

Pricing Electricity Derivatives Under Alternative Stochastic Spot Price Models

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Abstract

I propose several mean-reversion jump-diffusion models to describe spot prices of electricity. I incorporate multiple jumps, regime-switching and stochastic volatility into these models in order to capture the salient features of electricity prices due to the physical characteristics of electricity. Prices of various electricity derivatives are derived under each model using the Fourier transform methods. The implications of modeling assumptions to electricity derivative pricing are also examined.

Keywords: Electricity price modeling; Electricity derivative pricing, Jump diffusion process.

1 Introduction

Modeling the price behaviors of electricity is a very challenging task for researchers and practitioners due to the distinguishing characteristics of electricity. First of all, electricity can not be stored or inventoried economically once generated. Moreover, electricity supply and demand has to be balanced continuously so as to prevent the electric power networks from collapsing. Electricity spot prices are extremely volatile because the supply and demand shocks cannot be smoothed by inventories. As for how volatile the electricity prices can be, the wholesale prices of electricity fluctuated between \$0/MWh and \$7000/MWh in the Midwest of US during the summer of 1998. It is not uncommon to see a 150% implied volatility in traded electricity options. Figure (1) plots the implied volatility of electricity call options in the Midwestern US (Cinergy) across different strike prices at different points in time where the x-axis represents the “moneyness”, i.e. the strike prices divided by the corresponding forward prices. On top of

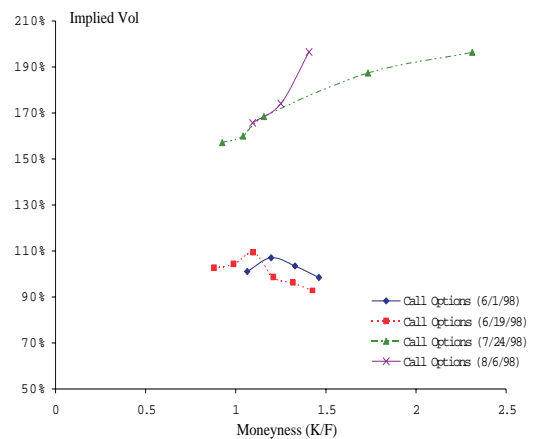


Figure 1: Implied Volatility of Call (Sept.) Options at Cinergy

the tremendous levels of volatility, the highly seasonal patterns of electricity prices also complicate the modeling issues.

The most noticeable features of electricity prices are mean-reversion and the presence of price jumps and spikes. Figure (2) shows the historical on-peak electricity spot prices in Texas (ERCOT) and at the California and Oregon border (COB). Mean-reversion is a common feature in the prices of many other traded commodities. The intuition behind mean-reversion is that when the price of a commodity is high, its supply tends to increase thus putting a downward pressure on the price; when the price is low, the supply of the commodity tends to decrease thus providing an upward lift to the price. As for the jumpy behavior in electricity spot prices, it is mainly attributed to the fact that a typical

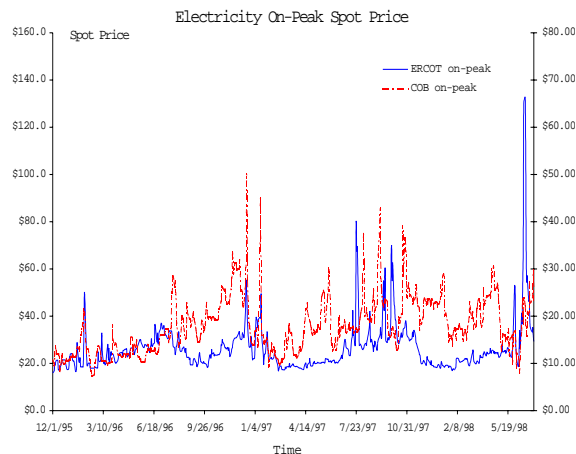


Figure 2: Electricity Historical Spot Prices

regional aggregate supply cost curve for electricity almost always has a kink at certain capacity level and the curve has a steep upward slope beyond that capacity level. A forced outage of a major power plant or a sudden surge in demand will make the regional electricity demand curve to cross the regional supply cost curve at its steep-rise portion thus causing a jump in the electricity price process. When the contingency making the electricity price to jump high is short-term in nature, the high price will quickly fall back down to the normal range as the contingency disappears thus causing a spike. In the summer of 1998, we observed the spot prices of electricity in the Eastern and the Midwestern US skyrocketing from \$50/MWh to \$7000/MWh because of the unexpected unavailability of some major power generation plants as well as the congestion on key transmission lines. Within a couple of days the prices fell back to the \$50/MWh range as the lost generation and transmission capacities were restored. Electricity prices may also exhibit regime-switching, caused by weather patterns and varying precipitation, in markets where the majority of installed electricity supply capacity is hydro power such as in the Nord Pool and the Victoria Pool.

There have been few studies on modeling electricity prices since electricity markets only came into existence a few years ago in US. Kaminski (1997) as well as Barz and Johnson (1998) are two papers on modeling electricity prices. Kaminski (1997) points out the needs of introducing jumps and stochastic volatility in modeling electricity prices. The Monte-Carlo simulation is used for pricing electricity derivatives under the jump-

diffusion price models. Barz and Johnson (1998) suggest the inadequacy of the Geometric Brownian motion and mean-reverting process in modeling electricity spot prices. With the objective of reflecting the key characteristics of electricity prices, they offer a price model which combines a mean-reverting process with a single jump process. However, they do not provide analytic results regarding derivative valuation under their proposed price model.

