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PRICING AND THE NEWSVENDOR PROBLEM: A REVIEW WITH EXTENSIONS

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In the newsvendor problem, a decision maker facing random demand for a perishable product decides how much of it to stock for a single selling period. This simple problem with its intuitively appealing solution is a crucial building block of stochastic inventory theory, which comprises a vast literature focusing on operational efficiency. Typically in this literature, market parameters such as demand and selling price are exogenous. However, incorporating these factors into the model can provide an excellent vehicle for examining how operational problems interact with marketing issues to influence decision making at the firm level. In this paper we examine an extension of the newsvendor problem in which stocking quantity and selling price are set simultaneously. We provide a comprehensive review that synthesizes existing results for the single period problem and develop additional results to enrich the existing knowledge base. We also review and develop insight into a dynamic inventory extension of this problem, and motivate the applicability of such models.

The newsvendor problem has a rich history that has been traced back to the economist Edgeworth (1888), who applied a variant to a bank cash-flow problem. However, it was not until the 1950s that this problem, like many other OR/MS models seeded by the war effort, became a topic of serious and extensive study by academicians. In its essential formulation, a decision maker facing random demand for a product that becomes obsolete at the end of a single period must decide how many units of the product to stock in order to maximize expected profit. The optimal solution to this problem is characterized by a balance between the expected cost of understocking and the expected cost of overstocking.

This simple problem, with its intuitively appealing optimal solution, is a crucial building block of a significant literature on stochastic inventory theory. Porteus (1990) provides an excellent review. Typically, the focus of this extensive literature is on operational efficiency to minimize expected cost. Demand or market parameters often are taken to be exogenous.

Whitin (1955) was the first to formulate a newsvendor model with price effects. In his model, selling price and stocking quantity are set simultaneously. Whitin adapted the newsvendor model to include a probability distribution of demand that depends on the unit selling price, where price is a decision variable rather than an external parameter. He established a sequential procedure for determining first the optimal stocking quantity as a function of price and then the corresponding optimal price. Mills (1959, 1962) refined the formulation by explicitly specifying mean demand as a function of the selling price. Yet, unlike the version of the newsvendor problem in which selling price is exogenous, this more strategic variant has received limited attention since the 1950s. This parallels in many ways the observation that since the 1950s, the practice of operations has emphasized functional efficiency at the expense of cross-functional effectiveness.

We believe that the newsvendor problem, because of its simple but elegant structure, can provide an excellent vehicle for examining how operational problems interact with marketing issues to influence decision-making at the firm level. The importance of such analysis is reinforced by the increasing prevalence of time-based competition (Stalk and Hout 1990) because as time-based competition intensifies, product life-cycles shrink so that more and more products acquire the attributes of fashion or seasonal goods. Consequently, we apply the newsvendor framework in Section 1 to analyze a firm who jointly sets a selling price and a stocking quantity prior to facing random demand in a single period. We review the history of the problem, generalize existing results, and provide a more

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integrated framework for investigating alternative formulations of the problem. Then, in Section 2 we review and provide additional insight into the dynamic inventory extension of the model presented in Section 1. We conclude this paper with a discussion to motivate and encourage the applications of the types of models presented in the first two sections.

1. THE SINGLE PERIOD PROBLEM

Consider a price-setting firm that stocks a single product, faces a random price-dependent demand function, and has the objective of determining jointly a stocking quantity, q, and selling price, p, to maximize expected profit. Randomness in demand is price independent and can be modeled either in an additive or a multiplicative fashion. Specifically, demand is defined as $D(p, \epsilon) = y(p) + \epsilon$ in the additive case (Mills 1959) and $D(p, \epsilon) = y(p)\epsilon$ in the multiplicative case (Karlin and Carr 1962), where $y(p)$ is a decreasing function that captures the dependency between demand and price, and ϵ is a random variable defined on the range $[A, B]$. Since different forms of $y(p)$ combine more naturally with different forms of $D(p, \epsilon)$, we let $y(p) = a - bp (a > 0, b > 0)$ in the additive case, but let $y(p) = ap^{-b} (a > 0, b > 1)$ in the multiplicative case. Both representations of $y(p)$ are common in the economics literature, with the former representing a linear demand curve and the latter representing an iso-elastic demand curve. One interpretation of this model is that the shape of the demand curve is deterministic while the scaling parameter representing the size of market is random. In order to assure that positive demand is possible for some range of p, we require that $A > -a$ in the additive case and $A > 0$ in the multiplicative case. However, from a practical standpoint, if a is large relative to the variance of ϵ, unbounded probability distributions such as the normal provide adequate approximations. For general purposes, we let $F(\cdot)$ represent the cumulative distribution function of ϵ, and $f(\cdot)$ the probability density function. Likewise, we define μ and σ as the mean and standard deviation of ϵ, respectively.

1.1. Additive Demand Case

In the additive demand case, $D(p, \epsilon) = y(p) + \epsilon$, where $y(p) = a - bp$. At the beginning of the selling period, q units are stocked for a cost of cq. If demand during the period does not exceed q, then the revenue is $pD(p, \epsilon)$ and each of the $q - D(p, \epsilon)$ leftovers is disposed at the unit cost h. Note that h may be negative ($h \geq -c$), in which case it represents a per-unit salvage value. Alternatively, if demand exceeds q, then the revenue is pq, and each of the $D(p, \epsilon) - q$ shortages is assessed the per-unit penalty cost s. The profit for the period, $\Pi(q, p)$, is the difference between sales revenue and the sum of the costs:

$$\Pi(q, p) = \begin{cases} pD(p, \epsilon) - cq - h[q - D(p, \epsilon)], & D(p, \epsilon) \leq q, \\ pq - cq - s[D(p, \epsilon) - q], & D(p, \epsilon) > q. \end{cases}$$

A convenient expression for this profit function is obtained by substituting $D(p, \epsilon) = y(p) + \epsilon$ and, consistent with Ernst (1970) and Thowsen (1975), defining $z = q - y(p)$:

$$\Pi(z, p) = \begin{cases} p[y(p) + \epsilon] - c[y(p) + z] - h[z - \epsilon], & \epsilon \leq z, \\ p[y(p) + z] - c[y(p) + z] - s[\epsilon - z], & \epsilon > z. \end{cases}$$

This transformation of variables provides an alternative interpretation of the stocking decision: If the choice of z is larger than the realized value of ϵ, then leftovers occur; if the choice of z is smaller than the realized value of ϵ, then shortages occur. The corresponding optimal stocking and pricing policy is to stock $q^* = y(p^*) + z^*$ units to sell at the unit price p^*, where z^* and p^* maximize expected profit.

Expected profit is:

$$E[\Pi(z, p)] = \int_A^z (p[y(p) + u] - h[z - u])f(u)\,du + \int_z^B (p[y(p) + z] - s[u - z])f(u)\,du - c[y(p) + z].$$

Defining $\Lambda(z) = \int_A^z (z - u)f(u)\,du$ and $\Theta(z) = \int_z^B (u - z)f(u)\,du$, we can write:

$$E[\Pi(z, p)] = \Psi(p) - L(z, p), \tag{1}$$

where

$$\Psi(p) \equiv (p - c)[y(p) + \mu], \tag{2}$$

and

$$L(z, p) \equiv (c + h)\Lambda(z) + (p + s - c)\Theta(z). \tag{3}$$

Equation (2) represents the riskless profit function (Mills 1959), the profit for a given price in the certainty-equivalent problem in which ϵ is replaced by μ. Equation (3) is the loss function (Silver and Peterson 1985), which assesses an overage cost $(c + h)$ for each of the $\Lambda(z)$ expected leftovers when z is chosen too high and an underage cost $(p + s - c)$ for each of the $\Theta(z)$ expected shortages when z is chosen too low. Expected profit is expressed by (1): the riskless profit, which would occur in the absence of uncertainty, less the expected loss that occurs as a result of the presence of uncertainty.

