PRICING ENERGY DERIVATIVES BY LINEAR PROGRAMMING: TOLLING AGREEMENT CONTRACTS

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Abstract

We introduce a new approach for pricing energy derivatives known as tolling agreement contracts. The pricing problem is reduced to a linear program. We prove that the optimal operating strategy for a power plant can be expressed through optimal exercise boundaries (similar to the exercise boundaries for American options). We find the boundaries as a byproduct of the pricing algorithm. The suggested approach can incorporate various real world power plant operational constraints. We demonstrate computational efficiency of the algorithm by pricing 1- and 10-year tolling agreement contracts.

Key words: energy derivatives, tolling agreement, dispatch policy, optimization, exercise boundaries, multiple stopping.

1 Introduction

Recently, the problem of pricing *tolling agreement contracts* (also known as pricing scheduling flexibility of electricity generating facilities) has been extensively researched in the academic literature. Under a tolling agreement contract a renter receives a right to operate a power plant for a fixed time horizon in exchange for a fixed payment. During the life of the contract the renter receives all cash inflows and outflows associated with the power plant. We analyze the optimal operating policies of the renter and estimate the value of the underlying tolling agreement contract. Tolling agreement contracts are popular exotic energy derivatives. The problem of pricing such contracts falls into the class of multiple optimal stopping problems and is extremely hard from the computational prospective.

Tolling agreement contracts have become popular since de-regulation of energy markets in the 1970s. To price the contracts practitioners used standard at the time discounted cash flow methods. Later, it was recognized in the literature (see, for example, Dixit and Pindyck (1994)) that the discounted cash flow approach is not suitable in a highly volatile price environment since it tends to underestimate the value of a contract. When researchers first became interested in this problem they tried to use well-developed approaches of option pricing. More specifically, they tried to represent scheduling flexibility of energy generating facilities as a sequence of so called *spark spread options* owned by a renter. A spark spread option gives its holder at a specified time in the future the right to exercise a profit equal to the non-negative part of the difference between the price of energy and the price of fuel multiplied by a coefficient called a *heat rate*, see Deng, Johnson and Sogomonian (1998), and Eydeland and Wolyniec (2003). An overview of methods based on spread options pricing can be found in Carmona and Durrleman (2003). The option based methods are extremely efficient in terms of computational time and provide much more accurate contract price estimates then the discounted cash flow methods (see, for example, Deng, Jonhnson and Sogomonian (1998) for comparison).

Further development of pricing algorithms led to creation of methods based on the stochastic dynamic programming framework. Contrary to the option based methods, the stochastic dynamic programming methods are flexible in incorporating real world power plant operational constraints. In addition to flexibility these methods also remain computationally feasible for contracts with relatively large horizons. One of the first attempts to use dynamic programming for pricing tolling agreement contracts was done by Deng and Oren (2003). In this paper the authors introduced an efficient dynamic programming algorithm that works with price processes defined on a lattice. A more general case and more rigorous theoretical treatment of the problem was done in the recent paper by Carmona and Ludkovski (2008). The paper builds a continuous-time stochastic control framework with general assumptions regarding underlying energy and fuel price dynamics. The approach is capable of dealing with most of the real world operational constraints. The authors also provide a rigorous convergence and efficiency analysis of the algorithm.

Although usually not easy to implement and having its own limitations, when there are no strict requirements on computational time and contract horizons are not too large, the dynamic programming framework seem to be a good alternative for pricing tolling agreement contracts.

Similarly to Carmona and Ludkovski (2008), a number of other authors tried to apply a dynamic optimal control (also called an optimal switching) setting to solve the optimal scheduling flexibility problem. They attempted to derive a closed-form solution of the problem. Interesting developments of this approach can be found in Dixit (1989), Brekke and Oksendal (1994), Bayraktar and Egami (2007), Pham and Ly Vath (2007), and references therein. A more general setup is considered in Hamadene and Jeanblanc (2007).

We finish the literature survey by referencing an interesting approach capable of incorporating various operational constraints found in Thompson (2004). In this work the authors consider a continuous optimal control space. They derive nonlinear partial-integro-differential equations (PIDEs) for the valuation and optimal operating strategies of energy generation facilities. Sophisticated numerical methods are available to solve the derived PIDEs.

In this paper we tried to look at the scheduling flexibility problem from a different optimization perspective and apply a different set of optimization tools. Although, like in any other framework, in order to model a working behavior of a real world power plant we had to make a number of simplifying assumptions, we believe that the developed framework may be a viable alternative for practitioners, especially in cases when computational time is critical or contract horizon is large. Also, the suggested framework is capable of working with price processes defined by a set of historical sample paths. This property may be useful in practical applications. Next, we outline some key features of the developed optimization framework. The suggested framework is robust, computationally efficient and produces contract price estimates with reasonable accuracy. The optimization procedure also provides a practical optimal dispatch strategy defined by a set of optimal exercise boundaries. Optimization problem is reduced to solving a linear programming (LP) problem with a number of variables and constraints independent of the number of sample paths or the time horizon used in the model. Computational speed of LP is the key factor of the efficiency of our framework. The suggested approach can price contracts with horizons as large as 10 years and longer. The robustness of the framework (resulting from explicit incorporation of shape properties of optimal exercise boundaries) allows us to obtain stable results with a small number of sample paths. Because of this property it is possible to use historical sample paths within the framework. The framework is flexible enough to incorporate various power plant operational constraints, such as start-up and shutdown costs, fixed renting costs, ramp up period time delay and ramp up period operational costs, as well as variable output capacity levels.

Section 2 provides a description of the problem and introduces notations. Section 3 derives a stochastic optimal control optimization problem that a renter of the power plant has to solve in order to find the optimal operating policy and, consequently, to find the price of the corresponding tolling agreement contract. Section 4 examines the properties of optimal operating strategies and proves theoretical results. The results of this section underly and justify the optimization framework developed in the following sections. Section 5 introduces a notion of optimal exercise boundaries and formulates an optimal operating policy in terms of optimal exercise boundaries. Section 6 describes an algorithm for finding time independent optimal exercise boundaries and a corresponding price of the tolling agreement contract. Section 7 generalizes the suggested approach to a case with time dependent optimal exercise boundaries. Section 8 provides results of numerical experiments. We investigate various computational aspects of the algorithm and compare the results with the corresponding results of dynamic programming algorithms. Section 9 summarizes the results.

2 Problem Description and Notation

Although the proposed methodology is quite general and may be applied to a wide class of tolling agreement contracts, this research considers an optimal policy of a renter leasing a two regime turbine power plant. We consider a combined cycle gas turbine (CCGT) power plant which can work in one of the two regimes: a *high capacity mode* and in a *low capacity mode*. Usually, the output capacity is measured in MWh (megawatt hours), i.e., megawatts (MW) of energy produced in one hour. To simplify the notation, we express the output capacity in megawatt hours per time period:

$$\text{MWh per period} = \text{MW} \cdot \text{ Number of hours in a period.}$$

We assume that in the high capacity mode the power plant's generating capacity is

$$\overline{Q} \text{ MWh per period.}$$

Consequently, we assume that in the low capacity mode the power plant's generation capacity is

$$\underline{Q} \text{ MWh per period, } (\overline{Q} > \underline{Q} > 0).$$

We consider a discrete time operating environment for a renter of the power plant. At each time point the renter has two options. She can either turn down the plant when its operation is not profitable (or keep the current state if the plant is already offline), or bring the plant online when its operation becomes profitable (or keep the current state if the plant is already online). The typical operation cycle: 1) a renter buys natural gas on the market, 2) converts it into electricity, and 3) sells the output energy on the market. The conversion rate is termed a *heat rate*. It is specified in British thermal units (MMBtu) of gas needed to produce one MWh of energy. In our model we assume an output dependent heat rate. Hence, in the high capacity mode the power plant has a heat rate

$$\overline{H} \text{ MMBtu/MWh,}$$

and in the low capacity mode the plant has a heat rate

$$\underline{H} \text{ MMBtu/MWh.}$$

For technical reasons we assume that

$$\overline{Q} \cdot \overline{H} \geq \underline{Q} \cdot \underline{H}.$$

The last constraint is non-restrictive for the real world power generation facilities, because usually the ratio $\frac{\overline{Q}}{\underline{Q}}$ is no less than $\frac{100}{60} = \frac{5}{3}$, and the ratio $\frac{\overline{H}}{\underline{H}}$ is usually no less than $\frac{10}{14} > \frac{3}{5}$ (a common case is $\frac{\overline{H}}{\underline{H}} > 1$). To bring the plant online the renter has to run it for a fixed period of time without energy output: *ramp up period*. Depending on the power plant the typical range for the ramp up period is from 2 to 12 hours. We make the length of one ramp up period as the smallest time unit in the model. With this assumption, we always have that the length of the ramp up period equals one. We also assume that during the ramp up period the power plant is in the low capacity mode. Thus, the total amount of gas consumed during the ramp up is

$$L = \underline{Q} \cdot \underline{H} \text{ MMBtu.}$$

Finally, we assume that to bring the plant online in addition to the ramp up costs, the renter also incurs fixed *startup costs*

$$C_s \ (C_s \geq 0).$$

To bring the plant down, she faces fixed *shutdown costs* equal to

$$C_d \ (C_d \geq 0).$$

There are also fixed *renting costs*

$$\overline{K} \text{ and } \underline{K} \ (\overline{K},\ \underline{K} \geq 0)$$

when the plant is operating in the high and low capacity modes correspondingly. For technical reasons we need to impose the constraint:

$$\underline{K} \leq C_s + C_d.$$

For practical applications the last constraint is not restrictive at all. We assume that switching between the high and low capacity regimes is costless and instantaneous. Finally, we denote the total number of time periods specified by the tolling contract as

$$N \ (N > 0),$$

price processes for energy and gas as

$$(P_i) \text{ and } (G_i)$$

respectively, and a one period risk free interest rate as r. We model fuel and energy price dynamics by continuous Markovian processes.

3 Stochastic Optimal Control Problem

Here we define renter's cash flows associated with the operation of the energy generation facility. When the plant is online, at any period i the renter has an option to exercise the profit:

$$\max\left(\overline{Q}\left(P_i - \overline{H} \cdot G_i\right) - \overline{K}, \underline{Q}\left(P_i - \underline{H} \cdot G_i\right) - \underline{K}\right),$$

called the *spark spread*. For simplicity let us denote the spark spread at time i as M_i. Therefore:

$$M_i = \max\left(\overline{Q}\left(P_i - \overline{H} \cdot G_i\right) - \overline{K}, \underline{Q}\left(P_i - \underline{H} \cdot G_i\right) - \underline{K}\right).$$

At any time the renter also has the right to turn down the plant. In this case he incurs fixed shutdown costs C_d. When bringing the plant online he faces fixed startup costs and ramp up period costs (cost of gas burnt during ramp up), so the total cash outflow is $(C_s + L \cdot G_i)$. To simplify reading we adopt the notation:

$$\hat{C}_s^i = (C_s + L \cdot G_i).$$

Therefore, at any time period i the renter has to make a decision either to turn on or turn off the power plant. In order to efficiently operate the plant, at each time period she has to solve a stochastic optimization problem of *optimal switching* (whether it is optimal to turn on or turn off the generation facility). We construct the renter's stochastic optimization problem below.

We start from introducing the stochastic framework. Let $(\Omega, \mathscr{F}, \mathsf{F} = (\mathscr{F}_i), \mathsf{P})$ be a stochastic basis, where F is a filtration $(\mathscr{F}_1 \subseteq \mathscr{F}_2 ... \subseteq \mathscr{F}_i ... \subseteq \mathscr{F}_N \subseteq \mathscr{F})$. We assume that P is a risk-neutral probability measure. We also assume that (P_i) and (G_i) are Markovian processes defined on the probability space above and are adapted to the filtration F. Assume that the current time period is 1. Let the vector $(\xi_1, ..., \xi_N)$ be the vector representing the renter's current switching decision, ξ_1, and his future switching decisions, $\xi_2, ..., \xi_N$. Let ξ_0 denote the initial state of the power plant. Any of the ξ_is can take one of two values, 0 or 1, meaning that the plant is *on* when $\xi_i = 1$ and is *off* when $\xi_i = 0$. Because we consider Markovian price processes only, the renter's decision at time

i depends only on the current state of the plant, ξ_{i-1}, and the vector of current gas and energy prices, (G_i, P_i). Therefore, we can represent ξ_i as the following function: $\xi_i(\xi_{i-1}, G_i, P_i)$. Looking from the time period 1, the switching decisions at time periods $2, 3, ..., N$ are random stochastic controls taking values 0 or 1, and only ξ_1 is a deterministic $0-1$ control variable.

The power plant is online (producing energy) at the beginning of a time period i only if it has been in the *"on"* state during the preceding time period. Therefore, the plant is online during the time period i if and only if $\xi_i \xi_{i-1} = 1$. The total profit function for the period i is

$$\phi_i = \xi_i \xi_{i-1} M_i - [\xi_i - \xi_{i-1}]^+ \hat{C}_s^i - [\xi_{i-1} - \xi_i]^+ C_d. \tag{1}$$

In the subsequent sections we also make use of the notation:

$$f_i(\xi_{i-1}, \xi_i) = [\xi_i - \xi_{i-1}]^+ \hat{C}_s^i + [\xi_{i-1} - \xi_i]^+ C_d.$$

Hence, we have

$$\phi_i = \xi_i \xi_{i-1} M_i - f_i(\xi_{i-1}, \xi_i).$$

Using the introduced notation, the cumulative profit for periods from j to N is

$$S(j, N) = \sum_{i=j}^{N} \phi_i e^{-r(i-j)}. \tag{2}$$

To find the value of the tolling contract and the optimal switching decision ξ_1 at time 1, the renter has to maximize the expected cumulative profit $S(1, N)$ over the set of all admissible $\mathscr{F}_i$-measurable stochastic controls $\xi_i(w)$. In other words, she needs to solve the stochastic optimization problem:

$$P_N^1: \ J_N^1 = \sup_{\xi_1, \ldots, \xi_N} E[S(1, N)|\mathscr{F}_1] = \sup_{\xi_1, \ldots, \xi_N} E[S(1, N)|G_1, P_1, \xi_0]. \tag{3}$$

Let $(\xi_1^*, \xi_2^*(w), ..., \xi_N^*(w))$ be an optimal solution of the above problem. The value of the contract is the optimal objective value J_N^1, and the optimal switching decision at time 1 is the optimal solution ξ_1^*. Similarly to the problem above, we can write an optimization problem for time j:

$$P_{N-j+1}^j: \ J_{N-j+1}^j = \sup_{\xi_j, \ldots, \xi_N} E[S(j, N)|\mathscr{F}_j] = \sup_{\xi_j, \ldots, \xi_N} E[S(j, N)|G_j, P_j, \xi_{j-1}]. \tag{4}$$

The upper index in J_{N-j+1}^j and P_{N-j+1}^j denotes the starting time period, and the lower index denotes the total number of periods in the optimization problem. To find an optimal operating decision at time j the renter has to solve the problem P_{N-j+1}^j and take the optimal solution ξ_j^* as his operating decision. Sometimes throughout this paper we indicate the conditioning on ξ_{j-1} through the notation:

$$J_{N-j+1}^j(\xi_{j-1}) \text{ and } P_{N-j+1}^j(\xi_{j-1}). \tag{5}$$

4 Optimal Operating Strategy

For the reader's convenience we derive all the theoretical results considering the optimization problem $P_N^1(\xi_0)$. Nevertheless, all the results hold without changes if we consider the generalized problem $P_{N-j+1}^j(\xi_{j-1})$ (only the variable indices have to be adjusted appropriately).

