Pricing Convexity Adjustment with Wiener Chaos

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Abstract

This paper presents an approximated formula of the convexity adjustment of Constant Maturity Swap rates, using Wiener Chaos expansion, for multi-factor lognormal zero coupon models.
We derive closed formulae for CMS bond and swap and apply results to various well-known one-factor models (Ho and Lee (1986), Amin and Jarrow(1992), Hull and White (1990), Mercurio and Moraleda(1996)). Quasi Monte Carlo simulations confirm the efficiency of the approximation. Its precision relies on the importance of second and higher order terms.

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1 Introduction

Due to the main role of interest rates swap rates in the determination of long term rates, it has been of great relevance to develop exotic options that incorporate swap rates. This has led to new products that use the rate of a Constant Maturity Swap (CMS) as an underlying rate. These are very diverse, ranging from CMS swaps and bonds to more complicated ones like CMS swaptions, caps and any traditional exotic fixed income derivatives. These CMS derivatives are tailored instruments for trading the steepening or flattening of the yield curve, since one receives/pays the swap rate (long term rate) in the future and lends/borrows at money market rates (short term rates) today. There are other products to trade the steepening or flattening of the yield curve, like in arrear derivatives and other products with embodied convexity. However, CMS derivatives have become more popular because they are more leveraged than their competitor derivatives and they correspond to long duration investment.

A main limitation for pricing and hedging these derivatives has been the inability to get closed formula within a standard term structure yield curve model. Usually, practitioners compare the CMS rate with the forward swap rate of the same maturity. In the CMS case, the investor pays/receives the swap rate only once, whereas in the case of the forward swap, during the whole life of the swap. Consequently, this modified schedule leads to a difference between the two rates, classically called convexity adjustment. The term convexity refers to the convexity of a receiver swap prices with respect to the swap rate. Traditionally, this adjustment is calculated assuming that swap rates behave according to the Black Scholes (1973) hypotheses.

There has been extensive research for the so called Black Scholes convexity adjustment. Brotherton-Ratcliffe and Iben (1993) first showed an analytic approximation for the convexity adjustment in the case of bond yield. Other works completed the initial formula: Hull (1997) extended it to swap rates, Hart (1997) gave a result with a better precision approximation, Kirikos and al (1997) showed how to adapt it to a Hull and White yield curve model. Recently Benhamou (2000) estimated the approximation error by means of a martingale approach.

However, when assuming that interest rates follow a diffusion process different from the Black-Scholes and Hull and White's ones, using the convexity adjustment in the Black Scholes setting is irrelevant. Indeed, since nowadays, almost all financial institutions rely on more realistic multi-factor term structure models, the traditional formula looks old-fashioned and inappropriate. In this paper, we offer a solution to it. Using approximations based on Wiener Chaos expansion, we provide an approximated formula for the convexity adjustment when assuming a multi-factor lognormal zero coupon model (Heath Jarrow hypotheses). This is consistent with most common term structure models.

The remainder of this paper is organized as follows. In section 2, we explain the intuition of the convexity adjustment as well as the products based on CMS rates. In section 3, we give explicit formulae of a coupon paying a CMS rate when assuming a log normal zero coupon bond model. In section 4, we explicit formulae for different term structure models and compare the closed form results with the ones given by a Quasi Monte Carlo method. We conclude briefly in section 5. In appendix, some key results on Wiener chaos expansion are presented as well as the approximation theorem proof.

2 Convexity: intuition and CMS products

In this section, we explain intuitively the nature of the convexity adjustment as well as the CMS products.

2.1 convexity of Swap rates

In the modern derivatives industry, two risks have emerged as intriguing and challenging for the management and control of secondary market risk: for equity derivatives, it has been the volatility smile and for fixed income derivatives, the convexity adjustment. Taking correctly these effects into account can provide competitive advantage for financial institutions.

Our paper focuses on swap rates. Since the receiver swap price is a convex function of the swap rate, it is not correct to say that the expected swap is equal to the forward swap rate, defined as the rate at which the forward swap has zero value. This can be seen with the figure 1.

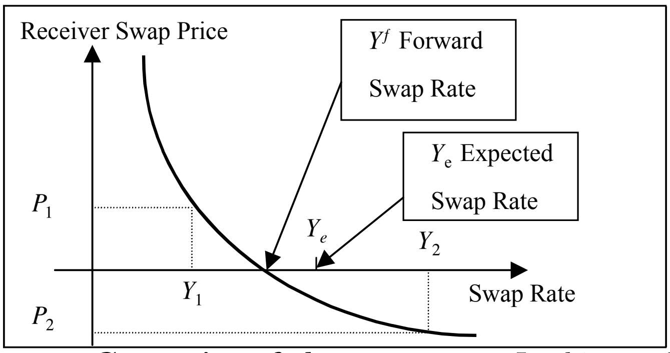

Figure 1: Convexity of the swap rate. In this graphic, we see that the convexity of the receiver swap price with respect to the swap rate leads to a higher expected swap rate than the forward swap rate, corresponding to a zero swap price.

Let see it by means of a simple model. In our economy, the world is binomial, with the prices of the swap equal to either P_1 or P_2 with equal probability $\frac{1}{2}$. The average price, calculated as the expected value of the future prices, leads to a zero value corresponding to a swap rate, Y^f, called forward swap rate. However, because of convexity of the receiver swap price with respect to the swap rate, the expected swap rate Y_e , equal to all the outcomes weighted by their corresponding probabilities ($Y_e = \frac{1}{2}Y_1 + \frac{1}{2}Y_2$) is higher than the forward swap Y^f. This little difference is called the convexity adjustment. In the rest of the paper, we will see how to determine the convexity adjustment when assuming more realistic description of interest rates' evolution .

2.2 CMS derivatives

Since their early creation in 1981, interest rates swap contracts have grown very rapidly. The swap market represents now hundreds of billions of dollars each year. Subsequently, investors have been and are potentially looking for new instruments to risk-manage and hedge their positions as well as to speculate on the steepening or flattening of the yield curve. Indeed, the main interest of investors has turned out to be speculation. Even if other products like in arrear derivatives enable to trade the flattening or the steepening of the yield curve, CMS derivatives are of particular interest since they are highly leveraged.

