Pricing Equity and Debt Tranches of Collateralized Fund of Hedge Funds Obligations

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Chapter 1

Introduction

Credit structuring technology has been employed to create the market for collateralized debt obligations (CDO). CDOs redistribute credit risk to meet investor demands for a wide range of rated securities with scheduled interest and principal payments, securitized by diversified pools of debt instruments. The equity components of CDOs are leveraged investments in the collateral portfolio that offer unique risk and return benefits and have created a market for alternative investments linked to the credit markets.

A relatively recent innovation in structuring technology is the advent of Collateralized Fund of Hedge Funds Obligations (CFO). In these structures, investors gain exposure to a diversified pool of hedge funds through a fund of funds manager. The capital structure of a CFO is similar to traditional CDOs, meaning that investors are offered different rated notes and equity interest. CFOs are structured as arbitrage market value CDOs. The fund of funds manager actively manages the fund to maximize total return while restraining price volatility within the guidelines of the structure.

The aim of this work is to provide a useful framework to evaluate Collateralized Fund of Hedge Funds Obligations, that is pricing the equity and the debt tranches of a CFO.

The basic idea of our pricing model is to evaluate each CFO liability as an option written on the underlying pool of hedge funds. The value of every tranche depends on the evolution of the collateral portfolio during the life of the contract. Furthermore, care is taken to distinguish between the fund of hedge funds and

its underlying hedge funds, each of which is itself a portfolio of various securities, debt instruments and financial contracts.
The models we propose incorporate skewness, excess-kurtosis and are able to capture more complex dependence structures among hedge fund log-returns than the simple correlation matrix. With these models it is possible to describe the impact of an equivalent change of probability measure not only on the marginal processes but also on the underlying dependence structure among hedge funds.

The theoretical problem
Consider a portfolio of financial products (in our case hedge funds) whose log-returns have distributions very far from the Normal. Assume that the value of this portfolio is regularly checked during the time with a fixed frequency, for example at the end of every month. Assume also that only historical data are available. Our task is to compute the *fair* price of an asset whose payoff is linked to the value of this portfolio at each date. To reach this purpose we can apply the martingale method: we have to compute the expected discounted payoff of the asset under a risk neutral probability measure.
The general problem is to define the joint distribution of the value of all the assets in the portfolio at each date of control under a suitable risk neutral probability measure.
To face this problem we consider separately two issues:

- the definition of the dependence structure under the physical probability measure;
- the choice of an appropriate equivalent martingale measure allowing to study the impact of the change of measure on the dependence structure

To solve the first problem we need to model both cross-section and temporal dependence among assets simultaneously.
The two extreme choices are:

- the direct specification of a multivariate stochastic process;
- the definition of the marginal distributions and the temporal dependence structure separately.

The first alternative has the advantage of the parsimony but can lack of flexibility. The second alternative is very flexible but this high flexibility could result inconsistent with the main principles of a pricing model. Our choice represents a possible compromise between these two extreme cases.
Finally, since no traded options are available for calibration purpose, we use an approach to change measure that allows to capture the impact of the change of measure on the marginal and joint parameters.

Methodological contribution
In this work three models are employed.
In the first model the physical dependence among hedge fund log-returns is introduced through a Gamma stochastic time change of a Multivariate Brownian motion with drift, with independent components. The idea is that the economy is driven by only one common factor, whose dynamic is given by a Gamma subordinator. A jump in the time-change leads to a jump in the price processes and so all jumps occur simultaneously. However the size of individual jumps are caused by the independent Brownian Motions.
In the second model we have both temporal and cross section sources of dependence. The dependence under the historical probability measure is obtained through a Gamma stochastic time change of a Multivariate Brownian motion with drift, whose components are correlated. In this case the Net Asset Value processes jump at the same time and the size of individual jumps are correlated.
In both cases the use of a common Gamma stochastic clock leads to the Multivariate Variance Gamma process to describe the joint dynamic of hedge fund log-returns.
In the third model we use a Multivariate Gamma subordinator to time-change a Multivariate Brownian motion with drift. The main feature of this model is that it allows to incorporate both a common time change, which can be interpreted as a measure of the global market activity, and an idiosyncratic time transform linked to the specific hedge fund and information update.
In all our models we have Variance Gamma margins under the physical probability measure but different dependence structures.
In the current setting, the market is incomplete. The risk due to jumps cannot be hedged and there is no more a unique risk neutral measure. Among the measures

equivalent to the historical one, we choose the Esscher measure for which the discounted Net Asset Value process of each hedge fund is a martingale. In particular, we change the probability measure using the *Multivariate Esscher Transform* and we show that under this new measure the first two models can be expressed again as Multivariate Brownian motion with drift, time changed with a Gamma stochastic clock identical to the physical one and with the same correlation matrix among the Brownian motions . We find also functional relations between real world and risk neutral parameters. In the third model the change of measure modifies both marginal and joint processes. In particular the marginal processes are no longer Variance Gamma. However, we show that under the Esscher Equivalent Martingale Measure the this process can be expressed as a Multivariate Brownian motion with drift time changed through a {new Multivariate subordinator. The use of the Multivariate Esscher Transform in our models represents a powerful tool to study the impact of the change of measure not only on the marginal price processes but also on the underlying dependence structure. This is another contribution of this paper.

The Application
These models are applied to evaluate the equity and the debt tranches of a CFO. The analysis is done starting from a simple CFO structure, which is then progressively complicated with the introduction of the structural features we encounter in typical CFOs. In this way, at each step of the evolution of the structure, the reader can understand the impact on the value, measured with respect to the first four moments of the distribution, and how this value is divided among the different tranches. The result is a useful schema that can provide some help in designing a CFO transaction. The analysis is also helpful for the CFO manager who usually invests in the equity tranche, because gives him some suggestions on how to increase the value of his investment.

The work is organized as follows. In Chapter 2 an introduction to Collateralized Fund of Hedge Funds Obligations is provided. Particular care is used to describe the structural features that influence both the Nav of the collateral portfolio and the payoff of each tranche. In Chapter 3 statistical properties of Hedge Fund monthly log-returns are analysed. In particular, it is shown that the evolution of hedge funds Net Asset Value cannot be described by a Geometric Brownian motion. Hedge funds monthly log-returns exhibit leptokurtic and usually negatively skewed distributions. In Chapter 4 is provided an introduction to Lévy processes and Exponential Lévy models with particular emphasis for the concepts, properties and instruments that are used in the sequel. In Chapter 5 we present three different models applied to describe the physical evolution of hedge fund log-returns. Then it is discussed the change of measure and its impact on marginal and joint processes for each model. In Chapter 6 the estimation methodology and the simulation procedure are illustrated. In Chapter 7 we discuss the pricing applications and the results. Finally, we report our conclusions and indicate a possible extension of our models.