Using (3) and the Bellman principle of optimality we derive a Bellman equation for $J_N^1(\xi_0)$:

$$J_N^1(\xi_0) = \sup_{\xi_1,...,\xi_N} \left(\xi_0\xi_1 M_1 - f_1(\xi_0,\xi_1) + E\left[\sum_{i=2}^{N} \left(\xi_i\xi_{i-1}M_i - f_i(\xi_{i-1},\xi_i)\right) e^{-r(i-1)} | \mathscr{F}_1 \right] \right) =$$
$$= \max_{\xi_1} \left(\xi_0\xi_1 M_1 - f_1(\xi_0,\xi_1) + e^{-r} E\left[\sup_{\xi_2,...,\xi_N} E\left[\sum_{i=2}^{N} \left(\xi_i\xi_{i-1}M_i - f_i(\xi_{i-1},\xi_i)\right) e^{-r(i-2)} | \mathscr{F}_2 \right] | \mathscr{F}_1 \right] \right) =$$
$$= \max_{\xi_1} \left(\xi_0\xi_1 M_1 - f_1(\xi_0,\xi_1) + e^{-r} E\left[J_{N-1}^2(\xi_1) | \mathscr{F}_1 \right] \right).$$

Summarizing, we have the following Bellman equation:

$$J_N^1(\xi_0) = \max_{\xi_1} \left(\xi_0\xi_1 M_1 - f_1(\xi_0,\xi_1) + e^{-r} E\left[J_{N-1}^2(\xi_1) | \mathscr{F}_1 \right] \right). \quad (6)$$

The next lemma formulates the necessary and sufficient conditions for $\xi_1^* = 1$ to be optimal in the problems $P_N^1(0)$ and $P_N^1(1)$, correspondingly.

Lemma 1 *The necessary and sufficient condition for $\xi_1^* = 1$ to be optimal in $P_N^1(1)$ is*

$$E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \geq e^r\left(-C_d - M_1\right).$$

The necessary and sufficient condition for $\xi_1^* = 1$ to be optimal in $P_N^1(0)$ is

$$E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \geq e^r \hat{C}_s^1.$$

Proof. See Appendix. □

Below we formulate and prove a simple but, nevertheless, important property.

Lemma 2

$$\forall\ i = \overline{1,N}: \quad \left(M_i + \hat{C}_s^i + C_d \geq 0 \right).$$

Proof. See Appendix. □

The next theorem states an important property of optimal solutions.

Theorem 1 *Let $\xi^{*1} = (\xi_1^{*1},...,\xi_N^{*1})$ and $\xi^{*0} = (\xi_1^{*0},...,\xi_N^{*0})$ be optimal solutions of the problems $P_N^1(1)$ and $P_N^1(0)$ correspondingly. $\forall N > 0$ the following relations hold*

$$\forall w \in \Omega: \ \xi_i^{*1}(w) \geq \xi_i^{*0}(w), \ i = \overline{1,N}.$$

Proof. See Appendex. □

The theorem below shows a *monotonicity* property of optimal solutions. This theorem is the main result which provides a basis for building efficient numerical algorithms in the subsequent sections. In the notation below we use the $'$ sign to denote the quantities corresponding to price processes $(G_i)'$ and $(P_i)'$. These quantities are defined in a similar way to the corresponding analogues without the $'$ sign. To prove the theorem we need the next two lemmas.

Lemma 3 *Consider two pairs of gas and energy price processes* $\{(G_i),(P_i)\}$ *and* $\{(G_i'),(P_i')\}$, *satisfying the condition:*

$$\forall\ w \in \Omega:\ P_i'(w) \geq P_i(w),\ G_i'(w) \leq G_i(w),\ i = \overline{1,N}.$$

The following inequalities hold

$$\forall\ i = \overline{1,N}:\ M_i' + \hat{C}_s^{i\prime} \geq M_i + \hat{C}_s^i.$$

Proof. See Appendix. □

Lemma 4 $\forall\ N > 0$ *the following inequalities hold*

$$\begin{aligned} 1)\ J_N^1(1) - J_N^1(0) &\leq M_1 + \hat{C}_s^1, \\ 2)\ J_N^1(1) - J_N^1(0) &\geq -C_d. \end{aligned}$$

Proof. See Appendix. □

Theorem 2 *Consider two pairs of gas and energy price processes* $\{(G_i),(P_i)\}$ *and* $\{(G_i'),(P_i')\}$, *satisfying the condition of Lemma 3. Let* ξ^{*1}, ξ^{*0}, $\xi^{*1\prime}$ *and* $\xi^{*0\prime}$ *be optimal solutions of the problems* $P_N^1(1)$, $P_N^1(0)$, $P_N^{1\prime}(1)$, *and* $P_N^{1\prime}(0)$, *correspondingly.* $\forall\ N > 0$ *the following statements are true:*

$$\begin{aligned} &1)\quad J_N^{1\prime}(1) - J_N^{1\prime}(0) \geq J_N^1(1) - J_N^1(0), \\ &2)\quad \xi_1^{*0} \leq \xi_1^{*0\prime},\ \text{and}\ \xi_1^{*1} \leq \xi_1^{*1\prime}. \end{aligned}$$

Proof. See Appendix. □

Corollary 2. 1 *Consider two pairs of gas and energy price processes* $\{(G_i),(P_i)\}$ *and* $\{(G_i'),(P_i')\}$ *with* (G_i) *and* (G_i'), (P_i) *and* (P_i') *following the same dynamics equations correspondingly. Let* (G_0,P_0) *and* (G_0',P_0') *be the corresponding initial points satisfying the property:*

$$P_0 \leq P_0',\ G_0 \geq G_0'.$$

If energy and fuel price processes follow one of the following dynamics (equations for energy and fuel prices may be different):

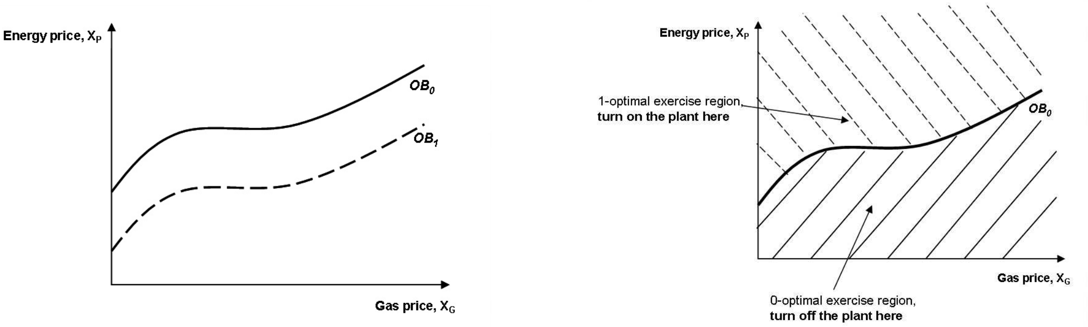
1) *Geometric Ornstein-Uhlenbeck process,*
2) *Geometric Brownian Motion process;*

then the optimal solutions of P_N^1 *and* $P_N^{1\prime}$ *have the following properties:*

$$\begin{aligned} &1)\quad \xi_1^{*0} \leq \xi_1^{*0\prime}, \\ &2)\quad \xi_1^{*1} \leq \xi_1^{*1\prime}. \end{aligned}$$

The assertion remains valid if we add a possibility of non-negative jumps to the price dynamics with i.i.d. intensities and magnitudes. We assume that jump intensities and magnitudes are independent of the price level as well.

Proof. See Appendix. □



(a) (b)

Figure 1. a) Optimal exercise boundaries. b) Optimal operating policy when current state of the plant is "off".

5 Optimal Exercise Boundary

As before, let ξ^{*0} and ξ^{*1} be optimal solutions of the problems $P_N^1(0)$ and $P_N^1(1)$, respectively. By (X_G, X_P), we denote the initial values $P_1 = X_P$ and $G_1 = X_G$ of energy and gas prices in the optimization problems $P_N^1(0)$ and $P_N^1(1)$. For $i = \overline{0,1}$ consider the sets below:

$$\begin{aligned} R_0^{N,i} &= \{(X_G, X_P) | \xi_1^{*i} = 0\}, \\ R_1^{N,i} &= \{(X_G, X_P) | \xi_1^{*i} = 1\}. \end{aligned}$$

We refer to $R_0^{N,0}$ and $R_0^{N,1}$ as an "off" and an "on" state *0-optimal exercise set*, correspondingly. Consistently, we call $R_1^{N,0}$ and $R_1^{N,1}$ as an "off" and an "on" state *1-optimal exercise sets*, respectively. If we plot points (X_G, X_P) on a plane where the y-coordinate represents X_P and the x-coordinate represents X_G, the boundary separating 0 and 1-optimal exercise sets on this plane is called an *optimal exercise boundary*. Clearly, there are two optimal exercise boundaries, one is for the "off" state (problem $P_N^1(0)$) and the other one is for the "on" state (problem $P_N^1(1)$) of the plant. From now on we concentrate on geometric Ornstein-Uhlenbeck (with and without jumps) price dynamics only. Although all the derived results are valid for other classes of stochastic processes (geometric Brownian Motion in particular), it is widely accepted in the literature that geometric Ornstein-Uhlenbeck processes are the most adequate to approximate energy and gas price dynamics. For the considered class of price dynamics the corollary of Theorem 2 implies monotonicity of the optimal exercise boundaries. Monotonicity of a boundary has to be understood in the following sense. If a point (X_G^1, X_P^1) belongs to the 0-optimal exercise set for some state of the plant, then any point (X_G^2, X_P^2), such that $X_P^1 \geq X_P^2$ and $X_G^1 \leq X_G^2$, has to belong to the 0-optimal exercise set for the same state of the plant as well. The monotonicity of the boundary also implies connectivity of the 0 and 1-optimal exercise sets. We denote the optimal exercise boundaries for the "off" and the "on" states of the plant as OB^0 and OB^1, correspondingly. Theorem 1 implies that the boundary OB^0 has to be always no lower than the boundary OB^1. Figure 1(a) summarizes properties of the optimal exercise boundaries. For any period of time the problem of finding the optimal exercise boundaries is equivalent to the renter's optimal switching problem. With the help of optimal exercise boundaries it is easy to formulate the renter's optimal switching policy. If the current point (X_G, X_P), representing current energy and gas prices, lies above the optimal exercise boundary for the current state of the plant, then the optimal renter's decision is to turn on the power plant (or leave it working if its current state is "on"). If the price point is below the optimal exercise boundary, then it is optimal to turn down the power plant (or leave it not working if its current state is "off"). If the optimal operating decision is to set the plant in the "on" state, then the optimal capacity regime is determined by instantaneous gains of the regime because switching between different capacity regimes is costless. Figure 1(b) explains the optimal operating behavior of the renter if the current state of the plant is "off"(she needs to use the boundary OB_0 in this case). Finding the optimal exercise boundaries for every time period is a timely and resource consuming process. To overcome this difficulty, we suggest to make use of the heuristic argument below. If a tolling agreement contract has a relatively large time horizon (one year is long enough), then it is not necessary to build the optimal exercise boundaries for every time period. A heuristic approach is to find a single pair of optimal exercise boundaries that can be used for all time periods. We refer to these optimal time independent boundaries as *optimal stationary exercise boundaries.* The proposed heuristic relies on a hypothesis that for most of the time periods except for a small fraction of the very last periods, stationary optimal exercise boundaries can be good approximations of the time dependent optimal exercise boundaries. The rationale for this is that for the time periods close to contract's expiration, infinite horizon may be a bad assumption. However, according to the hypothesis this should have little influence on the results since the fraction of such periods should be small relative to the total number of periods under the contract. Computational results reported later in the paper provide justification of the hypothesis for the considered setups. If needed, it is possible to extend the suggested approach on the case with time dependent boundaries. In the rest of the paper we concentrate on developing efficient optimization procedures for finding optimal stationary exercise boundaries. In one of the final sections we provide a short outline of how to generalize the developed approach for the time dependent case. We refer to the optimal stationary exercise boundary for the "off" state as SOB_0 and to the optimal stationary exercise boundary for the "on" state as SOB_1. It is natural to assume that the optimal stationary exercise boundaries inherit all the properties from the time period specific optimal exercise boundaries. Therefore, we restrict ourselves to the class of optimal stationary exercise boundaries having the monotonicity property discussed above and having the property that SOB_0 lies above SOB_1.

6 Finding Optimal Stationary Exercise Boundaries

6.1 Optimization on a Grid

In the preceding section we considered optimal exercise boundaries in a (X_G, X_P)-space. Here and throughout the rest of the paper we switch to a different $(\ln X_G, \ln X_P)$ space. Since the logarithm transformation is monotonic (in the sense that if $P_1(x_1, y_1), P_2(x_2, y_2)$ are some points in the old coordinates, and $P_1'(x_1', y_1'), P_2'(x_2', y_2')$ their respective mappings in the new coordinates, then the following property holds $(x_1 \leq x_2)\&(y_1 \geq y_2) \Leftrightarrow (x_1' \leq x_2')\&(y_1' \geq y_2')$), all the derived properties of optimal exercise boundaries hold in the new coordinates. The reason for switching to the logarithmic coordinates is that below in this section we work with uniform grids on the plane. Since we concentrate on geometric price process dynamics, to make the choice of uniform grids justified we need to convert geometric price processes into arithmetic ones, and, therefore, we use the logarithm transformation.

Let $\{(G_i^j, P_i^j)\}_{i=1,\dots,N}^{j=1,\dots,N_S}$ be a set of sample paths generated using geometric Ornstein-Uhlenbeck dynamics, where as before N is the total number of time periods in the contract, and N_S is the total number of sample paths. Also let $P_{\max} = \max_{(i,j)} \ln P_i^j$, $G_{\max} = \max_{(i,j)} \ln G_i^j$, $P_{\min} = \min_{(i,j)} \ln P_i^j$ and $G_{\min} = \min_{(i,j)} \ln G_i^j$. On the $(\ln X_G, \ln X_P)$-plane consider a rectangle with the top left vertex

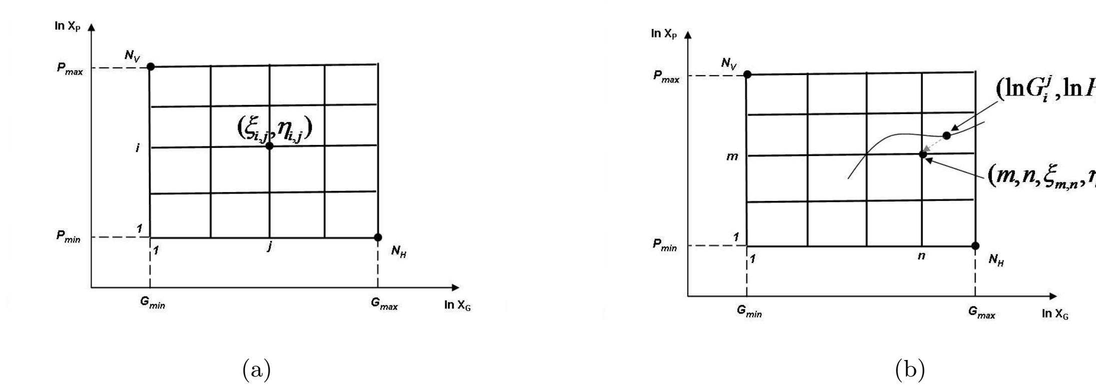


Figure 2. a) Grid on a logarithmic plane. b) Interpolation on a grid.

having coordinates $(G_{\min}, P_{\max})$ and the bottom right vertex having coordinates $(G_{\max}, P_{\min})$. Inside the constructed rectangle let us build a uniform grid with N_H horizontal nodes and N_V vertical nodes, see Figure 2(a). We numerate horizontal lines of the grid from 1 to N_V with numbers increasing to the top, and numerate vertical lines of the grid from 1 to N_H with numbers increasing to the right. The obtained uniform grid is our discretization of the initial $(\ln X_G, \ln X_P)$-plane. Below we develop algorithms for finding optimal exercise boundaries on the discretized $(\ln X_G, \ln X_P)$-plane. At any node (i, j) of the constructed grid we assign a pair of 0-1 variables $(\xi_{i,j}, \eta_{i,j})$, $i = \overline{1, N_V}$, $j = \overline{1, N_H}$, see Figure 2(a). The interpretation for these variables is the following: If at a grid point (i, j) $\xi_{i,j} = 1$, then the plane point corresponding to the node (i, j) lies above SOB_0. If $\xi_{i,j} = 0$, then the plane point corresponding to the node(i, j) lies below SOB_0. A similar interpretation is true for $\eta_{i,j}$s, the only difference is that the variables $\eta_{i,j}$ define SOB_1. Hence, the problem of finding optimal SOB_0 and SOB_1 is reduced to finding optimal values for $\xi_{i,j}$s and $\eta_{i,j}$s. Since SOB_0 and SOB_1 have the monotonicity property and SOB_0 lies above SOB_1, we need to impose some constraints on $\xi_{i,j}$s and $\eta_{i,j}$s in order to satisfy optimal exercise boundaries shape constraints explicitly. To do this, we need to add the following constraints:

$$\forall\ (N_V - 1) \geq i \geq 1, N_H \geq j \geq 1 \quad : \quad \xi_{i,j} \leq \xi_{i+1,j},\ \eta_{i,j} \leq \eta_{i+1,j}, \tag{7}$$

$$\forall\ N_V \geq i \geq 1, (N_H - 1) \geq j \geq 1 \quad : \quad \xi_{i,j} \geq \xi_{i,j+1},\ \eta_{i,j} \geq \eta_{i,j+1}, \tag{8}$$

$$\forall\ N_V \geq i \geq 1, N_H \geq j \geq 1 \quad : \quad \xi_{i,j} \geq \eta_{i,j}. \tag{9}$$