While some energy commodities, such as crude oil, may be properly modeled as traded securities, the non-storability of electricity makes such an approach inappropriate. Nevertheless, we can always view the spot price of electricity as a state variable or a function of several state variables. All the physical contracts/financial derivatives on electricity are therefore contingent claims on the state variables. In this paper, I examine a broad class of stochastic models which can be used to model price characteristics such as jump, stochastic volatility, as well as stochastic convenience yield. I feel that models with jumps and stochastic volatility are particularly suitable for modeling the electricity price processes.

I specify the electricity price processes as affine jump-diffusion processes¹ which were introduced in Duffie and Kan (1996). Affine jump-diffusion processes are flexible enough to allow me to capture the special characteristics of electricity prices such as mean-reversion, seasonality, and spikes². More importantly, I am able to compute the prices of various electricity derivatives under the assumed underlying affine jump-diffusion price processes by applying the transform analysis developed in Duffie, Pan and Singleton (1998). I consider not only the usual affine jump-diffusion models but also a regime-switching mean-reversion jump-diffusion model. The regime-switching model is used to model the random alternations between “abnormal” and “normal” equilibrium states of supply and demand for electricity³.

The remainder of this paper is organized as follows. In the next section, I propose three alternative electricity price models and compute the transform functions needed for contingent-claim pricing. In Section 3, I present illustrative examples of the models specified

¹For simplicity, I model the electricity price itself as a state variable. The extension of modeling the electricity price as an exponential-affine function of state variables is straightforward.

²“Spikes” refers to upward jumps followed shortly by downward jumps.

³This model is particularly suited for modeling electricity prices in regions where hydro electric power plants are the majority of the installed capacity.

in Section 2 and derive the pricing formulae of several electricity derivatives. The comparisons of the prices of electricity derivatives under different models are shown. Section 4 concludes the paper.

2 Mean-Reverting Jump Diffusion Price Models

Keeping in mind the objective of capturing prominent physical characteristics of electricity prices such as mean-reversion, regime-switching, stochastic volatility, and jumps/spikes, I examine the following three types of mean-reverting jump-diffusion electricity price models.

1. Mean-reverting jump-diffusion price process with deterministic volatility.
2. Mean-reverting jump-diffusion price process with regime-switching.
3. Mean-reverting jump-diffusion price process with stochastic volatility.

I consider two types of jumps in all of the above models. While this analytical approach could handle multiple types of jumps, I feel that, with properly chosen jump intensities, two types of jumps suffice in mimicking the jumps and/or spikes in the electricity price processes. The case of one type of jumps is included as a special case when the intensity of type-2 jump is set to zero.

In addition to the electricity price process under consideration, I also jointly specify another factor process which can be correlated with the underlying electricity price. This additional factor could be the price of the generating fuel such as natural gas, or something else, such as the aggregate physical demand of electricity. A jointly specified price process of the generating fuel is essential for risk management involving cross commodity risks between electricity and the fuel. There is empirical evidence demonstrating a positive correlation between electricity prices and the generating fuel prices in certain geographic regions during certain time periods of a year. In all models the risk free interest rate, r , is assumed to be deterministic.

2.1 Model 1: A mean-reverting deterministic volatility process with two types of jumps

I start with specifying the spot price of electricity as a mean-reverting jump-diffusion process with two types of jumps. Let the factor process X_t in (1) denote $\ln S_t^e$, where S_t^e is the price of the underlying electricity. Y_t is the other factor process which can be used to specify the logarithm of the spot price of a generating fuel, *e.g.* $Y_t = \ln S_t^g$ where S_t^g is the spot price of natural gas. In this formulation, I have type-1 jump representing the upward jumps and type-2 jump representing the downward jumps. By setting the intensity functions of the jump processes in a proper way, we can mimic the spikes in the electricity price process. Suppose the state vector process $(X_t, Y_t)'$ given by (1) is under the true measure⁴ and the risk premia⁵ associated with all state variables are linear functions of state variables. Assuming there exists a risk-neutral probability measure⁶ Q over the state space represented by the state variables⁷, the state vector process has the same form as that of (1) under the risk-neutral measure, but with different coefficients. For the ease of pricing derivatives in Section 3, I choose to directly specify the state vector process under the risk-neutral measure from here on with the assumption that the risk premia associated with all state variables are linear functions of state variables.

Assume that, under regularity conditions, X_t and Y_t are strong solutions to the following stochastic differential equation (SDE) under the risk-neutral measure Q ,

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt + \sum_{i=1}^2 \Delta Z_t^i$$

⁴True measure refers to the probability measure defining the statistical properties of the underlying price process observed in the real world.

⁵Risk premium is a quantity established by the capital markets in equilibrium. It is the amount subtracted from the mean return of a financial asset or a project in order to compensate the owner of the asset or the project for bearing the associated risks.

⁶Risk-neutral measure refers to the normalized Arrow-Debreu state prices over the states of the world. Since the sum or the integral of these positive normalized state prices equals one, they can be interpreted as a probability measure over the states of the world. The risk-neutral measure is, in general, not unique due to incomplete markets.