Chapter 2

Collateralized Fund of Hedge Funds Obligations

2.1 Collateralized Fund of Hedge Funds Obligations

Collateralized Debt Obligations are structured credit vehicles that redistribute credit risk to meet investor demands for a wide range of rated securities with scheduled interest and principal payments. CDOs are securitized by diversified pools of debt instruments. Relatively recent developments in credit structuring technology include the introduction of Collateralized Fund of Hedge Funds Obligations (CFOs). A CFO is created by using a standard securitization approach. Often a special purpose vehicle (SPV) issues multiple tranches of senior and subordinated notes that pay interest at fixed or floating rates and an equity tranche, and invests the proceeds in a portfolio of hedge funds. Picture 2.1 shows a schematic prototype of a CFO structure. The SPV, a new structure with no operating history, is set up as a bankruptcy-remote entity. This allows CFO investors to take only the risk of ownership of the assets but not the bankruptcy risk of the CFO's sponsor. The capital structure of a CFO is similar to that of a CDO. It consists of the collateral pool held in the SPV on the asset side and a group of notes having different priorities and payment obligations on the liability side. The asset-backed notes are expected to be redeemed at the scheduled maturity date with the liquidation proceeds of the collateral portfolio. The priority of payments among the different classes is sequential such that the Class A investors will be redeemed first. Following the full redemption of the Class A notes, the Class B notes will be redeemed. The Class C notes will repaid after the full redemption of the Class B notes. The Equity holders will receive all the residuals after the full redemption of the Class C notes[1]. The most senior tranche is usually rated AAA and is credit enhanced due to the subordination of the lower tranches. This means that in case of loss the lowest tranche, that is the equity tranche, absorbs losses first. When the equity tranche is exhausted, the next lowest tranche begins absorbing losses. A CFO may have a AAA rated tranche, an AA rated tranche, a single A rated tranche and an unrated equity tranche. The assets of the special purpose vehicle secure the notes issued under an indenture or deed of trust under which a trustee is appointed. CFOs tend to be structured as *arbitrage market-value* CDOs that invest in hedge funds. CFO assets are actively managed by an investment adviser or manager with fund of funds expertise in order to maximize total return while restraining price volatility within the guidelines of the structure, in return for management fees and incentive compensation. The leverage (the ratio of debt to equity issued) in a CFO typically ranges from two-to-one to five-to-one while a CDO may have leverage as high as twenty-five-to-one for investment grade assets. A CFO can also be regarded as a financial structure with equity investors and lenders where all the assets, equity and bond, are invested in a portfolio of hedge funds. The lenders earn a spread over interest rates and the equity holders, usually the manager of the CFO, earn the total return of the fund minus the financing fees.

CFOs typically have a stated term of three to seven years at the end of which all of the securities must be redeemed. Investors have limited redemption rights prior to maturity. Typically, redemptions before maturity are only possible if some predetermined events happen.

Detailed descriptions of real CFO structures can be found in Stone and Sizzu (2002) [121] and in some Moody's pre-sale reports [100, 101, 102, 103] for example.

[1] In the picture I, P, D, G indicate respectively interests, principal, dividends and capital gains. Notice that capital gains can be negative. In the worst case scenario G+P=0, i.e. equity holders lose all their invested capital.

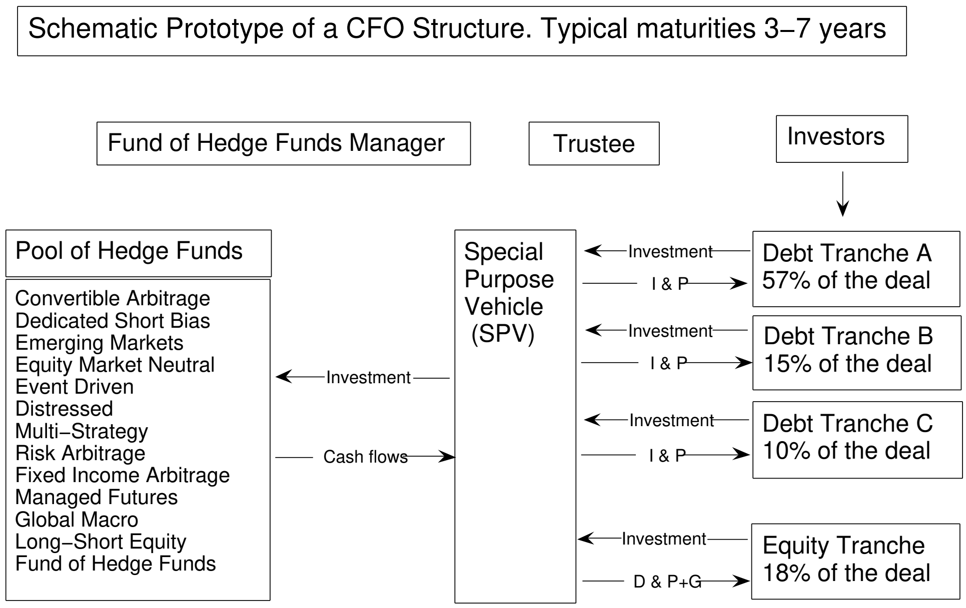

Figure 2.1: A CFO structure

2.2 Motivations for Investing in CFOs

The motivation for investment in CFOs requires explanation from the perspective of both equity and debt investors. The main reference on this topic is Mahadevan and Schwartz (2002) [84]. The interested reader can also see [87, 121, 127, 131].

Debt Investor

- A note of a CFO can improve the degree of diversification of any credit investor's portfolio. The cash flows of the debt tranches (coupons and principal payments) depend on the performance of the pool of the underlying hedge funds. Fund of hedge funds exhibit unique risk/return profile when compared to traditional asset classes and low correlation with the traditional credit markets.

- The fact that some collateral may consist of successful hedge funds that are closed to new investors is often cited as another motivation for CFO debt investment.

- The notes of a CFO carry credit ratings, making the investments available and suitable to a larger audience then the typical investor base for direct fund of funds investment.

- An important advantage for CFO debt investor over the direct fund of funds investment approach is the solid credit protection offered by CFOs structures. Trading restrictions, liquidation and early redemptions are procedures designed to safeguard cash flows for debt investors.

- CFO debt is usually less liquid then a direct investment in a fund of hedge funds because debtholders do not have early redemption options in their notes. This is of course a disadvantage for CFO debt investors. However, a source of liquidity for bondholders requiring early redemption may be represented by the secondary market for CDO notes.

- Debt holders avoid direct fees because equity holders pay them.

Equity Investor

- Some equity investors in funds of hedge funds seek to use leverage to increase their investment returns by funding some of the investment with a collateralized loan, much like the purchase of stock on margin. CFO equity investors gain access to a diversified portfolio of hedge funds with a nominal investment thanks to leverage provided by debtholders. Since it is raised directly from the capital markets, CFO debt also broadens and deepens the funding sources available to levered equity investors in funds of hedge funds. For example, in a CFO structure with a debt-equity ratio of 5:1, an investment in the equity tranche of $20 million results in a leveraged investment in the underlying fund of funds of $100 million. On a stand alone basis, it could be more difficult to invest in a well diversified portfolio of hedge funds with a nominal investment of $20 million.

- The fact that some collateral may consist of successful hedge funds that are closed to new investors is often cited as another motivation also for CFO equity investment.

- From a return on investment perspective, equity holders can benefit from a levered investment in the underlying fund of funds. As demonstrated in [84, 87] CFO equity holders take advantage from low break even rates versus the underlying collateral, helping equity investors when the fund of funds is able to outperform the performance hurdles set by the rated note investors.

- Term funding provided by bond investors gives equity investors a much more stable funding source than alternative that involve borrowing against collateral from banks and securities dealers, making CFO equity tranche possibly a less volatile investment vehicle for hedge funds. However, funding could not last the full term if triggers force early liquidation.

- CFO equity is less liquid then a direct investment in a fund of hedge funds because early redemption may be available to minority equity holders on a pro rata basis or at the option of the collateral manager. Often early redemptions are prohibited at all in order to avoid the deterioration of the credit quality of the rated notes. However, a source of liquidity for equity investors requiring early redemption may be represented by the secondary market for CDO equity.

- The relative lack of liquidity can be seen also as an advantage since it gives the CFO more flexibility to follow its strategies despite short-term market fluctuations.

- Equity investors directly pay underwriting fees, rating agencies fees and management fees. This is one of the drawbacks for hedge fund CFO equity investors. However, if equity holders are sufficiently lucky, the returns generated by their levered investment in the underlying pool of hedge funds will be high enough to compensate them for all the risks and expenses they face.

2.3 CFOs Risk Factors

Moody's defines a hedge fund as *an unrated and largely unregulated, privately organized and offered investment vehicle that is available to limited number of high net worth individuals and certain other qualified investors. Managers of hedge funds have the flexibility to take both long and short positions, use derivatives and leverage and invest in a wide range of instruments not limited to traditional stock and bond investments. The term hedge fund is often a misnomer as many hedge funds are not hedged at all* [131]. This definition reveals the potential volatility of an individual hedge fund and highlights the importance of having an experienced and skillful CFO portfolio manager.