Since the energy and gas price processes are continuous, and the $(\ln X_G, \ln X_P)$-plane is discrete, we need to choose an interpolation rule for assigning a pair of $(\xi_{i,j}, \eta_{i,j})$ for each sample path point (G_i^j, P_i^j). We pick the closest grid node in the Euclidian sense as our interpolation rule. Let (G_i^j, P_i^j) be a sample path point. Applying the logarithm transformation, we get a $(\ln G_i^j, \ln P_i^j)$ point on the logarithmic plane. Let (m, n) be the closest to $(\ln G_i^j, \ln P_i^j)$ grid node on the $(\ln X_G, \ln X_P)$-plane, then we assign $(\xi_{m,n}, \eta_{m,n})$ to (G_i^j, P_i^j) as the corresponding set of exercise boundary variables. To simplify the notation, for each sample path point (G_i^j, P_i^j) we denote the corresponding pair of exercise boundary variables as (ξ_i^j, η_i^j), see Figure 2(b). Using the assigned variables it is easy to formulate the optimal operating policy on the sample paths:

$$\begin{aligned}&1) \text{ If at a point } (G_i^j, P_i^j) \text{ the current state of the plant is "off"}, && (10)\\ &\qquad \text{then it is optimal to turn on the plant} \Leftrightarrow \ \xi_i^j = 1. \\ &2) \text{ If at a point } (G_i^j, P_i^j) \text{ the current state of the plant is "on"}, && (11)\\ &\qquad \text{then it is optimal to turn off the plant} \Leftrightarrow \ \eta_i^j = 0.\end{aligned}$$

6.2 Optimization with Two Optimal Stationary Exercise Boundaries

Here we construct an optimization problem to find optimal values of $\xi_{i,j}$s and $\eta_{i,j}$s. From (10)-(11) we see that depending on the current state of the plant, at any point (G_i^j, P_i^j) the optimal switching rule is either ξ_i^j or η_i^j. For the sake of simplicity, we introduce an auxiliary set of variables $\{\chi_i^j\}_{i=1,...,N}^{j=1,...,N_S}$. With the new variables, the optimal decision rule at any point (G_i^j, P_i^j) is χ_i^j. To follow the logic introduced by (10)-(11) the variables χ_i^js have to satisfy the constraints below:

$$\chi_i^j = (1-\chi_{i-1}^j)\xi_i^j + \chi_{i-1}^j \eta_i^j, \ i = 1, ..., N, \ j = 1, ..., N_S; \tag{12}$$

$$\chi_0^j = 0, \ j = 1, ..., N_S, \ \text{assuming the initial state of the plant is "off"}. \tag{13}$$

The logic behind these constraints is straightforward. At any point (G_i^j, P_i^j) the current state of the plant is determined by a variable χ_{i-1}^j. Whenever $\chi_{i-1}^j = 0$, meaning that the plant is currently "off", the optimal switching rule at (G_i^j, P_i^j) is ξ_i^j. Whenever $\chi_{i-1}^j = 1$, meaning that the plant is currently "on", the optimal switching rule at (G_i^j, P_i^j) is η_i^j. Similarly to (1), we can construct a profit function for each sample path j $(j = 1, ..., N_S)$ and each period i $(i = 1, ..., N)$:

$$\phi_i^j = e^{-r(i-1)} \left(\chi_i^j \chi_{i-1}^j M_i^j - [\chi_i^j - \chi_{i-1}^j]^+ C_{s,i}^j - [\chi_{i-1}^j - \chi_i^j]^+ C_d \right),$$

where

$$\begin{aligned} M_i^j &= \max\left(\overline{Q}\left(P_i^j - \overline{H}\cdot G_i^j\right) - \overline{K}, \underline{Q}\left(P_i^j - \underline{H}\cdot G_i^j\right) - \underline{K}\right) \text{ is the spark spread,} \\ C_{s,i}^j &= C_s + L\cdot G_i^j \text{ is the startup cost.}\end{aligned}$$

Parameters $C_d, C_s, L, \overline{Q}, \overline{H}, \underline{Q}, \underline{H}, \overline{K}$, and $\underline{K}$ have been introduced earlier in the paper. Below we formulate an optimization problem maximizing the expected profit of operating the power plant:

$$(P2B): \quad \max_{\xi_{i,j},\eta_{i,j}} \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N}\phi_i^j$$

s.t.

auxiliary variable constraints (12)-(13),

optimal exercise boundaries shape constraints (7)-(9),

$\xi_{i,j}, \eta_{i,j} \in \{0, 1\}, \ i = 1, ..., N_V, \ j = 1, ..., N_H.$

Although the optimization problem above solves the problem of finding optimal stationary exercise boundaries, the obtained formulation is computationally hard in general case, since it is a non-linear 0-1 optimization problem. A standard remedy in this case is to linearize the problem. The only obstacle for this remedy is that linearization would require introducing a large number of auxiliary variables and constraints making the resultant formulation computationally inefficient. The number of auxiliary variables and constraints in $(P2B)$ is of $O(N_S \cdot N)$ order. When the horizon of the contract is large enough (that is the most interesting case for us) and the number of sample paths is measured by at least hundreds, the linearized problem becomes computationally intractable because it requires introducing even more additional variables and constraints. Hence, we conclude that the optimization problem for the case with two stationary exercise boundaries is computationally inefficient in the form introduced here. The following subsection suggests a simplification of the formulated problem considering only one exercise boundary. Later, we return to the case with two boundaries suggesting an efficient heuristic.

6.3 Optimization with One Optimal Stationary Exercise Boundary

This subsection considers a simplification of the optimization problem formulated above. Since it turned out to be hard to construct a computationally efficient optimization problem for the case with two stationary exercise boundaries, here we build an optimization problem assuming that the renter uses only one stationary boundary. Using the introduced notation, we assume that independently of the current state of the plant the renter always uses the stationary boundary defined by $\xi_{i,j}$s. In this case we do not need to introduce the auxiliary variables, χ_i^js, and we can construct an optimization problem using only $\xi_{i,j}$ grid variables. The objective of the new problem is obtained from the objective of $(P2B)$ by replacing χ_i^js with ξ_i^js, and the feasible region of the new problem is defined by the optimal exercise boundary shape constraints. Therefore, the optimization problem for finding a single optimal stationary exercise boundary is

$$
\begin{aligned}
(P1B): \quad & \max_{\xi_{i,j}} \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N} e^{-r(i-1)}\left(\xi_i^j\xi_{i-1}^j M_i^j - [\xi_i^j - \xi_{i-1}^j]^+ C_{s,i}^j - [\xi_{i-1}^j - \xi_i^j]^+ C_d\right) \\
& s.t. \\
& \qquad \xi_{i,j} \le \xi_{i+1,j}, \ i = 1, ..., (N_V - 1), \ j = 1, ..., N_H, \\
& \qquad \xi_{i,j} \ge \xi_{i,j+1}, \ i = 1, ..., N_V, \ j = 1, ..., (N_H - 1), \\
& \qquad \xi_{i,j} \in \{0, 1\}, \ i = 1, ..., N_V, \ j = 1, ..., N_H.
\end{aligned}
$$

Using the identity:

$$\forall\, x, y \in \{0,1\}: \ xy = x - [x-y]^+, \tag{14}$$

we rewrite $(P1B)$ in the following equivalent form:

$$
\begin{aligned}
(P1B'): \quad & \max_{\xi_{i,j}} \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N} e^{-r(i-1)}\left(\xi_i^j M_i^j - [\xi_i^j - \xi_{i-1}^j]^+ \left(C_{s,i}^j + M_i^j\right) - [\xi_{i-1}^j - \xi_i^j]^+ C_d\right) \\
& s.t. \\
& \qquad \xi_{i,j} \le \xi_{i+1,j}, \ i = 1, ..., (N_V - 1), \ j = 1, ..., N_H, \\
& \qquad \xi_{i,j} \ge \xi_{i,j+1}, \ i = 1, ..., N_V, \ j = 1, ..., (N_H - 1), \\
& \qquad \xi_{i,j} \in \{0, 1\}, \ i = 1, ..., N_V, \ j = 1, ..., N_H.
\end{aligned}
$$

The problem $(P1B')$ is not linear, but it can be easily linearized. In order to do this, we need to introduce auxiliary variables, ν_i^js and μ_i^js. $(P1B')$ transforms into the following equivalent problem:

$$
\begin{aligned}
(P1BL): \quad & \max_{\xi_{i,j}} \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N} e^{-r(i-1)}\left(\xi_i^j M_i^j - \nu_i^j\left(C_{s,i}^j + M_i^j\right) - \mu_i^j C_d\right) && (15)\\
& s.t. \\
& \qquad \nu_i^j \geq \xi_i^j - \xi_{i-1}^j,\ \nu_i^j \geq 0,\ i=1,...,N,\ j=1,...,N_S, && (16)\\
& \qquad \nu_i^j \leq 1-\xi_{i-1}^j,\ \nu_i^j \leq \xi_i^j,\ i=1,...,N,\ j=1,...,N_S, && (17)\\
& \qquad \mu_i^j \geq \xi_{i-1}^j - \xi_i^j,\ \mu_i^j \geq 0,\ i=1,...,N,\ j=1,...,N_S, && (18)\\
& \qquad \mu_i^j \leq 1-\xi_i^j,\ \mu_i^j \leq \xi_{i-1}^j,\ i=1,...,N,\ j=1,...,N_S, && (19)\\
& \qquad \xi_{i,j} \leq \xi_{i+1,j},\ i=1,...,(N_V-1),\ j=1,...,N_H, && (20)\\
& \qquad \xi_{i,j} \geq \xi_{i,j+1},\ i=1,...,N_V,\ j=1,...,(N_H-1), && (21)\\
& \qquad \nu_i^j, \mu_i^j \in \mathscr{R},\ \xi_{i,j} \in \{0,1\},\ i=1,...,N_V,\ j=1,...,N_H, && (22)
\end{aligned}
$$

where constraints (16)-(19) ensure that $\nu_i^j = [\xi_i^j - \xi_{i-1}^j]^+$ and $\nu_i^j = [\xi_{i-1}^j - \xi_i^j]^+$. There is one major problem with writing $(P1BL)$ in the form given above. Although we obtained a linear problem, the notation states that we need to introduce $N \cdot N_S$ auxiliary variables. It can be shown that there is a linearization of $(P1B)$ that requires introducing no more than $(N_V^2 \cdot N_H^2)$ new variables. In this case, the total number of variables in the problem depends only on the number of nodes in the grid and not on the number of sample paths or time periods. To make $(P1B)$ linear we need to introduce an auxiliary variable for each of the pairs (ξ_i^j, ξ_{i-1}^j) and (ξ_{i-1}^j, ξ_i^j). Hence, the total number of auxiliary variables needed equals to the total number of different ordered pairs (ξ_i^j, ξ_{i-1}^j) and (ξ_{i-1}^j, ξ_i^j). Remembering the notation introduced earlier, by writing variables ξ_i^j we assume the following mapping for the indexes $\binom{j}{i}$ (we denote it by $\binom{j}{i}^\xi$ because it corresponds to variables ξ):

$$\binom{j}{i}^\xi : \{1,...,N\} \times \{1,...,N_S\} \to \{1,...,N_H\} \times \{1,...,N_V\}.$$

From the last mapping, it follows that the total number of different pairs considered above is no greater than $N_V^2 \cdot N_H^2$. The alternative linear equivalent of $(P1B)$ can be obtained from $(P1BL)$ by eliminating redundant ν_i^js and μ_i^js. From the above, each ν_i^j is associated with a pair of pairs $\left(\binom{j}{i}^\xi, \binom{j}{i-1}^\xi\right)$ and each μ_i^j is associated with a pair of pairs $\left(\binom{j}{i-1}^\xi, \binom{j}{i}^\xi\right)$. To eliminate the redundancy we need to assume that any ν_i^js or μ_i^js that are associated with the same pair of pairs are synonyms of the same auxiliary variable. With this assumption, we should think of $(P1BL)$ as a linear version of $(P1B)$ that is free of sample path dependent variables. This step of leaving the notation of $(P1BL)$ unchanged should ease the reader's understanding of the material below. Otherwise, the reader would have to spend a considerable amount of time to get through a much more complicated indexing system. Summarizing, from here and below the reader should think of $(P1BL)$ and its consequent transformations as free of sample path dependent variables bearing in mind the trick with variable synonyms discussed above.

$(P1BL)$ allows us to find a single optimal stationary exercise boundary. In general, $(P1BL)$ may not be easy to solve because it is a mixed integer programming problem. Thus, we have to work on finding a better formulation. Let us write the following chain of derivations:

$$\begin{aligned}
&\forall\ i=1,...,N,\ j=1,...,N_S: \\
&M_i^j+C_{s,i}^j=\max\left(\overline{Q}\left(P_i^j-\overline{H}\cdot G_i^j\right)-\overline{K}+C_{s,i}^j,\underline{Q}\left(P_i^j-\underline{H}\cdot G_i^j\right)-\underline{K}+C_{s,i}^j\right)\geq \\
&\geq \underline{Q}\left(P_i^j-\underline{H}\cdot G_i^j\right)-\underline{K}+\left(C_s+L\cdot G_i^j\right)= \\
&=\underline{Q}\left(P_i^j-\underline{H}\cdot G_i^j\right)-\underline{K}+\left(C_s+\underline{Q}\cdot\underline{H}\cdot G_i^j\right)=\underline{Q}\cdot P_i^j-\underline{K}+C_s\geq C_s-\underline{K}.
\end{aligned}$$

¿From the above, we have

$$\forall\ i=1,...,N,\ j=1,...,N_S:\ C_s\geq\underline{K}\Rightarrow M_i^j+C_{s,i}^j\geq 0. \tag{23}$$

The condition $C_s \geq \underline{K}$ is usually satisfied for all real life setups. It is almost always the case that the fixed costs incurred during the startup period are much bigger than the fixed renting costs when the plant is in an energy generating mode. If the condition $C_s \geq \underline{K}$ is satisfied, then we can write down a problem equivalent to $(P1BL)$ that has a smaller number of constraints:

$$\begin{aligned}
(P1BL'):\quad &\max_{\xi_{i,j}}\quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N}e^{-r(i-1)}\left(\xi_i^jM_i^j-\nu_i^j\left(C_{s,i}^j+M_i^j\right)-\mu_i^jC_d\right)\\
&s.t.\\
&\qquad \nu_i^j\geq\xi_i^j-\xi_{i-1}^j,\ \nu_i^j\geq 0,\ i=1,...,N,\ j=1,...,N_S,\\
&\qquad \mu_i^j\geq\xi_{i-1}^j-\xi_i^j,\ \mu_i^j\geq 0,\ i=1,...,N,\ j=1,...,N_S,\\
&\qquad \xi_{i,j}\leq\xi_{i+1,j},\ i=1,...,(N_V-1),\ j=1,...,N_H,\\
&\qquad \xi_{i,j}\geq\xi_{i,j+1},\ i=1,...,N_V,\ j=1,...,(N_H-1),\\
&\qquad \xi_{i,j}\in\{0,1\},\ i=1,...,N_V,\ j=1,...,N_H.
\end{aligned}$$

$(P1BL')$ is obtained from $(P1BL)$ by removing from its formulation the upper bound constraints for ν_i^js and μ_i^js. These two problems are equivalent because of the fact that it is never optimal for ν_i^js and μ_i^js to be on their upper bounds because we are solving a maximization problem and have the condition that $M_i^j+C_{s,i}^j\geq 0$ (from (23)) and $C_d\geq 0$. Now we have everything set to formulate the main result of this section.