CMS derivatives are called CMS because they use a Constant Maturity Swap rate as the underlying rate. They are very diverse ranging from CMS swaps, CMS bonds to CMS swaptions and all other types of CMS exotics. Two major products are mainly traded over the counter: CMS swap and CMS bond. Logically, a CMS swap is an agreement to exchange a fixed rate for a swap rate, the latter referring to a swap of constant maturity. Assuming that our CMS swap starts in five years, is annual and is based on a swap rate of five year maturity, this typical contract will be the following: in five years, the investor will receive the swap rate of the swap starting in five years from today maturing in ten years. The investor will pay in return a fixed rate agreed in advance in the contract. One year later, that is in six years from today, the investor will receive the swap rate of the swap starting this time in six years from today maturing in eleven years. Again, the investor will pay the fixed rate. We see that at each payment, the investor receives a swap rate of a different swap. All the swap have in common to be settled at the date of the payment and to have the same maturity. A CMS bond is very similar to a CMS swap. It is a bond with coupons paying a swap rate of constant maturity. Therefore a CMS bond is exactly equal to the swap leg paying the swap rate. Since the swap leg paying the swap rate can be decomposed into each different payment, to price the CMS swap or CMS bond, we only need to price one payment of a swap rate. The value of a swap rate paid only once is called CMS rate value. The difference in value between the forward swap rate and this CMS rate is called the convexity adjustment.

Indeed, other CMS derivatives can be priced using forward rates increased by the convexity adjustment. The rest of the paper will concentrate on the pricing of the CMS rate. Knowing these rates, one can use them to plug it into derivatives pricing formula to get an approached value of the CMS derivatives.

2.3 CMT bond and CMS swap

We consider a continuous trading economy with a trading interval $[0,\tau]$ for a fixed $\tau > 0$. The uncertainty in the economy is characterized by the probability space (Ω, F, Q) where Ω is the state space, F is the $\sigma-$algebra representing measurable events, and Q is the risk neutral probability measure uniquely defined in complete markets with no-arbitrage (Harrison, Kreps(1979) and Harrison, Pliska (1981)). We assume that information evolves according to the augmented right continuous complete filtration $\{F_t, t \in [0,\tau]\}$ generated by a standard (initialized at zero) $k-$dimensional Wiener Process (or Brownian motion). Let $(r_t)_{t<\tau}$ be the continuous spot rate, $B(t,T)_{t<\tau, T<\tau}$ the price at time t of a default-free forward zero coupon maturing at time T and $(y_T)_{T<\tau}$ the swap rate at time T. These three stochastic variables are supposed to be adapted to the information structure $(F_t)_{t\in[0,\tau]}$.

The i^{th} coupon of a CMS bond pays the swap rate y_{T_i}, with a constant maturity specified in the contract, determined at a fixing date T_i often equal (eventually prior) to the payment date T_i^p. Therefore, referring each coupon by the subscript variable i, the coupon value at time T_i^p is the swap rate times the nominal $y_{T_i}N$ while, at the fixing time, it is this value discounted by the forward zero coupon : $B(T_i, T_i^p)\, y_{T_i}N$. Assuming the no-arbitrage condition in a complete market, the value of one coupon C_i at time zero is obtained as the expectation under the risk neutral probability measure Q of the discounted payoff:

$$C_i = \mathbb{E}_Q\left[e^{-\int_0^{T_i} r_s ds} B(T_i, T_i^p)\, y_{T_i} N\right] \tag{1}$$

The total value at time zero of a N-nominal bond with m coupons with value at time zero $(C_i)_{i=1..m}$, with payment dates $(T_i^p)_{i=1..m}$, providing that the nominal N is paid at the end date T_m^p, is given by:

$$CMS\_Bond = \sum_{i=1}^{m} C_i + N * B(0, T_m^p) \tag{2}$$

In an interest rate CMS receiver swap, the fixed rate is received and the Constant Maturity Swap rate is paid. The different payment dates are also noted $T_1^p, ...T_m^p$. The fixed leg valuation is easy. Its total value, denoted by V_F, is equal to the sum of all the discounted cash flows equal to the fixed rate R_{fixed}:

$$V_F = \sum_{i=1}^{m} R_{fixed} B(0, T_i^p)$$

The fixing dates for the swap rates are denoted by $T_1, ...T_m$. The CMS leg can be valuated as the sum of all the different coupons with value at time T_i y_{T_i} and paid at time T_i^p. Its total value, denoted by V_{CMS}, is the sum of individual swap rate coupons:

$$V_{CMS} = \sum_{i=1}^{m} E_Q \left[e^{-\int_0^{T_i} r_s ds} B\left(T_i, T_i^p\right) y_{T_i} \right] \tag{3}$$

The price of the CMS swap is the difference of price between the two legs: $V_F - V_{CMS}$ for a receiver CMS swap and the opposite for a payer CMS swap. As a consequence, the rate R_{CMS_swap}, called the CMS swap rate, is the one which makes the value of the two legs equal:

$$R_{CMS\_swap} = \frac{V_{CMS}}{\sum_{i=1}^{m} B\left(0, T_i^p\right)} \tag{4}$$

The term of the denominator is classically called the sensitivity of the swap. The CMS swap rate is consequently the value of the CMS leg over the sensitivity of the swap.

As a conclusion of this subsection, CMS swap or CMS bonds are valued exactly with the same procedure. One needs to determine the exact value of a coupon paying the CMS rate. To calculate explicitly these quantities, we need to specify our interest rate model.

3 Calculating the convexity adjustment

In this section, we explain how to price the convexity adjustment with an approximated formula based on a Wiener Chaos expansion. Indeed, techniques based on perturbation theory or Kramers Moyal expansion could have also been used. Moreover, a recursive use of the Ito lemma gives exactly the same results. However, the framework given by Wiener Chaos expansion is much more powerfull and leads to a straightforward calculation instead of very tedious ones.

3.1 Pricing framework

We assume that default-free zero coupon bonds are modelled by a lognormal k-multi-factor model, with a k-dimensional deterministic volatility vector denoted by $V\left(t,T\right) = \left(v_1\left(t,T\right), ..., v_k\left(t,T\right)\right)'$ verifying the Novikov condition $\forall T < \tau, \quad e^{\frac{1}{2}\int_0^{\tau} \|V(s,T)\|^2 ds} < +\infty$. This enables us to use probability measure change since this condition is sufficient for the Girsanov theorem. The default-free $T-$maturity zero coupon bond price at time t is denoted by $B\left(t,T\right)$ and it is defined as the unique strong solution of the stochastic differential equation given under the risk neutral probability Q by:

$$\frac{dB\left(t,T\right)}{B\left(t,T\right)} = r_t dt + \left\langle V\left(t,T\right), dW_t \right\rangle \tag{5}$$

with $\langle V(t,T), dW_t\rangle = \sum_k v_k(t,T)\, dW_t^k$. The initial condition expresses that at maturity, the zero coupon bound is equal to the unity coupon $B(T,T) = 1$. Using traditional results (El-Karoui et al(1995)), we can define the forward neutral probability at time t, Q^t either by means of its Radon Nykodym derivatives with respect to the risk neutral probability measure or by the fact that $\widetilde{dW_s} = dW_s - V(s,t)\, ds$ is a standard Brownian motion under Q^t. We get that under this new probability measure, the bond price solution of the equation (5) can be written as a normalized Doleans martingale times the value of the forward zero coupon bond at time zero:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} e^{\int_0^t \left\langle V(s,T)-V(s,t), \widetilde{dW_s}\right\rangle - \frac{1}{2}\int_0^t \left(\|V(s,T)-V(s,t)\|^2\right) ds} \tag{6}$$