There are also other factors connected with hedge fund investment to take into account when structuring a CFO transaction.

An investment in a hedge fund does not result in a predictable cash flow to investors. Consequently, in structuring a CFO some provision must be made to ensure that rated notes are paid their promised coupons in cash and on time, if these promised cash flows cannot be capitalized.

A hedge fund investment is illiquid. Redemption of the investment is usually prohibited for a lock-out period lasting one to three years. At the expiration of the lock-out period, redemptions are available, but only on a limited basis such as monthly or quarterly and then only after the expiration of a notice period. In addition, hedge funds often reserve the right to suspend redemptions during periods of financial crisis.

Specific investments made by hedge funds are often carefully guarded secrets. This lack of transparency may make it difficult for a CFO manager to assess the CFO's aggregate exposure to a particular investment on a portfolio basis. It also presents a challenge to the CFO manager attempting to monitor a particular hedge fund's adherence to its advertised style or investment approach.

2.3.1 Liquidity

Liquidation of a hedge fund position involves three distinct temporal concepts:

- initial lock-out period
- redemption frequency
- notice period prior to redemption

As we have just said, redemption of the investment in hedge fund is usually not allowed for a lock-out period lasting one to three years, or in some extreme cases, for several years. At the expiration of this period, redemptions are possible, but only with a monthly or quarterly frequency and only after the expiration of a notice period. Typically, a notice period ranging from 30 to 90 days is required before the hedge fund manager is obligated to deliver funds to an investor. After the notice period, investments in the underlying hedge fund can be liquidated within days, weeks, months or years depending on its redemption frequency. Each underlying hedge fund in a CFO portfolio tends to have complicated and restrictive liquidity covenants. Indeed, a skilful manager must evaluate carefully all these features for each hedge fund when structuring the CFO collateral portfolio. Two further points should be underlined about hedge fund liquidity in the context of a CFO. First, most hedge funds retain the right to temporarily suspend redemption *(liquidity suspension)* under certain circumstances. Second, the terms of investments in hedge funds are sometimes negotiable. Terms of liquidation are no exception. At the most favourable extreme, the CFO portfolio manager contracts with a hedge fund manager to manage a portfolio but retains ownership of it. This arrangement is referred to as a managed account. Under this arrangement, the CFO manager has the legal right to directly liquidate the positions in the managed account. More typically, the CFO will have separate but specific liquidity provisions for each hedge fund investment.

2.3.2 Transparency

The NAV of the CFO portfolio, which is given by the sum of the market values of each underlying hedge fund[2] in the CFO portfolio. is frequently assessed and measured against predefined trigger levels. If the NAV falls below any of the trigger levels and the test violation is not cured within the prescribed time frame, the underlying hedge funds are redeemed to preserve the value of the rated notes. The administrator of each hedge fund have to report the fund's valuation to the CFO manager, usually on a monthly basis. However, due to the proprietary nature of current investments and the intense competition among hedge funds, the positions of the underlying hedge funds are typically not revealed. This lack of transparency implies that the CFO portfolio manager valuation depends on the accuracy and veracity of each hedge fund's reported value. Some protective measures are designed for the CFO portfolio. The financial statements of most hedge funds are audited on an annual basis. The accuracy of reported hedge fund NAV can also depend upon the degree to which the hedge fund administrator's valuation process is independent of the hedge fund manager. Another source of protection from the various risks posed by limited transparency is the due diligence process that CFO portfolio managers conduct on hedge funds prior to investment. This process can be very extensive, detailed and ongoing. Still, even with annual audits, independent administration, and diligent CFO managers, the limited transparency of hedge funds creates two distinct challenges for CFO managers as well as for rating agency models: fraud and style drift.

Fraud remains a rare event in the hedge fund industry, although it has happened in the past. The occurrence of liquidity suspension in a single hedge fund can have consequences similar to the incidence of fraud, that is the rapid total loss of a single hedge fund value.

Style drift is a more subtle consequence of the hedge funds' limited transparency. A particular hedge fund manager can choose to focus on one or more of these strategies. CFO managers typically covenant to maximum percentages within each strategy. For hedge fund investments that are not transparent, the diligent CFO portfolio manager can compare returns to associated indices and to similar

[2]The NAV of a hedge fund is the total value of the fund's portfolio less its liabilities. Its liabilities may be money owed to lending banks or fees owed to investment managers, for example

funds to gauge each fund's degree of adherence to its professed style. Evidence of possible style drift can be investigated by discussions with hedge fund managers. In fact, for the CFO portfolio manager, the best antidote to the risk of limited transparency is a set of strong relationships with various hedge fund managers and the knowledge gained from long years of investment experience in hedge funds. Most concerns about lack of transparency in a CFO are related to the issue of individual hedge fund collateral. One important point about transparency at the fund of fund level needs to be emphasized. Because of restrictive investment agreements, the names of each hedge fund in the CFO portfolio will not necessarily be revealed to investors.

2.4 CFOs Structural Features

This section describes the structural features of CFOs that have been created to address the risks described in the previous section.

- **Diversification**
  Portfolio diversification, across both strategies and funds, is an essential way to reduce the the risks arising from the unregulated and opaque nature of hedge funds. Probably the diversification across funds is the most important because of the potential for style drift and a rapid total loss of value in a single fund. Specific tests are used to measure, report and encourage both types of diversification. Limit on the maximum portfolio share for each fund ensures diversification across funds. Limitations on concentrations of funds within each styles encourage diversification across strategies. The diversification criteria are typically expressed as a percentage of the total NAV of the CFO portfolio. They can be structured in two different ways:
  a) the criteria have to be met only at the start of the transaction or at the time a fund is purchased
  b) the criteria have to be met on an on-going basis.

- **Over-Collateralization Tests/Minimum Net Worth Test**
  Over-Collateralization (OC) Tests are designed to give more credit protection to bond investors. In an OC Test, an advance rate is assigned for each class of rated notes. The CFO manager will test whether the value of the

assets (which are valued periodically), discounted by the advance rates relevant for each of the rated notes, exceeds indebtedness. If the discounted value is less than indebtedness, assets must be sold and debt repaid or less risky assets purchased in order to bring the portfolio back into compliance within a predetermined period, typically 3-6 months. Failure to cure an OC test breach within this period will trigger an event of default and subsequent acceleration of the rated notes. An advance rate reflects the protection provided against a decline in market value and is a function of the price volatility and the liquidity of the asset: the more volatile and less liquid the asset, the lower the advance rate.

The Minimum Net Worth (MNW) Test provides further protection to the note-holders against the decline of the portfolio NAV. If the adjusted net worth of the portfolio of hedge funds, given by the NAV minus the notional of outstanding liabilities, calculated on a monthly basis, is less than the required adjusted net worth, actions have to be taken to either cure the test by adding more equity or pay down liabilities. If this MNW Test is not cured within the predetermined time frame, all notes will be declared due immediately and the CFO portfolio assets will be liquidated to pay the liabilities.

- **Liquidity Profile Test**

  A Liquidity Profile Test is incorporated to prescribe the time interval between issuance of a redemption notice and receipt of the liquidation proceeds for each underlying hedge fund. Each separate time interval is associated with the minimum percentage of NAV. For example, a very simple CFO Liquidity Profile Test would be at least fifty percent of NAV within six months and one hundred percent within one year. The covenanted liquidity profile gives consideration to the notice period, redemption period and the initial lock out period.

- **Equity Distribution Rules**

  Equity distribution rules in a CFO transaction specify the timing of the distribution of gains to the CFO equity investors. Typically, some threshold return level has to be met before any gains are distributed to the equity holders. The higher the threshold, the more aligned are the interests of the equity and note holders and the more stable the debt ratings.