⁷A risk-neutral measure exists as long as the no-arbitrage conditions hold. Given the existence of a risk-neutral measure, the price of a contingent claim is just the expected value of its discounted payoff under the risk-neutral measure. See Duffie (1996) for more details.

$$+ \begin{pmatrix} \sigma_1(t) & 0 \\ \rho\sigma_2(t) & \sqrt{1-\rho^2}\sigma_2(t) \end{pmatrix} dW_t(1)$$

where $\kappa_1(t)$ and $\kappa_2(t)$ are the mean-reverting coefficients; $\theta_1(t)$ and $\theta_2(t)$ are the long term means; $\sigma_1(t)$ and $\sigma_2(t)$ are instantaneous volatility rates of X and Y ; ρ is the instantaneous correlation coefficient between X and Y which may be a function of time t ; W_t is a \mathcal{F}_t -adapted standard Brownian motion under Q in \mathbb{R}^2 ; Z^j is a compound Poisson process in \mathbb{R}^2 with the Poisson arrival intensity being $\lambda_j(t)$ ($j = 1, 2$). ΔZ^j denotes the random jump size in \mathbb{R}^2 with cumulative distribution function $v^j(z)$ ($j = 1, 2$). Let $\phi_J^j(c_1, c_2, t) \equiv \int_{\mathbb{R}^2} \exp(c \cdot z) dv^j(z)$ be the transform function of the jump size distribution of type- j jumps ($j = 1, 2$).

The transform function Define the generalized transform function as

$$\varphi(u, X_t, Y_t, t, T) \equiv E_t^Q[e^{-r\tau} \exp(u_1 X_T + u_2 Y_T)] \quad (2)$$

for any fixed time T where $\tau = T - t$ and $E_t^Q \equiv E^Q[\cdot | \mathcal{F}_t]$ is the expected value operator under the risk-neutral measure Q conditioning on the information at time t and $u \equiv (u_1, u_2) \in C^2$ (a set of 2-tuples of complex numbers). The transform function φ is well-defined at a given u under technical regularity conditions on $\kappa_i(t)$, $\theta_i(t)$, $\sigma_i(t)$, $\rho_i(t)$, $\lambda_i(t)$, and $\phi_J^i(c_1, c_2, t)$ ($i = 1, 2$).

Under the regularity conditions (see Duffie, Pan and Singleton (1998) or Deng (1999)), $\varphi \cdot e^{-rt}$ is a martingale under the risk-neutral measure Q since it is a \mathcal{F}_t -conditional expectation of a single random variable $\exp(u_1 X_T + u_2 Y_T)$. Therefore the drift term of $\varphi \cdot e^{-rt}$ is zero. Applying Ito's lemma for complex functions, we observe that φ needs to satisfy the following fundamental partial differential equation (PDE)

$$\mathcal{D}f - rf = 0 \quad (3)$$

where

$$\begin{aligned} \mathcal{D}f &\equiv \partial_t f + \mu_X \cdot \partial_X f + \frac{1}{2} \text{tr}(\partial_X^2 f \Sigma \Sigma^T) + \\ &\quad \sum_{j=1}^2 \lambda_j(t) \int_{\mathbb{R}^2} [f(X_t + \Delta Z_t^j, t) - f(X_t, t)] dv_j(z) \end{aligned}$$

The solution to (3) is⁸:

$$\varphi(u, X_t, Y_t, t, T) = \exp(\alpha(t, u) + \beta_1(t, u)X_t + \beta_2(t, u)Y_t)$$

⁸Details are given in Deng (1999).

where

$$\begin{aligned} \beta_1(t, [u_1, u_2]') &= u_1 \exp(-\int_t^T \kappa_1(s) ds) \\ \beta_2(t, [u_1, u_2]') &= u_2 \exp(-\int_t^T \kappa_2(s) ds) \\ \alpha(t, u) &= \int_t^T \left(\sum_{i=1}^2 [\kappa_i(s) \theta_i(s) \beta_i(s, u) \right. \\ &\quad \left. - r + \frac{1}{2} \sigma_i^2(s) \beta_i^2(s, u)] \right. \\ &\quad \left. + \rho(s) \sigma_1(s) \sigma_2(s) \beta_1(s, u) \beta_2(s, u) \right. \\ &\quad \left. + \sum_{j=1}^2 \lambda_j(s) (\phi_J^j(\beta(s, u), s) - 1) \right) ds \end{aligned} \quad (4)$$

2.2 Model 2: A regime-switching mean-reverting process with two types of jumps

To motivate this model, I consider the electricity prices in regions where the majority of power generation capacity is hydro-power. The level of precipitation causes the electricity price levels to alternate between “high” and “low” regimes. Other plausible scenarios for electricity prices to exhibit regime-switching are that the forced outages of power generation plants or unexpected contingencies in transmission networks often result in abnormally high electricity spot prices for a short time period and then a quick price fall-back. In order to capture the phenomena of spot prices switching between “high” and “normal” states, I extend model 1 to a Markov regime-switching model which I describe in detail below.

