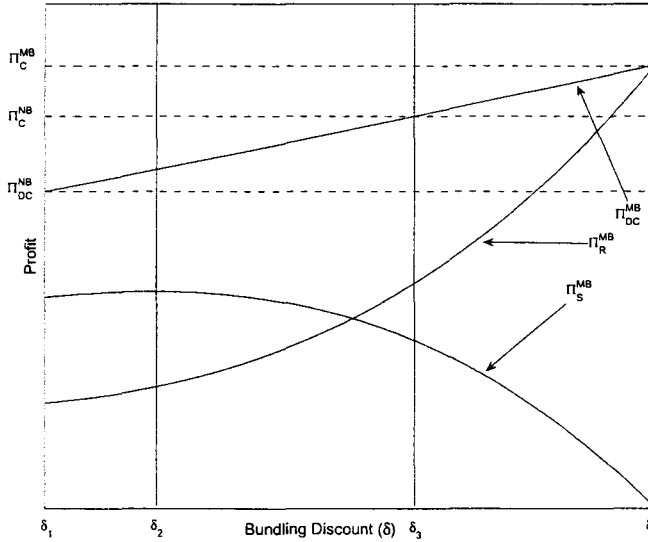


Figure 5.5: Performance of Mixed Bundles at different  $\delta$ .

### 5.3.3 Bundling Fee ( $\Theta$ )

Following the analysis in section 5.3.1 and 5.3.2 we summarize the the findings in Table 5.1. We can see that the supplier is worse off when the bundle discount is

Table 5.1: Summary of Results

Bundle Discount ( $\delta$ )	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$
Retailer	B.O.	B.O.	B.O.	B.O.
Supplier	B.O.	B.O.	W.O.	W.O.
Supply Chain	B.O.	B.O.	B.O.	B.O.

B.O.=Better Off, W.O.=Worse Off.

beyond  $\delta_2$ . Thus the supplier has no incentive to offer this bundling discount to the retailer beyond this point. However, the overall supply chain continues to do well and so does the retailer. Thus, bundling alone is not enough to guarantee higher profits and supply chain coordination. We propose that a bundling fee, ( $\Theta$ ), be used to improve profits. Like a franchise fee (Weng (1995)), the bundling fee can act as an incentive for the supplier to continue to offer a bundle discount to the retailer beyond  $\delta_2$ , even when she would be worse off. This fee ensures a pareto-efficient outcome, where the supplier and the retailer are both better off while bundling. This bundling fee will be charged by the supplier upfront, as a condition to engage in discounting beyond  $\delta_2$ . The exact range of values for this fee is given by the following proposition.

**Proposition 11** *In pure bundling, for  $\delta \in [\delta_2, \delta_4]$ , the range for the bundling fee ( $\Theta$ ) is given as  $\Theta \in [\Theta_L, \Theta_U]$ , where  $\Theta_L = \Pi_S^{PB}(\delta_2) - \Pi_S^{PB}(\delta|\delta \geq \delta_2)$  and  $\Theta_U = \Pi_R^{PB}(\delta|\delta \geq \delta_2) - \Pi_R^{PB}(\delta_2)$ .*

**Proof.** From Proposition 10 we know that

$$\Pi_S^{PB}(\delta_2) \geq \Pi_S^{PB}(\delta|\delta \geq \delta_2) \text{ and } \Pi_R^{PB}(\delta|\delta \geq \delta_2) \leq \Pi_R^{PB}(\delta_2).$$

For the bundling fee to act as a pareto-efficient coordination mechanism it has to satisfy

$$\begin{aligned} \Pi_S^{PB}(\delta_2) &\leq \Pi_S^{PB}(\delta|\delta \geq \delta_2) + \Theta \\ \Pi_R^{PB}(\delta_2) &\leq \Pi_R^{PB}(\delta|\delta \geq \delta_2) - \Theta \end{aligned}$$

After some algebra we can show that

$$\Theta \in [\Pi_S^{PB}(\delta_2) - \Pi_S^{PB}(\delta|\delta \geq \delta_2), \Pi_R^{PB}(\delta|\delta \geq \delta_2) - \Pi_R^{PB}(\delta_2)].$$

■

## 5.4 Numerical Example

We highlight the above analytical insights with the help of a numerical example. For this purpose we make use of the following parameter values:  $H_A = 7, H_B = 5, L_A = 3, L_B = 2, c_A = 3.2, c_B = 2.5, c = 0$  and  $M = 100$ . To have a meaningful comparison between the no bundling and bundling cases we are taking a conservative approach by assigning a high cost to the bundle, i.e., the cost saving parameter  $c = 0$  and  $c_{AB} = c_A + c_B = 5.7$ . As a benchmark, in Table 5.2 we include the prices, quantities and profits for a centralized and decentralized supply chain selling no bundles.