Theorem 3 *Consider the linear programming problem below:*

$$\begin{aligned}
(L1):\quad &\max_{\xi_{i,j}}\quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N}e^{-r(i-1)}\left(\xi_i^jM_i^j-\nu_i^j\left(C_{s,i}^j+M_i^j\right)-\mu_i^jC_d\right)\\
&s.t.\\
&\qquad \nu_i^j\geq\xi_i^j-\xi_{i-1}^j,\ \nu_i^j\geq 0,\ i=1,...,N,\ j=1,...,N_S,\\
&\qquad \mu_i^j\geq\xi_{i-1}^j-\xi_i^j,\ \mu_i^j\geq 0,\ i=1,...,N,\ j=1,...,N_S,\\
&\qquad \xi_{i,j}\leq\xi_{i+1,j},\ i=1,...,(N_V-1),\ j=1,...,N_H,\\
&\qquad \xi_{i,j}\geq\xi_{i,j+1},\ i=1,...,N_V,\ j=1,...,(N_H-1),\\
&\qquad 0\leq\xi_{i,j}\leq 1,\ i=1,...,N_V,\ j=1,...,N_H.
\end{aligned}$$

If the condition $C_s \geq \underline{K}$ is satisfied, then $(L1)$ has an optimal solution if and only if $(P1BL')$ has an optimal solution, and at least one optimal solution of $(L1)$ is optimal in $(P1BL')$.

This is an important theorem that reduces the problem of finding a single optimal stationary exercise boundary to solving a linear programming problem. Although not any optimal solution of $(L1)$ is optimal in $(P1BL')$, in the proof of the theorem we show that any optimal solution of $(L1)$ that is attained at a vertex of the feasible region is optimal in $(P1BL')$. The last conclusion has an important practical meaning. It says that if for solving $(L1)$ we use any linear programming solver that searches for optimal points only among vertices of the feasible region (as is the case with the simplex method based solvers), the optimal solution found is always a solution of $(P1BL')$. An important property of the derived linear program is that the maximal number of variables and constraints in it does not depend on the number of time periods or sample paths. This maximal number is always determined by the size of the grid. The number of variables and constraints in $(L1)$ is the same as the number of variables and constraints in $(P1BL')$ (the only difference is that $(L1)$ has box constraints for $\xi_{i,j}$s and $(P1BL')$ has integrality constraints instead). The proof of the theorem (see Appendix for the proof) relies on the result of the lemma below.

Lemma 5 *Consider a system of equations with n variables and n equations. Let $x_1, ..., x_n$ be the variables of the system. Assume that each equation of the system is one of the following types:*

$$
\begin{aligned}
&1)\ x_{i_1} = x_{i_2}, \quad \textit{for some } i_1, i_2 \in \{1, ..., n\},\\
&2)\ x_{i_1} = c_{i_1}, \quad \textit{for some } i_1 \in \{1, ..., n\}, \textit{ and some constant } c_{i_1} \in C \subseteq R.
\end{aligned}
$$

If the system has a unique solution $(x_1^*, ..., x_n^*)$, then the solution can be only of the following form:

$$\forall\ i \in \{1, ..., n\}: \ x_i^* = c_i, \ \textit{where } c_i \in C.$$

Proof. See Appendix. □

We finish this subsection by pointing out some key features of the introduced approach. The first point is that the suggested algorithm does not have an issue known in stochastic programming as anticipativity. Since there are no sample path dependent variables involved and the algorithm is based on variables defined on a grid with a rigid structure (imposed by monotonicity constraints), the algorithm is non-anticipative by construction. As it was mentioned earlier, the theoretical maximum number of variables and constraints in the constructed linear optimization problem has an order of $O(N_H^2 \cdot N_V^2)$. $O(N_H^2 \cdot N_V^2)$ is just a theoretical upper bound. For the real world problems the number of variables and constraints in the linear problem usually has an order of $O(N_H \cdot N_V)$. We can derive this estimate using simple arguments. For most of the real world price dynamics of energy and gas it is reasonable to assume that if (i, j) is the current price point on the grid, then a price point at the next time period will be in the vicinity of (i, j). In other words, if (i, j) is the current price point on the grid and (m, n) is a price point at the next period of time, then with a probability close to 1 we can claim that there holds

$$|i - m| \leq R, \ |j - n| \leq R,$$

for some fixed constant R. If this is the case, then the total number of auxiliary variables needed in the model has an order of $O(R \cdot N_H \cdot N_V) = O(N_H \cdot N_V)$ (it is a linear function of the number of nodes in the grid). Therefore, there are $O(N_H \cdot N_V)$ grid variables and $O(N_H \cdot N_V)$ auxiliary variables in the model. From the last statement, we get that the total number of variables in the model should have an order of $O(N_H \cdot N_V)$. The time of constructing the linear program has an order of $O(N \cdot N_S)$ since we have to go through each time period of every sample path.

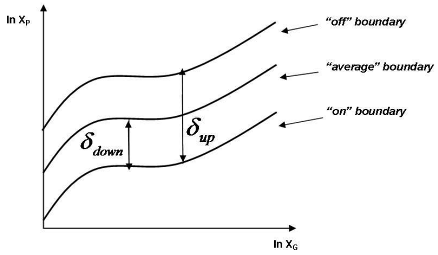


Figure 3. Heuristic for finding two optimal exercise boundaries.

6.4 Heuristic with Two Stationary Exercise Boundaries

The previous subsection suggested an algorithm for finding a single optimal stationary exercise boundary. Here and further throughout the paper we refer to this boundary as an optimal "average" boundary (this boundary is used for all states of the plant, that is why the boundary is "average" in some sense). Above, we also suggested an optimization problem ($P2B$) for finding optimal "off" and "on" state exercise boundaries, but it turned out that the suggested formulation is hard to solve. This subsection develops a heuristic for finding a suboptimal solution of ($P2B$) that we claim to be a good substitute for an optimal solution of ($P2B$). In the section presenting numerical results below, we examine the benefits of using suboptimal exercise boundaries given by the heuristic as compared to using an optimal "average" boundary given by ($L1$). The idea of the heuristic is to look for optimal "on" and "off" exercise boundaries in the class of exercise boundaries parallel to the "average" optimal exercise boundary given by ($L1$). By "parallel" we mean boundaries that can be obtained from the "average" boundary by a parallel shift (an up shift for the "off" boundary and a down shift for the "on" boundary). The heuristic works in two stages (see also Figure 3):

1) Assume that the "average" boundary found by solving ($L1$) is an optimal "off" boundary and find an optimal "on" boundary that is parallel to it (find an optimal size δ_{down} of the down shift).

2)Assume that the boundary found on the previous stage is an optimal "on" boundary and find an optimal "off" boundary that is parallel to it (find an optimal size δ_{up} of the up shift).

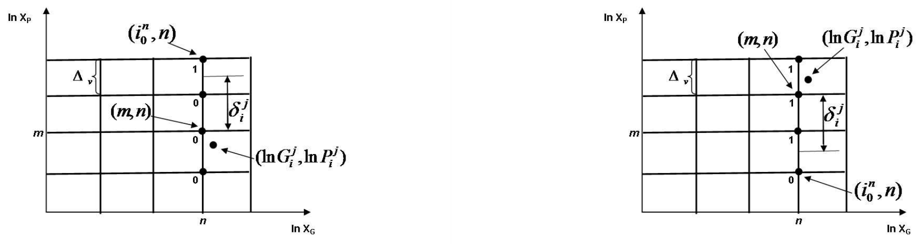
Using the notation introduced earlier in this section:

$$\chi_i^p = \begin{cases} \xi_i^p, \text{ if the current state of the plant is "off" and } i \neq 0, \\ \eta_i^p, \text{ if the current state of the plant is "on" and } i \neq 0, \\ 0, \text{ if } i = 0, \text{ assuming the initial state of the plant is "off"}. \end{cases}$$

Below, we make use of the following lemma.

Lemma 6

$$\forall\ i = \overline{1, N},\ p = \overline{1, N_S}:\ \chi_i^p = \sum_{k=1}^{i} \xi_k^p \prod_{l=k+1}^{i} (\eta_l^p - \xi_l^p).$$



(a) (b)

Figure 4. a) Distance between a point and the "off" boundary. b) Distance between a point and the "on" boundary.

Proof. See Appendix. □

Now assume that an optimal "off" boundary is known and equal to the "average" boundary found by solving $(L1)$. Using the notation introduced earlier, the last assumption means that the grid variables $\{\xi_{i,j}\}$ $i = \overline{1, N_V}$ and $j = \overline{1, N_H}$ are known. The heuristic approach suggests to find an "on" boundary as an optimal boundary that can be obtained by shifting the found "off" boundary downwards. Therefore, the task is to find an optimal size of the shift $\delta \geq 0$. Let $\delta \geq 0$ be an arbitrary shift size. For each point (G_i^j, P_i^j) let us also define δ_i^j, a vertical distance from a point $(\ln G_i^j, \ln P_i^j)$ to the "off" boundary. Let (m, n) be the closest grid node to the point $(\ln G_i^j, \ln P_i^j)$ on the logarithmic plane. Since we use the closest grid node as an interpolation rule on the grid, we approximate δ_i^j with a value equal to the distance between the "off" boundary and the node (m, n). Let i_0^n be the smallest index in the n^{th} column, such that $\xi_{i_0^n, n} = 1$, and $\triangle_v$ be the distance between two adjacent vertical nodes. Then,

$$\delta_i^j = y(i_0^n) - \frac{\triangle_v}{2} - y(m),$$

where $y(i)$ denotes the y-coordinate of points in the i^{th} row of the grid, see Figure 4(a). Because we consider only $\delta \geq 0$, to make the algorithm computationally more efficient we make the following adjustment to the computation of δ_i^j. We compute δ_i^j as defined above, and if we obtain $\delta_i^j < 0$ then we set $\delta_i^j = -\epsilon$ $(\epsilon > 0)$. Any positive value can be taken as the value of ϵ. In cases when $\xi_{N_V, n} = 0$, meaning that there are no 1s in the n^{th} column, we set $\delta_i^j = +\infty$. Since the "on" boundary is obtained from the "off" boundary by a shift of size δ, the following is true:

$$\forall\ i, j :\ \eta_i^j = sgn\left([\delta - \delta_i^j]^+\right), \tag{24}$$

where $sgn(\cdot)$ is a sign function. It is easy to notice, that for $\delta = 0$ we get $\eta_i^j = \xi_i^j$. For $\forall\ i, j$ we also define a set of indexes S_i^j and the maximal element k_i^j of the set S_i^j:

$$\begin{aligned} S_i^j &= \{k : \xi_k^j = 1,\ k \leq i,\ \xi_l^j = 0,\ l = \overline{k+1, i}\}, \\ k_i^j &= \begin{cases} \max_{k \in S_i^j}, \text{ if } S_i^j \neq \varnothing, \\ N+1, \text{ if } S_i^j = \varnothing. \end{cases} \end{aligned}$$

It can be shown that

$$|S_i^j| \leq 1,$$

and the following formula is true:

$$\forall\ j,\ \forall\ i > 1:\ k_i^j = \begin{cases} i, & \text{if } \xi_i^j = 1, \\ k_{i-1}^j, & \text{otherwise.} \end{cases}$$

From Lemma 6, we have

$$\forall\ i = \overline{1,N},\ j = \overline{1,N_S}:\ \chi_i^j = \begin{cases} \prod_{l=k_i^j+1}^{i} \eta_l^j,\ \text{if } k_i^j < i, \\ 1,\ \text{if } k_i^j = i, \\ 0,\ \text{if } k_i^j > i. \end{cases} \tag{25}$$

Applying (24) to (25):

$$\forall\ i = \overline{1,N},\ j = \overline{1,N_S}:\ \chi_i^j = \begin{cases} \prod_{l=k_i^j+1}^{i} sgn\left([\delta - \delta_l^j]^+\right) = sgn\left([\delta - \delta_{i,j}^*]^+\right),\ \text{if } k_i^j < i, \\ 1,\ \text{if } k_i^j = i, \\ 0,\ \text{if } k_i^j > i, \end{cases} \tag{26}$$

where $\delta_{i,j}^* = \max_{l=\overline{k_i^j+1,i}}\{\delta_l^j\}$. If we extend the definition of $\delta_{i,j}^*$:

$$\forall\ i = \overline{1,N},\ j = \overline{1,N_S}:\ \delta_{i,j}^* = \begin{cases} \max_{l=\overline{k_i^j+1,i}}\{\delta_l^j\} = \max\{\delta_{i-1,j}^*, \delta_i^j\},\ \text{if } k_i^j < i, \\ -\epsilon,\ \text{if } k_i^j = i, \\ +\infty,\ \text{if } k_i^j > i, \end{cases}$$

then (26) can be rewritten:

$$\forall\ i = \overline{1,N},\ j = \overline{1,N_S}:\ \chi_i^j = sgn\left([\delta - \delta_{i,j}^*]^+\right). \tag{27}$$

To find an optimal δ we need to solve a version of $(P2B)$ without the boundary shape constraints. We do not need the shape constraints because the "off" boundary is already known, and the shape of the "on" boundary is completely determined by the shape of the "off" boundary (since we are looking for a parallel boundary). Summarizing, we need to solve the problem:

$$\begin{aligned} &\max_\delta \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N} e^{-r(i-1)}\left(\chi_i^j\chi_{i-1}^j M_i^j - [\chi_i^j - \chi_{i-1}^j]^+ C_{s,i}^j - [\chi_{i-1}^j - \chi_i^j]^+ C_d\right) \\ &s.t. \\ &\qquad \delta \geq 0,\ (27)\ \text{constraints.} \end{aligned}$$

With (14) the last problem transforms into an equivalent problem:

$$\begin{aligned} (H1):\quad &\max_\delta \quad \frac{1}{N_S}\sum_{j=1}^{N_S}\sum_{i=1}^{N} e^{-r(i-1)}\left(\chi_i^j M_i^j - [\chi_i^j - \chi_{i-1}^j]^+\left(C_{s,i}^j + M_i^j\right) - [\chi_{i-1}^j - \chi_i^j]^+ C_d\right) \\ &s.t. \\ &\qquad \delta \geq 0,\ (27)\ \text{constraints.} \end{aligned}$$

(27) implies

$$[\chi_i^j - \chi_{i-1}^j]^+ = \begin{cases} 0,\ \text{if } \delta_{i,j}^* \geq \delta_{i-1,j}^*, \\ sgn\left([\delta - \delta_{i,j}^*]^+\right) - sgn\left([\delta - \delta_{i-1,j}^*]^+\right),\ \text{otherwise.} \end{cases} \tag{28}$$

$$[\chi_{i-1}^j - \chi_i^j]^+ = \begin{cases} 0, \text{ if } \delta_{i-1,j}^* \geq \delta_{i,j}^*, \\ sgn\left([\delta - \delta_{i-1,j}^*]^+\right) - sgn\left([\delta - \delta_{i,j}^*]^+\right), \text{ otherwise.} \end{cases} \tag{29}$$

From (27), (28), and (29) it follows that $(H1)$ reduces to a problem of the type:

$$\max_{\delta \geq 0} \sum_{i=1}^{M} a_i \cdot sgn\left([\delta - \beta_i]^+\right), \tag{30}$$

for some constants $a_i \in R$, $\beta_i \geq 0$, and integer $M > 0$. The last problem is easily solvable. Without loss of generality we assume $0 \leq \beta_1 \leq \beta_2 \leq ... \leq \beta_M$ (change variables if needed). We calculate

$$S_0 = 0, \; S_i = \sum_{j=1}^{i} a_j, \; i = \overline{1, M},$$
$$i^* = argmax_{i=\overline{0,M}}\{S_i\}.$$

Now we write down an optimal solution of the optimization problem above:

$$\delta^* = \begin{cases} \beta_{i^*+1}, \text{ if } i^* < M, \\ \beta_M(1+\epsilon), \; \forall \epsilon > 0, \text{ if } i^* = M. \end{cases}$$

At the second stage of the heuristic we assume that an optimal "on" boundary is known and equal to the boundary found at the first stage. Using our notation, the last assumption means that the grid variables $\{\eta_{i,j}\}$ $i = \overline{1, N_V}, j = \overline{1, N_H}$ are known. The heuristic suggests finding an "off" boundary as an optimal boundary that can be obtained by shifting the "on" boundary upwards. Thus, the task is to find an optimal size of the shift $\delta \geq 0$. To do this we need to follow steps similar to the steps we performed at the first stage. Again, we need to define a distance between a point on the logarithmic plane and the "on" boundary. Consider an arbitrary sample point (G_i^j, P_i^j). Let (m, n) be the closest grid node to the point $(\ln G_i^j, \ln P_i^j)$ on the logarithmic plane, i_0^n be the largest index in the n^{th} column, such that $\eta_{i_0^n,n} = 0$, and $\triangle_v$ be the distance between two adjacent vertical nodes. Then, a distance between $(\ln G_i^j, \ln P_i^j)$ and the "on" boundary is approximated as follows:

$$\delta_i^j = y(m) - y(i_0^n) - \frac{\triangle_v}{2},$$

where $y(i)$ denotes the y-coordinate of points in the i^{th} row of the grid, see Figure 4(b). As at the fist state, we make a similar adjustment of the formula above. If we have $\delta_i^j < 0$ then we set $\delta_i^j = -\epsilon$, where ϵ can be any positive constant. In cases when there are no 0s in the n^{th} column, we set $\delta_i^j = +\infty$. Let $\delta \geq 0$ be an arbitrary shift size. Then, $\forall i = \overline{1, N}, \; j = \overline{1, N_S}$ it is true that:

$$\xi_i^j = sgn\left([\delta_i^j - \delta]^+\right). \tag{31}$$

Below, we make use of the following lemma.