Let U_t be a continuous-time two-state Markov chain

$$dU_t = 1_{U_t=0} \cdot \delta(U_t) dN_t^{(0)} + 1_{U_t=1} \cdot \delta(U_t) dN_t^{(1)} \quad (5)$$

where 1_A is an indicator function for event A , $N_t^{(i)}$ is a Poisson process with arrival intensity $\lambda^{(i)}$ ($i = 0, 1$) and $\delta(0) = -\delta(1) = 1$. I next define the corresponding compensated continuous-time Markov chain $M(t)$ as

$$dM_t = -\lambda(U_t) \delta(U_t) dt + dU_t \quad (6)$$

The joint specification of electricity and the generating fuel price processes under the risk-neutral measure Q is given by:

$$\begin{aligned} d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1-\rho(t)^2}\sigma_2(t) \end{pmatrix} dW_t \\ &\quad + \sum_{j=1}^2 \Delta Z_t^j + \iota(U_{t-}) dM_t \end{aligned} \quad (7)$$

where W_t is a \mathcal{F}_t -adapted standard Brownian motion under Q in \mathbb{R}^2 ; $\{\iota(i) \equiv (\iota_1(i), \iota_2(i))'; i = 0, 1\}$ denotes the sizes of the random jumps in state variables when regime-switching occurs. Let $\phi_{\iota(i)}(c_1, c_2, t) \equiv \int_{\mathbb{R}^2} \exp(c \cdot z) dv_{\iota(i)}(z)$ be the transform function of the regime-jump size distribution $v_{\iota(i)}$ ($i = 1, 2$). Z^j , ΔZ^j and $\phi_J^j(c_1, c_2, t)$ are similarly defined as those in Model 1. Strong solutions to (7) exist under regularity conditions.

The transform functions $F^i(\bar{x}, t)$ ($i = 0, 1$) are defined as

$$E[e^{-r\tau} \exp(u_1 X_T + u_2 Y_T) | X_t = x, Y_t = y, U_t = i] \quad (8)$$

where $\tau = T - t$ and U_t is the Markov regime state variable. It turns out that $F^i(\bar{x}, t)$ ($i = 0, 1$) are exponential affine functions of the underlying state variables. The coefficients of the transform functions are obtained through solving a system of complex-valued ordinary differential equations⁹.

2.3 Model 3: A mean-reverting stochastic volatility process with two types of jumps

I consider a three-factor affine jump-diffusion process with two types of jumps in this model (9). Let X_t and Y_t denote the logarithm of the spot prices of electricity and a generating fuel, e.g. natural gas, respectively. V_t represents the stochastic volatility factor. There is empirical evidence alluding to the fact that the volatility of electricity price is high when the aggregate load (or, demand) is high and vice versa. Therefore, V_t can be thought as a factor which is proportional to the regional aggregate demand process for electricity. Jumps may appear in both X_t and V_t since weather conditions such as unusual heat waves may cause simultaneous jumps in both the electricity price and the aggregate load. The state vector process $(X_t, V_t, Y_t)'$ is specified by (9). Under proper regularity conditions, there exists a Markov process which is the strong solution to the following SDEs under the risk-neutral measure Q .

$$\begin{aligned} d \begin{pmatrix} X_t \\ V_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_V(t)(\theta_V(t) - V_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt \\ &\quad + \Sigma dW_t + \sum_{i=1}^2 \Delta Z_t^i \end{aligned} \quad (9)$$

⁹Details are given in Deng (1999).

where Σ is given by

$$\Sigma \equiv \begin{pmatrix} \sqrt{V_t} & 0 & 0 \\ \rho_1(t)\sigma_2(t)\sqrt{V_t} & \sqrt{1-\rho_1^2(t)}\sigma_2(t)\sqrt{V_t} & 0 \\ \rho_2(t)\sigma_3(t)\sqrt{V_t} & 0 & \sigma_3(t) \end{pmatrix};$$

W is a \mathcal{F}_t -adapted standard Brownian motion under Q in \mathbb{R}^3 ; Z^j is a compound Poisson process in \mathbb{R}^3 with the Poisson arrival intensity being $\lambda^j(X_t, V_t, Y_t, t)$ ($j = 1, 2$). I model the spiky behavior by assuming that the intensity function of type-1 jumps is only a function of time t , denoted by $\lambda^{(1)}(t)$, and the intensity of type-2 jumps is a function of V_t , i.e. $\lambda^{(2)}(V_t, t) = \lambda_2(t)V_t$. Let $\phi_J^j(c_1, c_2, c_3, t) \equiv \int_{\mathbb{R}^3} \exp(c \cdot z) dv^j(z)$ denote the transform function of the jump-size distribution of type- j jumps, $v^j(z)$, ($j = 1, 2$).

The transform function Following similar arguments to those used in Model 1, we know that the transform function

$$\varphi(u, X_t, V_t, Y_t, t, T) \equiv E_t^Q[e^{-r\tau} \exp(u_1 X_T + u_2 V_T + u_3 Y_T)]$$

is of form

$$\varphi(u, X_t, V_t, Y_t, t, T) = \exp(\alpha(t, u) + \beta(t, u) \cdot \bar{X}_T) \quad (10)$$

where $\tau = T - t$, $E_t^Q[\cdot] \equiv E^Q[\cdot | \mathcal{F}_t]$, $\bar{X}_t \equiv [X_t, V_t, Y_t]'$; $\alpha(u, t)$ and $\beta(t, u) \equiv [\beta_1(u, t), \beta_2(u, t), \beta_3(u, t)]'$ are solutions to a system of ordinary differential equations¹⁰.