To price a CMS swap/bond, we need to determine the value of one coupon, knowing that the total value of the swap/bond is the sum of the individual swap coupons. The core of the pricing problem is to determine the value at time zero, Π_0, of a contingent claim that at a payment time T, gives the swap rate y_T fixed at time T, of a vanilla interest rate swap. The underlying interest rate swap has n equally separated payment dates : $T_1, ...T_n$. As proved for example in Musiela Ruttkowski(1997) page 389 equation (16.4)) the no-arbitrage condition gives a simple expression of the swap rate y_T with respect to the zero coupon bonds $(B(T,T_i))_{i=0..n}$

$$y_T = \frac{B(T,T_0) - B(T,T_n)}{\sum_{i=1}^n B(T,T_i)} \tag{7}$$

We then adopt the following definition of the CMS rate:

Definition 1 *CMS rate is the expected value under the forward risk neutral probability measure at the payment time T of the swap rate y_T*

$$CMS\_Rate = \mathbb{E}_{Q_T}(y_T) \tag{8}$$

When payment time T^p is different from fixing time T^f, the above formula is modified in $CMS\_Rate = \mathbb{E}_{Q_{T^p}}\left(y_{T_f}\right)$

The guiding idea of the paper is to obtain an approximation formula for the expression above, by means of Wiener Chaos expansion. Let us introduce some notations. We call B_{T_i} the forward zero coupon:

$$B_{T_i} = \frac{B(0,T_i)}{B(0,T)}$$

Let the forward volatility $V_s^{(T,T_i)}$ be the volatility of a $T-$forward zero coupon maturing at time T_i:

$$V_s^{(T,T_i)} = V(s,T_i) - V(s,T)$$

let $C(T_i, T_j)$ denote the (symmetric) correlation term between the return of the zero coupon bonds (mathematically between the logarithm of zero coupon bonds)

$$C(T_i, T_j) = \int_0^T \left\langle V_s^{(T,T_i)}, V_s^{(T,T_j)} \right\rangle ds$$

and K the sensitivity of the forward swap defined as the sum of the forward zero coupon bonds $K = \sum_{i=1}^n B_{T_i}$.

Definition 2 *Convexity adjustment CA is the difference between the CMS rate and the value today of the forward swap rate:*

$$CA = CMS\ rate - y^{forward} \tag{9}$$

The value today of the forward swap rate is given by the equation (7) with the time considered being zero leading to $y^{forward} = \frac{B_{T_0} - B_{T_n}}{K}$.

3.2 Closed formulae

The paper's result is the following approximation theorem. By means of approximations based on Wiener chaos, we can get a closed formula for the CMS rate.

Theorem 1 *Under the above assumptions, the convexity adjustment denoted CA can be expressed as a sum of different correlation terms, plus an error term expressed with Landau notation as an $O\left(\|V_s(.,.)\|^4\right)$:*

$$CA = \begin{pmatrix} \frac{\sum_{i=1}^n B_{T_i}\left(B_{T_n} C(T_i,T_n) - B_{T_0} C(T_i,T_0)\right)}{K^2} \\ +y^{forward} \frac{\sum_{i,j=1}^n B_{T_i} B_{T_j} C(T_i,T_j)}{K^2} \end{pmatrix} + O\left(\|V_s(.,.)\|^4\right) \tag{10}$$

Proof: see section 6.2 page 17.□

This theorem shows us that the convexity adjustment on a swap rate is a simple function of correlation terms. Interestingly, it is a linear function of the forward swap rate $y^{forward}$. The terms $B_{T_i} B_{T_n} C(T_i, T_n)$ respectively $B_{T_i} B_{T_0} C(T_i, T_0)$ can be interpreted as the convexity adjustment between the zero coupon bonds $B(T, T_i)$ and $B(T, T_n)$ respectively $B(T, T_i)$ and $B(T, T_0)$ as the following proof states it:

Proposition 1 *The convexity adjustment CA between two zero coupons bonds can be expressed as a simple expression of the correlation term*

$$\begin{aligned} CA &= \mathbb{E}^{Q_T}\left[B(T,T_i)\, B(T,T_j)\right] - \mathbb{E}^{Q_T}\left[B(T,T_i)\right] \mathbb{E}^{Q_T}\left[B(T,T_i)\right] \\ &= B_{T_i} B_{T_j} C(T_i, T_j) + O\left(\|V_s(.,.)\|^4\right) \end{aligned} \tag{11}$$

Proof: Plugging in the expression of the zero coupon bond (6), the convexity adjustment can be expressed as the value at time zero of the forward zero coupons $B_{T_i}B_{T_j}$ times an expectation:

$$CA = B_{T_i}B_{T_j}\mathbb{E}^{Q_T}\left[e^{\int_0^t \left\langle V_s^{(T,T_i)}+V_s^{\left(T,T_j\right)},\widetilde{dW_s}\right\rangle-\frac{1}{2}\int_0^t\left(\left\|V_s^{(T,T_i)}\right\|^2+\left\|V_s^{\left(T,T_j\right)}\right\|\right)ds}-1\right]$$

Using the fact that $e^{\int_0^t\left\langle f(s),\widetilde{dW_s}\right\rangle-\frac{1}{2}\int_0^t\left(\|f(s)\|^2\right)ds}$ is a martingale for any deterministic function $f(.)$, this expression simplifies to $B_{T_i}B_{T_j}\left(e^{\int_0^t\left\langle V_s^{(T,T_i)},V_s^{(T,T_i)}\right\rangle ds}-1\right)$, which leads to the result (11) when taking a Taylor expansion up to the first order.□

Corollary 1 *When the underlying CMS swap is a spot CMS swap:* $T=T_0$ *and the formula simplifies to*

$$CA = \left(\begin{array}{c}\frac{B_{T_n}\sum_{i=1}^n B_{T_i}C(T_i,T_n)}{K^2}\\+y^{forward}\frac{\sum_{i,j=1}^n B_{T_i}B_{T_j}C(T_i,T_j)}{K^2}\end{array}\right) \quad (12)$$

Proof: When the CMS swap is a spot CMS swap, the correlation term $C(T_0,T_i)$ (convexity term due to the fact that we have a forward swap) becomes zero.□

In this latter case, equation (12), the convexity adjustment is always positive. This result can be easily derived within an elementary term structure model (since we notice that the rate of a forward bond should always be above the forward rate). Put another way, for this CMS, it is pure convexity.