Lemma 7

$$\forall \delta \geq 0, \; \forall i = \overline{1, N}, \; j = \overline{1, N_S} : \; \chi_i^j = sgn\left([\gamma_{i,j} - \delta]^+\right),$$

where $\gamma_{i,j}$s are some constants that do not depend on δ.

Proof. See Appendix. □

Lemma 7 establishes a result similar to (27). From the proof of the lemma we deduce an explicit formula for computing $\gamma_{i,j}$:

$$\forall i > 1, \forall j: \ \gamma_{i,j} = \begin{cases} -\epsilon, \text{ if } \eta_i^j = 0, \\ \max\{\gamma_{i-1,j}, \delta_i^j\}, \text{ otherwise.} \end{cases}$$

For $i = 1$ we have

$$\forall j: \ \gamma_{1,j} = \delta_1^j.$$

Similarly to the first stage, we come to a problem analogous to (30):

$$\max_{\delta \geq 0} \sum_{i=1}^{M} a_i \cdot sgn\left([\beta_i - \delta]^+\right), \tag{32}$$

for some constants $a_i \in R, \beta_i \geq 0$, and some integer $M > 0$. Without loss of generality we assume $\beta_1 \geq \beta_2 \geq ... \geq \beta_M \geq 0$ (change variables if needed). We calculate

$$S_0 = 0, \ S_i = \sum_{j=1}^{i} a_j, \ i = \overline{1, M},$$
$$i^* = argmax_{i=\overline{0,M}}\{S_i\}.$$

Now we write down an optimal solution of the optimization problem above:

$$\delta^* = \begin{cases} \beta_{i^*+1}, \text{ if } i^* < M, \\ \beta_M(1-\epsilon), \ \forall 1 \geq \epsilon > 0, \text{ if } i^* = M. \end{cases}$$

Therefore, we showed both stages of the heuristic. As one can see, at each stage of the heuristic the optimization problem that has to be solved is easily solvable. Moreover, we derived explicit solutions of those formulations. The most time consuming operation when computing solutions is sorting of the coefficients. Since the maximal number of coefficients is bounded from above by a constant of an order $O(N_H \cdot N_V)$, the sorting time is bounded by $O(N_H \cdot N_V \cdot \ln(N_H \cdot N_V))$. The time of problem construction at each stage is $O(N \cdot N_S)$. Hence, the total time of the heuristic is $O(N_H \cdot N_V \cdot \ln(N_H \cdot N_V)) + O(N \cdot N_S)$, and usually it is of the same order as $O(N \cdot N_S)$.

7 Time Dependent Optimal Exercise Boundaries

In the previous section we developed algorithms for finding optimal stationary (time-independent) exercise boundaries. In this section we want to extend our approach for the case of time-dependent exercise boundaries. The underlying idea is simple. In the case with stationary exercise boundaries we projected all sample paths on the $(\ln X_G, \ln X_P)$ plane and built optimal exercise boundaries on that plane. Instead of considering just one $(\ln X_G, \ln X_P)$ plane we can consider as many planes as we want, each plane corresponding to a different time point. In this case, we consider a three-dimensional space $(\ln X_G, \ln X_P, t)$, and time cuts in this space. A time cut is a plane perpendicular to the time axes (t) and intersecting it at some point t_i. To define a time cut we only need to define t_i, a point of intersection with the time axes. Furthermore, we refer to a time cut, intersecting the time axes at some point t_i, as (t_i). Let us assume that we have n such time cuts:

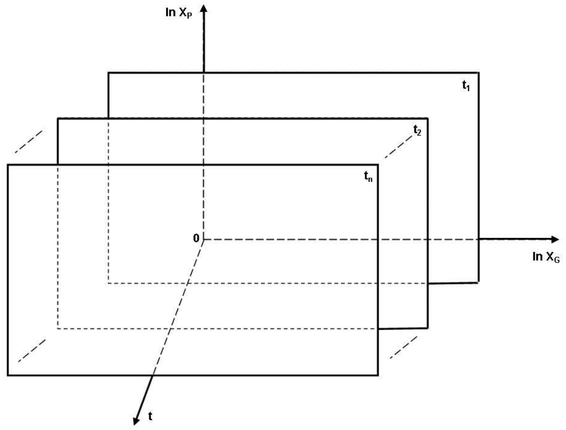


Figure 5. A sequence of time cuts for points $0 \leq t_1 < t_2 < ... < t_n \leq T$.

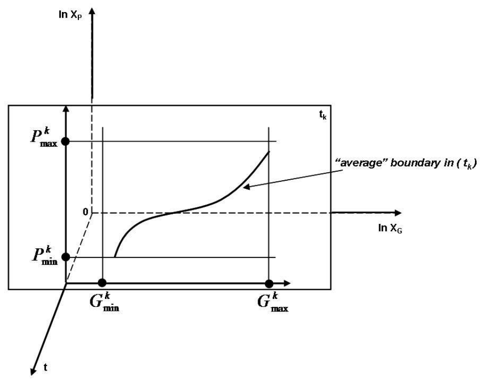


Figure 6. "Average boundary" in a cut (t_k).

$0 \leq t_1 < t_2 < ... < t_n \leq T$, where T is the length of the contract, see Figure 5. Now we can use algorithms from the previous section to build separate exercise boundaries for each time cut, (t_k). We need to follow the same steps as before. The main difference in this case is that different sample path points may be projected onto different planes (cuts). We apply the closest distance criterium in mapping sample path points onto the cuts. Consider an arbitrary sample path point, (G_i^j, P_i^j). Let an index i correspond to a time moment, T_i. Hence, in our three dimensional space the sample path point, (G_i^j, P_i^j), has coordinates, $(\ln G_i^j, \ln P_i^j, T_i)$. In this case, we project (perpendicularly) the point $(\ln G_i^j, \ln P_i^j, T_i)$ onto a cut (t_k), such that $k = argmin_{l=\overline{1,n}} |T_i - t_l|$. Now let us consider a case when we want to build a separate "average" exercise boundary in each (t_k). As before, at the first step, in each plane (t_k) we need to build a uniform grid in a similar way as we did before. Then, on each grid we need to introduce variables $\xi^k_{i,j}$s defining an "average" exercise boundary. Finally, using a given set of sample paths (G_i^j, P_i^j), $i = \overline{1,N}, j = \overline{1,N_S}$ we can build and solve an optimization problem similar to $(L1)$. As a result, we obtain n different exercise boundaries, one for each (t_k), see Figure 6. The points $t_1, ..., t_n$ do not have necessarily to be uniformly distributed

Table 1. Parameters of geometric Ornstein-Uhlenbeck processes.

a_e	3.0	a_g	2.25
b_e	3.2553	b_g	0.87
σ_e	0.79	σ_g	0.6
ρ	0.3		

Initial price of gas=$3.16, initial price of energy=$21.7.

on the time axes. A good strategy may be to use less points further from T, and more points closer to T, since time-dependence is more significant when there are relatively few periods remaining until the expiration of the contract.

Although we leave a detailed consideration of the subject beyond the scope of this paper, we would like to mention that it is also possible to extend the algorithms for building independent "on" and "off" boundaries by shifting an "average" boundary to the case with multiple time cuts.

8 Numerical Case Studies

In this section we consider case studies for two different power plant setups. The first setup is similar to the one considered in Deng and Oren (2003). We price a 10-year tolling agreement contract and compare the obtained results with the results reported in Deng and Oren (2003). The second setup is similar to the setup considered in Deng and Xia (2005). We price a 1-year tolling agreement contract and compare our results with the results reported in Deng and Xia (2005). We also investigate various characteristics of the introduced algorithms such as robustness with respect to the grid size, computational effects of the infinite horizon assumption, etc. At the end of the section we briefly discuss a procedure used for sample paths generation and provide final remarks regarding the computational results. We conducted all our experiments on a desktop PC equipped with Pentium(R) 4 CPU 3.80 GHz and 2.00 GB of RAM. We used CPLEX (R) 9.1 solver for solving linear programs.

8.1 Long Horizon Case Study

In this subsection we price a 10-year tolling agreement contract. We consider a power plant setup similar to the one used in Deng and Oren (2003). In this setup, the power plant has a ramp up period equal to 1 day. Hence, the length of one period in our model is set to 1 day. The power plant operating profit is computed on a 16 hours a day basis. We consider operation of the power plant during peak hours only (06:00 to 22:00). We use this modelling constraint in order to match the setup from Deng and Oren (2003). Keeping the notation introduced earlier, the remaining

Table 2. Prices (in million dollars) of 10-year tolling agreement contracts: calculated using the algorithm with an "average" exercise boundary versus the values reported in Deng and Oren (2003).

$\overline{H}$	Average	Std. Dev.	Deng	Difference (%)
7.5	40.02	0.30	40.8	1.9
8.5	30.71	0.33	32.12	4.4
9.5	23.00	0.25	24.82	7.3
10.5	16.81	0.21	18.88	10.9

$\overline{H}$=heat rate in the high capacity mode, Average=average contract price given by 20 runs of the algorithm with an "average" exercise boundary and using 1000 sample paths, Std. Dev.=standard deviation of the contract price estimates, Deng=contract price reported in Deng and Oren (2003), Difference (%)=(Deng-Average)/Deng.

parameters of the setup are given below:

$$\begin{aligned} N &= 3650, \\ \overline{Q} &= 100 * 16 = 1600 \text{ MWh per period}, \\ \underline{Q} &= 60 * 16 = 960 \text{ MWh per period}, \\ C_s &= \$8000, \\ r &= 4.5\%, \\ C_d &= \overline{K} = \underline{K} = 0. \end{aligned}$$

We conduct case studies for various levels of $\underline{H}$ and $\overline{H}$, but we keep the following ratio constant:

$$\overline{H} : \underline{H} = 1 : 1.38.$$

In Deng and Oren (2003), the ramp up costs function has an additional term that we do not have in our model. Although this term is non-significant, it can be modelled within our framework by adding the length of one period $\triangle_t$ to the fixed startup costs. Finalizing the setup description, we assume that the risk-neutral dynamics of the energy and the gas price processes are given by the following geometric Ornstein-Uhlenbeck processes:

$$\begin{aligned} d\ln P_t &= a_e\left(b_e - \ln P_t\right)dt + \sigma_e dW_t^1, \\ d\ln G_t &= a_g\left(b_g - \ln G_t\right)dt + \sigma_g dW_t^2, \\ dW_t^1 dW_t^2 &= \rho dt. \end{aligned}$$

The parameters for the Ornstein-Uhlenbeck processes are given in Table 1. Having defined all the model parameters, we proceed with pricing a 10-year tolling agreement contract. Below we provide numerical results for setups with four different pairs of heat rate levels $\overline{H}$ and $\underline{H}$. For each setup, we find the contract's price estimate using the developed algorithm with an optimal "average" stationary exercise boundary. We apply the suggested heuristic algorithm to find the suboptimal "off" and "on" stationary exercise boundaries. Then, we compare the price estimates given by both algorithms with the prices reported in Deng and Oren (2003). To generate one price estimate we make 20 independent runs of the algorithms using 1000 paths. Table 2 summarizes the results

Table 3. Prices (in million dollars) of 10-year tolling agreement contracts: calculated using the heuristic algorithm with two exercise boundaries versus the values reported in Deng and Oren (2003) and versus the prices given by the algorithm with an “average” exercise boundary.

$\overline{H}$	Average	Std. Dev.	Deng Diff. (%)	1 Boundary Diff. (%)
7.5	40.63	0.29	0.4	1.5
8.5	31.70	0.33	1.3	3.2
9.5	24.30	0.24	2.1	5.7
10.5	18.39	0.20	2.6	9.4

$\overline{H}$=heat rate in the high capacity mode, Average=average contract price given by 20 runs of the heuristic algorithm with two exercise boundaries and using 1000 sample paths, Std. Dev.=standard deviation of the contract price estimates, Deng Diff. (%)=(Deng from Table 2 -Average)/Deng from Table 2, 1 Boundary Diff. (%)=(Average-Average from Table 2)/Average from Table 2.

of the algorithm with one “average” exercise boundary and corresponding results of the algorithm presented in Deng and Oren (2003). Table 3 summarizes the results obtained by applying the heuristic algorithm using the “average” boundary found by the initial algorithm. This table also compares the obtained prices with the prices from Deng and Oren (2003). In addition, we show the advantage of using the heuristic algorithm by comparing the heuristic results with the results of the initial algorithm.

Table 2 shows that the initial algorithm with one “average” exercise boundary provides good estimates for contract prices in setups with the most efficient power plants ($\overline{H} = 7.5$ and $\overline{H} = 8.5$). In setups with less efficient power plants the error exceeds the 5% level but, nevertheless, the largest error is only slightly bigger than the 10% level. Overall, the algorithm generates reasonable lower bounds for contract prices in all considered scenarios. Table 3 presents computational results for the heuristic with two exercise boundaries. The heuristic turned out to be effective in reducing the computational error of the initial ”average” boundary algorithm. The largest error has been reduced from the 10.9% level to the mere 2.6% level. As reported in the table, the strategy with two exercise boundaries increases the operating profit on average from 1.5% to 9.4% depending on the efficiency of the power plant. Remarkably, all the computational benefits of the heuristic come almost at no additional computational costs. The heuristic does not require generating any additional sample paths, and the algorithms determining the optimal up and down shifts of the “average” boundary are time efficient.

The results show that with only 1000 sample paths and 20 independent runs we are able to achieve low levels of the standard deviation. The largest standard deviation reported in the tables reaches only 1.3% (when $\overline{H} = 10.5$) of the corresponding average estimate for the algorithm with one exercise boundary and only 1.1% (when $\overline{H} = 10.5$) for the algorithm with two exercise boundaries. When we increased the number of sample paths to 2000 the largest standard deviation sharply dropped to 0.6%. We do not report here the results of the experiments with 2000 sample paths because standard deviation is the only significantly changed quantity.

Table 4. Prices (in million dollars) of 10-year tolling agreement contracts: calculated on an 80x80 grid versus calculated on a 50x50 grid.