Table 5.2: No Bundles

	$w_A$	$w_B$	$Q_A$	$Q_B$	$P_A$	$P_B$	$\Pi_R$	$\Pi_S$	$\Pi_R + \Pi_S$
Centralized	3.20	2.50	47.50	41.66	5.10	3.75	142.34	0.00	142.34
Decentralized	5.10	3.75	23.75	20.83	6.05	4.37	35.58	71.16	106.75

The corresponding data for pure bundle is reported in Table 5.3. We see that when the supplier and the retailer engage in pure bundling, the supply chain achieves the total profit of a decentralized supply chain selling no bundles at  $\delta_1$ . The supplier maximizes her profit at  $\delta_2$  and has no incentive to engage in further discounting. The pure bundle achieves the profit of a centralized supply chain selling no bundle at  $\delta_3$ . But now the supplier needs an incentive in terms of bundling fee  $\Theta$  to engage in bundling. Similarly at  $\delta_4$ , the supply chain achieves the performance of a centralized supply chain selling pure bundles.

Table 5.3: Coordination through Pure Bundles

$\delta$	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$
	3.97	4.20	5.79	6.30
$w_{AB}$	8.03	7.80	6.21	5.70
$Q_{AB}$	29.18	32.66	53.62	60.00
$P_{AB}$	9.35	9.20	8.35	8.10
$\Pi_R^{PB}$	38.62	45.73	115.03	144.00
$\Pi_S^{PB}$	68.00	68.60	27.31	0.00
$\Pi_{DC}^{PB}$	106.62	114.33	142.34	144.00
$\Theta$	[0.00, 0.00]	[0.00, 0.00]	[36.69, 69.30]	[68.60, 98.27]

We see that when the supplier and retailer engage in mixed bundling, the supply chain achieves the total profit of a decentralized supply chain selling no bundles at  $\delta_1$ . The supplier maximizes her profit at  $\delta_2$  and has no incentive to engage in further discounting. The mixed bundle achieves the profits of a centralized supply

chain selling no bundles at  $\delta_3$ . But now the supplier needs an incentive in terms of bundling fee  $\Theta$  to engage in bundling. Similarly at  $\delta_4$ , the supply chain achieves the performance of a centralized supply chain selling mixed bundles. Note that in the case of mixed bundling, different values of the wholesale and retail prices can yield the same solution. In this numerical example, we present one such case. The data for mixed bundle is reported in Table 5.4.

Table 5.4: Coordination through Mixed Bundles

$\delta$	$\delta_1$	$\delta_2$	$\delta_3$	$\delta_4$
	0.00	0.67	1.75	6.10
$\alpha$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$
	0.00	0.96	1.99	1.17
$w_A$	5.10	5.10	5.10	3.20
$w_B$	3.75	3.75	3.75	2.50
$w_{AB}$	8.85	8.18	7.10	5.70
$Q_A$	23.75	10.46	2.92	12.22
$Q_B$	20.83	9.78	5.22	9.52
$Q_{AB}$	0.00	18.40	37.00	40.32
$P_A$	6.05	6.17	6.37	5.47
$P_B$	4.37	4.47	4.54	4.12
$P_{AB}$	10.42	9.68	8.92	8.42
$\Pi_R^{MB}$	35.58	45.78	75.17	152.89
$\Pi_S^{MB}$	71.16	77.75	63.87	0.00
$\Pi_{DC}^{MB}$	106.75	123.53	139.04	152.89
$\Theta$	[0.00, 0.00]	[0.00, 0.00]	[13.88, 29.39]	[77.75, 107.11]

Because the supplier can be potentially worse off while offering a bundling discount beyond a certain point, as we can see from the preceding analysis, a bundling fee  $\Theta$  is required to ensure pareto-efficient results, i.e., both the supplier and the retailer can always be better off while bundling. Depending upon the bargaining power of each agent in the supply chain, the bundling fee charged by the supplier to the retailer may lie in the range  $[\Theta_L, \Theta_U]$ . If the supplier has greater bargaining power than the retailer, then the bundling fee will tend toward  $\Theta_U$  while if the retailer has greater bargaining power than the supplier, then it will tend toward  $\Theta_L$ .