The previous results are approximation formulae. Specifying the error term as the difference between the intractable expression of the convexity adjustment and the closed formula obtained by Weiner Chaos, we can stipulate an upper boundary for the error term. Indeed, the use of Wiener Chaos expansion provides that the error term is dominated by the following quantity $O_3 = O\left(\left(\int_{s_1=0}^T\int_{s_2=0}^T\int_{s_3=0}^T\left\|V_{s_1}^{(T,T_i)}\right\|^2...\left\|V_{s_3}^{(T,T_i)}\right\|^2 ds_1...ds_3\right)^{1/2}\right)$. This indicates that our approximation is all the more efficient than the volatility is small.

3.3 Extension

It turns out that some CMS rate are with a delayed adjustment. The case is more complicated to handle. However, the same methodology gives a closed formula for the price.

Theorem 2 *In the case of a payment date T^p different from the fixing time T, the above expression gets additional terms due to delayed adjustment. The convexity adjustment is then given by:*

$$CA = \left(\begin{array}{c} \frac{\sum_{i=1}^{n} B_{T_i}\left(B_{T_n} C(T_i,T_n) - B_{T_0} C(T_i,T_0)\right)}{K^2} \\ +y^{forward}\left(\frac{\sum_{i,j=1}^{n} B_{T_i} B_{T_j} (C(T_i,T_j) - C(T_i,T^p))}{K^2}\right) \end{array} \right) \quad (13)$$

Proof: The proof goes along the same lines as the one of theorem (1) and can be done using thee same techniques. □

Corollary 2 *The convexity adjustment can also be expressed as:*

$$CA = \frac{\sum_{i,j=1}^{n} B_{T_i} B_{T_j} \left(\begin{array}{c} B_{T_n} \left(C\left(T_i, T_n\right) - C\left(T_i, T_j\right) + C\left(T_i, T^p\right)\right) \\ -B_{T_0} \left(C\left(T_i, T_0\right) - C\left(T_i, T_j\right) + C\left(T_i, T^p\right)\right) \end{array} \right)}{K^3} \quad (14)$$

The interpretation is simple. This formula expresses the convexity adjustment as the difference of correlation terms. Since these terms are small, this suggests already that the convexity adjustment is small. This a posteriori justifies our approached method where we cut the Wiener Chaos expansion after the second order. Indeed, the theoretical justification of the limitation of the expansion until the second order can be found as well in the theorem of Pawula which states that a positive transition probability, the Kramers-Moyal expansion (similar to the Wiener Chaos one) may be stopped either after the first term or after the second term. If it does not stop after the second term, it must contain an infinite number of terms.

For the interpretation of this convexity adjustment, we assume that the correlation term $C\left(T_i, T_j\right)$ is an increasing function of both T_i and T. Let us assume that the payment date T^p is prior to the different payment dates of the underlying swap $(T_i)_{i=1..n}$, i.e., $T_i > T^p$ for every i. Consequently, the first term in the RHS of equation (13) $y^{forward} S_1$, of the same sign as $\sum_{i=1}^{n} B_{T_i} B_{T_j} \left(C\left(T_i, T_j\right) - C\left(T_i, T^p\right)\right)$ is positive. The other term is closely connected to the sign of

$$\sum_{i=1}^{n} B_{T_i} \left(B_{T_n} \left(C\left(T_i, T_n\right) - C\left(T^p, T_n\right)\right) - B_{T_0} \left(C\left(T_i, T_0\right) - C\left(T^p, T_0\right)\right)\right)$$

This leads to think that this expression, expressed as a difference, should be relatively small and in many cases, smaller than the first correction term. In the case it is non positive, it should be slightly negative. This result is of great significance since it states that under non-classical conditions, the expected swap rate can be lower than its corresponding forward swap rate, mainly due to a negative delayed adjustment.

4 Application and results

In this section, we apply the formula to different types of stochastic interest rate model.

4.1 Application to different models

In this section, we apply our closed formula to various one-factor interest rates model. Therefore, for all of them, the number of factors k is one.

4.1.1 Ho and Lee model

Among the early one-factor interest rate term structure model, the Ho and Lee (1986) model was originally in the form of a binomial tree of bond prices. After the Heath Jarrow Morton formalism, this model has been rewritten in the form of a diffusion of the zero coupons bonds:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma (T-t) dW_t$$

It has been observed that the volatility of zero coupons bonds was decreasing with time. This model assume a linear decrease. The forward volatility as well as the correlation have consequently simple form:

$$V_s^{(T,T_i)} = \sigma (T_i - T)$$

and $C(T_i, T_j) = \sigma^2 (T_i - T)(T_j - T)\dot{T}$. The convexity adjustment formula (13) can than be expressed as a function of forward zero coupon and the volatility:

$$convexity = \begin{pmatrix} \sigma^2 \frac{\left(\sum_{i=1}^n B_{T_i} T(T_i - T^p)\right)\left(B_{T_n}(T_n - T) - B_{T_0}(T_0 - T)\right)}{K^2} \\ +y^{forward} \sigma^2 \frac{\sum_{i,j=1}^n B_{T_i} B_{T_j} (T_i - T)(T_j - T^p) T}{K^2} \end{pmatrix}$$

4.1.2 Amin and Jarrow model

The purpose of the Amin and Jarrow (1992) model is to take into account a phenomenon called the volatility hump. Basically, the volatility of zero coupons bonds is first increasing and then decreasing. Amin and Jarrow offered to model the volatility as a second order polynomial given by $\sigma_0 (T-t) + \sigma_1 \frac{(T-t)^2}{2}$. This leads to the following expression for the zero coupons bonds diffusion

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \left(\sigma_0 (T-t) + \sigma_1 \frac{(T-t)^2}{2}\right) dW_t$$

The forward volatility is expressed as a second order polynomial expression of the different maturities $V_s^{(T,T_i)} = \left(\sigma_0 (T_i - T) + \sigma_1 \frac{\left[(T_i-t)^2-(T-t)^2\right]}{2}\right)$ as well as for the correlation term, which is more complicated and is expressed in this particular case as a sun of four terms:

$$C(T_i, T_j) = A_1 + A_2 + A_3 + A_4$$

with

$$\begin{array}{l} A_1 = \sigma_0^2 (T_i - T)(T_i - T) T \\ A_2 = \sigma_0 \sigma_1 (T_i - T) \frac{1}{2} \left[\frac{T_j^3 - (T_j - T)^3}{3} - \frac{T^3}{3} \right] \\ A_3 = \sigma_0 \sigma_1 (T_j - T) \frac{1}{2} \left[\frac{T_i^3 - (T_i - T)^3}{3} - \frac{T^3}{3} \right] \\ A_4 = \sigma_1^2 (T_i - T)(T_i - T) \left[\frac{1}{4} \left(T T_i T_j + \frac{1}{3} T^3 \right) \right] \end{array}$$

The convexity is then calculated thanks to the convexity adjustment formula (13).