- **Liquidity Facility**
  The underlying assets of a CFO transaction are not typically coupon-bearing instruments. The source of the coupon payments for the CFO liabilities could be the proceeds from gains or liquidations set aside by the CFO manager. If any rated liability tranche is not pikable[3], a structural enhancement, such as a reserve account or external liquidity facility is then required to ensure timely interest payments on the non-pikable CFO liabilities.

2.5 Rating Methodology

Credit ratings for CFO debt tranches are based on many factors related to the underlying hedge funds, their managers and CFO manager. Rating agencies take into account concentration limitations, risk-adjusted returns, and the volatility of the net asset value. Advance rates, which are metrics used by rating agencies to determine how much of an asset can be used as collateral to issue rated debt, are the basis for determining the rating of debt issued by CFOs. Advance rates are positively correlated with the credit quality and stability of an asset. At the same time advance rates are inversely related to the risk of the debt, implying that the lower the desired credit rating of debt, the more one can borrow against the collateral.

2.5.1 Moody's Approach

The ratings assigned by Moody's reflect the expected losses posed to liability holders. To determine these expected losses, Moody's employs a Monte Carlo simulation approach. In particular, a hypothetical portfolio is created based on a *worst case* portfolio composed of the most volatile and correlated strategies according to the guidelines of the CFO. Moody's generate the time series of the returns of the underlying hedge funds (which correspond to these worst case strategies) based on a random process. The monthly net asset value of the portfolio can then be calculated and applied in the waterfall, including the Over Collateralisation tests, such that an interest and principal shortfall can be calculated for each note in that

[3]This means that coupons cannot be capitalized but they must be paid in cash and on time

given scenario. Moody's repeat this calculation for a significant number of runs and then calculate the average of these shortfalls in order to compute an expected loss. The rating is then derived by benchmarking the expected loss calculated to Moody's idealised expected loss table based on the expected average life of the notes.

2.6 Pricing Methodology

As we have already explained in the Introduction of this work, our aim is to provide a useful framework to evaluate Collateralized Fund of Hedge Funds Obligations, that is pricing the equity and the debt tranches of a CFO. The fair price of each tranche is computed as its expected discounted payoff under a suitable risk neutral probability measure.
The payoff of every tranche is linked to

- the risk neutral evolution of the CFO portfolio NAV, which depends on the temporal behaviour of all its underlying hedge funds;
- the structural features of the CFO such as Over-Collateralization test, priority of payment waterfall (which includes all coupon payments), equity distribution rules, liquidity profile and so on.

To compute the collateral portfolio NAV under the selected risk neutral probability measure at each time of control t, it is necessary to model the joint risk neutral evolution of the underlying hedge funds. At the same time we have to consider any anticipated payments, Over-Collateralization test and the CFO liquidity profile.

Chapter 3

Statistical Properties of Hedge Fund Returns

Hedge fund returns usually differ substantially from returns of standard asset classes and there is growing evidence that, as a result, they may be of interest to investors. Important issues include identifying the right portion to invest in hedge funds and how to construct a portfolio of hedge funds. Usually, skewness and larger kurtosis, relative to standard asset classes, are observed in monthly log-return series of hedge funds. In addition, positive autocorrelation is sometimes observed. This leads to underestimation of the true volatility, and as a consequence, an overestimation of the Sharpe ratio, if it is used as a performance measure. Therefore, standard methods used for portfolio construction like mean-variance approach can be inadequate for hedge funds. Interested readers on the subject can see [105] in which the authors propose an efficient method for fund of hedge funds construction under downside risk measures.
In our pricing procedure care is taken to distinguish between the fund of hedge funds and its underlying hedge funds, each of which is itself a portfolio of various securities, debt instruments and financial contracts.
In this chapter we analyse the statistical properties of hedge fund indices employed to build the collateral portfolio for our applications.

3.1 The Data

We got hedge fund index data from Credit Suisse/Tremont Hedge Index[1]. Credit Suisse/Tremont maintains monthly NAV and simple return data for a Global hedge fund index and for the following 13 indices corresponding to different styles: convertible arbitrage, dedicated short bias, emerging markets, equity market neutral, event driven, ED distressed, ED multi-strategy, ED risk arbitrage, fixed income arbitrage, global macro, long/short equity, managed futures and multi-strategy. Contrary to other hedge fund indices, the Credit Suisse/Tremont indices reflect the monthly net of fee NAV on an asset-weighted basket of funds. Large funds therefore have a larger influence on the index than smaller funds. Most indices are affected by some form of survivorship bias. In order to minimize this effect, Credit Suisse/Tremont does not remove hedge funds in the process of liquidation from an index, and therefore captures all of the potential negative performance before a fund ceases to operate.
Our sample covers the period from January 1994 through May 2008 , for a total of 173 monthly log-return data for each hedge fund index.

3.1.1 Summary Statistics

Descriptive statistics are reported in Table 3.1. A brief examination of the last two columns of this table indicates that hedge fund returns are not Gaussian. Twelve hedge fund indices over fourteen exhibit a negative skewness. All index display excess kurtosis. However, the degree of asymmetry and fat tails is quite different among hedge funds. These results are similar to those reported in [2, 3, 66] obtained using different hedge fund indices and in [105] obtained employing CS/T indices on a shorter time period. Negative skewness and excess kurtosis are due to hedge fund use of derivatives, leverage and short selling. However, some statistics are really extreme and at least in part this can be explained also by the presence of some outliers in the sample. Figure 3.1 shows boxplots for two indices. A boxplot summarizes the distribution of a set of data by displaying the centring and spread of the data using a few primary elements. The box portion of a boxplot represents the first and third quartiles. The median is depicted using a line through the

[1]http://hedgeindex.com

centre of the box, while the mean is drawn using a black star.

Table 3.1:
Summary Statistics of Monthly Log-returns for CS/Tremont Indices
Period January 1994-May 2008 (**Smoothed Data**)

Index	Mean %	Median %	Max %	Min %	Std.Dev. %	Skew.	Kurt.
CS/T Global Index	0,78	0,76	7,94	-7,98	2,12	-0,04	5,73
Convertible Arbitrage	0,58	0,86	3,45	-5,80	1,38	-1,64	7,64
Dedicated Short Bias	-0,21	-0,36	20,2	-9,36	4,75	0,56	4,11
Emerging Markets	0,70	1,38	15,3	-26,2	4,50	-1,18	10,4
Equity Market Neutral	0,71	0,67	3,19	-1,27	0,76	0,36	3,90
Event Driven	0,83	1,02	3,84	-12,6	1,61	-3,58	30,1
ED Distressed	0,93	1,11	4,08	-13,4	1,78	-3,15	26,1
ED Multi-Strategy	0,78	0,86	4,29	-12,3	1,74	-2,65	20,9
ED Risk Arbitrage	0,55	0,55	3,58	-6,48	1,16	-1,29	10,4
Fixed Income Arbitrage	0,40	0,59	2,18	-7,25	1,16	-3,19	19,2
Global Macro	1,02	1,04	10,5	-12,2	2,97	-0,14	6,69
Long/Short Equity	0,86	0,94	12,0	-12,2	2,79	-0,13	7,31
Managed Futures	0,47	0,32	9,30	-9,71	3,39	-0,12	3,15
Multi-Strategy	0,65	0,75	3,57	-4,93	1,22	-1,11	5,89

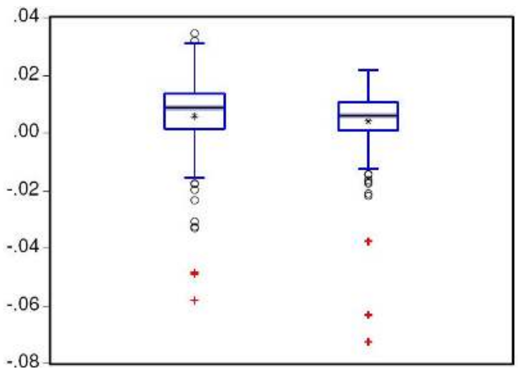

Figure 3.1: Box Plots for Convertible Arbitrage and Fixed Income Arbitrage

The inner fences are defined as the first quartile minus 1.5*IQR (interquartile range) and the third quartile plus 1.5*IQR. The inner fences are not drawn, but graphic elements known as whiskers and staples show the values that are outside the first and third quartiles, but within the inner fences. The staple is a line drawn at the last data point within (or equal to) each of the inner fences. Whiskers are lines drawn from each hinge to the corresponding staple. Data points outside the inner fence are known as outliers. To further characterize outliers, we define the outer fences as the first quartile minus 3.0*IQR and the third quartile plus 3.0*IQR. Data between the inner and outer fences are termed near outliers (circles), and those outside the outer fence are referred to as far outliers (pluses).