$\overline{H}$	50x50 Average	50x50 Std. Dev.	80x80 Average	80x80 Std. Dev.
7.5	40.63	0.29	40.66	0.30
8.5	31.70	0.33	31.74	0.35
9.5	24.30	0.24	24.31	0.23
10.5	18.39	0.20	18.40	0.26

$\overline{H}$=heat rate in the high capacity mode, 50x50 Average=average contract price given by 20 runs of the heuristic algorithm using 1000 sample paths and calculated on a 50x50 grid, 50x50 Std. Dev.=standard deviation of the contract price estimates calculated on a 50x50 grid, 80x80 Average= average contract price given by 20 runs of the heuristic algorithm using 1000 sample paths and calculated on an 80x80 grid, 80x80 Std. Dev.=standard deviation of the contract price estimates calculated on an 80x80 grid.

We have not discussed yet the granularity of the grid in the experiments. Robustness with respect to the grid granularity is one of the most important properties of any algorithm defined on a grid. Lack of this type of robustness makes the algorithm too sensitive to the particular choice of the grid, and the obtained experimental results cannot be considered reliable. The results above were obtained from running the algorithms on a grid with 50 vertical and 50 horizontal nodes. To test the robustness of the reported results we conducted analogous case studies with a grid having 80 vertical and 80 horizontal nodes. The results of the experiments with two exercise boundaries are presented in Table 4. The table shows that increasing the granularity of the grid has little influence on the results. Numerical experiments show that the suggested algorithm is robust with respect to the granularity of the grid. The algorithm is robust because of explicit incorporation of the exercise boundary shape constraints into the optimization model.

To estimate the time efficiency of the algorithm, Table 5 reports computational time of one run of the optimization algorithm finding an "average" boundary for the setup with 1000 sample paths and a 50x50 grid. The choice of $\overline{H}$ has no influence on computational time, but for the sake of determination we took $\overline{H} = 7.5$. For the same setup we also provide computational time of running the heuristic that determines the suboptimal "off" and "on" boundaries. The table shows that the algorithm needs 13 seconds to find an optimal "average" boundary, and then it takes 4 seconds for the heuristic to determine the suboptimal "on" and "off" boundaries. Summarizing, using the introduced computational scheme it takes only 17 seconds to generate one estimate of a 10-year horizon tolling agreement contract price, and 20 runs of the algorithm provides an accurate estimate with a standard deviation as low as 1%.

Concluding the subsection, we would like to stress that the results generated by our algorithm closely match the results reported in Deng and Oren (2003). Unfortunately, it is not possible to do a true comparison of our results with the results in Deng and Oren (2003) because the theoretical contract prices are not known. In favor of the approach suggested in Deng and Oren (2003) is the fact that the authors found time-dependent optimal exercise boundaries. The authors discretized the underlying energy and gas price processes on a lattice and then applied a dynamic programming algorithm. From our view, the main difficulty in this approach is to estimate the error coming from

Table 5. Computational times (in seconds) of pricing a 10-year tolling agreement contract using 1000 sample paths and a 50x50 grid.

"Average" Boundary		"On" Boundary	"Off" Boundary
Forming Problem	Solving Problem		
7	6	2	2

"Average" Boundary=time of finding the optimal "average" boundary, Forming Problem=time of constructing an optimization problem for finding the optimal "average" boundary, Solving Problem=time of solving a linear programming problem determining the optimal "average" boundary, "On" Boundary=time of the heuristic finding the optimal down shift determining the suboptimal "on" boundary, "Off" Boundary=time of the heuristic finding the optimal up shift determining the suboptimal "off" boundary.

the convergence speed of a lattice process to the corresponding continuous process. The authors in Deng and Oren (2003) do not provide such estimates, but since in the considered setup the size of a time period in the lattice used is small relative to the contract's horizon, we expect the results to be in the vicinity of the corresponding theoretical values.

For the setups where it is possible to represent price processes on a lattice and the error of such a representation is small, the dynamic programming based methods are the preferable alternatives when time-dependent boundaries are needed and there are no strict constraints in time efficiency. When time efficiency is needed or when it is hard represent price processes on a lattice (in the case of price dynamics exhibiting jumps or defined by a set of historical sample paths) we see our method as a valuable alternative. The reported results show that when contract horizon is large the error coming from the approximation of time-dependent boundaries with a time-independent one is small.

8.2 Short Horizon Case Study

This subsection considers pricing of a 1-year tolling agreement contract under the setup similar to the one described in Deng and Xia (2005). We assume the power plant operates 24 hours a day. In our model one time period equals 12 hours. Keeping our usual notation, the other parameters of the power plant are:

$$\begin{aligned} N &= 730, \\ \overline{Q} &= 150 * 12 = 1800 \text{ MWh per period}, \\ \underline{Q} : \overline{Q} &= 0.2 : 1, \\ C_s &= \$2000, \\ r &= 5\%, \\ C_d &= \$1000, \\ \overline{K} &= \underline{K} = 0. \end{aligned}$$

We consider various levels of $\underline{H}$ and $\overline{H}$, but we keep the following ratio constant:

$$\overline{H} : \underline{H} = 1 : 1.38.$$

Table 6. Parameters of geometric Ornstein-Uhlenbeck processes for the short horizon case study.

a_e	0.0651	a_g	0.0087
b_e	3.5527	b_g	1.3638
σ_e	0.1507	σ_g	0.0468
ρ	0.177		

Note: Initial price of gas=\$3, initial price of energy=\$34.7.

Table 7. Prices (in million dollars) of 1-year tolling agreement contracts calculated using an "average" exercise boundary.

$\overline{H}$	Average	Std. Dev.
7.5	15.83	0.15
8.0	13.92	0.13
10.5	4.59	0.12
13.5	0.09	0.03

Note: $\overline{H}$=heat rate in the high capacity mode, Average=average contract price given by 30 runs of the algorithm with an "average" exercise boundary and using 500 sample paths, Std. Dev.=standard deviation of the contract price estimates.

As in the previous case study, in Deng and Xia (2005) the ramp-up costs function has an additional term that is not present in our model. Although this term is non-significant, it can be easily modelled within our framework by adding the length of one period $\triangle_t$ to the fixed startup costs. Finalizing the setup description we assume that the risk-neutral dynamics of the energy and gas price processes are given by the following geometric Ornstein-Uhlenbeck processes:

$$\begin{aligned} d\ln P_t &= a_e\left(b_e - \ln P_t\right)dt + \sigma_e dW_t^1, \\ d\ln G_t &= a_g\left(b_g - \ln G_t\right)dt + \sigma_g dW_t^2, \\ dW_t^1 dW_t^2 &= \rho dt. \end{aligned}$$

The parameters for the Ornstein-Uhlenbeck processes are given in Table 6. Below we provide numerical results of pricing a 1-year tolling agreement contract with four different pairs of heat rate levels, $\overline{H}$ and $\underline{H}$. For each setup we find the contract's price estimate for the case with an optimal "average" stationary exercise boundary. Then, we apply the heuristic algorithm to find the suboptimal "off" and "on" stationary exercise boundaries. To generate one price estimate we make 30 independent runs of the algorithms using 500 paths.

Tables 7 and 8 contain the estimates of 1-year contract prices calculated using the algorithm with one "average" exercise boundary and the heuristic with two exercise boundaries, correspondingly. The results reported in the tables are almost identical and the heuristic algorithm provides almost no advantage compared to the base algorithm with one boundary. This can be explained by low switching costs in the model and a small number of switches between different regimes required by the optimal operating strategy.

Table 9 reports the computational time of one run of the algorithm for finding an "average" boundary for the setup with 500 sample paths and a 50x50 grid. The choice of $\overline{H}$ has no influence

Table 8. Prices (in million dollars) of 1-year tolling agreement contracts.

$\overline{H}$	2b. Avg.	2b. Std.	Deng Avg.	Deng Std.	No Costs Avg.	No Costs Std.
7.5	15.83	0.15	16.29	0.32	15.86	0.005
8.0	13.92	0.13	15.08	0.32	13.93	0.004
10.5	4.61	0.12	8.91	0.29	4.64	0.005
13.5	0.11	0.03	4.87	0.2	0.18	0.001

Note: $\overline{H}$=heat rate in the high capacity mode, Avg.=average contract price given by 30 runs of the heuristic algorithm with two exercise boundaries and using 500 sample paths, Std.=standard deviation of the contract price estimates given in Avg., Deng Avg.=average contract price reported in Deng and Xia (2005), Deng Std.=standard deviation of the contract price estimates given in Deng Std., No Costs Avg.=average contract prices for the idealized no-costs power plant setup calculated using Monte Carlo simulation with 1000 sample paths and 50 independent runs, No Costs Std.=standard deviation of the contract price estimates given in No Costs Std..

Table 9. Computational times (in milliseconds) of pricing a 1-year tolling agreement contract using 500 sample paths and a 50x50 grid.

"Average" Boundary		"On" Boundary	"Off" Boundary
Forming Problem	Solving Problem		
547	625	280	280

Note: "Average" Boundary=time of finding the optimal "average" boundary, "Forming Problem"=time of constructing an optimization problem for finding the optimal "average" boundary, "Solving Problem"=time of solving a linear programming problem determining the optimal "average" boundary, "On" Boundary=time of the heuristic finding the optimal down shift determining the suboptimal "on" boundary, "Off" Boundary=time of the heuristic finding the optimal up shift determining the suboptimal "off" boundary.

on computational time, but for the sake of determination we took $\overline{H} = 13.5$. For the same setup we also provide computational time for the heuristic determining suboptimal "off" and "on" boundaries. The table shows that it takes 1.172 seconds to find an optimal "average" boundary, and then it takes 0.56 seconds for the heuristic to determine the suboptimal "on" and "off" boundaries.

Our initial reason for choosing the particular power plant setup was to compare the results obtained by our algorithms with results reported in Deng and Xia (2005) (see Table 8). Case study showed that the prices from Deng and Xia (2005) are significantly higher than the prices produced by our algorithms. The difference in the prices takes an extreme value in the setups with $\overline{H} = 10.5$ and $\overline{H} = 13.5$. To investigate this disparity, we ran another case study. We considered an idealized version of the previous power plant setting. We eliminated the ramp-up period constraint and all the costs from the setup. In this idealized setting the optimal operating policy is trivial: run the plant in the high capacity mode whenever the spark spread is positive and shut down the plant in all other cases. In order to get an estimate of the contract price in this setting it is enough to conduct a simple Monte Carlo simulation.

Table 10. Prices (in million dollars) of 1-year tolling agreement contracts calculated using the heuristic algorithm with two exercise boundaries versus the prices computed for the idealized power plant setup without costs.

$\overline{H}$	Price with Costs	Price without Costs	Error Upper Bound (%)
7.5	15.83	15.86	0.19
8.5	13.92	13.93	0.07
9.5	4.61	4.64	0.65
10.5	0.11	0.18	63.6

Note: $\overline{H}$=heat rate in the high capacity mode, Price with Costs=2b. Avg. from Table 8, Price without Costs =No Costs Avg. from Table 8, Error Upper Bound (%)=upper bound for the pricing error produced by the heuristic with two boundaries ((Price without Costs-Price with Costs)/Price with Costs).

Table 8 reports the results of the conducted Monte Carlo simulations. To generate one estimate we used 1000 sample paths and 50 independent runs. Analyzing the results, one should expect the contract prices for the idealized power plant setup to be the upper bounds for the contract prices for the power plant setup with operational costs. We found that the results reported in Deng and Xia (2005) significantly exceed the results computed for the setup without costs. There may be a number of potential explanations for this finding. The simplest explanation could be that we are comparing mismatched setups due to our misinterpretation of some parts of the setup described in Deng and Xia (2005). Other potential explanations include typos in the specification of the power plant setup parameters or typos in the documentation of the numerical results in Deng and Xia (2005). It is also likely that potential problems with computational results in Deng and Xia (2005) come from the modelling approach. Deng and Xia use a simple Euler scheme to generate sample paths given by the geometric Ornstein-Uhlenbeck dynamics. A number of researchers warned about potential problems with the Euler scheme. Discretization errors associated with this scheme may lead to various kinds of computational instabilities. One can find discussions of this topic in Glasserman (2004). Deng and Xia also use the Tsitsiklis and Van Roy form (Tsitsiklis and Van Roy (2001)) of the regression algorithm for the dynamic programming equations. Carmona and Ludkovski (2008) mention that empirical evidence shows poor convergence properties of the Tsitsiklis and Var Roy form of the regression.

Contrary to the results from Deng and Xia (2005), the price estimates computed using our algorithms never exceed the contract price estimates for the idealized setup. Analyzing the results from Table 10, we can conclude that for all the heat rates but one ($\overline{H} = 13.5$) our heuristic produces accurate contract prices, since the pricing errors are bounded from above by 0.65% (such proximity to the prices from the idealized setup is explained by small number of switches needed). In the case of $\overline{H} = 13.5$, the operating costs are significant compared to the expected profit and the price for the idealized setup can no longer serve as a good upper bound.

We finish this subsection by investigating the effect of using time-dependent exercise boundaries. For these purposes we conduct a case study using the algorithm with four time-dependent exercise boundaries. This algorithm is a generalization of the algorithm with one stationary exercise boundary and was described in the preceding section.

Table 11. Prices (in million dollars) of 1-year tolling agreement contracts calculated using the algorithm with time-dependent exercise boundaries.

$\overline{H}$	Average	Std. Dev.
7.5	15.87	0.11
8.0	13.91	0.12
10.5	4.60	0.10
13.5	0.11	0.02

Note: $\overline{H}$=heat rate in the high capacity mode, Average=average contract price given by 30 runs of the algorithm with four time-dependent optimal exercise boundaries and using 1000 sample paths, Std. Dev.=standard deviation of the contract price estimates.

Table 11 reports numerical results of the conducted case study. We used 30 runs and 1000 sample paths to generate one estimate. We consider time cuts at the following points on the time axis: 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1.

Looking at Table 11 and Table 7 we can find little or no difference in the reported results. Therefore, the numerical experiment shows that infinite horizon is a reasonable assumption for the considered setup.

Summarizing, this subsection priced a short horizon tolling agreement contract similar to the one considered in Deng and Xia (2005). We showed the accuracy of the obtained price estimates by comparing them with the corresponding price estimates for the idealized setup without operational constraints. We also investigated effects of the infinite horizon assumption and showed validity of the approach for the considered setup.

8.3 Sample Path Simulation

Here we would like to give some comments regarding the sample path simulation procedure we used in our experiments. When generating Ornstein-Uhlenbeck sample paths we were thoroughly following the procedures described in Glasserman (2004). The simulation algorithm is relying on the explicit form of the solution to the stochastic differential equation followed by an Ornstein-Uhlenbeck process. Contrary to the Euler scheme, the simulation algorithm we adopted is free of convergence problems associated with the size of the discretization step. For more detail see Glasserman (2004).

8.4 Final Remarks

In this subsection we would like to provide some final thoughts on the problem of time dependency of optimal exercise boundaries and applicability of the introduced algorithm.

First of all, we would like to stress that the optimal exercise boundary time independency assumption in the generic version of the algorithm limits applicability of the algorithm to pricing only contracts with relatively large horizons. The reported results showed that the algorithm performed well in a long horizon case study closely matching the prices generated by a dynamic programming algorithm with time dependent boundaries. Although we confirmed no evidence of algorithm instabilities in the conducted short horizon case study, in general we expect the algorithm to be more successful in pricing contracts with longer horizons. Our intuition suggests that contract horizons of 2 years and more should be long enough to justify the use of time-independent boundaries. We expect that when horizon is long enough, time-independent boundaries can potentially lead to sub-optimal operating decisions only at a small fraction of time points that are close to the expiration and this should have only a small impact on the profit objective function.

Section 7 presents the first step toward building an efficient algorithm with time dependent optimal exercise boundaries. In the short horizon case study we applied the discussed version of the time-dependent algorithm to test sensitivity of the results to the number of optimal exercise boundaries used. Although it remained outside the scope of the paper, it is also possible to generalize the heuristic that builds separate "on" and "off" exercise boundaries from a single "average" exercise boundary on the time dependent case.