4.1.3 Hull and White model

This model represents a significant breakthrough compared to the Ho&Lee model. It is a one factor model, extendable to a two factors or more version, that enables both to incorporate deterministically mean-reverting features and to allow perfect matching of an arbitrary yield curve. It has become very popular among practitioners since there exits closed forms for vanilla interest rates derivatives like cap/floor and swaption (on factor version). This implies a quick calibration. The form with the time-dependent volatility has been advocated to be unstable and is consequently not used in practice. We will give here the convexity adjustment for the classic Hull and White (1990) model with a constant volatility σ and constant mean reverting parameter λ. In this model, in his formulation on zero coupons, zero coupons bonds follow a diffusion given by

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma \frac{1 - e^{-\lambda(T-t)}}{\lambda} dW_t.$$

The volatility structure is realistic since it is decreasing with time. It does not allow for the hump which can be seen as the main drawback of this model. In this case,

$$V(s,t) = \sigma \frac{1 - e^{-\lambda(T-t)}}{\lambda}$$

and the forward volatility is given by $V_s^{(T,T_i)} = \sigma \frac{e^{-\lambda T} - e^{-\lambda T_i}}{\lambda} e^{\lambda s}$ where as the correlation term is becoming

$$C(T_i, T_j) = \sigma^2 \frac{1 - e^{-\lambda(T_j - T)}}{\lambda} \frac{1 - e^{-\lambda(T_i - T)}}{\lambda} \frac{1 - e^{-2\lambda T}}{2\lambda}$$

It is worth noticing that this model assume a lower correlation between the different rates than the Ho&Lee model. We get the following convexity adjustment formula $convexity = HW_1 + HW_2$

$$
\begin{aligned}
HW_1 &= \sigma^2 y^{forward} \frac{\sum_{i,j=1}^{n} B_{T_i} B_{T_j} \frac{1-e^{-2\lambda T}}{2\lambda} \frac{1-e^{-\lambda(T_i-T)}}{\lambda} \left(\frac{e^{-\lambda(T^p-T)} - e^{-\lambda\left(T_j-T\right)}}{\lambda} \right)}{K^2} \\
HW_2 &= \sigma^2 \frac{\sum_{i=1}^{n} B_{T_i} \frac{1-e^{-2\lambda T}}{2\lambda} \left(\frac{e^{-\lambda(T^p-T)} - e^{-\lambda(T_i-T)}}{\lambda} \right) \left(B_{T_n} \frac{1-e^{-\lambda(T_n-T)}}{\lambda} - B_{T_0} \frac{1-e^{-\lambda(T_0-T)}}{\lambda} \right)}{K^2}
\end{aligned}
$$

or for the simplified version $T = T_0 = T^p$

$$
\begin{aligned}
HW_1 &= \sigma^2 \frac{\sum_{i=1}^{n} B_{T_i} \frac{1-e^{-2\lambda T}}{2\lambda} \frac{1-e^{-\lambda(T_i-T)}}{\lambda}}{K^2} \sum_{j=1}^{n} B_{T_j} \left(\frac{1 - e^{-\lambda(T_j-T)}}{\lambda} \right) y^{forward} \\
HW_2 &= \sigma^2 \frac{\sum_{i=1}^{n} B_{T_i} \frac{1-e^{-2\lambda T}}{2\lambda} \left(\frac{1-e^{-\lambda(T_i-T)}}{\lambda} \right)}{K^2} \left(B_{T_n} \frac{1 - e^{-\lambda(T_n-T)}}{\lambda} \right)
\end{aligned}
$$

4.1.4 Mercurio and Moraleda model

Last but not least, we examine the case of the Mercurio and Moraleda (1996) model. This model has been introduced like the Amin and Jarrow model to take account of the volatility hump. Mercurio and Moraleda (1996) suggested to use a combination of Ho and Lee and Hull and White volatility form to get another volatility in which the hump would be modelled more realistically with still analytical tractability. This leads to the following diffusion for the zero coupons bonds:

$$
\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} + \gamma \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda^2} - \frac{(T-t)\, e^{-\lambda(T-t)}}{\lambda} \right) \right) dW_t
$$

In this particular case, the volatility structure takes the following form: $V_s^{(T,T_i)} = g(s,T_i) + f(s,T_i)$

$$
\begin{aligned}
g(s,T_i) &= \sigma \frac{e^{-\lambda T} - e^{-\lambda T_i}}{\lambda} e^{\lambda s} \\
f(s,T_i) &= \gamma\sigma \left(\frac{(T_i-s)e^{-\lambda(T_i-s)} - (T-s)e^{-\lambda(T-s)}}{\lambda} + e^{\lambda s} \frac{e^{-\lambda T} - e^{-\lambda T_i}}{\lambda^2} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
C(T_i,T_j) &= M_{21} + M_{22} + M_{23} + M_{24} \\
M_{21} &= \sigma^2 \frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda} \frac{1-e^{-\lambda(T_i-T)}}{\lambda} \frac{1-e^{-2\lambda T}}{2\lambda} \\
M_{22} &= \int_0^T f(s,T_i)\, f(s,T_j)\, ds \\
M_{23} &= \int_0^T g(s,T_i)\, f(s,T_j)\, ds \\
M_{24} &= \int_0^T g(s,T_j)\, f(s,T_i)\, ds
\end{aligned}
$$

or after simplification

$$\begin{aligned} M_{22} &= \psi\left(T_i,T_j\right) \\ M_{23} &= \alpha\left(i\right)\beta\left(j\right) \\ M_{24} &= \alpha\left(j\right)\beta\left(i\right) \end{aligned}$$

$$\alpha\left(i\right) = \gamma\sigma^2 \frac{1-e^{-\lambda\left(T_i-T\right)}}{\lambda}$$

$$\beta\left(j\right) = \left(\begin{array}{c} \left(\frac{T_j e^{-\lambda\left(T_j-T\right)}-T}{\lambda} \frac{1-e^{-2\lambda T}}{2\lambda} + \frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda} \frac{2\lambda T-1+e^{-2\lambda T}}{4\lambda^2} \right) \\ + \frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda^2} \frac{1-e^{-2\lambda T}}{2\lambda} \end{array} \right)$$