The objective is to maximize expected profit:

$$\underset{z;p}{\text{Maximize}}\ E[\Pi(z, p)]. \tag{4}$$

Consider, then, the first and second partial derivatives of $E[\Pi(z, p)]$ taken with respect to z and p:

$$\frac{\partial E[\Pi(z, p)]}{\partial z} = -(c + h) + (p + s + h)[1 - F(z)], \tag{5}$$

$$\frac{\partial^2 E[\Pi(z, p)]}{\partial z^2} = -(p + s + h)f(z), \tag{6}$$

$$\frac{\partial E[\Pi(z, p)]}{\partial p} = 2b(p^0 - p) - \Theta(z),$$

$$\text{where } p^0 = \frac{a + bc + \mu}{2b}, \tag{7}$$

$$\frac{\partial^2 E[\Pi(z, p)]}{\partial p^2} = -2b. \tag{8}$$

The term p^0 denotes the optimal riskless price, which is the price that maximizes $\Psi(p)$.

Notice from (6) that $E[\Pi(z, p)]$ is concave in z for a given p. Thus, it is possible to reduce (4) to an optimization problem over the single variable p by first solving for the optimal value of z as a function of p and then substituting the result back into $E[\Pi(z, p)]$. This method was introduced by Whitin (1955) and yields the familiar fractile rule for determining z, $[1 - F(z^*)] = (c + h)/(p + s + h)$), which is the standard newsvendor result when p is fixed (Porteus 1990). Similarly, from (8), $E[\Pi(z, p)]$ is concave in p for a given z, thereby validating Zabel's (1970) method of first optimizing p for a given z, and then searching over the resulting optimal trajectory to maximize $E[\Pi(z, p^*)]$. Both sequential procedures yield the same conclusions, but only the latter approach is presented here.

Lemma 1 follows directly from (7) and (8):

Lemma 1. *For a fixed z, the optimal price is determined uniquely as a function of z:*

$$p^* \equiv p(z) = p^0 - \frac{\Theta(z)}{2b}.$$

Since $\Theta(z)$ is nonnegative, $p^* \leq p^0$. This relationship was demonstrated first by Mills (1959).

Substituting $p^* = p(z)$ into (4), the optimization problem becomes a maximization over the single variable z:

$$\underset{z}{\text{Maximize}}\ E[\Pi(z, p(z))].$$

Therefore, the effort required to compute the optimal stocking and pricing policy depends on the shape of $E[\Pi(z, p(z))]$. However, as Theorem 1 demonstrates, $E[\Pi(z, p(z))]$ might have multiple points that satisfy the first-order optimality condition, depending on the parameters of the problem.

Theorem 1. *The single-period optimal stocking and pricing policy for the additive demand case is to stock $q^* = y(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 1 and z^* is determined according to the following:*

(*a*) *If $F(\cdot)$ is an arbitrary distribution function, then an exhaustive search over all values of z in the region $[A, B]$ will determine z^*.*

(*b*) *If $F(\cdot)$ is a distribution function satisfying the condition $2r(z)^2 + dr(z)/dz > 0$ for $A \leq z \leq B$, where $r(\cdot) \equiv f(\cdot)/[1 - F(\cdot)]$ is the hazard rate, then z^* is the largest z in the region $[A, B]$ that satisfies $dE[\Pi(z, p(z))]/dz = 0$.*

(*c*) *If the condition for (b) is met AND $a - b(c - 2s) + A > 0$, then z^* is the unique z in the region $[A, B]$ that satisfies $dE[\Pi(z, p(z))]/dz = 0$.*

Proof. See the appendix.

The second condition in (c) guarantees that $E[\Pi(z, p(z))]$ is unimodal in z when $F(\cdot)$ is a distribution function satisfying $2r(\cdot)^2 + r'(\cdot) > 0$. Ernst (1970) reached a similar conclusion under the more restrictive assumption that $F(\cdot)$ is a member of the PF_2 family of distributions and Young (1978) expanded the set of applicable distributions to include the log-normal. However, our articulation of Theorem 1 generalizes the conditions for which the optimal solution to the single period problem can be identified analytically because PF_2 distributions and the log-normal distribution have nondecreasing hazard rates (Barlow and Proschan 1975), and all nondecreasing hazard rate distributions satisfy the condition in (b).

1.2. Multiplicative Demand Case

In the multiplicative demand case, $D(p, \epsilon) = y(p)\epsilon$, where $y(p) = ap^{-b}$. Analogous results for this section also apply when $y(p) = ae^{-b}$, another representation of demand common in the economics literature, but we omit the parallel analysis in order not to distract from the presentation. By substituting $D(p, \epsilon) = y(p)\epsilon$ and $z \equiv q/y(p)$, the single period profit function can be written conveniently as:

$$\Pi(z, p) = \begin{cases} py(p)\epsilon - cy(p)z - hy(p)[z - \epsilon], & \epsilon \leq z, \\ py(p)z - cy(p)z - sy(p)[\epsilon - z], & \epsilon > z. \end{cases}$$

Although we define z differently for the multiplicative demand case than we do for the additive demand case, the effect is the same: If z is larger than the realized value of ϵ, leftovers occur; if z is smaller, shortages occur. This suggests that although z is defined primarily for mathematical convenience depending on the uncertainty structure of the demand curve, a consistent managerial interpretation for z exists. We provide one such interpretation in Section 1.3.

Analogous to the additive demand case, the optimal stocking and pricing policy is to stock $q^* = y(p^*)z^*$ units to sell at the unit price p^*, where z^* and p^* jointly maximize expected profit. And, as in the additive demand case, expected profit can be written as follows:

$$E[\Pi(z, p)] = \Psi(p) - L(z, p). \tag{9}$$

But, now

$$\Psi(p) \equiv (p - c)y(p)\mu, \tag{10}$$

and

$$L(z, p) \equiv y(p)[(c + h)\Lambda(z) + (p + s - c)\Theta(z)]. \tag{11}$$

Consequently, expected profit again is interpreted as riskless profit, $\Psi(p)$, less an expected loss due to uncertainty, $L(z, p)$. However, in this case $\Lambda(z)y(p)$ represents expected leftovers and $\Theta(z)y(p)$ represents expected shortages.

To maximize $E[\Pi(z, p)]$, we follow the same sequential procedure detailed in the previous section. First, the optimal selling price is established as a function of z ($p^* = p(z)$). Then that price is substituted back into the expected profit function, thereby reducing the problem to a maximization over a single variable.

Lemma 2. *For a fixed z, the optimal price is determined uniquely as a function of z:*

$$p^* \equiv p(z) = p^0 + \frac{b}{b-1}\left[\frac{(c+h)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right],$$

where $p^0 = \frac{bc}{b-1}$.

Proof. See the appendix.

By assumption, $b > 1$ and $A > 0$. In addition, it can be shown that $\Theta(z)$ is nonincreasing in z, which implies that $\mu - \Theta(z) \geq \mu - \Theta(A) = A > 0$. Therefore, $p^* \geq p^0$, where p^0 denotes the optimal riskless price. This relationship, which was demonstrated first by Karlin and Carr (1962), is opposite of the corresponding relationship found to be true by Mills (1959) for the additive demand case. We address this issue in Section 1.3. Also note that in the multiplicative demand case, p^0 does not depend in any way on the characterization of ϵ; whereas in the additive demand case, p^0 is a linear function of the mean of ϵ.

As in the additive model, the shape of $E[\Pi(z, p(z))]$ depends on the parameters of the problem.

Theorem 2. *The single-period optimal stocking and pricing policy for the multiplicative demand case is to stock* $q^* = y(p^*)z^*$ *units to sell at the unit price* p^*, *where* p^* *is specified by Lemma 2 and* z^* *is determined according to the following:*

(*a*) *If* $F(\cdot)$ *is an arbitrary distribution function, then an exhaustive search over all values of z in the region* $[A, B]$ *will determine* z^*.