3.1.2 Kernel Density Estimators

The simplest nonparametric density estimate of a distribution of a series is the histogram. To estimate the empirical densities we use kernel density estimators. The kernel density estimator replaces the boxes in a histogram by bumps that are smooth [119]. Smoothing is done by putting less weight on observations that are further from the point being evaluated.

More technically, the kernel density estimate of a series Y at a point y is estimated by:

$$\hat{f}(y) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{y - Y_i}{h}\right)$$

where N is the number of observations, h is the bandwidth (or smoothing parameter) and $K(u)$ is a kernel weighting function that integrates to one. Different choices of the function $K(u)$ are possible. Among them we may cite Epanechnikov, Triangular, Uniform, Gaussian, Biweight, Triweight and Cosinus kernel functions[2].

The Gaussian kernel is

$$K(u) = \frac{\exp\left(-\frac{1}{2}u^2\right)}{\sqrt{2\pi}} \tag{3.2}$$

The bandwidth h controls the smoothness of the density estimate; the larger the bandwidth, the smoother the estimate. Bandwidth selection is of crucial importance in density estimation.

[2] E-Views Help Guide reports a detailed description of all these kernel functions

Figures 3.2 and 3.3 show the Gaussian kernel density estimates for two hedge fund indices using different bandwidths. Each graph also reports the Normal density function (black line with dots). The density estimate with optimal bandwidth h is in blue. Estimates for smaller ($0.5h$) and larger ($2h$) bandwidths are in red and green.

Empirical densities give us further evidence against normality. These empirical distributions are in general skewed and have higher peak and fatter tails than the Gaussian distribution with the same mean and variance.

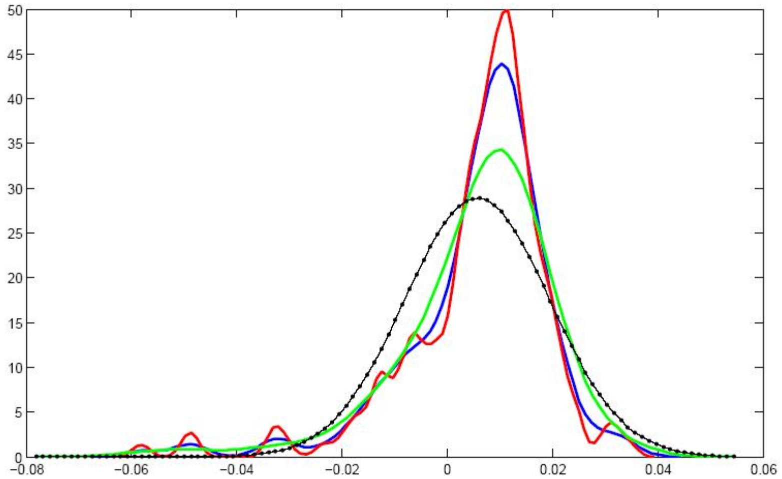

Figure 3.2: Kernel density estimation for Convertible Arbitrage

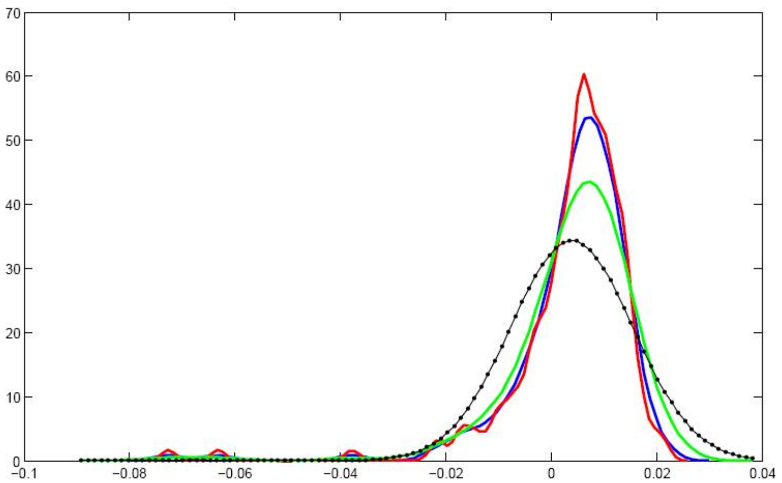

Figure 3.3: Kernel Density Estimation for Fixed Income Arbitrage

3.1.3 Quantile-Quantile Plots

The quantile-quantile (QQ)-plot is a simple yet powerful tool for comparing two distributions. QQ-plot shows the quantiles of the chosen series against the quantiles of another series or a theoretical distribution. If the two distributions are the same, the QQ-plot should lie on a straight line. If the QQ-plot does not lie on a straight line, the two distributions differ along some dimension. The pattern of deviation from linearity provides an indication of the nature of the mismatch.
Each graph in figure 3.4 plots the quantiles of the chosen series of hedge fund log-returns against the theoretical quantiles of the Gaussian distribution. These plots suggest that it is unlikely that observed log-return time series come from normally distributed random variables. In particular, the mismatch is related to the tails, especially the left one.

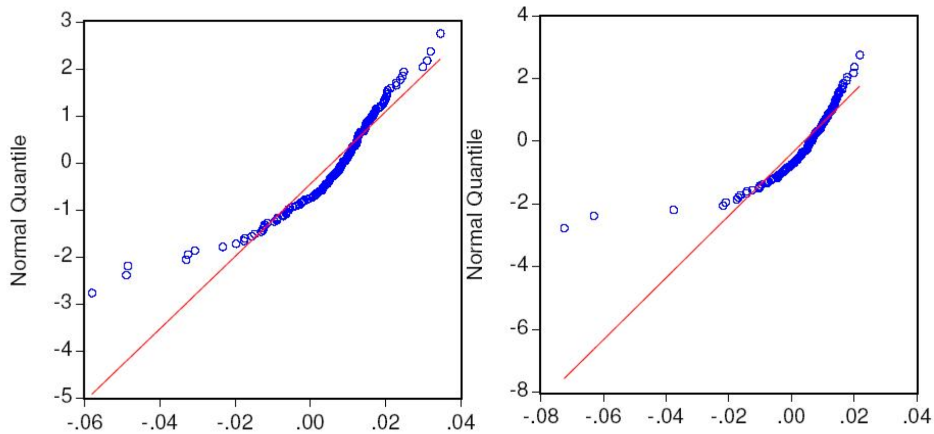

Figure 3.4: Q-Q Plots for Convertible Arbitrage and Fixed Income Arbitrage

3.1.4 Normality Tests

First column of Table 3.2 reports the Jarque-Bera Statistic Test for each hedge fund index. This statistic test is asymptotically distributed as a chi-square with two degrees of freedom under the null hypothesis. Only for the Managed Future Index the normality hypothesis cannot be rejected. In all other cases the normality assumption is clearly refused [3].

[3] Three stars (***) indicate a p-value less than 1%, two stars (**) than 5%, one star (*) than 10%.