## 5.5 Conclusions

Bundling has been found to be an useful marketing tool to boost sales and profit in economics and marketing literature over many years. Bundling literature usually focuses on studying the feasibility of the bundle in a retail setting. However, these studies fail to account for the fact that the product being bundled and sold can be manufactured and supplied by different entities. This is because the studies usually consider a retailer selling the goods without considering a supplier, which is analogous to a centralized supply chain in our study. Thus, in a multi-entity or decentralized supply chain, we were able to show that bundling is not always a feasible or a very profitable strategy. This is especially true if the products or the bundle are discounted beyond a certain point, because it may make one party worse off while making the other better off. This reduces the effectiveness of the bundling strategy in a supply chain setting.

In the two tier supply chain that we studied, we found that the retailer and the supply chain can simultaneously improve their profits by offering bundled goods to the consumers and achieve performance of a coordinated supply chain (single entity) by setting an appropriate bundling fee. The bundling discount (offered by the supplier to the retailer ( $\delta$ ) and by the retailer to the customers ( $\alpha$ )) augments the market for the two products, by targeting customers based on their valuations. Thus individual components are sold to individuals who value only one product very highly and sell bundles to customers with high overall valuations, thereby extracting maximum profit for the supply chain. The resultant increase in sales of product  $A$  and  $B$  either as components or in a bundle  $AB$ , coupled with the potential savings in costs such as transaction costs and set up costs, would lead to greater profits for both agents in the supply chain compared to the one where only components are sold in a decentralized setting.



# Chapter 6

## Concluding Remarks

### 6.1 Thesis Summary

In this thesis, we have focused on studying the problem of a retailer facing three different scenarios involving supply uncertainty, stockout based substitution and product bundling.

In chapter 3 we studied a retailer facing uncertain supply, who has to make pricing and stocking decisions. We analyze the case of simultaneous pricing, whereby the retailer jointly decides on the order size and the retail price before observing the yield and the demand. We also look at the case of postponed pricing whereby the retailer makes the pricing decision after observing the value of the yield. From our analysis, we can see that postponing the pricing decision can help produce more profits for the retailer even when the demand is price-sensitive and stochastic. This is due to greater uncertainty, when in addition to the demand, the supply is also uncertain. Thus, from an operational standpoint, it is in the interest of the retailer facing unreliable supply and stochastic demand to postpone the pricing decision till the supply uncertainty is resolved, if possible. However, postponement may not always be a feasible response to the supply uncertainty problem as the advertising and marketing effort required by the retailer may have to begin well in advance of the receipt of goods by the retailer and thus price postponement although, more profitable, may not be the feasible response. In such case, the retailer can optimally select a price and quantity such that the expected profit is maximized. Furthermore, we see that the higher expected profit for the retailer under postponed pricing case is not due to higher expected revenue but due to minimizing the costs associated with salvage and shortages when the retailer either incorrectly prices or orders. The sensitivity analysis also helped us gain insights into the role of price and order quantity decisions and their relationship to the supply uncertainty. As the supply uncertainty decreases in terms of a higher expected yield ( $\mu_r$ ), the expected profits for both simultaneous and postponed pricing scenarios tend to converge and the marginal difference between them decreases. Reducing the supply variability ( $\sigma_r$ ), does not seem to have a

significant impact on the order quantity  $Q$  for both the simultaneous and postponed pricing case. However as variability reduces, the expected profits for both simultaneous and postponed pricing tend to converge and the marginal difference between them decreases.