Despite the conducted case study, the discussed version of the time-dependent algorithm remains rather theoretical than practical in its current form. In order to make the time-dependent version practical more research is needed in identifying time-dependent shape properties of optimal exercise boundaries. An important step would be to find at what time cut planes it is optimal to build optimal exercise boundaries and what is the minimal number of time cut planes needed to efficiently price a contract. With this regard, we would like to reference an interesting paper on a closely related problem of pricing American put options. In AitSahlia (1999) the authors show that the optimal exercise boundary in the American put option problem can be efficiently approximated with piecewise linear functions (in a transformed space) using a small number of knots. Ideas from AitSahlia (1999) could potentially lead to a development of an efficient extension of our algorithm to the time dependent case.

9 Summary

We introduced a new approach for pricing tolling agreement contracts. The suggested approach can incorporate various power plant operational constraints such as startup and shutdown costs, fixed renting costs, variable output capacity levels, output capacity dependent heat rates, ramp up period delay, and ramp up period costs. The developed framework does not rely on any particular energy or fuel price process dynamics. The approach can deal with a general class of underlying price dynamics that includes Ornstein-Uhlenbeck processes with and without jumps and dynamics defined by historical sample paths.

The introduced algorithm reduces the problem of pricing tolling agreement contracts to a single linear programming problem. The existence of computationally efficient techniques for solving linear programming problems is the primary reason for computational efficiency of the algorithm. The developed algorithm remains efficient for contract horizons of 10 years and larger. An important property is that the maximum number of variables and constraints in the linear programming problem does not depend on the length of the contract or the number of sample paths in the model. The algorithm also provides an easy to use power plant operating strategy described by exercise boundaries.

The algorithm is computationally stable and produces low variance contract price estimates even with a relatively small number of sample paths. It is also robust with respect to the granularity of the grid on which the optimal exercise boundaries are defined. In the conducted case studies we also investigated computational effects of optimal exercise boundaries time independence assumption. One of the important factors of the numerical stability of the algorithm is an implicit incorporation of shape properties of optimal exercise boundaries into the linear optimization problem via a set of monotonicity constraints on grid variables.

Of course, the suggested approach has its own limitations and simplifying assumptions. One of the most important issues that remains open is theoretical investigation of the infinite horizon assumption for the optimal exercise boundaries. Although the assumption proved to be reasonable in the conducted cases studies, it would be important to understand under what conditions the assumption fails to be justified. It is also important to theoretically estimate the error coming from the heuristic that builds "on" and "off" exercise boundaries from a single "average" exercise boundary using a parallel shift.

In cases when time dependent boundaries are needed it is possible to use a time-depended version of the introduced algorithm. In this paper we briefly outlined how to generalize the algorithm for finding a single "average" exercise boundary on the time-dependent case but further research is needed in order to build a more efficient algorithm for the case of time dependent optimal exercise boundaries. We provided a brief outline of research in this direction.

Appendix

Proof of Lemma 1 . From (6) it flows that $\xi_1^* = 1$ is optimal in $P_N^1(1)$ if and only if:

$$M_1 - f_1(1,1) + e^{-r} E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] \geq -f_1(1,0) + e^{-r} E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \Leftrightarrow$$
$$\Leftrightarrow E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \geq e^r\left(-C_d - M_1\right).$$

Similarly, from (6) it flows that $\xi_1^* = 1$ is optimal in $P_N^1(0)$ if and only if:

$$-f_1(0,1) + e^{-r} E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] \geq -f_1(0,0) + e^{-r} E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \Leftrightarrow$$
$$\Leftrightarrow E\left[J_{N-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{N-1}^2(0)|\mathscr{F}_1\right] \geq e^r \hat{C}_s^1.$$

The lemma is proved. □

Proof of Lemma 2 . The proof is straightforward:

$$M_i + \hat{C}_s^i + C_d = \max\left(\overline{Q}\left(P_i - \overline{H}\cdot G_i\right) - \overline{K} + \hat{C}_s^i + C_d, \underline{Q}\left(P_i - \underline{H}\cdot G_i\right) - \underline{K} + \hat{C}_s^i + C_d\right).$$

It is also true that:

$$\underline{Q}\left(P_i - \underline{H}\cdot G_i\right) - \underline{K} + \hat{C}_s^i + C_d = \underline{Q}\cdot P_i - \underline{Q}\cdot\underline{H}\cdot G_i - \underline{K} + \left(C_s + L\cdot G_i\right) + C_d =$$
$$= \underline{Q}\cdot P_i + \left(L - \underline{Q}\cdot\underline{H}\right)G_i + \left(C_s + C_d - \underline{K}\right) \geq \underline{Q}\cdot P_i \geq 0.$$

Here we used the facts that $L = \underline{Q}\cdot\underline{H}$ and $C_s + C_d \geq \underline{K}$ (see Section 2). Since the second expression under the max sign is always nonnegative, the whole max expression is always nonnegative. Hence, the lemma is proved. □

Proof of Theorem 1 . We work the proof using the method of mathematical induction by N.
1). First, we verify the statement for $N = 1$. In this case the problem $P_N^1(0)$ takes the form:

$$\max_{\xi_1}\left(-\hat{C}_s^1\xi_1\right).$$

Since $\hat{C}_s^1 \geq 0$, the optimal solution of the problem above is $\xi_1^{*0} = 0$. Hence, whatever the optimal solution of $P_N^1(1)$ is, it is always true: $\xi_1^{*0} \leq \xi_1^{*1}$.
2). Assume the statement is true for $N = K - 1$ $(K > 1)$. Let us prove the theorem's result for $N = K$. Using (6):

$$P_K^1(0): \quad \max_{\xi_1}\left(-f_1(0,\xi_1) + e^{-r}E\left[J_{K-1}^2(\xi_1)\right]\right)$$
$$P_K^1(1): \quad \max_{\xi_1}\left(M_1\xi_1 - f_1(1,\xi_1) + e^{-r}E\left[J_{K-1}^2(\xi_1)\right]\right)$$

Let $\xi_1^{*0} = 1$ be optimal in $P_K^1(0)$, then from Lemma 1 it has to be satisfied:

$$E\left[J_{K-1}^2(1)\right] - E\left[J_{K-1}^2(0)\right] \geq e^r\hat{C}_s^1. \tag{33}$$

Using the same lemma, the necessary and sufficient condition for $\xi_1^{*1} = 1$ to be optimal in $P_K^1(1)$ is

$$E\left[J_{K-1}^2(1)\right] - E\left[J_{K-1}^2(0)\right] \geq e^r\left(-C_d - M_1\right). \tag{34}$$

From Lemma 2:

$$\hat{C}_s^1 \geq -C_d - M_1. \tag{35}$$

Applying (35) to (33) we get (34), hence it is true that $\xi_1^{*1} = 1$ is optimal in $P_K^1(1)$. Thus, we claim

$$\xi_1^{*1} \geq \xi_1^{*0}. \tag{36}$$

For any realized values of G_2 and P_2 let us consider the problem:

$$P_{K-1}^2 : \sup_{\xi_2,\dots,\xi_K} E[S(2,K)|\mathscr{F}_2].$$

Denote by $\xi^{*(K-1)}(w,1)$ and $\xi^{*(K-1)}(w,0)$ optimal solutions of the problems $P_{K-1}^2(1)$ and $P_{K-1}^2(0)$, respectively. Since P_{K-1}^2 has $(K-1)$ time periods, applying the induction's hypothesis we have

$$\forall\ w \in \Omega,\ \forall\ i = \overline{2,K} :\ \xi_i^{*(K-1)}(w,1) \geq \xi_i^{*(K-1)}(w,0). \tag{37}$$

From (36) and (37):

$$\forall\ w \in \Omega,\ \forall\ i = \overline{2,K} :\ \xi_i^{*(K-1)}(w,\xi_1^{*1}) \geq \xi_i^{*(K-1)}(w,\xi_1^{*0}). \tag{38}$$

Applying the principle of optimality:

$$\begin{aligned} \xi_i^{*(K-1)}(w,\xi_1^{*1}) &= \xi_i^{*1}(w), && (39) \\ \xi_i^{*(K-1)}(w,\xi_1^{*0}) &= \xi_i^{*0}(w). && (40) \end{aligned}$$

Finally, from (38), (39) and (40):

$$\forall\ w \in \Omega,\ \forall\ i = \overline{2,K} :\ \xi_i^{*1}(w) \geq \xi_i^{*0}(w) \tag{41}$$

Combining (41) with (36), we get the theorem's statement. Therefore, the induction step is shown and this completes the proof of the theorem. □

Proof of Lemma 3 . For $\forall\ i = \overline{1,N}$ consider the derivations below.

If $\underline{Q}\,(P_i - \underline{H} \cdot G_i) - \underline{K} > \overline{Q}\,\left(P_i - \overline{H} \cdot G_i\right) - \overline{K}$, then it is true:

$$\begin{aligned} M_i &= \underline{Q}\,(P_i - \underline{H} \cdot G_i) - \underline{K} \Rightarrow \\ M_i' - M_i &\geq \left(\underline{Q}\left(P_i' - \underline{H} \cdot G_i'\right) - \underline{K}\right) - \left(\underline{Q}\,(P_i - \underline{H} \cdot G_i) - \underline{K}\right) = \\ &= \underline{Q}\left(P_i' - P_i\right) + \underline{Q} \cdot \underline{H}\left(G_i - G_i'\right) \Rightarrow \\ M_i' - M_i &\geq \underline{Q} \cdot \underline{H}\left(G_i - G_i'\right) = L\left(G_i - G_i'\right). \end{aligned}$$

Analogously, if $\underline{Q}\,(P_i - \underline{H} \cdot G_i) - \underline{K} \leq \overline{Q}\,\left(P_i - \overline{H} \cdot G_i\right) - \overline{K}$ we get:

$$\begin{aligned} M_i' - M_i &\geq \overline{Q} \cdot \overline{H}\left(G_i - G_i'\right) \Rightarrow \\ M_i' - M_i &\geq \underline{Q} \cdot \underline{H}\left(G_i - G_i'\right) = L\left(G_i - G_i'\right). \end{aligned}$$

Above we used a model constraint: $\overline{Q} \cdot \overline{H} \geq \underline{Q} \cdot \underline{H}$. Hence, we proved

$$M_i' - M_i \geq L\left(G_i - G_i'\right). \tag{42}$$

Finally, because $\hat{C}_s^i - \hat{C}_s^{i\prime} = L\,(G_i - G_i')$, and using (42) we come to

$$M_i' + \hat{C}_s^{i\prime} \geq M_i + \hat{C}_s^i.$$

Hence, the lemma is proved. □

Proof of Lemma 4. Let ξ^{*1} and ξ^{*0} be optimal solutions of the problems $P_N^1(1)$ and $P_N^1(0)$, respectively.

1) If $\xi_1^{*1} = 0$ then from Theorem 1 $\xi_1^{*0} = 0$. Using (6):

$$J_N^1(1) - J_N^1(0) = -C_d.$$

Then, applying Lemma 2:

$$M_1 + \hat{C}_s^1 \geq J_N^1(1) - J_N^1(0).$$

Summarizing, it is true:

$$M_1 + \hat{C}_s^1 \geq J_N^1(1) - J_N^1(0) \geq -C_d.$$

2) Now we consider the case when $\xi_1^{*1} = 1$. If $\xi_1^{*0} = 1$ then applying (6) we have

$$J_N^1(1) - J_N^1(0) = M_1 + \hat{C}_s^1.$$

Then, from Lemma 2:

$$J_N^1(1) - J_N^1(0) \geq -C_d.$$

Summarizing, it is true:

$$M_1 + \hat{C}_s^1 \geq J_N^1(1) - J_N^1(0) \geq -C_d. \tag{43}$$

In the case $\xi_1^{*0} = 0$ we apply the following arguments. Since $\xi_1^{*0} = 1$ is suboptimal in $P_N^1(0)$, then it is true:

$$J_N^1(1) - J_N^1(0) \leq M_1 + \hat{C}_s^1. \tag{44}$$

Because $\xi_1^{*1} = 0$ is suboptimal in $P_N^1(1)$, we get the inequality:

$$J_N^1(1) - J_N^1(0) \geq -C_d. \tag{45}$$

From (43),(44), and (45) we get the assertion of the lemma for case 2). Hence, the lemma is proved. □

Proof of Theorem 2. We work the proof using the method of mathematical induction by N.
1). First, we verify the statements for $N = 1$.

$$J_1^1(1) - J_1^1(0) = \left(\sup_{\xi_1} \xi_1 M_1 - [1-\xi_1]^+ C_d\right) - \left(\sup_{\xi_1} -\xi_1 \hat{C}_s^1\right) = \sup_{\xi_1} \xi_1 M_1 - [1-\xi_1]^+ C_d.$$

Similarly:

$$J_1^{1\prime}(1) - J_1^{1\prime}(0) = \sup_{\xi_1'} \xi_1' M_1' - [1-\xi_1']^+ C_d'.$$

It is obvious that $\xi_1^{*0} = \xi_1^{*0\prime} = 0$ are optimal solutions in $J_1^1(0)$ and $J_1^{1\prime}(0)$ respectively. There are two possible cases for $J_1^1(1)$:

a) If $\quad M_1 \geq -C_d \Rightarrow \xi_1^{*1} = 1$ is optimal in $J_1^1(1) \Rightarrow J_1^1(1) - J_1^1(0) = M_1$.
b) If $\quad M_1 < -C_d \Rightarrow \xi_1^{*1} = 0$ is optimal in $J_1^1(1) \Rightarrow J_1^1(1) - J_1^1(0) = -C_d$.

In case a) we also have

$$M_1' \geq M_1 \geq -C_d = -C_{d'} \Rightarrow \xi_1^{*1\prime} = 1 \text{ is optimal in } J_1^{1\prime}(1) \text{ and:}$$
$$J_1^{1\prime}(1) - J_1^{1\prime}(0) = M_1' \geq M_1 = J_1^1(1) - J_1^1(0).$$

In case b):

$$J_1^{1\prime}(1) - J_1^{1\prime}(0) = \max\left(M_1', -C_d'\right) \geq -C_d' = -C_d = J_1^1(1) - J_1^1(0).$$

Summarizing a) and b):

$$J_1^{1\prime}(1) - J_1^{1\prime}(0) \geq J_1^1(1) - J_1^1(0),$$
$$\xi_1^{*1\prime} \geq \xi_1^{*1},$$
$$\xi_1^{*0\prime} \geq \xi_1^{*0}.$$

2). Assume the statements are true for $N = K-1$ ($K > 1$). Let us prove the theorem's result for $N = K$. For any realizations of $G_2,\ P_2,\ G_2'$ and P_2' we can consider the problems P_{K-1}^2 and $P_{K-1}^{2\prime}$. Applying the induction's assumptions to these problems we have

$$J_{K-1}^{2\prime}(1) - J_{K-1}^{2\prime}(0) \geq J_{K-1}^2(1) - J_{K-1}^2(0). \tag{46}$$

Since $G_2,\ P_2,\ G_2'$ and P_2' are random at time 1, by taking the expectations of both sides in (46), we derive

$$E\left[J_{K-1}^{2\prime}(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^{2\prime}(0)|\mathscr{F}_1\right] \geq E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right]. \tag{47}$$

We continue the proof conditioning on the optimal solution of J_K^1.
a) Assume $\xi_1^{*0} = 1$ is optimal, then from Theorem 1 we get $\xi_1^{*1} = 1$. Using (6):

$$J_K^1(1) - J_K^1(0) = M_1 + \hat{C}_s^1. \tag{48}$$

From Lemma 1, $\xi_1^{*0} = 1$ implies:

$$E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right] \geq e^r \hat{C}_s^1. \tag{49}$$

Combining (47) with (49) and using $G_1' \leq G_1$:

$$E\left[J_{K-1}^{2\prime}(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^{2\prime}(0)|\mathscr{F}_1\right] \geq e^r \hat{C}_s^1 \geq e^r \hat{C}_s^{1\prime}.$$