(*b*) *If* $F(\cdot)$ *is a distribution function satisfying the condition* $2r(z)^2 + dr(z)/dz > 0$ *for* $A \leq z \leq B$, *and* $b \geq 2$, *then* z^* *is the unique z in the region* $[A, B]$ *that satisfies* $dE[\Pi(z, p(z))]/dz = 0$.

Proof. See the appendix.

Once again, our articulation of this theorem provides a slight generalization of the existing literature. Zabel (1970) first demonstrated the uniqueness of z^* in the multiplicative demand case of the single period problem, but he assumed that $s = 0$ and considered only two special forms for $F(\cdot)$: the exponential distribution and the uniform distribution. Earlier, Nevins (1966) reached a similar conclusion regarding z^* for the case when $F(\cdot)$ is a normal distribution when he performed a simulation experiment. Young (1978) extended the result to cases in which $F(\cdot)$ either is a log-normal distribution or a member of the PF_2 family of distributions. However, assuming that the product is elastic enough (i.e., that $b \geq 2$), our proof requires only that $F(\cdot)$ is such that $2r(\cdot)^2 + r'(\cdot) > 0$, which is more general yet.

1.3. Unified Framework for Additive and Multiplicative Demand Cases

A basic difference between the additive and the multiplicative representations of demand is the manner in which the pricing decision contributes to demand uncertainty. To demonstrate, let $E[\,\cdot\,]$ and $\text{VAR}[\,\cdot\,]$ denote the expectation and variance operators, respectively; and consider the first and second moments of the random variable, $D(p, \epsilon)$:

$$E[D(p, \epsilon)] = \begin{cases} y(p) + \mu & \text{if additive demand case,} \\ y(p)\mu & \text{if multiplicative demand case,} \end{cases}$$

and

$$\text{VAR}[D(p, \epsilon)] = \begin{cases} \sigma^2 & \text{if additive demand case,} \\ y(p)^2\sigma^2 & \text{if multiplicative demand case.} \end{cases}$$

Thus, the variance of demand is independent of price in the additive demand case but is a decreasing function of price in the multiplicative demand case. However, the demand coefficient of variation, $\sqrt{\text{VAR}[D(p, \epsilon)]}/E[D(p, \epsilon)]$, is an increasing function of price in the additive demand case, while it is independent of price in the multiplicative demand case.

This distinction is important because it establishes an analytical basis for explaining differences in structure that arise in the results of the joint stocking and pricing problem when the two modeling alternatives for demand are analyzed. Recall that Mills (1959) defined as a benchmark the riskless price, which represents the optimal selling price for the special situation in which there is no variation of demand from its mean. Given this definition, Mills found that $p^* \leq p^0$ if randomness in demand is modeled within an additive context; but Karlin and Carr (1962) found that $p^* \geq p^0$ if randomness in demand is modeled within a multiplicative context.

Young (1978) verified both of these results by analyzing a model that combines both additive and multiplicative effects, defining the demand function as $D(p, \epsilon) = y_1(p)\epsilon + y_2(p)$. This formulation corresponds to the additive demand case when $y_1(p) = 1$ and to the multiplicative demand case when $y_2(p) = 0$. Given this specification, demand variance is $y_1(p)^2\sigma^2$ and demand coefficient of variation is $y_1(p)\sigma/[y_1(p)\mu + y_2(p)]$. Using these two measures of uncertainty, we identify three possibilities:

(i) variance is decreasing in p while coefficient of variation is increasing in p;

(ii) variance is decreasing in p while coefficient of variation is nonincreasing in p; and

(iii) variance is nondecreasing in p while coefficient of variation is increasing in p.

The fourth alternative, a situation in which variance is nondecreasing in p while coefficient of variation is nonincreasing in p, is not possible under the assumption that expected demand is decreasing in p.

Young analyzed two of these three possibilities, concluding that $p^* \geq p^0$ if (ii) is satisfied; but $p^* \leq p^0$ if (iii) is satisfied. Since the multiplicative demand case satisfies (ii) and the additive demand case satisfies (iii), his results are consistent with both Mills' and Karlin and Carr's. Curiously, Young omits a prescription for a scenario in which (i) is satisfied; and although he identified how the relationship between p^* and p^0 depends on the method in which randomness is incorporated in demand, he did not provide an explanation for the apparent contradiction between the results of the additive and multiplicative demand cases. We next offer one possible explanation.

In a deterministic setting (assuming $\epsilon = \mu$), there is no risk of overstocking or understocking and the optimal course of action is to choose the riskless price. With uncertainty, there is the risk of overstocking or understocking, but pricing provides an opportunity to reduce that risk. Given that variance and coefficient of variation represent two common measures of uncertainty, ideally price could be used to decrease both. Unfortunately, that is not possible in either the additive or the multiplicative demand case. But, in the additive demand case, it is possible to decrease the demand coefficient of variation without adversely affecting the demand variance by choosing a lower price; and in the multiplicative case, it is possible to decrease the demand variance without adversely affecting the demand coefficient of variation by choosing a higher price. Consequently, from this perspective, it is intuitive that $p^* \leqslant p^0$ in the additive demand case while $p^* \geqslant p^0$ in the multiplicative demand case. Note that this intuitive explanation leaves unresolved the relationship between p^* and p^0 for a scenario in which demand variance is decreasing in price while the demand coefficient of variation is increasing in price. We conjecture that in such a case either the price dependency of demand variance or of demand coefficient of variation will take precedence, thereby ensuring a determinable direction for the relationship. We leave this conjecture as a possible direction for continued research.

Although we provide this explanation to justify why the pricing strategy appears to differ depending on how randomness is incorporated into the demand function, one of our goals for this paper is to develop a unified framework for understanding the results of the joint stocking and pricing problem, regardless of the form of demand uncertainty. Our intent is to develop consistent insight for both the additive and the multiplicative demand cases. We proceed by developing two key ideas. First, we provide a managerial significant interpretation for z, demonstrating that although z is defined differently for each of the two cases, its meaning is consistent for both: z represents a stocking factor that we define as a surrogate for safety factor Silver and Peterson (1985). Then, we define a new pricing benchmark that we refer to as the base price. Our notion of this base price extends Mills' idea of the riskless price, although we feel that the base price is more amenable to environments in which demand is uncertain because it takes into account the fact that expected sales differ from expected demand when uncertainty exists. In the special case in which there is no variation in demand, base price and riskless price are equivalent. We develop the concept of base price because it establishes a new frame of reference that is convenient for resolving the contradiction in optimal pricing strategies that occurs as a result of the choice between the additive and the multiplicative models. We find that, in general, the optimal pricing strategy is to charge a premium over the base price, where the amount of the premium depends on the risk of overstocking or understocking. But, since demand variance is independent of price in the additive demand case, it turns out that the premium associated with the optimal selling price is zero for that case.

From Equations (1)–(3) and (9)–(11), expected profit for the single period can be expressed as:

$$\begin{aligned} E[\Pi(z, p)] = {} & (p - c)E[D(p, \epsilon)] \\ & - \{(c + h)E[\text{Leftovers}(z, p)] \\ & + (p - c + s)E[\text{Shortages}(z, p)]\}, \end{aligned} \tag{12}$$

which we interpreted as the difference between riskless profit and the loss function. This expression is convenient for analysis when p is fixed because then the objective of finding the stocking quantity that maximizes expected profit simplifies to the equivalent problem of finding the stocking quantity that minimizes the loss function. Consequently, it is common in the literature on the newsvendor problem. However, since price is not fixed in the joint stocking and pricing problem, we find it more insightful to apply the identity, E[Sales] = E[Demand] − E[Shortages], and express the profit function as follows:

$$\begin{aligned} E[\Pi(z, p)] = {} & (p - c)E[\text{Sales}(z, p)] \\ & - \{(c + h)E[\text{Leftovers}(z, p)] \\ & + sE[\text{Shortages}(z, p)]\}. \end{aligned} \tag{13}$$

We interpret (13) similarly to (12): Expected profit is the difference between the total contribution expected from sales and the expected loss resulting from the inevitable occurrence of either leftovers or shortages. This form of the profit function allows us to develop the notion of base price as a pricing strategy benchmark, which, together with the concept of stocking factor as a decision variable, provides an interpretive framework that is consistent for both the additive and the multiplicative demand cases.