In chapter 4 we study a retailer selling two substitutable goods to consumers who, upon not finding their preferred product *may* substitute. Retailers that operate on very low margins can lose as much as 4% of their annual revenues due to stockouts (Gruen et al. (2002)), thus stockout based substitution and resultant cannibalization can have an impact on the retailer's bottom line and should be incorporated in the overall pricing and ordering decisions when applicable. We were able to show that the problem is concave in prices for a given inventory policy. The prices for the two products tend to move in opposite directions as the rate of substitution changes. We compare our findings to a model that does not consider stockout related substitution and the resultant cannibalization of inventory and find that the model that does not consider substitution tends to overestimate the expected profit for low degrees of substitution and tends to underestimate the expected profit for high degree of substitution. Furthermore, the prices charged and the quantity ordered by the retailer for each product, tend to be suboptimal. The total quantity stocked in general, for both products, is lower when we account for substitution and cannibalization. Although cannibalization also plays an important role in the overall problem as it tends to reduce the retailers expected profit, this effect is not as significant as that of substitution. Overall, with prudent management of the pricing and ordering process the retailer can boost the expected profit for the retailer by reducing the amount of inventory that the retailer holds as well as charging an optimal price for the goods.

In chapter 5 we study a two tier supply chain with a retailer and a supplier selling two products to consumers. Bundling has been found to be a useful marketing tool to boost sales and profit for the retailer in a centralized supply chain setting. However, in a decentralized supply chain, this may not always be the case. This is because the manufacturing, distribution and the retailing in a decentralized supply chain is usually undertaken by different entities trying to maximize their own profits without focusing on the profitability of the overall supply chain. This myopic behavior leads to double marginalization that prevents the supplier and the retailer from maximizing the overall profit. We found that the retailer and the supplier can simultaneously improve their profits by offering bundled goods to the consumers and achieve performance of a coordinated supply chain. The bundling discount augments the market for the two products, by targeting customers based on their valuations, thereby extracting maximum profit for the supply chain. Most economics and marketing literature that extols the virtues of bundling fail to consider the operational and supply chain aspect of bundling i.e., beyond a certain bundling discount the supplier and retailer are not proportionately better off in terms of share of overall profit and thus in order to coordinate, a bundling fee is required to ensure cooperation from both parties such that none of the parties are worse off.

## 6.2 Scope for Future Work

For future work, we have identified several avenues of research for all three problems.

The implications of unreliable supply to the supplier are higher production and shipping costs as well as the cost of having unsalable goods leftover at the end of a selling period when the uncertainty is high. For the retailer, it means higher shortage or salvage losses. Furthermore, postponement may not always be a feasible response to the supply uncertainty problem as the advertising effort required by the retailer may have to begin well in advance of the receipt of goods by the retailer. This opens a window of opportunity for the supplier to improve the expected process yield  $\mu_r$  and/or decrease supply variability  $\sigma_r$ . Another factor that can be considered, is supplier diversification where one supplier is more reliable than the other and charges a higher price. The retailer then can optimally not only choose the price and quantity but can also make a decision as to the allocation of orders to its supplier thereby reducing the supply uncertainty as well as minimizing the costs and losses.

For the case of product substitution, it would be important to study this problem in a supply chain context, where the pricing and stocking decisions are made independently. Specifically, studying the case where the inventory is vendor managed (VMI) vs. the case where they are retailer managed (RMI). Although the retailer makes the retail pricing decision, the vendor sets the wholesale price of the product under the framework presented in this thesis. Although similar studies have been done under very specific scenarios (e.g., Mishra and Raghunathan (2004)), a generalized model needs to be developed incorporating two-way substitution with cannibalization of two asymmetric products. Another important avenue worth pursuing is analyzing this problem under retail competition.

For the problem of bundling, one avenue worth exploring is to look at ways of using bundling as a lever for dynamic pricing. For example, a bundle is offered only occasionally, if the sales lag estimates or as a means for clearing inventory at the end of a season. Finally, we think it will be worthwhile to look at the role of bundling in a competitive environment where retailers compete with each other not just on price but also on product formats. For example, one retailer sells pure bundles and the other mixed bundles. In particular, it would be interesting to study the scenario where the retailer may choose to ‘defect’ and sell mixed bundles while buying pure bundles from the supplier.

# Appendix A

## Proofs from Chapter 5

Proof of Theorem 2.

**Proof.** Given that  $V_A \sim U[L_A, H_A]$  and  $V_B \sim U[L_B, H_B]$ , from equations (5.1) and (5.2) we obtain

$$Q_j = M \cdot \frac{H_j - P_j}{H_j - L_j}, \quad j = \{A, B\}. \quad (\text{A.1})$$

Substituting equation (A.1) in (5.3), we get:

$$\Pi_C^{NB} = M \sum_{j=A,B} (P_j - c_j) \cdot \frac{H_j - P_j}{H_j - L_j}. \quad (\text{A.2})$$

The Hessian of  $\Pi_C^{NB}$  is  $\begin{bmatrix} \frac{-2M}{H_A - L_A} & 0 \\ 0 & \frac{-2M}{H_B - L_B} \end{bmatrix}$  which is negative definite since  $M > 0$ .