According to Lemma 1, the last inequality is equivalent to optimality of $\xi_1^{*0\prime} = 1$ in $P_K^{1\prime}(0)$. Using Theorem 1 we also get $\xi_1^{*1\prime} = 1$ optimal in $P_K^{1\prime}(1)$. With the help of (6) we compute the difference:

$$J_K^{1\prime}(1) - J_K^{1\prime}(0) = M_1' + \hat{C}_s^{1\prime}. \tag{50}$$

Since the condition of Lemma 3 is satisfied, then using the lemma and taking into account (48) and (50):

$$J_K^{1\prime}(1) - J_K^{1\prime}(0) \geq J_K^1(1) - J_K^1(0).$$

The last relation completes the proof of the induction step for case a).
b) Assume $\xi_1^{*1} = 0$ is optimal in $P_K^1(1)$. Automatically, from Theorem 1 we have that $\xi_1^{*0} = 0$ is optimal in $P_K^1(0)$. Using (6):

$$J_K^1(1) - J_K^1(0) = -C_d. \tag{51}$$

From Lemma 4 and (51):

$$J_K^{1\prime}(1) - J_K^{1\prime}(0) \geq -C_d' = -C_d = J_K^1(1) - J_K^1(0).$$

Since for any optimal solutions, $\xi_1^{*1\prime}$ and $\xi_1^{*0\prime}$, it is always true that $\xi_1^{*1\prime} \geq \xi_1^{*1}$ and $\xi_1^{*0\prime} \geq \xi_1^{*0}$, therefore, the induction step for case b) is shown.
c) The last case left is when $\xi_1^{*1} = 1$ and $\xi_1^{*0} = 0$. Using (6) we compute

$$J_K^1(1) - J_K^1(0) = M_1 + e^{-r}\left(E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right]\right). \tag{52}$$

From the optimality of $\xi_1^{*1} = 1$ and Lemma 1:

$$E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right] \geq e^r\left(-C_d - M_1\right). \tag{53}$$

Combining (47), (53), and the condition that $M_1' \geq M_1$:

$$\begin{aligned} &E\left[J_{K-1}^{2\prime}(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^{2\prime}(0)|\mathscr{F}_1\right] \geq E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right] \geq \\ &\geq e^r\left(-C_d - M_1\right) \geq e^r\left(-C_d - M_1'\right) = e^r\left(-C_d' - M_1'\right). \end{aligned} \tag{54}$$

Using (54) and Lemma 1 again, we find that $\xi_1^{*1\prime} = 1$ is optimal in $P_K^{1\prime}(1)$. Therefore, the monotonicity of optimal solutions is shown. It is left to show the inequality part of the induction hypothesis. If $\xi_1^{*0\prime} = 0$ is optimal, then using (6), (47), $M_1' \geq M_1$ and (52):

$$\begin{aligned} &J_K^{1\prime}(1) - J_K^{1\prime}(0) = M_1' + e^{-r}\left(E\left[J_{K-1}^{2\prime}(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^{2\prime}(0)|\mathscr{F}_1\right]\right) \geq \\ &\geq M_1 + e^{-r}\left(E\left[J_{K-1}^2(1)|\mathscr{F}_1\right] - E\left[J_{K-1}^2(0)|\mathscr{F}_1\right]\right) = J_K^1(1) - J_K^1(0). \end{aligned}$$

If $\xi_1^{*0\prime} = 1$ is optimal, then using (6):

$$J_K^{1\prime}(1) - J_K^{1\prime}(0) = M_1' + \hat{C}_s^{1\prime}. \tag{55}$$

Applying Lemma 3 and Lemma 4 to (55), we get:

$$J_K^{1\prime}(1) - J_K^{1\prime}(0) \geq M_1 + \hat{C}_s^1 \geq J_K^1(1) - J_K^1(0).$$

Hence the induction step for case c) is completed and this finishes the proof of the theorem. □

Proof of Corollary 2.1 . First, consider the case of a geometric Ornstein-Uhlenbeck process. The process is defined by the equation:

$$dX_t = X_t\left(a\left(b - \ln X_t\right)dt + \sigma dW_t\right). \tag{56}$$

Applying Ito's Lemma:

$$\begin{aligned} d\ln X_t &= \frac{dX_t}{X_t} - \frac{1}{2}\sigma^2 dt, \\ \frac{dX_t}{X_t} &= d\ln X_t + \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Substituting the latter equation into (56), and setting $S_t = \ln X_t$ and $\theta = b - \frac{\sigma^2}{2a}$:

$$dS_t = a\left(\theta - S_t\right) + \sigma dW_t. \tag{57}$$

Therefore, $S(t)$ is a standard arithmetic Ornstein-Uhlenbeck process. The solution to (57) is

$$S(t) = e^{-at}S(0) + a\int_0^t e^{-a(t-s)}\theta(s)ds + \sigma\int_0^t e^{-a(t-s)}dW_s. \tag{58}$$

If (P_t) and (P'_t) follow the same geometric Ornstein-Uhlenbeck dynamics, then from the condition $P_0 \leq P'_0$ and (58) we have

$$\forall\ t > 0,\ \forall\ w \in \Omega : \quad \ln P_t(w) \leq \ln P'_t(w) \Rightarrow \tag{59}$$

$$\forall\ t > 0,\ \forall\ w \in \Omega : \quad P_t(w) \leq P'_t(w). \tag{60}$$

Analogously, if (G_t) and (G'_t) have the same geometric Ornstein-Uhlenbeck dynamics then from $G_0 \geq G'_0$:

$$\forall\ t > 0,\ \forall\ w \in \Omega :\ G_t(w) \geq G'_t(w). \tag{61}$$

For a process X_t following the geometric Brownian Motion we have

$$X(t) = X(0)e^{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW_s}. \tag{62}$$

Using (62) and the corollary's assumptions we derive (60) and (61).

We can also allow jumps with independent intensities and magnitudes in the price dynamics. Indeed, if we add a jump component ξdJ to the dynamics equation of the price process $X(t)$, then the solution of the equation changes to:

$$X'(t) \quad = \quad X(t) \cdot \prod_{i=1}^{N(t)} (1 + \xi_i),\ \text{for geometric price dynamics,}$$

where $N(t)$ is the total number of jumps before time t, ξ_is are realizations of jump magnitudes, and $X(t)$ is a solution of the same dynamics equation but without jumps. It can be shown that if (60)- (61) like relations are satisfied for $X(t)$ then they are also satisfied for the process $X'(t)$.

Using (60), (61) and applying Theorem 2 we get the assertion of the corollary. □

Proof of Lemma 5 . Let k be the number of equations of type 2) in the system. Then for some $\{i_1, ..., i_k\} \subseteq \{1, .., n\}$ and some $\{c_{i_1}, ..., c_{i_k}\} \subseteq C$, the following equations are in the system:

$$\begin{cases} x_{i_1} = c_{i_1}, \\ ..., \\ x_{i_k} = c_{i_k}. \end{cases}$$

Without loss of generality, we assume $i_j = j,\ j = 1, ..., k$ (change variables if needed). Therefore, in the solution we always have $x^*_i = c_i,\ i = 1, ...k$. Let the rest $(n - k)$ equations of the system be

$$\begin{cases} x_{i_1} = x_{j_1}, \\ ..., \\ x_{i_{n-k}} = x_{j_{n-k}}, \end{cases}$$

where $\{i_1,...,i_{n-k}\} \subseteq \{1,..,n\}$ and $\{j_1,...,j_{n-k}\} \subseteq \{1,..,n\}$. These $(n-k)$ equations can be rewritten as a number of chains of equalities of the form:

$$\begin{cases} x_1^1 = ... = x_{l_1}^1, \\ ..., \\ x_1^m = ... = x_{l_m}^m, \end{cases}$$

where $2 \le l_i \le (n-k),\ i = 1,...,m$ (since $(n-k)$ is the maximal possible length of a chain), and the following conditions apply

$$\forall\ i,j = \overline{1,m},\ i \ne j:\ \ \{x_1^i,...,x_{l_i}^i\} \cap \left\{x_1^j,...,x_{l_j}^j\right\} = \varnothing,$$
$$\bigcup_{i=1}^{m} \{x_1^i,...,x_{l_i}^i\} = \bigcup_{p=1}^{n-k} \{x_{i_p}, x_{j_p}\}.$$

The initial system has a unique solution if and only if the next condition is satisfied:

$$\forall\ i = \overline{1,m}\ \exists p:\ 1 \le p \le l_i,\ \exists j:\ 1 \le j \le k \text{ such that } :\ x_p^i = x_j.$$

Hence, the solution of the system is

$$\begin{cases} x_1^* = c_1,...,x_k^* = c_k, \\ x_1^{1*} = ... = x_{l_1}^{1*} = c_{p_1}, \\ ..., \\ x_1^{m*} = ... = x_{l_m}^{m*} = c_{p_m}, \end{cases}$$

where $\{p_1,...,p_m\} \subseteq \{1,..,k\}$ and $\{x_{k+1}^*,...,x_n^*\} \subseteq \bigcup_{i=1}^m \{x_1^{i*},...,x_{l_i}^{i*}\}$. From the last result, if the system has a unique solution it has to be of the form specified in the statement of the lemma. Therefore, the lemma is proved. $\square$

Proof of Theorem 3 . Since the problems $(P1BL')$ and $(L1)$ have box constraints for $\xi_{i,j}$ variables, lower bounds on ν_i^j and μ_i^j variables, and it is always optimal for ν_i^js and μ_i^js to be on their lower bounds (because we solve maximization problems and always have $C_d \ge 0$ and $C_s \ge \underline{K} \Rightarrow (C_{s,i}^j + M_i^j) \ge 0$), $(P1BL')$ and $(L1)$ always have optimal solutions (the problems are bounded). As $(L1)$ is a continuous relaxation of $(P1BL')$, to prove the theorem it is enough to show that $(L1)$ always has an integer optimal solution. Utilizing the fact that if a linear problem has an optimal solution, at least one of the solutions is attained at a vertex of the feasible region (the optimal solution set may consist of more than one point). Below we prove that all vertexes of the feasible region of $(L1)$ may have only 0-1 coordinates. Let P be the total number of variables in $(L1)$. Since we have a P-dimensional space of variables, in this space any vertex of the feasible region has to be a unique solution of a system of P equations (an intersection of P hyperplanes) and has to satisfy all the feasibility constraints of the problem. Analyzing the constraints constituting the feasible region of $(L1)$, we find that the feasible region is constrained by hyperplanes defined by equations of one of the following types:

$$\nu_i^j = \xi_i^j - \xi_{i-1}^j,\ \nu_i^j = 0,\ \mu_i^j = \xi_i^j - \xi_{i-1}^j,\ \mu_i^j = 0,\ \xi_{i,j} = \xi_{m,l},\ \xi_{i,j} = 0,\ \xi_{i,j} = 1.$$

Therefore, any vertex has to be a unique solution of a system of P equations, and each of the equations has to be of one of the types specified above. Let us consider such a system. For each variable ν_i^j there are only two possible equations that can be present in the system and have ν_i^j in them. These equations are

$$\nu_i^j = \xi_i^j - \xi_{i-1}^j \text{ and } \nu_i^j = 0.$$

For each ν_i^j there are three possible cases written below.

1) Only $\nu_i^j = 0$ is present in the system.

2) Only $\nu_i^j = \xi_i^j - \xi_{i-1}^j$ is present in the system.

3) Both $\nu_i^j = 0$ and $\nu_i^j = \xi_i^j - \xi_{i-1}^j$are present in the system.

In case 1) we eliminate the variable v_i^j from the system and the newly obtained system still has to have a unique solution. To get the solution of the initial system we just need to add $\nu_i^j = 0$ to the solution of the obtained system.

In case 2) we also eliminate the variable ν_i^j and the corresponding equation from the system. The resultant system has one less variable and one less equation than the initial system. As the initial system has a unique solution, the new system has to inherit this property. Let $\xi_i^j = \xi_i^{j*}$ and $\xi_{i-1}^j = \xi_{i-1}^{j*}$ be a part of the solution of the new system. We can get the solution of the initial system by adding $\nu_i^j = \xi_i^{j*} - \xi_{i-1}^{j*}$ to the solution of the obtained smaller system.

In case 3) we perform similar steps. We eliminate the variable ν_i^j and the two corresponding equations. The only difference in this case is that we also add a new constraint $\xi_i^j = \xi_{i-1}^j$ (this equation can be easily derived from the two equations with ν_i^j) to the system. As in the previous case the obtained system has one less variable, one less constraint, and it inherits the property that it has to have a unique solution. The solution of the initial system can be obtained by adding $\nu_i^j = 0$ to the solution of the newly constructed system.

Applying the described steps we can reduce the original system with P equations to a smaller system that does not have variables ν_i^j in it. From the above, we have that the constructed system has to have a unique solution. We can also apply a similar logic to eliminate variables μ_i^j from the system. Ultimately, we can get a system that has only variables $\xi_{i,j}$ in it. This system has to have a unique solution, and can consist only of equations of the types:

$$\xi_{i,j} = \xi_{m,l},\ \xi_{i,j} = 0,\ \xi_{i,j} = 1.$$

The variable elimination procedure guarantees that the number of variables in the obtained system equals to the number of equations. Applying Lemma 5 with $C = \{0, 1\}$ to the system with $\xi_{i,j}$s only, we find that its unique solution has a 0-1 representation. From the variable elimination procedure we also get that $\nu_i^j,\ \mu_i^j \in \{-1, 0, 1\}$, but since $\nu_i^j = -1$ and $\mu_i^j = -1$ are not feasible in $(L1)$ we get the result that the vertexes of the feasible region can only have 0-1 coordinates. The theorem is proved. □

Proof of Lemma 6 . We use the method of mathematical induction to prove the formula.
1) For $i = 1$ the formula simplifies to the following: $\chi_1^p = \xi_1^p$. The last identity is true by the definition of χ_i^p since the initial state of the plant is "off".
2)Assume the formula is true for $\forall\ i \leq (K-1)$ and let us prove it for $i = K$. From the definition of χ_i^p we have

$$\chi_K^p = (1 - \chi_{K-1}^p)\xi_K^p + \chi_{K-1}^p \eta_K^p = \xi_K^p + \chi_{K-1}^p(\eta_K^p - \xi_K^p).$$

Using the induction assumption we expand the last equality as follows:

$$\begin{aligned}\chi_K^p &= \xi_K^p + \left(\sum_{k=1}^{K-1} \xi_k^p \prod_{l=k+1}^{K-1} (\eta_l^p - \xi_l^p)\right)(\eta_K^p - \xi_K^p) = \xi_K^p + \sum_{k=1}^{K-1} \xi_k^p \prod_{l=k+1}^{K} (\eta_l^p - \xi_l^p) = \\ &= \sum_{k=1}^{K} \xi_k^p \prod_{l=k+1}^{K} (\eta_l^p - \xi_l^p).\end{aligned}$$

From the last equality, the formula is true for $i = K$. Therefore, the induction step is shown and this finishes the proof of the lemma. □

Proof of Lemma 7. Consider an arbitrary $j = \overline{1, N_S}$. We prove the lemma using an induction by i.

1) If $i = 1$ then using (31):

$$\chi_1^j = \xi_1^j = sgn\left([\delta_1^j - \delta]^+\right).$$

In this case $\gamma_{1,j} = \delta_1^j$.

2) Suppose that the statement of the lemma is true for $\forall i \leq K - 1$, for some $K \leq N$. From the definition of χ_K^j we have

$$\chi_K^j = (1 - \chi_{K-1}^j)\xi_K^j + \chi_{K-1}^j \eta_K^j.$$

If $\eta_K^j = 0$, then we also have $\xi_K^j = 0$ (since the "off" boundary always lies no lower than the "on" boundary). Therefore:

$$\chi_K^j = 0 = sgn\left([0 - \delta]^+\right).$$

In this case we have $\gamma_{K,j} = 0$. If $\eta_K^j = 1$, then:

$$\chi_K^j = (1 - \chi_{K-1}^j)\xi_K^j + \chi_{K-1}^j = \max\{\chi_{K-1}^j, \xi_K^j\}.$$

Using the induction hypothesis we derive

$$\begin{aligned}\chi_K^j &= \max\left\{sgn\left([\gamma_{K-1,j} - \delta]^+\right), \xi_K^j\right\} = \max\left\{sgn\left([\gamma_{K-1,j} - \delta]^+\right), sgn\left([\delta_K^j - \delta]^+\right)\right\} = \\ &= sgn\left([\gamma_{K,j} - \delta]^+\right),\end{aligned}$$

where $\gamma_{K,j} = \max\{\gamma_{K-1,j}, \delta_K^j\}$. Therefore, the induction step is shown and the lemma is proved. □

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