This Table displays also Anderson-Darling and Cramer von Mises statistic tests. Both statistic tests can be used to test if a sample of data came from a population with a specific distribution. They are modifications of the Kolmogorov-Smirnov test and give more weight to the tails than does the K-S test. The K-S test is distribution free in the sense that the critical values do not depend on the specific distribution being tested. These tests make use of the specific distribution in calculating critical values. This has the advantage of allowing a more sensitive test and the disadvantage that critical values must be calculated for each distribution. Both tests belong to the class of empirical distribution function tests (EDF). These tests are based on the evaluation of the observed *distance* between the empirical distribution and the specified theoretical distribution function. For a comprehensive survey of empirical distribution function tests, also called smooth goodness of fit tests, a good reference is [34]. For a general introduction to Goodness of Fit test the interest reader can see [67].
The Anderson-Darling statistic test is computed as

$$A^2 = -N - \frac{1}{N}\sum_{i=1}^{N}(2i-1)\left[\ln F(Y_i) + \ln F(1 - Y_{N-i+1})\right] \tag{3.3}$$

while the Cramer von Mises statistic test is calculated as

$$W^2 = \frac{1}{12N} + \sum_{i=1}^{N}\left[\frac{2i-1}{2N} - F(Y_i)\right]^2 \tag{3.4}$$

where N is the number of observations, Y_i is the i-th observed value in increasing order, and F is the cumulative distribution function of the specified distribution . Under the null hypothesis hedge fund index log-returns come from a Gaussian distribution with mean and standard deviation unknown[4]. Both statistic tests allow to reject the normality hypothesis for twelve hedge funds. The Gaussian assumption is not refused for two indices: Managed Futures and Dedicated Short Bias[5].

[4]These tests were performed using E-views. This software estimates mean and standard deviation with MLE using the specified distribution under the null hypothesis. Critical values are different if the true parameters are assumed to be known

[5]In this last case the Jarque-Bera test provides an absolutely different conclusion

Table 3.2:
Normality Test Statistics for CS/Tremont Indices
Period January 1994-May 2008 (**Smoothed Data**)

Index	Jarque-Bera Stat-Test	Anderson-Darling Stat-Test	Cramer von Mises Stat-Test
CS/T Global Index	49,64***	2,411***	0,385***
Convertible Arbitrage	220,6***	5,198***	0,936***
Dedicated Short Bias	16,82***	0,528	0,063
Emerging Markets	412,9***	3,004***	0,555***
Equity Market Neutral	8,96**	0,767**	0,117*
Event Driven	5333***	4,456***	0,701***
ED Distressed	3920***	3,262***	0,482***
ED Multi-Strategy	2386***	3,971***	0,686***
ED Risk Arbitrage	419,4***	1,589***	0,243***
Fixed Income Arbitrage	2085***	7,949***	1,349***
Global Macro	91,62***	4,07***	0,721***
Long/Short Equity	124,8***	1,665***	0,251***
Managed Futures	0,549	0,242	0,036
Multi-Strategy	90,85***	2,091***	0,299***

3.1.5 Serial Correlation

All the previous results are obtained under the implicit assumption of IID observations. The first three columns of Table 3.3 report hedge fund autocorrelations up to order three. In particular, we note that all first order autocorrelations are positive and nine of them are significantly different from zero. The last column reports P-values of Ljung-Box statistic test for the joint relevance of autocorrelations up to order twelve. As already noted by C. Brooks, H. M. Kat and S. Lu [17, 66], also in our sample, Convertible Arbitrage and ED Distressed indices seems to be among the most affected by first order and general serial correlation.

Table 3.3:
Autocorrelations up to order 3
Ljung-Box Autocorrelation Tests with lags up to order 12
Period January 1994-May 2008 (**Smoothed Data**)

Index	AC(1)	AC(2)	AC(3)	Ljung-Box-Q(12) P-Value
CS/T Global Index	0,099	0,014	-0.026	0,661
Convertible Arbitrage	0,484***	0,284	0,113	0,000
Dedicated Short Bias	0,099	-0,037	-0,072	0,248
Emerging Markets	0,275***	0,020	0,002	0,033
Equity Market Neutral	0,227**	0,095	0,031	0,081
Event Driven	0,282***	0,135	-0,001	0,038
ED Distressed	0,282***	0,137	0,019	0,025
ED Multi-Strategy	0,251**	0,142	0,009	0,098
ED Risk Arbitrage	0,220**	-0,090	-0,158	0,002
Fixed Income Arbitrage	0,280***	0,006	0,016	0,048
Global Macro	0,057	0,018	0,088	0,029
Long/Short Equity	0,145*	0,024	-0,083	0,036
Managed Futures	0,057	-0,154	-0,076	0,071
Multi-Strategy	0,041	0,050	0,077	0,958

3.2 Unsmoothed Data

The observed positive autocorrelation is quite a unique property and seems inconsistent with the notion of efficient markets. According to C. Brooks, H. M. Kat and S. Lu [17, 66] one possible explanation is that the nature of hedge funds' strategies leads their returns to be inherently related to those of preceding months. As this implies lags in the major systematic risk factor, however, this is not the most plausible explanation. An alternative and more likely explanation lies in the difficulty for hedge fund managers to obtain up-to-date valuations of their positions in illiquid and complex over-the-counter securities. When confronted with this problem, hedge fund managers either use the last reported transaction price or an estimate of the current market price which may easily create lags in the evolution of the net asset value. This would explain why the different convertible arbitrage and distressed indices, employed in their work, exhibit the most significant autocorrelations.

One possible method for evaluating the effect of this autocorrelation stems from the real estate finance literature. Do to smoothing in appraisal and infrequent valuations of properties, the returns of direct property investment indices suffer from similar problems as hedge fund index returns. The approach employed in this literature has been to *unsmooth* the observed returns to create a new set of returns which are more volatile and whose characteristics are believed to more accurately capture the characteristics of the underlying property value. Geltner *et al.* [46, 47, 48, 49] give an extensive discussion of the motivations and methodologies to unsmooth the series of returns. Following this tradition, the observed value of a hedge fund index each month can be expressed as a weighted average of the underlying true value and the observed value of the hedge fund in the previous month.

Given these assumptions, it is possible to get the unsmoothed series with approximately zero first order autocorrelation:

$$y_t = \frac{y_t^* - \alpha y_{t-1}^*}{1-\alpha} \tag{3.5}$$

where y_t and y_t^* are the true unobservable underlying return and the observed return at time t. The parameter α is set equal to the first order autocorrelation coefficient of the time series.

We apply this procedure to get unsmoothed log-return series for each hedge fund index and repeat the previous statistical analyses with these new data to evaluate the impact of the unsmoothing procedure.

3.2.1 Summary Statistics

Descriptive statistics are reported in Table 3.4. The most interesting result is shown in the fifth column. All the unsmoothed time series exhibit a greater standard deviation, with a mean increment of 23%. The biggest increment is reached by Convertible Arbitrage, with an increase of 70%. As evidenced by [66], the unsmoothing procedure has also a relatively small impact on the skewness and kurtosis of each hedge fund, but the direction of these changes is not clear. The last two columns of table 3.4 clearly evidence that the normality hypothesis for the distribution of hedge fund log-returns is still unlikely. Figure 3.5 shows box plots for Convertible Arbitrage and Fixed Income Arbitrage. Unsmoothed series exhibit some outliers as the observed (smoothed) series.