Thus,  $\Pi_C^{NB}$  is jointly concave in  $P_A$  and  $P_B$ . Since the prices in (5.4) satisfy  $H_j \geq P_j \geq c_j$  and the first order optimality conditions, they are optimal. Substituting (5.4) in (A.1) we get (5.5). The optimal profit in formula (5.6) is obtained by substituting (5.4) and (5.5) in (5.3) ■

Proof of Theorem 3.

**Proof.** Consider the second derivative of the profit function with respect to  $P_{AB}$ :

$$\frac{d^2 \Pi_C^{PB}}{dP_{AB}^2} = \begin{cases} \frac{2(L_A + L_B) + c_{AB} - 3P_{AB}}{(H_A - L_A)(H_B - L_B)} \leq 0, & \text{if } P_{AB} \in I_1, \\ \frac{-2}{H_A - L_A} \leq 0, & \text{if } P_{AB} \in I_2, \\ \frac{3P_{AB} - 2(H_A + H_B) - c_{AB}}{(H_A - L_A)(H_B - L_B)}, & \text{if } P_{AB} \in I_3. \end{cases}$$

Thus, the profit function is concave for  $P_{AB} \in I_1 \cup I_2$ . When  $P_{AB} \in I_3$  the profit is a concave-convex function: concave for  $H_B + L_A \leq P_{AB} \leq \frac{2(H_A + H_B) + c_{AB}}{3}$  and convex for  $\frac{2(H_A + H_B) + c_{AB}}{3} \leq P_{AB} \leq H_A + H_B$ . Noting that  $P_{ABi}$  satisfies the first order necessary optimality conditions for  $P_{AB} \in I_i$ ,  $i \in \{1, 2, 3\}$  respectively, we conclude that if

$P_{ABi} \in I_i$  then  $P_{ABi}$  is a global maximum for  $\Pi_C^{PB}$  when  $P_{AB}$  is constrained to be in  $P_i$ . We will now show that depending on  $c_{AB}$  only one  $P_{ABi} \in I_i$  will be satisfied for only one  $i$ . We note that the profit function is continuous on  $I_1 \cap I_2 \cap I_3$  and consider the following three cases.

**Case 1:**  $c_{AB} \in I_4$ , i.e.,  $L_A + L_B \leq c_{AB} \leq \frac{3H_B + 4L_A - 2H_A - L_B}{2}$ . With some algebra it can be verified that in this case  $P_{AB1} \in I_1$ , but  $P_{AB2} \notin I_2$  and  $P_{AB3} \notin I_3$ . We will show that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB1})$  for  $P_{AB} \in [c_{AB}, H_A + H_B] = I_1 \cap I_2 \cap I_3$ . Since  $P_{AB1} \in I_1$ , it follows from concavity that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB1})$  for  $P_{AB} \in I_1$ . To look at the shape of the profit function in  $I_2 \cap I_3$  we first consider

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_B+L_A} = \frac{L_B + 2H_A - 4L_A - 3H_B + 2c_{AB}}{2(H_A - L_A)} \leq 0.$$

Therefore, given the concavity of the profit function in  $I_2$  and that  $P_{AB2} \geq H_B + L_A$  we conclude that it is decreasing for all  $P_{AB} \in I_2$  and so  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB1})$  in the interval  $I_2$ . Now, given the concave-convex nature of the profit function in  $I_3$  and considering the facts that

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_A+L_B} = \frac{H_B - 3L_B - 2H_A + 2c_{AB}}{2(H_A - L_A)} \leq 0,$$

$$P_{AB3} \leq H_A + L_B, \text{ and } \Pi_C^{PB}(P_{AB} = H_A + H_B) = 0,$$

we also conclude that the profit function is decreasing for all  $P_{AB} \in I_3$  and so  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB1})$  in the interval  $I_3$  as well.