$$\psi\left(T_i,T_j\right) = \left(\gamma\sigma\right)^2 \left(\begin{array}{c} \left(\frac{1-e^{-\lambda\left(T_i-T\right)}}{\lambda}\right) \left(\frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda}\right) \left(\frac{2\lambda^2T^2-2T\lambda+1-e^{-2T\lambda}}{4\lambda^3}\right) \\ + \left(\frac{T_i e^{-\lambda\left(T_i-T\right)}-T}{\lambda} + \frac{1-e^{-\lambda\left(T_i-T\right)}}{\lambda^2}\right) \left(\frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda}\right) \frac{2T\lambda-1+e^{-2T\lambda}}{4\lambda^2} \\ + \left(\frac{T_j e^{-\lambda\left(T_j-T\right)}-T}{\lambda} + \frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda^2}\right) \left(\frac{1-e^{-\lambda\left(T_i-T\right)}}{\lambda}\right) \frac{2T\lambda-1+e^{-2T\lambda}}{4\lambda^2} \\ + \left(\begin{array}{c} \frac{T_i e^{-\lambda\left(T_i-T\right)}-T}{\lambda} \\ + \frac{1-e^{-\lambda\left(T_i-T\right)}}{\lambda^2} \end{array} \right) \left(\begin{array}{c} \frac{T_j e^{-\lambda\left(T_j-T\right)}-T}{\lambda} \\ + \frac{1-e^{-\lambda\left(T_j-T\right)}}{\lambda^2} \end{array} \right) \frac{1-e^{-2T\lambda}}{2\lambda} \end{array} \right)$$

The convexity is then calculated thanks to the convexity adjustment formula (13)

4.2 Results for a standard contract

In this section, we give some results with a Ho and Lee model, a one factor Hull and White model, and a Mercurio and Moraleda model. We compare them to the results we get from a Quasi Monte Carlo simulation with 10,000 random draws. We got that the difference between our formula and the Quasi Monte Carlo simulation was negligeable. These results are summarized in the four tables given in the appendix section: table 1, 2, 3 and 4. Interestingly, convexity adjustment are different depending on the model but very closed one to another.

5 Conclusion

In this paper, we have seen that Wiener Chaos theory provides closed formulae which are very good approximations of the correct result. The interesting point is that this methodology is quite general and could also be applied for many other products where the payoff function is a non linear function of lognormal variables.

Indeed, there are many extensions to this paper. One is to extend to other convexity adjustment our methodology: convexity adjustment of futures contracts to forwards one. A second development, quite promising, is to apply Wiener chaos technique to other option pricing problem.

6 Annex

6.1 Introduction to Wiener Chaos

6.1.1 Intuition

Introduced in finance by Lacoste (1996) (in an paper about transaction costs) and Brace and Musiela (1995), Wiener Chaos expansion could be intuitively thought of the generalization of Taylor's expansion to stochastic processes with some martingale considerations. This representation of stochastic processes initially proved for the Brownian motion by Wiener (1938) and later for Levy process (see Ito 1956) has been recently refocused, motivated by the contemporary development of the Malliavin calculus theory and its application not only to probability theory but also to mechanics, economics and finance (1995).

More precisely, we present in this section the basic properties of the chaotic representation for a given fundamental martingale. Let M be a square-integrable martingale according to an appropriate filtration called F_t with deterministic Doob Meyer brackets $\langle M \rangle_t$ (defined through the requirement that $(M_t^2 - \langle M \rangle_t)$ be a martingale). The latter property is vital for obtaining the chaotic orthogonal representation of the space $\mathcal{L}^2(\mathcal{F}_\infty)$. Let

$$C_n = \{(s_1, ..., s_n) \in \mathbb{R}^n, 0 < s_1 < ... < s_n < t\}$$

be the set of strictly increasingly-ordered n-uplets. Let $(\Phi_n)_{n \in \mathcal{N}}$ be the morphisms from $\mathcal{L}^2(C_n)$ to $\mathcal{L}^2(\mathcal{F}_\infty)$

$$\begin{aligned} \Phi_n(f) &: \mathcal{L}^2(C_n) \to \mathcal{L}^2(\mathcal{F}_\infty) \\ \Phi_n(f) &= \int_0^t ... \int_0^{s_{n-1}} f(s_1, ..., s_n)\, dM_{sn}\, dM_{s_1} \end{aligned}$$

The interesting property of the series of the images of $\mathcal{L}^2(C_n)$ by the morphisms $(\Phi_n)_{n \in \mathcal{N}}$ is the orthogonal decomposition of the space $\mathcal{L}^2(\mathcal{F}_\infty)$.

$$\mathcal{L}^2(\mathcal{F}_\infty) = \overset{\perp}{\underset{n}{\oplus}} \Phi_n\left(\mathcal{L}^2(C_n)\right)$$

This fundamental decomposition of the space $\mathcal{L}^2(\mathcal{F}_\infty)$ into sub-spaces called M-chaos subspaces leads to the interesting representation of any function F of $\mathcal{L}^2(\mathcal{F}_\infty)$ into a series of terms resulting from the orthogonal projection of the function F on the series of M-chaos subspaces.

$$F = \sum_n \Phi_n(f) = \sum_n \int_{C_n} f_n(s_1, ..., s_n)\, dM_{s_n} dM_{s_1}$$

where $f_n \in L^2(C_n)$. Deriving the Wiener Chaos expansion of a function f element of $L^2(F_\infty)$ is very simple as the following theorem proves it:

6.1.2 Theorem and proposition

Theorem 3 *Decomposition in Wiener Chaos*
Let $D^n F$ represent the nth derivative of function F according to its second variable. The M-chaos decomposition of the process $(F(t, M_t))_{t\geq 0}$ gives, for all $t \geq 0$,

$$F(t, M_t) = \mathbb{E}[F(t, M_t)] + \sum_{n=1}^{\infty} \mathbb{E}[D^n F(t, M_t)] \int_{C_n} dM_{s_n} ... dM_{s_1}$$

Proof : See Lacoste (1996) Theorem 3.1 p 201.

The following two propositions refer to important facts about Wiener Chaos, heavily used in the rest of the paper.

Proposition 2 *Orthogonality of the different chaos*
The fundamental properties used are the orthogonality of the different chaos. Let $f_n \in L^2(C_n)$ and $f_m \in L^2(C_m)$ and let $(M)_{t\in\mathbf{R}^+}$ be a martingale process defined as in the previous section

$$\begin{aligned} & \mathbb{E}\left[\int_{C_n} f_n(s_1, ..., s_n)\, dM_{s_n} dM_{s_1} \int_{C_m} f_m(s_1, ..., s_m)\, dM_{s_m} dM_{s_1}\right] \\ = & \ \delta_{n,m} \int_{C_n} f_n(s_1, ..., s_n)\, f_m(s_1, ..., s_m)\, ds_1 ... ds_n \end{aligned}$$

with $\delta_{n,m}$ the Kronecker delta.

$$\begin{aligned} \delta_{n,m} &= 1 \qquad if \quad n = m \\ &= 0 \qquad otherwise \end{aligned}$$

The other result we used is the decomposition of a geometric Brownian motion (or a Doleans martingale).