1.3.1. Stocking Factor. Silver and Peterson (1985) define safety factor, SF, as the number of standard deviations that stocking quantity deviates from expected demand:

$$\text{SF} \equiv \frac{q - E[D(p, \epsilon)]}{\text{SD}[D(p, \epsilon)]},$$

$$\text{where } \text{SD}[D(p, \epsilon)] \equiv \sqrt{\text{VAR}[D(p, \epsilon)]}. \tag{14}$$

This definition provides the basis for the following theorem.

Theorem 3. *For both the additive and the multiplicative demand cases, the variable z represents the stocking factor, defined as follows:*

$$z = \mu + \text{SF}\sigma.$$

Proof. First consider the additive demand case: $E[D(p, \epsilon)] = y(p) + \mu$, $\text{VAR}[D(p, \epsilon)] = \sigma^2$, and $z \equiv q - y(p)$. Thus, from (14):

$$z = (E[D(p, \epsilon)] + \text{SF} \cdot \text{SD}[D(p, \epsilon)]) - y(p)$$
$$= (y(p) + \mu + \text{SF}\sigma) - y(p) = \mu + \text{SF}\sigma.$$

Similarly, for the multiplicative demand case: $E[D(p, \epsilon)] = y(p)\mu$, $\text{VAR}[D(p, \epsilon)] = y(p)^2\sigma^2$, and $z \equiv q/y(p)$. Thus:

$$z = \frac{E[D(p, \epsilon)] + \text{SF} \cdot \text{SD}[D(p, \epsilon)]}{y(p)}$$
$$= \frac{y(p)\mu + \text{SF}y(p)\sigma}{y(p)} = \mu + \text{SF}\sigma. \quad \square$$

The stocking factor interpretation of z also applies to Young's specification of demand, which includes both additive and multiplicative effects. Given Young's definition of $D(p, \epsilon)$, the mean and standard deviation of demand is $E[D(p, \epsilon)] = y_1(p)\mu + y_2(p)$ and $\text{SD}[D(p, \epsilon)] = y_1(p)\sigma$, respectively. Correspondingly, we define $z = [q - y_2(p)]/y_1(p)$, which implies that:

$$z = \frac{E[D(p, \epsilon)] + \text{SF} \cdot \text{SD}[D(p, \epsilon)] - y_2(p)}{y_1(p)}$$
$$= \frac{y_1(p)\mu + \text{SF}y_1(p)\sigma}{y_1(p)} = \mu + \text{SF}\sigma,$$

thereby completing the illustration.

Finally, we note that for the additive demand case, z also can be interpreted as a surrogate for safety stock, since safety stock is defined as the deviation of stocking quantity from expected demand (i.e., safety stock $\equiv q - E[D(p, \epsilon)] = z - \mu$). However, this interpretation of z does not hold for the multiplicative demand case; hence, it is less appealing than the stocking factor interpretation. Given that z represents the stocking factor, the joint stocking and pricing problem can be transformed into an equivalent optimization problem in which the joint decision can be interpreted as having to choose a selling price and a stocking factor, rather than a selling price and a stocking quantity, regardless of whether the problem is formulated as the additive demand or the multiplicative demand case. This is important because substituting z for q provides analytical tractability. Theorem 3, then, merely ensures that one need not sacrifice managerial understanding for mathematical convenience.

1.3.2. Base Price.

For a given value of z, we define base price, $p_B(z)$, as the price that maximizes the function $J(z, p) \equiv (p - c)E[\text{Sales}(z, p)]$, which represents the expected sales contribution.

Lemma 3. *For both the additive and the multiplicative demand cases, $p_B(z)$ is the unique value of p, given z, that satisfies*:

$$p = c + \left(- \frac{E[\text{Sales}(z, p)]}{\partial E[\text{Sales}(z, p)]/\partial p} \right).$$

Proof. By definition, $p_B(z)$ maximizes $J(z, p) = (p - c)E[\text{Sales}(z, p)]$. Therefore, it satisfies the following first order, optimality condition:

$$\frac{\partial J(z, p)}{\partial p} = E[\text{Sales}(z, p)] + (p - c)\frac{\partial E[\text{Sales}(z, p)]}{\partial p} = 0$$
$$\Rightarrow p = c + \left(- \frac{E[\text{Sales}(z, p)]}{\partial E[\text{Sales}(z, p)]/\partial p} \right).$$

To demonstrate that the value of p that satisfies this equation is unique and corresponds to the maximum of $J(z, p)$, consider the additive and the multiplicative demand cases separately.

For the additive demand case,

$$E[\text{Sales}(z, p)] \equiv E[D(p, \epsilon)] - E[\text{Shortages}(z, p)]$$
$$= y(p) + \mu - \Theta(z),$$

where $y(p) = a - bp$; and consequently, $\partial E[\text{Sales}(z, p)]/\partial p = -b$. Thus,

$$\frac{\partial J(z, p)}{\partial p} = [y(p) + \mu - \Theta(z)] - b(p - c)$$
$$= a + bc + \mu - \Theta(z) - 2bp,$$

which is a linearly decreasing function of p. Notice also that $\partial J(z, p)/\partial p > 0$ for small values of p since $\mu - \Theta(z) \geqslant A > -a$. This means that $\partial J(z, p)/\partial p$ crosses zero exactly once, thereby changing its sign from positive to negative. Therefore, the unique p satisfying the equation $\partial J(z, p)/\partial p = 0$ corresponds to the maximum of $J(z, p)$.

For the multiplicative demand case,

$$E[\text{Sales}(z, p)] \equiv E[D(p, \epsilon)] - E[\text{Shortages}(z, p)]$$
$$= y(p)\mu - y(p)\Theta(z) = y(p)[\mu - \Theta(z)],$$

where $y(p) = ap^{-b}$; and consequently,

$$\partial E[\text{Sales}(z, p)]/\partial p = -bap^{-b-1}[\mu - \Theta(z)].$$

Thus,

$$\frac{\partial J(z, p)}{\partial p} = y(p)[\mu - \Theta(z)] - bap^{-b-1}[\mu - \Theta(z)](p - c)$$
$$= a(\mu - \Theta(z))p^{-b-1}[bc - (b - 1)p].$$

For $p < \infty$, the term in [ ] determines the sign of $\partial J(z, p)/\partial p$. Since the term in [ ] is linearly decreasing in p, the logic applied in the additive demand case also applies here. $\square$

From Lemmas 1 and 3, we observe that for a given z in the additive demand case, $p^0 \geqslant p^* = p_B(z)$. However, from Lemmas 2 and 3, we observe that for a given z in the multiplicative demand case, $p^* \geqslant p^0 = p_B(z)$. We summarize these observations in the next theorem.

Theorem 4. *For both the additive and the multiplicative demand cases, $p^* \geqslant p_B(z)$. Thus, given z, p^* is interpreted as the sum of the base price and a premium.*

Since the premium component of p^* is zero in the additive demand case, we focus on the multiplicative demand case in order to develop an economic interpretation. From Lemma 2:

$$\text{Premium} = p^* - p_B(z) = p^* - p^0$$
$$= \frac{b}{b-1}\left[\frac{(c+h)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right];$$

or, multiplying both the numerator and denominator by $y(p)$, we can write:

$$\text{Premium} = \frac{b}{b-1}$$
$$\cdot\left[\frac{(c+h)E[\text{Leftovers}(z,p)] + sE[\text{Shortages}(z,p)]}{E[\text{Sales}(z,p)]}\right].$$

Thus, the idea behind the per-unit premium is to recoup, on a per-sale basis, the total expected cost resulting from the management of inventory that is used as a buffer against uncertainty in demand. That is, the premium in selling price is based on a formula that takes the total expected leftover cost ($(c+h)E[\text{Leftovers}(z,p)]$), adds to it the total expected shortage cost due to penalties in excess of the net margin lost ($sE[\text{Shortages}(z,p)]$), and then spreads the sum over total expected sales. The quotient then is adjusted by $b/(b-1)$, which is a weighting that is specific to the form of the demand function and is related to its price elasticity.