**Case 2:**  $c_{AB} \in I_5$ , i.e.,  $\frac{3H_B + 4L_A - 2H_A - L_B}{2} \leq c_{AB} \leq \frac{2H_A + 3L_B - H_B}{2}$ . In this case  $P_{AB2} \in I_2$ , but  $P_{AB1} \notin I_1$  and  $P_{AB3} \notin I_3$ . We will show that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB2})$  for  $P_{AB} \in [c_{AB}, H_A + H_B] = I_1 \cap I_2 \cap I_3$ . Since  $P_{AB2} \in I_2$ , it follows from concavity that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB2})$  for  $P_{AB} \in I_2$ . To look at the shape of the profit function in  $I_1 \cap I_3$  we first consider

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_B+L_A} = \frac{L_B + 2H_A - 4L_A - 3H_B + 2c_{AB}}{2(H_A - L_A)} \geq 0.$$

Therefore, given the concavity of the profit function in  $I_1$  and that  $P_{AB1} \geq H_B + L_A$ , we conclude that it is increasing for all  $P_{AB} \in I_1$  and so  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB2})$  in the interval  $I_1$ . Now, given the concave-convex nature of the profit function in  $I_3$  and considering the facts that

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_A+L_B} = \frac{H_B - 3L_B - 2H_A + 2c_{AB}}{2(H_A - L_A)} \leq 0,$$

$$P_{AB3} \leq H_A + L_B, \text{ and } \Pi_C^{PB}(P_{AB} = H_A + H_B) = 0,$$

we conclude that the profit function is decreasing for all  $P_{AB} \in I_3$  and so  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB2})$  in the interval  $I_3$  as well.

**Case 3:**  $c_{AB} \in I_6$ , i.e.,  $\frac{2H_A+3L_B-H_B}{2} \leq c_{AB} \leq H_A + H_B$ . In this case  $P_{AB3} \in I_3$ , but  $P_{AB1} \notin I_1$  and  $P_{AB2} \notin I_2$ . We will show that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB1})$  for  $P_{AB} \in [c_{AB}, H_A + H_B] = I_1 \cap I_2 \cap I_3$ . Since  $P_{AB3} \in I_3$  and  $\Pi_C^{PB}(P_{AB} = H_A + H_B) = 0$ , it follows from the concave-convex nature of the profit function in  $I_3$  that  $\Pi_C^{PB}(P_{AB}) \leq \Pi_C^{PB}(P_{AB3})$  for  $P_{AB} \in I_3$ . To look at the shape of the profit function in  $I_1 \cap I_2$  we first consider

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_B+L_A} = \frac{L_B + 2H_A - 4L_A - 3H_B + 2c_{AB}}{2(H_A - L_A)} \geq 0.$$

Therefore, given the concavity of the profit function in  $I_1$  and that  $P_{AB1} \geq H_B + L_A$  we conclude that it is increasing for all  $P_{AB} \in I_1$  and so  $\Pi_C^{PB}(P_{AB1}) \leq \Pi_C^{PB}(P_{AB} = H_B + L_A)$  in the interval  $I_1$ . Now, given the convexity of the profit function in  $I_2$  and considering the facts that

$$\left. \frac{d\Pi_C^{PB}}{dP_{AB}} \right|_{P_{AB}=H_A+L_B} = \frac{H_B - 3L_B - 2H_A + 2c_{AB}}{2(H_A - L_A)} \geq 0$$

and

$$P_{AB2} \geq H_A + L_B,$$

we also conclude that the profit function is increasing for all  $P_{AB} \in I_2$  and so  $\Pi_C^{PB}(P_{AB2}) \leq \Pi_C^{PB}(P_{AB} = H_A + L_B)$  in the interval  $I_3$  as well. ■

Proof of Proposition 8

**Proof.**

(a) As per Theorem 3, depending on the value of  $c_{AB}$  we have three cases:

(i)  $c_{AB} \in I_4$ . Thus,  $P_{AB}^{CPB} \in I_1$  implying that

$$P_{AB}^{CPB} \leq \min(H_A + L_B, H_B + L_A) = H_B + L_A.$$

Now,

$$\begin{aligned} P_A^{CNB} + P_B^{CNB} - (H_B + L_A) &= \frac{H_A - H_B}{2} + \frac{c_A + c_B}{2} - L_A \\ &\geq \frac{H_A - H_B}{2} + \frac{L_A + L_B + c}{2} - L_A \\ &= \frac{H_A - L_A - (H_B - L_B)}{2} + \frac{c}{2} \\ &\geq 0. \end{aligned}$$

where in the first inequality we made use of the fact that  $c_A + c_B - c = c_{AB} \in I_4$  and the last inequality follows from the fact that  $H_B - L_B \leq H_A - L_A$  and  $c \geq 0$ . Therefore,  $P_{AB}^{CPB} \leq P_A^{CNB} + P_B^{CNB}$ .