3 Electricity Derivative Pricing

Having specified the mean-reverting jump-diffusion price models and demonstrated how to compute the generalized transform functions of the state vector at any given time T , the prices of European-type contingent claims on the underlying electricity under the proposed models can then be obtained through the inversion of the transform functions. Suppose \bar{X}_t is a state vector in \mathbb{R}^n and $u \in \mathbb{C}^n$ (a set of n -tuples of complex numbers) and the generalized transform function is given by

$$\begin{aligned} \varphi(u, \bar{X}_t, t, T) &\equiv E^Q[e^{-r(T-t)} \exp(u \cdot \bar{X}_T) | \mathcal{F}_t] \\ &= \exp[\alpha(t, u) + \beta(t, u) \cdot \bar{X}_t] \end{aligned} \quad (11)$$

Let $G(v, X_t, Y_t, t, T; \bar{a}, \bar{b})$ denote the time- t price of a contingent claim which pays out

$$\exp(\bar{a} \cdot \bar{X}_T), \text{ if } \bar{b} \cdot \bar{X}_T \leq v \text{ is true at time } T$$

¹⁰Details are given in Deng (1999).

where \bar{a}, \bar{b} are vectors in R^n and $v \in R^1$, then we have (see Duffie, Pan and Singleton [6] for a formal proof):

$$\begin{aligned} G(v, \bar{X}_t, t, T; \bar{a}, \bar{b}) &= E^Q[e^{-r\tau} \exp(\bar{a} \cdot \bar{X}_T) \mathbf{1}_{\bar{b} \cdot \bar{X}_T \leq v} | \mathcal{F}_t] \\ &= \frac{\varphi(\bar{a}, \bar{X}_t, t, T)}{2} - M \end{aligned} \quad (12)$$

where

$$M = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\varphi(\bar{a} + iw\bar{b}, \bar{X}_t, t, T)e^{-iwv}]}{w} dw$$

For properly chosen constants v , \bar{a} , and \bar{b} , $G(v, \bar{X}_t, t, T; \bar{a}, \bar{b})$ serves as a building block in pricing contingent claims such as forwards/futures, call/put options, and cross-commodity spread options. To illustrate this point, I take some concrete examples of the models proposed in Section 2 and compute the prices of several commonly traded electricity derivatives. Specifically, model 1a is a special case of model I ($I = 1, 2, 3$). Closed-form solutions of the derivative securities (up to the Fourier inversion) are provided whenever available.

3.1 Illustrative Models

The illustrative models presented here are obtained by setting the model parameters to be constants in the proposed three general models. The jumps appear in the electricity price process and the volatility process (model 3a) only. The jump sizes are distributed as independent exponential random variables in R^n thus having the following transform function:

$$\phi_J^j(\bar{c}, t) \equiv \prod_{k=1}^n \frac{1}{1 - \mu_J^k c_k} \quad (13)$$

The simulated price paths under the three illustrative models are shown in Figure (3) for parameters given in Table (1). The x-axis represents the simulation time horizon in the number of years while the y-axis represents the electricity price level in dollars.

3.1.1 Model 1a

Model 1a is a special case of (1) with all parameters being constants. The jumps are in the logarithm of the electricity spot price, X_t . The size of a type- j jump ($j = 1, 2$) is exponentially distributed with mean μ_J^j ($j = 1, 2$). The transform function of the jump-size distribution is $\phi_J^j(c_1, c_2, t) \equiv \frac{1}{1 - \mu_J^j c_1}$ ($j = 1, 2$).

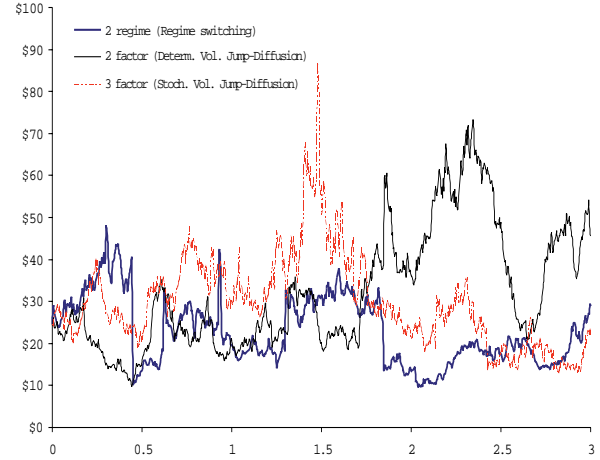


Figure 3: Simulated Spot Prices under the Three Models

3.1.2 Model 2a

Model 2a is a regime-switching model with the regime-jumps appearing only in the electricity price process. For instance, this model is suitable for modeling the occasional price spikes in the electricity spot prices caused by forced outages of the major power generation plants or line contingency in transmission networks. I assume for simplicity that there are no jumps within each regime. The sizes of regime-jumps are assumed to be distributed as independent exponential random variables and the transform functions of the regime-jump sizes are $\phi_i(c_1, c_2, t) \equiv \frac{1}{1 - \mu_i c_1}$ ($i = 0, 1$) where $\mu_0 \geq 0$ (i.e. upward jumps) and $\mu_1 \leq 0$ (i.e. downward jumps).

3.1.3 Model 3a

Model 3a is a stochastic volatility model in which the type-1 jumps are simultaneous jumps in the electricity spot price and the volatility, and the type-2 jumps are in the electricity spot price only. All parameters are constants. Type- J jump Z^J ($J = 1, 2$) is a compound Poisson process in \mathbb{R}^3 . The Poisson arrival intensity functions for the jump processes are $\lambda^1(X_t, V_t, Y_t, t) = \lambda_1$ and $\lambda^2(X_t, V_t, Y_t, t) = \lambda_2 V_t$, respectively. The transform functions of the jump-size distributions are

$$\begin{aligned} \phi_J^1(c_1, c_2, c_3, t) &\equiv \frac{1}{(1 - \mu_1^1 c_1)(1 - \mu_1^2 c_2)} \\ \phi_J^2(c_1, c_2, c_3, t) &\equiv \frac{1}{1 - \mu_2^1 c_1} \end{aligned}$$

where μ_J^k is the mean size of the type- J ($J = 1, 2$) jump in factor k ($k = 1, 2$).