We conclude that the premium charged depends wholly on how the selling price affects expected leftovers and shortages beyond the effect that is incorporated in the stocking factor, which itself depends on the selling price. Since, for a given z, expected leftovers and expected shortages are constant in the additive demand case, the effect is like that of a fixed cost and consequently, nothing is passed to the customer in the form of a premium. However, since for a given z, expected leftovers and expected shortages still depend on p in the multiplicative demand case, the effect is like that of a marginal cost and consequently, that cost is passed to the customer in the form of a premium. Intuitively, in the additive demand case, the stocking factor serves as the only means available to guard against the risk of having leftovers or shortages. In the multiplicative demand case, a price increase provides an additional hedge.

1.4. The Value of Information

As the results above demonstrate, setting prices in addition to selecting stocking quantities provides more refinement in terms of managing the effect of uncertainty. It therefore is natural to consider the economic impact of uncertainty. We provide one such estimate by computing the expected value of perfect information for the additive demand case, omitting a similar calculation for the multiplicative demand case.

We define perfect information to mean that the value of ϵ is known before the pricing and stocking decisions are made. In such a case, $D(p, \epsilon)$ is a deterministic function, which implies that there is no risk of leftovers or shortages. Correspondingly, the optimal selling price is $p_\epsilon = \text{argmax}\{(p - c)D(p, \epsilon)\}$ and the optimal stocking quantity is $q_\epsilon = D(p_\epsilon, \epsilon)$. The expected value of perfect information is defined as the difference between the expected optimal profit in the case with perfect information and the expected optimal profit in the case with uncertainty.

Theorem 5. *The expected increase in profit that results from obtaining perfect information in the additive demand case is given by*:

$$E[\textit{value of perfect information}]$$
$$= \frac{\sigma^2 + \Theta(z^*)^2}{4b} + L(z^*, p^*).$$

Proof. See the appendix.

The first term in the expression for the expected value of perfect information represents the increase in expected net revenue that results from selling expected demand at the optimal perfect information price (p_ϵ) rather than selling the expected demand at the optimal imperfect information price (p^*). The second term denotes the savings from the elimination of operating costs associated with uncertainty. Note that if selling price is fixed at p, the E[value of perfect information] $= L(z^*, p)$, which indicates the cost of uncertainty associated with a stock-setting/non-price-setting firm. If stocking quantity is set after demand is observed, then E[value of perfect information] $= \sigma^2/4b$, which indicates the cost of uncertainty associated with a price-setting/non-stock-setting firm. Just as uncertainty links the selling price and the stocking quantity decisions in a single-period setting, the prospect of learning more about the demand function provides a link between periods in a multiple-period setting. However, such an extension is beyond the scope of this paper. We refer the reader to Petruzzi and Dada (1997) for further development of learning in this context and focus next on a more natural multiple-period extension of the single-period problem.

2. THE MULTIPLE PERIOD PROBLEM

In Section 1, we reviewed and developed new insight into the joint stocking and pricing problem under demand uncertainty. In brief, the results prescribe optimal buffer levels in the form of a stocking factor and a pricing premium to hedge against the risk of uncertainty. A natural extension of this problem is a corresponding management situation involving multiple periods, where units left over from one period are available to meet demands in subsequent periods. Conceptually, this extension affects the determination of the optimal buffer levels because the existence of multiple selling opportunities changes the economic risks associated with understocking and overstocking. We review and develop insight into this extension next.

Applying the notation from Section 1 and defining α as a discounting factor, a generic formulation of the multiple period problem can be written as follows. This formulation assumes that unmet demand is lost because that is the predominant case considered in the literature, but it is

Table I
Specifications for $D(p, \epsilon)$, Leftovers (z, p) and Shortages (z, p)

	Additive Demand	Multiplicative Demand
$D(p, \epsilon)$	$y(p) + \epsilon$	$y(p)\epsilon$
Leftovers(z, p)	$[z - \epsilon]^+$	$y(p)[z - \epsilon]^+$
Shortages(z, p)	$[\epsilon - z]^+$	$y(p)[\epsilon - z]^+$

straightforward to adapt the model to include backorders. Let

$G_t(x) \equiv$ maximum discounted expected profit over a t-period horizon, given that x is the initial inventory on-hand,

and let

$$J_t(z, p) \equiv E[\Pi(z, p)] + \alpha\left[\int_A^z G_{t-1}(\text{Leftovers}(z, p)) f(u)\, du + \int_z^B G_{t-1}(0) f(u)\, du\right], \tag{15}$$

where, recall, $E[\Pi(z, p)]$ is given by (12) and denotes the single-period expected profit given that z is the stocking factor, p is the selling price, and c is the per-unit stocking cost for each unit stocked. Then,

$$G_t(x) = \max_{z,p}\{J_t(z, p) + cx\}. \tag{16}$$

The term cx reflects the savings from not having to pay the per-unit stocking cost for units left over from the previous period—a cost that is included in $J_t(z, p)$. Specific forms for $D(p, \epsilon)$, Leftovers(z, p), and Shortages(z, p) depend on whether demand uncertainty is expressed as additive or multiplicative. Table I itemizes these expressions for each of the two cases.

To develop the form of the optimal policy for this problem, define (z_t, p_t) as the unconstrained optimal decision vector for period t. Assuming (z_t, p_t) is an interior point, then it satisfies:

$$\left.\frac{\partial J_t(z, p)}{\partial z}\right|_{\substack{z=z_t\\ p=p_t}} = \left.\frac{\partial J_t(z, p)}{\partial p}\right|_{\substack{z=z_t\\ p=p_t}} = 0. \tag{17}$$

If (z_t, p_t) is a feasible alternative in period t, then it is optimal. But, typically the formulation given by (16) is interpreted to include the constraint $q(z, p) \geq x$, where x is the beginning inventory, left over from the previous period, and $q(z, p)$ is the stocking quantity (i.e., $q(z, p) = y(p) + z$ if demand uncertainty is modeled as additive and $q(z, p) = y(p)z$ if demand uncertainty is modeled as multiplicative). A constraint such as this is common in stochastic inventory theory and reflects an implicit assumption that disposal is costly. In other words, the constraint implies that if at the beginning of an arbitrary period t, the amount of stock left over from the previous period, x, is greater than the ideal stocking quantity for period t, say q_t, then it is too restrictive simply to discard the excess $x - q_t$ and begin the period with the ideal stocking amount q_t.

Under the implicit assumption of costly disposal, the primary focus of the limited research done on this problem has been to establish sufficient conditions for which $J_t(z, p)$ is guaranteed to have a unique interior point maximum for all t. Under such circumstances, the optimal stocking policy takes the familiar form:

$$q_t^* = \begin{cases} q(z_t, p_t) & \text{if } q(z_t, p_t) \geq x, \\ x & \text{otherwise,} \end{cases} \tag{18}$$

where (z_t, p_t) is the unique maximum of $J_t(z, p)$. The corresponding optimal pricing policy is

$$p_t^* = \begin{cases} p_t & \text{if } q(z_t, p_t) \geq x, \\ p_t(x) & \text{otherwise,} \end{cases} \tag{19}$$

where $p_t(x) = \text{argmax}_p\{J_t(z(x, p), p)\}$ and $z(x, p)$ reflects the binding constraint $q_t^* = x$; that is, $z(x, p) \equiv x - y(p)$ in the additive demand case and $z(x, p) \equiv x/y(p)$ in the multiplicative demand case. Ernst (1970) was the first to establish such sufficiency conditions. He demonstrated that, for each t, (17) has a unique solution that indeed corresponds to the maximum of $J_t(z, p)$ when the following are satisfied:

(i) demand uncertainty is modeled as additive;
(ii) $s = 0$;
(iii) $a + A > bc$; and
(iv) ϵ is drawn from a continuous member of the PF_2 family of densities.