Proposition 3 *Wiener Chaos decomposition of a geometric multidimensional Brownian motion*
The geometric multidimensional Brownian motion denoted by A_{T_k} can be expanded as the Hilbertian sum of orthogonal terms called Wiener Chaos of order i, denoted by I_i:

$$A_{T_k} = e^{\int_0^T \left\langle V_s^{(T,T_k)}, d\widetilde{W}_s \right\rangle - \frac{1}{2} \int_0^T \left\| V_s^{(T,T_k)} \right\|^2 ds} \tag{15}$$

$$= \sum_{i=0}^{\infty} I_i(V, T, T_k) \tag{16}$$

with

$$I_0(V, T, T_k) = 1$$

$$I_{i,i>0}(V, T, T_k) = \frac{\int_{s_1=0}^{T} ... \int_{s_i=0}^{T} \left\langle V(s_1, T, T_k), d\widetilde{W}_{s_1} \right\rangle ... \left\langle V(s_i, T, T_k), d\widetilde{W}_{s_i} \right\rangle}{i!}$$

Proof: see either (1997) exercise p1.2.d. page 19 or(1996) page 201 Theorem 3.1.□

6.2 Proof of the theorem

This appendix section gives the proof of therorem 1.

6.2.1 Finding the convexity adjustment

We remind some notations for the proof. We denote by K the sensitivity of the forward swap, $K = \sum_{i=1}^{n} B_{T_i}$. We write down as well that a zero coupon bond can be written as a normalized Doleans martingale times its value at time zero, leading to the following notation: $B_T^{(T,T_i)} = B_{T_i} A_{T_i}$ with $A_{T_i} = e^{\int_0^T \left\langle V_s^{(T,T_i)}, d\widetilde{W}_s \right\rangle - \frac{1}{2}\int_0^T \left\| V_s^{(T,T_i)} \right\|^2 ds}$ and $B_{T_i} = \frac{B(0,T_i)}{B(0,T)}$. We need to calculate the following quantity:

$$\Pi_0 = B\left(0,T\right) \mathbb{E}_{Q_T}\left(\frac{B_{T_0}A_{T_0} - B_{T_n}A_{T_n}}{\sum_{i=1}^{n} B_{T_i}A_{T_i}}\right)$$

Using the linearity of the expectation operator, we get the above expression can be separated into two terms

$$\frac{\Pi_0}{B\left(0,T\right)} = B_{T_0}\mathbb{E}_{Q_T}\left(\frac{A_{T_0}}{\sum_{i=1}^{n} B_{T_i}A_{T_i}}\right) - B_{T_n}\mathbb{E}_{Q_T}\left(\frac{A_{T_n}}{\sum_{i=1}^{n} B_{T_i}A_{T_i}}\right)$$

Using the technical lemma (by means of Wiener chaos expansion) proved below, we get that the two expectations can be approached by the following expression

$$\mathbb{E}_{Q_T}\left(\frac{A_{T_j}}{\sum_{i=1}^{n} B_{T_i}A_{T_i}}\right) = \frac{1}{K} - \frac{\sum_{i=1}^{n} B_{T_i}C\left(T_j,T_i\right)}{K^2} + \frac{\sum_{i,k=1}^{n} B_{T_i}B_{T_k}C\left(T_i,T_k\right)}{K^3} + O_3$$

with the signification of O_3 explained in the technical lemma. Rearranging the term, we get that the price of the expected swap rate could be written as a simple expression

$$\begin{aligned} & \frac{\Pi_0}{B\left(0,T\right)} \\ = & \frac{B_{T_0} - B_{T_n}}{K} + \frac{\sum_{i=1}^{n} B_{T_0}B_{T_i}C\left(T_0,T_i\right) - B_{T_n}B_{T_i}C\left(T_n,T_i\right)}{K^2} + \frac{\sum_{i,k=1}^{n} B_{T_i}B_{T_k}C\left(T_i,T_k\right)}{K^2} \end{aligned}$$

which leads to the final result.□

6.2.2 Approximation using Wiener Chaos

In this section, we want to prove the following technical lemma. Using a simplified version of Landau notation, O_3 denotes a negligeable quantity with respect to the $\left\|V_{\cdot}^{(T,T_i)}\right\|_{L^2}^3$, i.e.

$$O_3 = O\left(\left(\int_{s_1=0}^{T}\int_{s_2=0}^{T}\int_{s_3=0}^{T}\left\|V_{s_1}^{(T,T_i)}\right\|^2 ... \left\|V_{s_3}^{(T,T_i)}\right\|^2 ds_1...ds_3\right)^{1/2}\right)$$

Lemma 1 *Using the notation as above the expected value of the non linear stochastic expression* $\frac{A_{T_j}}{\sum_{i=1}^n B_{T_i}A_{T_i}}$ *can be given by a simple function of the correlation terms:* $\mathbb{E}_{Q_T}\left(\frac{A_{T_j}}{\sum_{i=1}^n B_{T_i}A_{T_i}}\right) = \frac{1}{K} - \frac{\sum_{i=1}^n B_{T_i}C(T_j,T_i)}{K^2} + \frac{\sum_{i,k=1}^n B_{T_i}B_{T_k}C(T_i,T_k)}{K^3} + \varepsilon$ *where the error term,* ε*, denotes a negligeable quantity with respect to the* $\left\|V_{\cdot}^{(T,T_i)}\right\|_{L^2}^3$*, i.e.* $\varepsilon = O_3$.