(ii)  $c_{AB} \in I_5$ . Thus,  $P_{AB}^{CPB} = \frac{2H_A + H_B + L_B + 2c_{AB}}{4}$  and so  $P_A^{CNB} + P_B^{CNB} - P_{AB}^{CPB} = \frac{H_B - L_B}{4} + \frac{c}{2}$ .

(iii)  $c_{AB} \in I_6$ . Thus,  $P_{AB}^{CPB} = \frac{H_A + H_B + 2c_{AB}}{3}$  and so  $P_A^{CNB} + P_B^{CNB} - P_{AB}^{CPB} = \frac{H_A + H_B - c_{AB}}{6} + \frac{c}{2}$ .

(b) To compare the pure bundling profit to the no bundling profit we consider  $G(c) = \Pi_C^{PB} - \Pi_C^{NB}$ . We will show that  $G \geq 0$  if and only if  $c \geq \bar{c}$ . We look at the two cases, depending on the value of  $c_{AB}$ :

(i)  $c_{AB} \in I_5$ . Here we have

$$G(c) = M \frac{(2H_A + H_B + L_B - 2c_A - 2c_B + 2c)^2}{16(H_A - L_A)} - \Pi_C^{NB}$$

is quadratic convex in  $c$  and increasing. It is easy to show that  $G(c) \geq 0$  if and only if  $c \geq \bar{c} = \frac{2}{\sqrt{M}} \sqrt{(H_A - L_A) \Pi_C^{NB}} + c_A + c_B - H_A - \frac{1}{2}(H_B + L_B) \geq 0$ .

(ii)  $c_{AB} \in I_6$ . Here we have  $G(c) = \frac{2M}{27} \frac{(H_A + H_B - c_A - c_B + c)^3}{(H_A - L_A)(H_B - L_B)} - \Pi_C^{NB}$  is cubic and increasing. It is easy to show that  $G(c) \geq 0$  if and only if  $c \geq \bar{c} = 3 \sqrt[3]{\frac{1}{2M} \Pi_C^{NB} (H_A - L_A)(H_B - L_B) + c_A + c_B - H_A - H_B} \geq 0$ .

■

Proof of Theorem 4.

**Proof.** Differentiating the profit function (5.7) w.r.t.  $P_A$  and  $P_B$  and performing some algebra we obtain

$$\frac{\partial \Pi_C^{MB}}{\partial P_i} = M \frac{2(H_i + c_i - P_i)(H_j - L_j) + 3\alpha^2 + (4H_j - 6P_j + 2c_j - 2c)\alpha + 2c(P_j - H_j)}{2(H_i - L_i)(H_j - L_j)}, i, j \in \{A, B\}, i \neq j.$$

Solving for the first order optimality conditions yields equation (5.8). The Hessian of the profit function for a given value of  $\alpha$  is given as

$$\begin{bmatrix} \frac{-2M}{(H_A - L_A)} & \frac{M(c - 3\alpha)}{(H_A - L_A)(H_B - L_B)} \\ \frac{M(c - 3\alpha)}{(H_A - L_A)(H_B - L_B)} & \frac{-2M}{(H_B - L_B)} \end{bmatrix}$$

from which we obtain the following condition to ensure that the Hessian is negative definite in  $P_A$  and  $P_B$  for a given value of  $\alpha$ :

$$(c - 3\alpha)^2 \leq 4(H_A - L_A)(H_B - L_B)$$

or

$$\frac{c}{3} - \frac{2}{3} \sqrt{(H_A - L_A)(H_B - L_B)} \leq \alpha \leq \frac{c}{3} + \frac{2}{3} \sqrt{(H_A - L_A)(H_B - L_B)}.$$

Given that  $\alpha$  is constrained to  $[0, \min(H_A - L_A, H_B - L_B)]$ , it follows that  $\Pi_C^{MB}$  is jointly concave in  $P_A$  and  $P_B$  when  $\max(\frac{3}{2} \min(H_A - L_A, H_B - L_B) - \frac{c}{2}, \frac{c}{2}) \leq$

$\sqrt{(H_A - L_A)(H_B - L_B)}$  and  $\alpha$  is fixed. Since  $\frac{\partial^2 \Pi_C^{MB}}{\partial P_i^2} = \frac{-2M}{H_i - L_i} < 0, i \in \{A, B\}$ , it follows that  $\Pi_C^{MB}$  is concave-quadratic in both  $P_A$  and  $P_B$ . Therefore, to solve the constrained problem, for a given  $\alpha$ , it is sufficient to compare the prices that solve the first order optimality conditions to the price bounds  $c_i$  and  $H_i$  as in the statement of the theorem. ■

Proof of Proposition 10.