3.1.4 Transform functions

The transform functions in Model 1a, 2a and 3a are denoted by φ_{1a} , φ_{2a} , and φ_{3a} , respectively. φ_{1a} can be solved in closed-form while φ_{2a} and φ_{3a} are solved numerically.

3.2 Electricity Derivatives

In this subsection, I derive the pricing formulae for the futures/forwards, call options, spark spread options, and locational spread options. The derivative prices are calculated using the parameters given in Table (1). I compare the derivative prices under different models as well.

3.2.1 Futures/Forward Price

A futures (forward) contract promising to deliver one unit of electricity S^i at a future time T for a price of F has the following payoff at time T

$$\text{Payoff} = S_T^i - F$$

Since no initial payment is required to enter into a futures contract, the futures price F at time t is given by

$$F(S_t^i, t, T) = E^Q[S_T^i | \mathcal{F}_t] \quad (14)$$

$$= e^{r\tau} E^Q[e^{-r\tau} \cdot \exp(X_T^i) | \mathcal{F}_t] \quad (15)$$

where $E^Q[\cdot | \mathcal{F}_t]$ is the conditional expectation under the risk-neutral measure Q . We thus have

$$F(S_t^i, t, T) = e^{r\tau} \cdot \varphi(\bar{e}_i^T, \bar{X}_t, \tau) \quad (16)$$

where $\varphi(u, \bar{X}_t, \tau)$ is the transform function given by (11); $\tau = T - t$; \bar{e}_i is the vector with i^{th} component being 1 and all other components being 0.

Proposition 1 *In Model 1a, the futures price of electricity S_t at time t with delivery time T is*

$$\begin{aligned} F(S_t, t, T) &= e^{r\tau} \cdot \varphi_{1a}([1, 0]', X_t, Y_t, \tau) \\ &= \exp[X_t \exp(-\kappa_1 \tau) + \frac{a_1 \sigma_1^2}{4\kappa_1} + j(\tau)] \end{aligned} \quad (17)$$

where $\tau = T - t$, $X_t = \ln(S_t)$, $a_1 = 1 - \exp(-2\kappa_1 \tau)$ and $j(\tau) = \theta_1(1 - \exp(-\kappa_1 \tau)) - \sum_{j=1}^2 \frac{\lambda_j^j}{\kappa_1} \ln \frac{\mu_j^j - 1}{\mu_j^j \exp(-\kappa_1 \tau) - 1}$.

Note that the futures price in this model is simply the scaled-up futures price in the Ornstein-Uhlenbeck mean-reversion model with the scaling factor being $\exp(-\sum_{j=1}^2 \frac{\lambda_j^j}{\kappa_1} \ln \frac{\mu_j^j - 1}{\mu_j^j \exp(-\kappa_1 \tau) - 1})$. If we interpret the spikes in the electricity price process as upward jumps followed shortly by downward jumps of similar sizes, then over a long time horizon both the frequencies and the average sizes of the upwards jumps and the downwards jumps are roughly the same, i.e. $\lambda_j^1 = \lambda_j^2$ and $\mu_j^1 \approx -\mu_j^2$. One might intuitively think that the up-jumps and down-jumps would offset each other's effect in the futures price. What (17) tells us is that this intuition is not quite right and indeed, in the case where $\lambda_j^1 = \lambda_j^2$ and $\mu_j^1 = -\mu_j^2$, the futures price is definitely higher than that corresponding to the no-jump case.

Futures price (model 2a) The futures price at time t in model 2a is

$$F(S_t, t, T) = e^{r\tau} \cdot \varphi_{2a}([1, 0]', X_t, Y_t, \tau) \quad (i = 0, 1) \quad (18)$$

where φ_{2a}^i ($i = 0, 1$) are the transform functions; i is the Markov regime state variable; and $\tau = T - t$.

Futures price (model 3a) The futures price at time t in model 3a is

$$F(S_t, t, T) = e^{r\tau} \cdot \varphi_{3a}([1, 0, 0]', X_t, V_t, Y_t, \tau) \quad (19)$$

where φ_{3a} is the transform function and $\tau = T - t$.

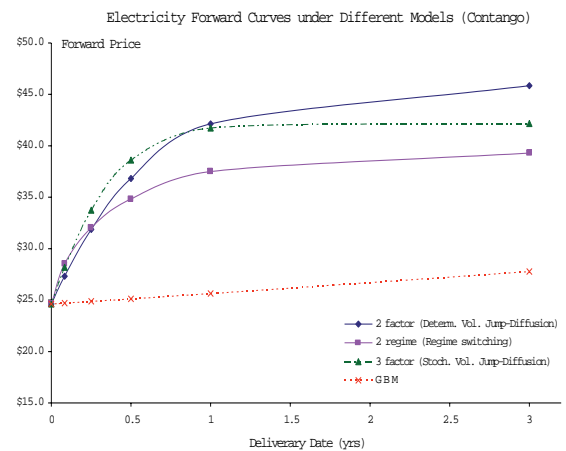


Figure 4: Forward Curves under Different Models (Contango)