Table 3.4:
Summary Statistics of Monthly Log-returns for CS/Tremont Indices
Period January 1994-May 2008 (**Unsmoothed Data**)

Index	Mean %	Median %	Max %	Min %	Std.Dev. %	Skew.	Kurt.
CS/T Global Index	0,78	0,72	8,28	-8,94	2,35	-0,08	5,52
Convertible Arbitrage	0,58	0,80	8,39	-10,1	2,34	-1,13	8,38
Dedicated Short Bias	-0,20	-0,44	22,2	-10,8	5,25	0,55	4,05
Emerging Markets	0,63	1,59	18,9	-36,2	5,89	-1,47	11,5
Equity Market Neutral	0,72	0,64	3,83	-2,29	0,96	0,27	4,34
Event Driven	0,81	1,05	4,29	-17,6	2,14	-3,86	33,5
ED Distressed	0,91	1,04	5,69	-18,7	2,37	-3,40	29,7
ED Multi-Strategy	0,77	0,99	5,32	-16,38	2,24	-2,68	21,7
ED Risk Arbitrage	0,55	0,66	4,79	-8,17	1,45	-1,17	10,4
Fixed Income Arbitrage	0,40	0,63	5,49	-8,61	1,55	-2,30	15,7
Global Macro	1,03	1,02	11,0	-12,7	3,15	-0,14	6,50
Long/Short Equity	0,86	0,82	12,6	-14,4	3,23	-0,18	6,89
Managed Futures	0,47	0,37	9,94	-10,3	3,59	-0,17	3,18
Multi-Strategy	0,65	0,81	3,87	-5,17	1,28	-1,12	5,98

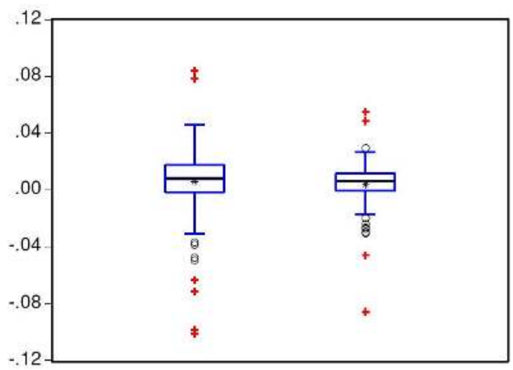

Figure 3.5: Box Plots for Convertible Arbitrage and Fixed Income Arbitrage

3.2.2 Kernel Density Estimators

Figures 3.6 and 3.7 report Normal kernel density estimates for smoothed (blue line) and unsmoothed (red line) series of log-returns for the same two indices. These estimates are based on the optimal bandwidth h. In addition each graph shows the Gaussian density with expectation and standard deviation equal to their empirical unsmoothed counterparts (black line with dots).

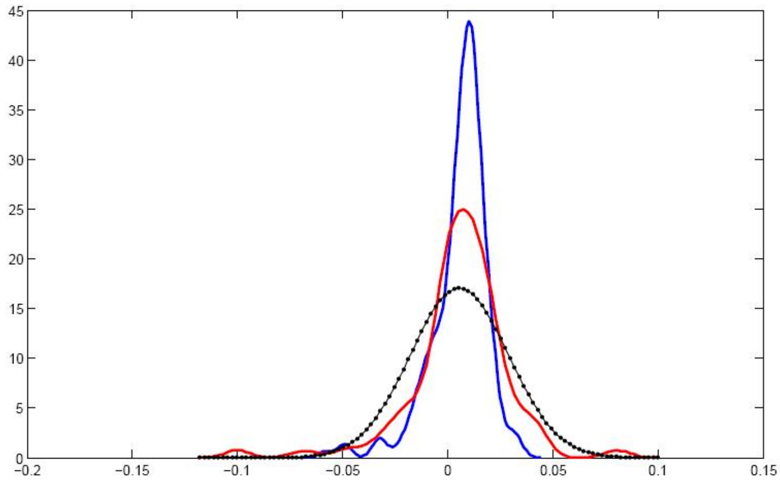

Figure 3.6: Normal Kernel Density Estimation for Convertible Arbitrage

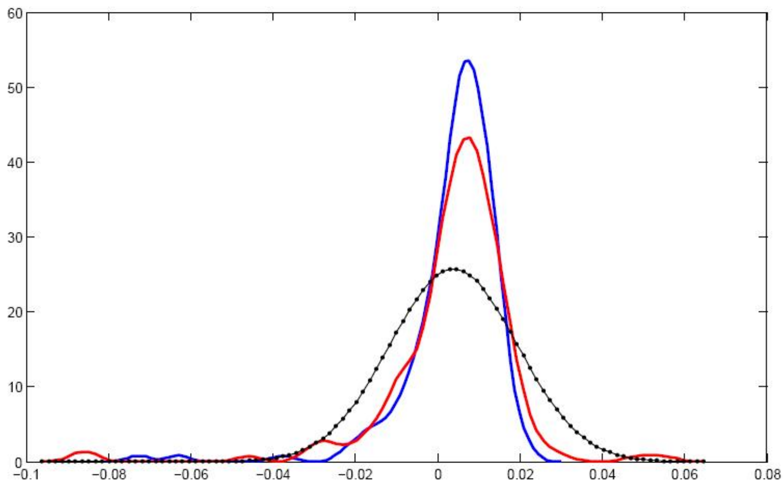

Figure 3.7: Normal Kernel Density Estimation for Fixed Income Arbitrage

Both empirical densities for unsmoothed log-returns exhibit negative skewness

and fatter tails than the Gaussian density.
A comparison between blue and red lines shows that the unsmoothing procedure generates empirical distributions with lower peaks and greater dispersion around the mean. Bigger is the first order autocorrelation, larger is the variance and smaller is the peak of the unsmoothed empirical density with respect to the observed one.

3.2.3 Quantile-Quantile Plots

Figure 3.8 provides further evidence against the normality assumption. Comparing Figures 3.4 and 3.8 it is possible to see different patterns of deviation from linearity. In Figure 3.4 the convex shape of the QQ-plots for Convertible Arbi-

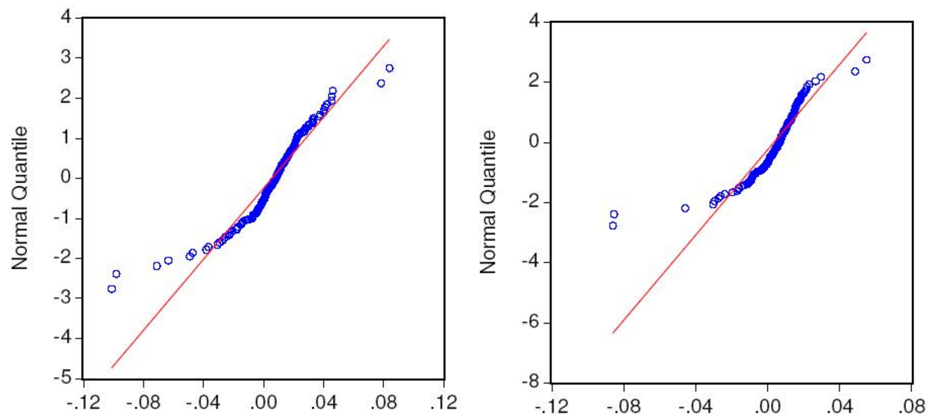

Figure 3.8: Q-Q Plots for Convertible Arbitrage and Fixed Income Arbitrage

trage and Fixed Income Arbitrage indicate that the distribution of both indices is negatively skewed with a longer left tail and a shorter rigth tail than the Normal distribution. See figures 3.2 and 3.3. Notice that if the shape were concave, it would indicate that the distribution is positively skewed.
QQ-plots that fall on a straight line in the middle but curve upward at the left end and curve downward at the right end indicate that the distribution is leptokurtic and has thicker tails than the Gaussian distribution. Figure 3.8 depicts a similar situation for both indices. Both tails for each index are heavier than the tails of the Gaussian distribution. See figures 3.6 and 3.7. It should be noted that if the plot curves downward at the left, and upward at the right, it is an indication that the distribution is platykurtic and has thinner tails than the normal distribution.

The unsmoothing procedure for this two indices has produced a reduction in absolute value of the skewness (still negative) and has increased the length of the right tail. However, this result is not valid for all CS/T indices.