**Proof.** We will prove the results for the case when  $c_{AB} \in I_6$  and  $P_{AB} \in I_3 \cup I_6$  such that the retailer will order  $Q_{AB} = \frac{M}{2} \frac{(H_A + H_B - P_{AB})^2}{(H_A - L_A)(H_B - L_B)}$  and charge price  $P_{AB} = \frac{H_A + H_B + 2w_{AB}}{3}$ . The other cases can be proved in a similar way. The wholesale price is given as  $w_{AB} = H_A + H_B - \delta$ . The supplier, retailer and the total supply chain profits are

$$\begin{aligned}\Pi_S^{PB} &= (w_{AB} - c_{AB}) Q_{AB}, \\ \Pi_R^{PB} &= (P_{AB} - w_{AB}) Q_{AB}, \\ \Pi_{DC}^{PB} &= \Pi_S^{PB} + \Pi_R^{PB}.\end{aligned}$$

Their derivatives with respect to  $\delta$  are

$$\begin{aligned}\frac{\partial \Pi_S^{PB}}{\partial \delta} &= \frac{2 M \delta (2 (H_A + H_B - c_{AB}) - 3 \delta)}{9 (H_A - L_A) (H_B - L_B)} \\ \frac{\partial \Pi_R^{PB}}{\partial \delta} &= \frac{2 M \delta^2}{9 (H_A - L_A) (H_B - L_B)} \\ \frac{\partial \Pi_{DC}^{PB}}{\partial \delta} &= \frac{4 M \delta (H_A + H_B - \delta - c_{AB})}{9 (H_A - L_A) (H_B - L_B)}.\end{aligned}$$

First, we will establish the following facts:

- Fact 1:  $\Pi_C^{NB} \leq \Pi_C^{PB}$  since  $c \geq \bar{c}$ .
- Fact 2:  $\Pi_{DC}^{NB} \leq \Pi_{DC}^{PB}$ . This has been shown in (5.20).
- Fact 3: Since  $\frac{\partial \Pi_R^{PB}}{\partial \delta} > 0$  we get that  $\Pi_R$  is increasing in  $\delta$ .
- Fact 4:  $\Pi_S^{PB}$  is concave in  $\delta$ : it is increasing in  $[\delta_0, \delta_2]$  and decreasing in  $[\delta_2, \delta_4]$ .
- Fact 5:  $\Pi_{DC}^{PB}$  is increasing for  $\delta \in [\delta_0, \delta_4]$ . This follows because  $\frac{\partial \Pi_{DC}^{PB}}{\partial \delta} \geq 0$  for  $\delta \in [0, H_A + H_B - c_{AB}]$ .

Now, we consider each of the four cases separately:

- (a) Region I: That  $\Pi_S^{PB}$ ,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing follows from Facts 3–5. The inequalities follow from Facts 1 and 2 and the fact that  $\delta_1$  is such that  $\Pi_{DC}^{PB} = \Pi_{DC}^{NB}$ .



- (b) Region II: That  $\Pi_s^{PB}$ ,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing follows from Facts 3–5. The inequalities follow from Facts 1 and 2 and the fact that in Region II  $\delta \geq \delta_1$ .
- (c) Region III: That  $\Pi_s^{PB}$  is decreasing,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing follows from Facts 3–5. The inequalities follow from Facts 1 and 2 and the fact that in Region III  $\delta \geq \delta_1$ .
- (d) Region IV: That  $\Pi_s^{PB}$  is decreasing,  $\Pi_R^{PB}$  and  $\Pi_{DC}^{PB}$  are increasing follows from Facts 3–5. The inequalities follow from Facts 1 and 2 and the fact that in Region IV  $\delta \geq \delta_3$ .

■

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