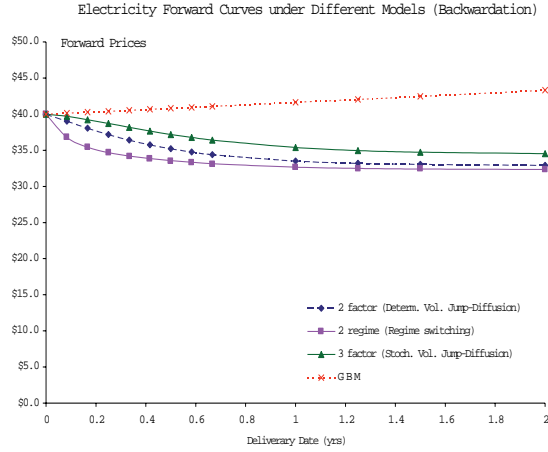


Figure 5: Forward Curves under Different Models (Backwardation)

Forward curves Using the parameters in Table (1) for modeling the electricity spot price in the Midwestern US (Cinergy to be specific), I obtain forward curves at Cinergy under each of the three illustrative models. The jointly specified factor process is the spot price of natural gas at Henry Hub. For the initial values of $S_e = \$24.63$, $S_g = \$2.105$, $V = 0.5$, $U = 0$ and $r = 4\%$, Figure (4) illustrates three forward curves of electricity, which are all in contango form since the initial value S_e is lower than the long-term mean value given by θ_e . Figure (5) plots three electricity forward curves in backwardation when the initial electricity price S_e is set to be \$40 which is higher than the corresponding long-term mean value. The forward curves under the Geometric Brownian motion (GBM) price model are also shown in the two figures. Under the GBM price model, the forward prices always exhibit a fixed rate of growth.

3.2.2 Call Option

A “plain vanilla” European call option on electricity S^i with strike price K has the payoff of

$$C(S_T^i, K, T, T) = \max(S_T^i - K, 0)$$

at maturity time T . The price of the call option at time t is given by

$$\begin{aligned} C(S_t^i, K, t, T) &= E^Q[e^{-r(T-t)} \max(S_T^i - K, 0) | \mathcal{F}_t] \\ &= G_1 - K \cdot G_2 \end{aligned} \quad (20)$$

where $\tau = T - t$ and G_1, G_2 are obtained by setting $\{a = \bar{e}_i, b = -\bar{e}_i, v = -\ln K\}$ and $\{a = \bar{0}, b = -\bar{e}_i,$

$v = -\ln K\}$ in (12), respectively.

$$\begin{aligned} G_1 &= E^Q[e^{-r\tau} \exp(X_T) 1_{X_T \geq \ln K} | \mathcal{F}_t] \\ &= F_t^i e^{-r\tau} \left(\frac{1}{2} - M_1 \right) \end{aligned} \quad (21)$$

where

$$M_1 = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\varphi([1 - w \cdot i, 0], \bar{X}_t, \tau) \cdot e^{(r\tau + i \cdot w \ln K)}]}{w F_t^i} dw$$

$F_t^i = e^{r\tau} \cdot \varphi(\bar{e}_i^T, \bar{X}_t, \tau)$ is the time- t forward price of commodity S^i with delivery time T .

$$\begin{aligned} G_2 &= E^Q[e^{-r\tau} 1_{X_T \geq \ln K} | \mathcal{F}_t] \\ &= e^{-r\tau} \left(\frac{1}{2} - M_2 \right) \end{aligned} \quad (22)$$

where

$$M_2 = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[\varphi(i \cdot w \bar{e}_i, \bar{X}_t, t, T) e^{r \cdot \tau + i \cdot w \ln K}]}{w} dw$$

Call option price Substituting φ_{1a} , φ_{2a} , and φ_{3a} into (21) and (22) we have the call option price given by (20) under **Model 1a**, **2a** and **3a**, respectively.

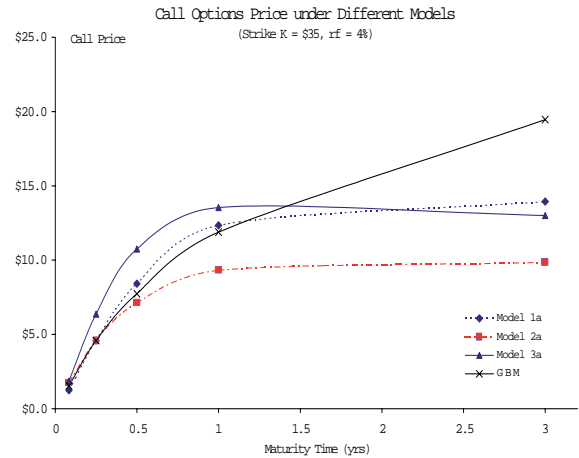


Figure 6: Call Options Price under Different Models

Volatility smile With the model parameters given in Table (1), Figure (6) plots the call option values with different maturity time under different models. The call option prices under a Geometric Brownian motion (GBM) model are also plotted for comparison purpose. Note that, as maturity time increases, the value of a call option converges to the underlying electricity spot

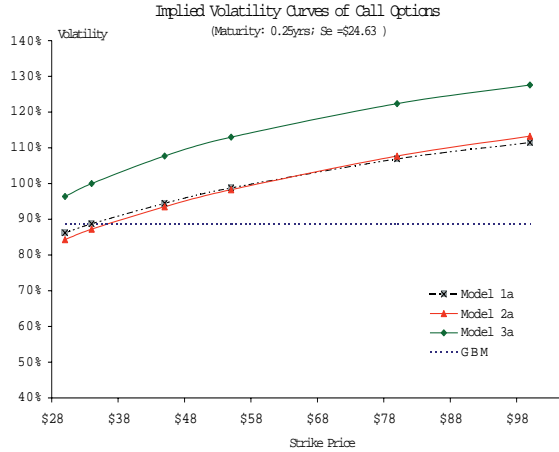


Figure 7: Volatility-Smile under Different Models

price under the GBM price model (with no convenience yields). However, under the three proposed price models, the mean-reversion effects cause the value of a call option to converge to a long-term value, which is most likely to be depending on fundamental characteristics of electricity supply and demand, rather than the underlying spot price. Figure (7) illustrates the implied volatility¹¹ curves under the three illustrative models, which all exhibit the similar kinds of volatility “smile” or “smirk” to the market implied volatility curves as shown in Figure (1).