Proof: let us introduce some notations $U_0 = 1$, $U_1 = \frac{\sum_{i=1}^n B_{T_i}I_1(V,T,T_i)}{K}$, $U_2 = \frac{\sum_{i=1}^n B_{T_i}I_2(V,T,T_i)}{S}$. By a Wiener Chaos expansion theorem 3, and result (16), we can expand the term A_{T_i} and we get:

$$\begin{aligned}&\sum_{i=1}^n B_{T_i}A_{T_i}\\ =\;& \sum_{i=1}^n B_{T_i} + \sum_{i=1}^n B_{T_i}I_1(V,T,T_i) + \sum_{i=1}^n B_{T_i}I_2(V,T,T_i) + \varepsilon_1\end{aligned}$$

where the error term ε_1 is a negligeable quantity with respect to the $\|V(T,T_i)\|_{L^2}^3$ ($\varepsilon_1 = O_3$). The simple Taylor expansion $\frac{1}{1+x} = 1 - x + x^2 + o(x^3)$ gives that we can rewrite the denominator of the function in the expectation as now linear terms

$$\begin{aligned}&\frac{1}{\sum_{i=1}^n B_{T_i}A_{T_i}}\\ =\;& \frac{1}{K} - \frac{\sum_{i=1}^n B_{T_i}I_1(V,T,T_i)}{K^2} - \frac{\sum_{i=1}^n B_{T_i}I_2(V,T,T_i)}{K^2} + \frac{1}{K}\left(\frac{\sum_{i=1}^n B_{T_i}I_1(V,T,T_i)}{\sum_{i=1}^n B_{T_i}}\right)^2 + \varepsilon_2\end{aligned} \tag{17}$$

where the error term ε_2 is a negligeable quantity with respect to the $\left\|V_{\cdot}^{(T,T_i)}\right\|_{L^2}^3$ ($\varepsilon_2 = O_3$). In the expectation to calculate $\mathbb{E}_{Q_T}\left(\frac{B_{T_j}A_{T_j}}{\sum_{i=1}^n B_{T_i}A_{T_i}}\right)$, the term A_{T_j} can be seen as a change of probability measure. We denote by Q^{T,T_j} the new probability measure defined by its Radon Nikodym derivative with respect to the forward neutral probability measure Q_T, and W_s^{T,T_j} the Q^{T,T_j} standard Brownian motion:

$$\begin{aligned}\frac{dQ^{T,T_i}}{dQ_T} &= e^{\int_0^T \left\langle V_s^{\left(T,T_j\right)}, d\widetilde{W}_s\right\rangle - \frac{1}{2}\int_0^T \left\|V_s^{\left(T,T_j\right)}\right\|^2 ds}\\ dW_s^{T,T_j} &= d\widetilde{W}_s - V_s^{(T,T_j)}ds\end{aligned}$$

Then the measure change eliminates the numerator term and simplifies the expectation to calculate as only a function of $\frac{1}{\sum_{i=1}^{n} B_{T_i} A_{T_i}}$ in a new probability measure Q^{T,T_j}. By linearity of the expectation operator and using the approximation (17) , we get

$$\begin{aligned}
& \mathbb{E}_{Q^{T,T_j}}\left(\frac{1}{\sum_{i=1}^{n} B_{T_i} A_{T_i}}\right) \\
= & \frac{1}{K} - \mathbb{E}_{Q^{T,T_j}}\left(\frac{\sum_{i=1}^{n} B_{T_i} I_1(V,T,T_i)}{K^2}\right) - \mathbb{E}_{Q^{T,T_j}}\left(\frac{\sum_{i=1}^{n} B_{T_i} I_2(V,T,T_i)}{K^2}\right) \\
& + \frac{1}{K}\mathbb{E}_{Q^{T,T_j}}\left(\left(\frac{\sum_{i=1}^{n} B_{T_i} I_1(V,T,T_i)}{K}\right)^2\right) + \varepsilon_3
\end{aligned}$$

where the error term ε_3 is a negligeable quantity with respect to the $\left\|V_{\cdot}^{(T,T_i)}\right\|_{L^2}^3$ ($\varepsilon_3 = O_3$). One can conclude by successively proving that

$$\begin{aligned}
\mathbb{E}_{Q^{T,T_j}}\left(I_1(V,T,T_i)\right) &= C(T_i,T_j) \\
\mathbb{E}_{Q^{TP,T_j}} I_2(V,T,T_i) &= O_3 \\
\mathbb{E}_{Q^{T,T_j}}\left(\left(\sum_{i=1}^{n} B_{T_i} I_1(V,T,T_i)\right)^2\right) &= \sum_{i,k=1}^{n} B_{T_i} B_{T_k} C(T_i,T_k) + O_3
\end{aligned}$$

□

6.3 Results of the Quasi Monte Carlo simulation

This annex sub-section shows results of a Quasi Monte Carlo simulation for the four different models. The simulation was done using 10,000 draws. The convexity term was calculated on an interest rate curved dated September, 2, 1999. Interestingly, convexity adjustment are different depending on the model but very closed one to another.

Year	forward Swap Rates	CMS Swap	QMC price	convexity adjustement in basis point
0	4.163826	4.163826	4.163826	0
1	4.385075	4.43604	4.436145	5.57
2	4.600037	4.699187	4.699212	9.91
3	4.80722	5.951161	5.951101	14.39
5	5.13929	5.36107	5.36087	22.18
7	5.366385	5.649873	5.649921	28.35
10	5.586253	5.935744	5.935735	34.95

Table 1: Convexity adjustment for Ho and Lee model
Result obtained with $\sigma = 1\%$

Year	forward Swap Rates	CMS Swap	QMC price	convexity adjustement in basis point
0	4.163826	4.163826	4.163826	0
1	4.385075	4.400307	4.400318	1.52
2	4.600037	4.635506	4.635521	3.55
3	4.80722	4.868121	4.868136	6.09
5	5.13929	5.266523	5.266514	12.72
7	5.366385	5.579279	5.579263	21.29
10	5.586253	5.959299	5.959281	37.30

Table 2: Convexity adjustment for Amin and Jarrow model
Results obtained with $\sigma_0 = 0.1\%$ and $\sigma_1 = 0.1\%$

Year	forward Swap Rates	CMS Swap	QMC price	convexity adjustement in basis point
0	4.163826	4.163826	4.163821	0.00
1	4.385075	4.441479	4.441467	5.64
2	4.600037	4.708704	4.708715	10.87
3	4.80722	4.963449	4.963459	15.62
5	5.13929	5.375376	5.375363	23.61
7	5.366385	5.662372	5.662368	29.60
10	5.586253	5.940745	5.940736	35.45

Table 3: Convexity adjustment for Hull and White model
Results obtained with $\sigma = 1.1\% \quad \lambda = 1\%$

Year	forward Swap Rates	CMS Swap	QMC price	convexity adjustement in basis point
0	4.163826	4.440826	4.440826	0.00
1	4.385075	4.440826	4.440812	5.58
2	4.600037	4.707352	4.707347	10.73
3	4.80722	4.961371	4.961368	15.42
5	5.13929	5.371831	5.371820	23.25
7	5.366385	5.657425	5.657414	29.10
10	5.586253	5.933928	5.933938	34.77

Table 4: Convexity adjustment for Mercurio and Moraleda model
Results obtained with $\sigma = 0.9\% \quad \lambda = 1\% \gamma = 0.11\%$

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