The proof follows from (15) and (16) using a standard induction argument. In a similar analysis, Zabel (1972) expanded Ernst's result by providing an alternative set of conditions. He followed Ernst's precedence in assuming (i) and (ii), and he required that ϵ be drawn from either an exponential or a uniform p.d.f., which is more restrictive than (iv), but he demonstrated the desired result for the more general case in which $y(p)$ is concave rather than linear and the marginal cost of stocking an additional unit is nondecreasing rather than constant. Thowsen (1975) extended both Ernst's and Zabel's list to include the case of a convex holding cost function. To our knowledge, similar conditions have not been identified for the multiplicative demand case, although Zabel (1972) identifies some of the analytical roadblocks that creep into such an analysis.

Understandably, finding sufficient conditions for which $J_t(z, p)$ is guaranteed to have a unique interior point maximum for all t is not an easy endeavor. From (15), $J_t(z, p)$ represents the combined immediate and discounted future payoffs associated with the decisions made in period t. Theorems 1 and 2 from Section 1 allude to the difficulty in assessing the immediate payoff of these decisions even without considering future effects; when the future is included, the analysis is all that more difficult.

It is not our intent to reproduce here the work of previous authors, nor is it necessarily our intent to expand on the conditions under which the analysis of the multiple period problem is tractable. Instead, we choose to revisit the implicit costly disposal assumption, argue why the assumption ought not to apply to the case of a price-setting firm, and demonstrate the resulting simplicity of the problem, both in terms of tractability and computation, when the implicit assumption is revised.

What if disposal were not costly? Indeed, what if instead, a salvage market existed such that unwanted excess inventory could be disposed for revenue? Taking this concept one step further, suppose that the salvage market is such that each leftover remaining at the end of a period can be sold for αc (or equivalently, each can be sold immediately at the beginning of the subsequent period, prior to the new order being placed, for c). This assumption provides the underpinning for Veinott's (1965) seminal result indicating that a myopic policy is optimal for the analogous dynamic problem in which selling price is exogenous. The same result prevails when selling price is a decision variable: the multiple-period problem reduces to a sequence of identical single-period problems.

To demonstrate, consider that if such a salvage market exists, then it can be interpreted as a second *supply* source of the product. This is because one management decision at the beginning of a new period is the determination of how many of the leftovers from the previous period should be sold to the salvage market for a per-unit revenue of c. Or equivalently, this decision question can be stated as: How many of the leftovers should not be sold to the salvage market, understanding that the cost for each unit not sold is c due to lost revenue? Thus, at the beginning of a new period, the stocking decision is determined as the sum of the number of units not sold in the salvage market (for an implicit per-unit cost of c) and the number of units explicitly purchased/produced (for a per unit cost of c). Since the per-unit cost of the product is the same regardless of the source, the two sources can be thought of as a single source and the activity sequence at the beginning of a period can be interpreted simply as follows: First, all leftovers are salvaged for c each and then the total stocking quantity for the new period is purchased/produced for c each. Therefore, if a salvage market exists in which units can be sold for their purchase price, then we can write:

$$G_t(x) = cx + G_t(0). \tag{20}$$

If we assume further that the salvage market continues to exist even at the end of the problem horizon, then (20) holds for all t, including $t = 0$. Alternatively, if the horizon is infinite, then (20) always holds, but the subscript no longer is necessary. From (15), then, this implies for all t:

$$J_t(z, p) = E[\hat{\Pi}(z, p)] + \alpha E[G_{t-1}(0)], \tag{21}$$

where $E[\hat{\Pi}(z, p)]$ is the same as $E[\Pi(z, p)]$, as given by (12), except that the parameter $\hat{h} = h - \alpha c$ is substituted for h.

From (21), $J_t(z, p)$ is maximized at $(\hat{z}, \hat{p})$, which is independent of time and maximizes $E[\hat{\Pi}(z, p)]$. Thus, under the assumptions made in this section regarding the existence of a salvage market, the solution to the dynamic inventory problem for a price-setting firm is stationary and myopic, which implies that it can be determined by the techniques presented in Section 1. Moreover, Theorems 1 and 2 continue to apply. Interestingly, although these conclusions require the existence of a salvage market, in practice this market actually is not required. Since the policy is stationary and demand is nonnegative, the number of leftovers remaining from a given period always will be no greater than the desired stocking level for the subsequent period. Consequently, the action taken at the beginning of a new period always will be to purchase additional units in order to bring the stocking level *up to* the base stock; never will it be necessary to salvage units in order to bring the stocking level *down to* the base stock. The mere assumption of the existence of a salvage market as described here is sufficient for ensuring the uselessness of its existence.

In the discussion above, we tacitly assumed that the initial inventory on hand at the start of the problem horizon is smaller than the first period's desired stocking quantity. If that were not the case, then in fact the optimality of a stationary myopic policy would require the use of a salvage market at the beginning of the first period. But, were such a contingency to occur, the firm could create its own salvage market simply by marking down the price temporarily at the beginning of the period, thereby stimulating new demand only for as long as there continued to be an overstock. The temporary sale would continue until the desired stocking quantity was reached, at which time the desired selling price, that is, the selling price associated with the desired stocking quantity, would be charged once again. The trick is to establish a *temporary* sales price: If there is an overstock situation, rather than discounting the selling price for the entire quantity stocked at the beginning of a period, which is a consequence of the optimal policy given by (18) and (19), the selling price need be discounted only for a designated portion of the entire stocking quantity.

Under reasonable technical assumptions such as the market being large enough, such a policy can be, and often is, used in practice. A simple example is the popular "kick-off" sale advertised by retailers at the beginning of new selling seasons, particularly at the start of the important Christmas retail season. Such promotions might take the following form: The first n customers who purchase Product X will receive a Y% discount off the regular selling price. Granted, competitive factors beyond the scope of this paper also may play an integral role in developing promotions of this type; but nevertheless, one effect of such a sale is the creation of a salvage market for as many as n units. (In reality, the size of the salvage market might be less than n because the demand of the product associated with the regular selling price most likely will be cannibalized as a result of the sale price.) Another example in which a temporary sales price is used widely is the "early bird special." Although a primary motivation for this popular sale is to shift demand to fit a fixed capacity level better, the effect is the same. Over the course of a given evening, a restaurant might have available more aggregate units of its product (say, seat-hours) than the total evening demand for that product; however, by offering a discount on the first *n* seat-hours that it sells, the restaurant effectively creates a salvage market in which it can sell a unit that otherwise it would not have been able to sell. We consider early bird sales to be a salvage market because the restaurant still sells the same number of seat hours at the regular price that it would have sold in the absence of the early bird special. Thus, the early bird sales are not in lieu of higher-priced sales of the same product. Instead, they are in lieu of not making a sale at all. Apparently, universities are adopting a similar mentality in selling a unit of their product: an admission slot. In a recent article in *Smart Money* magazine about how financial aid packages can attract potential incoming students, Amy Virshup (1997) writes: "... colleges, like airlines, discovered that selling a seat at a discount is better than not selling it at all."