3.2.3 Cross Commodity Spread Option

In electricity markets, cross commodity derivatives play crucial roles in risk management. The spark spread and locational spread options are good examples. The spark spread options, which are derivatives on electricity and the fossil fuels used to generate electricity, can have various applications in risk management for utility companies and power marketers. Moreover, such options are essential in asset valuation for fossil fuel electricity generating plants (e.g. Deng et al. (1998)).

I define a general *cross-commodity spread call* option as an option with the following payoff at maturity time

¹¹Implied volatility refers to the volatility parameter corresponding to a given call option value through the Black-Scholes call option pricing formula. It is obtained by inverting the Black-Scholes pricing formula using the given call option value and other known parameters as inputs.

T ,

$$CSC(S_T^1, S_T^2, K, T) = \max(S_T^1 - K \cdot S_T^2, 0) \quad (23)$$

where S_T^i is the spot price of commodity i ($i = 1, 2$) and K is a scaling constant associated with the spot price of commodity two. The interpretation of K is different in different examples. For instance, K represents the strike heat rate H in a spark spread option, and it represents the loss factor L in a locational spread option.

The time- t value of a European cross-commodity spread call option on two commodities is given by

$$\begin{aligned} CSC(S_t^1, S_t^2, K, t) &= E_t^Q[e^{-r\tau} \max(S_T^1 - K \cdot S_T^2, 0)] \\ &= G_1 - G_2 \end{aligned} \quad (24)$$

where $\tau = T - t$,

$$\begin{aligned} G_1 &= G(0, \ln S_t^1, \ln(K \cdot S_t^2), t, T; [1, \bar{0}]', [-1, 1, \bar{0}]') \\ G_2 &= G(0, \ln S_t^1, \ln(K \cdot S_t^2), t, T; [0, 1, \bar{0}]', [-1, 1, \bar{0}]') \end{aligned} \quad (25)$$

and recall that G is defined in (12).

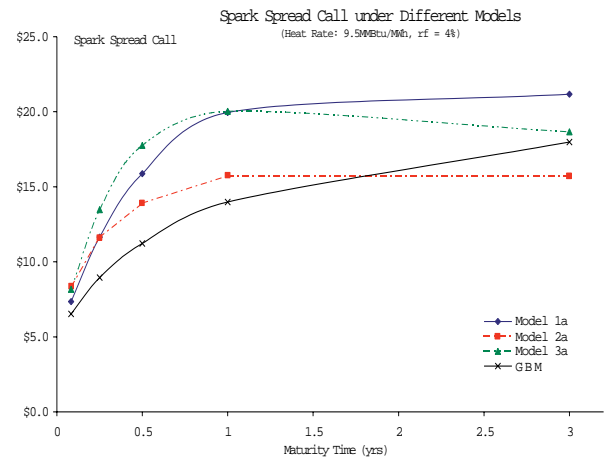


Figure 8: Spark Spread Call Price under Different Models

Cross commodity spread call option price under each of the three models are obtained by substituting φ_{1a} , φ_{2a} , and φ_{3a} into (24) and (25).

The spark spread call option values with strike heat rate $H = 9.5$ for different maturity time are shown in Figure (8). Again, the spark spread call option value converges to the underlying spot price under the GBM price model (with no convenience yields). However,

under the mean-reversion jump-diffusion price models, it converges to a long-term value which is most likely to be depending on fundamental characteristics of supply and demand.

	Model 1a	Model 2a	Model 3a
κ_1	1.70	1.37	2.17
κ_2	1.80	1.80	3.50
κ_3	N/A	N/A	1.80
θ_1	3.40	3.30	3.20
θ_2	0.87	0.87	0.85
θ_3	N/A	N/A	0.87
σ_1	0.74	0.80	N/A
σ_2	0.34	0.34	0.80
σ_3	N/A	N/A	0.54
ρ_1	0.20	0.20	0.25
ρ_2	N/A	N/A	0.20
λ_1	6.08	6.42	6.43
μ_{11}	0.19	0.26	0.23
μ_{12}	N/A	N/A	0.22
λ_2	7.00	8.20	5.00
μ_{21}	-0.11	-0.20	-0.14

Table 1: Parameters for the Illustrative Models

4 Conclusion

In this paper, I propose three types of mean-reversion jump-diffusion models for modeling electricity spot prices with jumps and spikes. I demonstrate how the prices of the electricity derivatives can be obtained by means of transform analysis. The market anticipation of jumps and spikes in the electricity spot price processes explains the enormous implied volatility observed from market prices of traded electricity options. Contrary to the commonly used Geometric Brownian motion price model, the proposed mean-reversion jump-diffusion spot price models yield call option values that approximate the market values of short-maturity out-of-the-money options very well. The applications of the electricity price models and the corresponding electricity derivative pricing results are illustrated in Deng (1998).

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