3. APPLICABILITY OF MODELS

In this paper, we reviewed the literature on incorporating pricing into the newsvendor model. In the single-period model, a comparison with the benchmark deterministic model reveals that the structure of the optimal policy depends on how uncertainty is introduced in the model. If uncertainty enters in an additive form, then the optimal price is no higher than that in the deterministic model; alternatively, if uncertainty enters in a multiplicative form, the optimal price is no lower than that in the deterministic model. We reconcile this apparent contradiction by introducing the notion of a base price and demonstrating that the optimal price can be interpreted as the base price plus a premium.

In multiple period versions of the problem, the pricing and stocking decision in each period is linked to successive periods through leftover inventory. In stark contrast to the research on the single-period model, the literature on multiple period models does not provide structural results of the optimal policy that yield managerial insight; rather the emphasis has been on technical properties that speed up computation of optimal policies. We believe that the difficulty in obtaining structural results can be traced to the assumption that leftovers cannot be disposed. We show how revising this assumption and allowing for the possibility of salvaging leftovers is sufficient to yield a stationary myopic policy for the multiple period problem; but the revised assumption appears innocuous because such a market is never really needed when the optimal policy is used. Consequently, all results and managerial insight available for the single-period model apply directly to the multiple-period model.

Although we believe that the practical implementation of the models developed in this paper are motivated primarily by the simplicity, usefulness, and applicability of the single-period results, we also believe that the full adoption of such models first requires overcoming challenges in integrating sales tracking systems with inventory systems, and then incorporating the price sensitivities of various classes of customers. However, it appears that the first step in this process is well underway: Ongoing innovations in information technology are increasing the potential for practical implementation of newsvendor-like problems that exploit improved knowledge regarding inventory availability and demand uncertainty.

For example, consider a retailer who sells a seasonal good. Most textbooks in operations research or management science would consider the decision problem in the context of a single period. However, if this retailer has a point-of-sales (POS) system that tracks sales electronically, it can determine if demand is being depleted at an adequate rate. In the event that sales are slower than expected, then rather than wait until the end of the season to mark down the product, the retailer can run a temporary promotion to bring remaining inventory in line with the target for the rest of the season. Effectively, the retailer dynamically can revise downward its estimate of the demand curve. Therefore, it can revise downward its target inventory level and offer a sale such as that discussed at the end of Section 2 in order to create a salvage market for any excess inventory. Moreover, the retailer can be viewed as implementing a form of yield management because inventory (capacity) is fixed at the beginning of the season and temporary markdowns keep remaining capacity in line with estimates of remaining demand.

Recent models along these lines that adapt prices during the season include work by Federgruen and Heching (1997) and Gallego and van Ryzin (1994). An empirical analysis is reported by Gallego et al. (1997). Their discussion indicates the potential benefit of implementing newsvendor-like models. Hence, variants of the newsvendor model like those described in this paper, along with continuing developments in information technology, offer promising opportunities for researchers to collaborate with practitioners to improve the efficacy of industrial pricing policies and inventory management.

APPENDIX

Proof of Theorem 1. From the chain rule and Lemma 1:

$$\frac{dE[\Pi(z, p(z))]}{dz} = -(c+h) + \left[p^0 + s + h - \frac{\Theta(z)}{2b}\right][1 - F(z)].$$

To identify values of z that satisfy this first-order optimality condition, let $R(z) \equiv dE[\Pi(z, p(z))]/dz$ and consider finding the zeros of $R(z)$:

$$\frac{dR(z)}{dz} = \frac{d}{dz}\left[\frac{dE[\Pi(z, p(z))]}{dz}\right] = -\frac{f(z)}{2b}\left\{2b(p^0 + s + h) - \Theta(z) - \frac{1 - F(z)}{r(z)}\right\},$$

where $r(\cdot) \equiv f(\cdot)/[1 - F(\cdot)]$ denotes the hazard rate (Barlow and Proschan 1975). Also,

$$\frac{d^2R(z)}{dz^2} = \left[\frac{dR(z)/dz}{f(z)}\right]\frac{df(z)}{dz} - \frac{f(z)}{2b} \cdot \left\{[1 - F(z)] + \frac{f(z)}{r(z)} + \frac{[1 - F(z)][dr(z)/dz]}{r(z)^2}\right\}$$

$$\Rightarrow \frac{d^2R(z)}{dz^2}\bigg|_{dR(z)/dz=0} = -\frac{f(z)[1 - F(z)]}{2br(z)^2}\left\{2r(z)^2 + \frac{dr(z)}{dz}\right\}.$$

If $F(\cdot)$ is a distribution satisfying the condition $2r(z)^2 + dr(z)/dz > 0$, then it follows that $R(z)$ either is monotone or unimodal, implying that $R(z) = dE[\Pi(z, p(z))]/dz$ has at most two roots. Further, $R(B) = -(c + h) < 0$. Therefore, if $R(z)$ has only one root, it indicates a change of sign for $R(z)$ from positive to negative, and thus it corresponds to a local maximum of $E[\Pi(z, p(z))]$; if it has two roots, the larger of the two corresponds to a local maximum and the smaller of the two corresponds to a local minimum of $E[\Pi(z, p(z))]$. In either case, $E[\Pi(z, p(z))]$ has only one local maximum, identified either as the unique value of z that satisfies $R(z) = dE[\Pi(z, p(z))]/dz = 0$ or as the larger of two values of z that satisfies $R(z) = 0$. And since $E[\Pi(z, p(z))]$ is unimodal if $R(z)$ has only one root (still assuming that $2r(z)^2 + dr(z)/dz > 0$), a sufficient condition for unimodality of $E[\Pi(z, p(z))]$ is $R(A) > 0$ or, equivalently, $2bR(A) > 0$, where:

$$\begin{aligned} 2bR(A) &= -2b(c + h) + [2b(p^0 + s + h) - \Theta(A)] \cdot [1 - F(A)] \\ &= -2b(c + h) + [(a + bc + \mu) + 2b(s + h) - (\mu - A)] \\ &= a - b(c - 2s) + A. \quad \square \end{aligned}$$

Proof of Lemma 2. For the multiplicative demand case, p^0 is defined as the p that maximizes $\Psi(p) = (p - c)y(p)\mu = (p - c)(ap^{-b})\mu$. Hence, consider:

$$\begin{aligned} \frac{d\Psi(p)}{dp} &= [ap^{-b} - b(p - c)ap^{-b-1}]\mu \\ &= -(b - 1)ap^{-b-1}\left[p - \frac{bc}{b - 1}\right]\mu. \end{aligned}$$

Since $(b - 1)ap^{-b-1} > 0$ for $p < \infty$, the function $\Psi(p)$ is increasing for $0 < p < bc/(b - 1)$ and is decreasing for $bc/(b - 1) < p < \infty$. Therefore, $\Psi(p)$ reaches its maximum at $p^0 = bc/(b - 1)$.

Next, recall from (9)–(11):

$$E[\Pi(z, p)] = y(p)[(p - c)\mu - (c + h)\Lambda(z) - (p + s - c)\Theta(z)].$$

Thus:

$$\frac{\partial E[\Pi(z, p)]}{\partial p} = (b - 1)\frac{y(p)}{p}[\mu - \Theta(z)] \cdot \left\{p^0 + \frac{b}{b - 1}\left[\frac{(c + h)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right] - p\right\}.$$

Since $\mu - \Theta(z) \geq A > 0$, $E[\Pi(z, p)]$ is increasing if and only if the term in { } is positive. Therefore, given z, $E[\Pi(z, p)]$ reaches its maximum at

$$p^0 + \frac{b}{b - 1}\left[\frac{(c + h)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right]. \quad \square$$

Proof of Theorem 2. This proof is similar to that of Theorem 1. From (9)–(11):

$$\frac{dE[\Pi(z, p(z))]}{dz} = y(p(z))[1 - F(z)]\left[p(z) + s + h - \frac{c + h}{1 - F(z)}\right].$$

Define $R(z) = (p(z) + s + h) - (c + h)/[1 - F(z)]$. Then, since $y(p(z))[1 - F(z)] > 0$ for all values of z other than the boundary value $z = B$, $R(z)$ identifies the behavior of $E[\Pi(z, p(z))]$ as follows: $E[\Pi(z, p(z))]$ is increasing for any z that satisfies $R(z) > 0$, decreasing for any z that satisfies $R(z) < 0$, and has a local optimum for any z that satisfies $R(z) = 0$. Thus, analyzing $R(z)$ is sufficient for determining the shape of $E[\Pi(z, p(z))]$.

First, notice, using Lemma 2, that $R(B) < 0$ and $R(A) > 0$:

$$R(B) = (p(B) + s + h) - \frac{c + h}{0} \to -\infty < 0,$$

$$\begin{aligned} R(A) &= (p(A) + s + h) - (c + h) \\ &= \frac{bc}{b - 1} + \frac{b}{b - 1}\left[\frac{s(\mu - A)}{A}\right] + s - c \\ &= \frac{1}{b - 1}\left[c + s\left(b\frac{\mu}{A} - 1\right)\right] > 0. \end{aligned}$$

The last inequality follows because $b > 1$ and $\mu > A$.

Next, recall that $r(\cdot) = f(\cdot)/[1 - F(\cdot)]$ and consider how $R(z)$ behaves in z:

$$\frac{dR(z)}{dz} = \frac{dp(z)}{dz} - \frac{(c + h)r(z)}{1 - F(z)},$$

and

$$\frac{d^2R(z)}{dz^2} = \frac{d^2p(z)}{dz^2} - (c + h)\left[\frac{dr(z)/dz}{1 - F(z)} + \frac{r(z)^2}{1 - F(z)}\right];$$

where, from Lemma 2:

$$\frac{dp(z)}{dz} = \frac{b}{b - 1}\left[\frac{(c + h)[F(z)z - \Lambda(z)] - s[1 - F(z)]\mu}{[\mu - \Theta(z)]^2}\right],$$

and

$$\begin{aligned} \frac{d^2p(z)}{dz} &= \frac{b}{b - 1}f(z)\left[\frac{(c + h)z + s\mu}{[\mu - \Theta(z)]^2}\right] - \frac{2[1 - F(z)]}{\mu - \Theta(z)}\frac{dp(z)}{dz} \\ &= \frac{b}{b - 1}\frac{(c + h)r(z)}{\mu - \Theta(z)} - \left[\frac{2[1 - F(z)]}{\mu - \Theta(z)} + r(z)\right]\frac{dp(z)}{dz}. \end{aligned}$$

Thus, by substitution:

$$\frac{d^2R(z)}{dz^2} = -(c+h)\left[\frac{b-2}{b-1}\frac{r(z)}{\mu-\Theta(z)} + \frac{2r(z)^2 + dr(z)/dz}{1-F(z)}\right] - \left[\frac{2[1-F(z)]}{\mu-\Theta(z)} + r(z)\right]\frac{dR(z)}{dz}$$

$$\Rightarrow \left.\frac{d^2R(z)}{dz^2}\right|_{dR(z)/dz=0} < 0$$

if $2r(z)^2 + dr(z)/dz > 0$ and $b \geqslant 2$.

This last inequality means that $R(z)$ is unimodal in z, first increasing and then decreasing. Therefore, given that $2r(z)^2 + dr(z)/dz > 0$ and $b \geqslant 2$, $E[\Pi(z, p(z))]$ reaches its maximum at the unique value of $z \neq B$ that satisfies

$$dE[\Pi(z, p(z))]/dz = y(p(z))[1 - F(z)]R(z) = 0. \quad \square$$

Proof of Theorem 5. First consider the profit if a value for ϵ were revealed before the optimal stocking and pricing decisions were made. In the additive demand case, the optimal stocking quantity then would equal the known demand, $y(p) + \epsilon$, and the optimal selling price, p_ϵ, would maximize the function $\tilde{\Pi}(p) = (p - c)[y(p) + \epsilon]$:

$$p_\epsilon = \frac{a + bc + \epsilon}{2b} = p^0 + \frac{\epsilon - \mu}{2b}.$$

Therefore, the optimal expected profit for a firm operating with perfect information is:

$$\begin{aligned} E[\tilde{\Pi}(p_\epsilon)] &= E[(p_\epsilon - c)(y(p_\epsilon) + \epsilon)] \\ &= (p^0 - c)(a - bp^0 + \mu) + \frac{E[\epsilon^2] - \mu^2}{4b} \\ &= \Psi(p^0) + \frac{\sigma^2}{4b}. \end{aligned}$$

Next, consider $E[\Pi(z^*, p^*)]$. From (1), (2), and Theorem 1:

$$\begin{aligned} E[\Pi(z^*, p^*)] &= \Psi(p^*) - L(z^*, p^*) \\ &= (p^* - c)[y(p^*) + \mu] - L(z^*, p^*) \\ &= \Psi(p^0) - \frac{\Theta(z^*)^2}{4b} - L(z^*, p^*). \end{aligned}$$

The expected increase in profit that results from obtaining perfect information is the difference between $E[\tilde{\Pi}(p_\epsilon)]$ and $E[\Pi(z^*, p^*)]$:

E[value of perfect information]

$$= \frac{\sigma^2 + \Theta(z^*)^2}{4b} + L(z^*, p^*). \quad \square$$

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REFERENCES

Barlow, R. E., F. Proschan. 1975. *Statistical Theory of Reliability and Life Testing: Probability Models.* Holt, Rinehart, and Winston, New York.

Edgeworth, F. 1888. The mathematical theory of banking. *J. Royal Statist. Soc.* **51** 113–127.

Ernst, R. L. 1970. A linear inventory model of a monopolistic firm. Ph.D. Dissertation, Department of Economics, University of California, Berkeley, CA.

Fedegruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Oper. Res.*

Gallego, G., A. Heching, G. van Ryzin. 1997. Markdown pricing: an empirical analysis of policies and revenue potential at one apparel retailer. Working Paper, Columbia University, New York.

——, G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* 40 999–1020.

Karlin, S., C. R. Carr. 1962. Prices and optimal inventory policy. K. J. Arrow, S. Karlin, and H. Scarf (Eds.). *Studies in Applied Probability and Management Science.* Stanford University Press, Stanford, CA, 159–172.

Mills, E. S. 1959. Uncertainty and price theory. *Quart. J. Econom.* **73** 116–130.

——. 1962. *Price, Output and Inventory Policy.* John Wiley, New York.

Nevins, A. J. 1966. Some effects of uncertainty: simulation of a model of price. *Quart. J. Econom.* **80** 73–87.

Petruzzi, N. C., M. Dada. 1997. Dynamic pricing and inventory control with learning. Working Paper, University of Illinois.

Porteus, E. L. 1990. Stochastic inventory theory. D. P. Heyman and M. J. Sobel (Eds.). *Handbooks in OR & MS, Vol. 2.* Elsevier, North-Holland, 605–652.

Silver, E. A., R. Peterson. 1985. *Decision Systems for Inventory Management and Production Planning.* John Wiley, New York.

Stalk, G., Jr., T. M. Hout. 1990. *Time-Based Competition.* Free Press, New York.

Thowsen, G. T. 1975. A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Res. Logistics Quart.* **33** 461–476.

Veinott, A., Jr. 1965. Optimal policy for a multi-product, dynamic non-stationary inventory problem. *Management Sci.* **12** 206–222.

Virshup, A. 1997. Have they got a deal for you. *Smart Money* **6** (12) 120–131.

Whitin, T. M. 1955. Inventory control and price theory. *Management Sci.* **2** 61–68.

Young, L. 1978. Price, inventory and the structure of uncertain demand. *New Zealand Oper. Res.* **6** 157–177.

Zabel, E. 1970. Monopoly and uncertainty. *Rev. Econom. Studies* **37** 205–219.

——. 1972. Multiperiod monopoly under uncertainty. *J. Econom. Theory* **5** 524–536.