3.2.4 Normality Tests

Table 3.5 reports formal normality tests. The null hypothesis is always completely rejected, with the exception of the Managed Futures Index. The situation of the Dedicated Short Bias Index is again ambiguous; the response of the Jarque-Bera test and the EDF tests is clearly opposite.

Table 3.5:
Normality Test Statistics for CS/Tremont Indices
Period January 1994-May 2008 (**Unsmoothed Data**)

Index	Jarque-Bera Stat-Test	Anderson-Darling Stat-Test	Cramer-von Mises Stat-Test
CS/T Global Index	42,35***	2,105***	0,330***
Convertible Arbitrage	231,12***	4,390***	0,757***
Dedicated Short Bias	15,61***	0,475	0,059
Emerging Markets	554,8***	3,288***	0,618***
Equity Market Neutral	13,94***	0,914**	0,134**
Event Driven	6745***	4,334***	0,657***
ED Distressed	5149***	3,502***	0,529***
ED Multi-Strategy	2590***	2,950***	0,483***
ED Risk Arbitrage	403,8***	1,829***	0,275***
Fixed Income Arbitrage	1249***	7,727***	1,332***
Global Macro	82,25***	3,922***	0,696***
Long/Short Equity	101,9***	1,415***	0,211***
Managed Futures	1,013	0,283	0,039
Multi-Strategy	95,20***	2,237***	0,325***

3.2.5 Serial Correlation

Table 3.6 shows that these unsmoothed time series are not affected by first order autocorrelation. Notice, however some problems of general serial correlation for two indices. Ljung-Box Statistics LB-Q(12) for ED Risk Arbitrage and Global Macro do not allow to reject the hypothesis that autocorrelations up to order 12 are different from zero at significance level of 5%. Further analyses, not reported in this work, reveals that in both cases this is due to autocorrelation of order 5.

Table 3.6:
Autocorrelations up to order 3
Ljung-Box Autocorrelation Tests with lags up to order 12
Period January 1994-May 2008 (**Unsmoothed Data**)

Index	AC(1)	AC(2)	AC(3)	Ljung-Box Q(12)/P-Value
CS/T Global Index	0,003	0,010	-0,017	0.725
Convertible Arbitrage	-0,032	0,085	-0,085	0,810
Dedicated Short Bias	0,005	-0,038	-0,057	0,575
Emerging Markets	0,025	-0,045	0,035	0,634
Equity Market Neutral	-0,012	0,036	0,018	0,718
Event Driven	-0,008	0,085	-0,036	0,978
ED Distressed	-0,008	0,083	-0,023	0,910
ED Multi-Strategy	-0,017	0,100	-0,019	0,880
ED Risk Arbitrage	0,033	-0,116	-0,128	0,023
Fixed Income Arbitrage	0,034	-0,077	-0,007	0,889
Global Macro	-0,003	0,008	0,090	0,013
Long/Short Equity	0,000	0,018	-0,072	0,192
Managed Futures	0,009	-0,154	-0,068	0,115
Multi-Strategy	-0,004	0,044	0,083	0,966

To conclude this Chapter we summarize the main results on statistical properties of hedge funds' log-returns:

- the distributions of monthly hedge funds' log-returns are usually not symmetric and negatively skewed;
- these distributions have fatter tails than the Normal;
- often hedge fund log-returns exhibit first order serial correlation. However,

this is very likely a result of the appraisal procedure, and so observed data do not reflect exactly the true values that are unobserved. In other words, the true generating hedge fund log-returns process can be considered as a process with uncorrelated increments[6];

- the evolution of hedge funds' Net Asset Value in general cannot be described by a Geometric Brownian motion.

To model the temporal behaviour of hedge funds' log-returns more flexible stochastic processes than Brownian motion are therefore necessary. More general Lévy processes can represent a possible solution.

[6]See previous discussion about smoothed and unsmoothed data

Chapter 4

Lévy processes

In the previous chapter we have shown that monthly log-returns of hedge funds are not Normally distributed. Other more flexible distributions are therefore needed to take into account skewness and excess-kurtosis. Furthermore, not only more flexible static distributions are necessary, but in order to model the behaviour through time of hedge funds' log-returns more flexible stochastic processes than Brownian motion are required. In the late 1980s and 1990s, different processes were proposed for modelling financial data. To describe the evolution of log-returns Brownian Motion was replaced by more general Lévy process. The Gaussian distribution was substituted by more flexible infinitely divisible distributions able to take into account skewness and excess kurtosis. Examples of such distribution are the Normal Inverse Gaussian (See Barndorff-Nielsen(1995-1998) [9, 10]), the Symmetric Variance Gamma (See Madan and Seneta(1990) [80]), the Variance Gamma (See Madan *et al.* (1998) [82]), the Hyperbolic (See Eberlein and Keller (1995), [37]) the Generalized Hyperbolic (See Eberlein and Prause (1998-1999) [38, 107]), the CGMY (See Carr, Geman, Madan and Yor (2002) [23] and the Meixner distribution (Schoutens (2001-2002) [114, 115]).

This chapter contains a short review of the properties of Lévy processes and Exponential Lévy models, that are used in the sequel. General reference works on Lévy processes are by Bertoin (1996) [14], Sato (1999) [109] and Applebaum (2003) [1]. For a general reference on Lévy processes in finance the best choice is undoubtedly Cont and Tankov (2004) [33]. Persons more interested in practical applications of financial models with jumps can find Schoutens (2003) [112] very useful.

4.1 Definition and Properties

A Lévy process on a probability space $(\Omega, \Im,\ P)$ with values in R is a right continuous with left limit stochastic process $\{X_t,\ t \geq 0\}$ with the following properties:

- $X_0 = 0$;
- Independent increments: for every increasing sequence of times to $t_0, \ldots t_n$, the random variables $X_{t_0},\ X_{t_1} - X_{t_0}, \ldots,\ X_{t_n} - X_{t_{n-1}}$ are independent;
- Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t;
- Stochastic continuity: $\forall \epsilon > 0$, $\lim_{h \to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$

The last condition does not imply in any way that the sample paths are continuous. It ensures that for given time t, the probability of seeing a jump at t is zero: discontinuities occur only at random times.
A very important property of Lévy processes is the infinitely divisibility. If $\{X_t,\ t \geq 0\}$ is a Lévy process then for every t, X_t has an infinitely divisible distribution. Conversely, if F is an infinitely divisible distribution then there exists a Lévy process $\{X_t,\ t \geq 0\}$ such that the distribution of X_1 is given by F. This means that the Characteristic Function of X_t can be obtained from the Characteristic Function of X_1 and vice versa:

$$\Psi_{X_t}(u) = E[\exp(iuX_t)] = E[\exp(iuX_1)]^t = (\Psi_{X_1}(u))^t \quad \forall u \in R \tag{4.1}$$

In particular, the Characteristic Function of X_t can be computed as

$$E[\exp(iuX_t)] = \exp(t[\zeta(u)]) \quad \forall u \in R \tag{4.2}$$

where $\zeta(u) = \phi_{X_1}(u) = \ln \Psi_{X_1}(u)$ corresponds to the Cumulant Characteristic Function of X_1 and is called Characteristic Exponent of X_t.
Notice that for a Lévy process the Cumulant Characteristic Function varies linearly in t: $\phi_{X_t}(u) = t\phi_{X_1}(u) = t\zeta(u)$. The law of X_t is therefore determined by the knowledge of the law of X_1 : the only degree of freedom we have in specifying a Lévy process is to specify the distribution of X_t for a single time (say, $t = 1$).
The Characteristic Exponent $\zeta(u)$ satisfies the so called *Lévy-Khintchin* formula:

$$\zeta(u) = i\gamma u - \frac{1}{2}\eta^2 u^2 + \int_R (\exp(iux) - 1 - iux1_{|x| \leq 1})\upsilon(dx) \tag{4.3}$$

where $\gamma \in$ R, $\eta \geq 0$, and υ is a measure on R-$\{0\}$ with $\int_R (1 \wedge x^2)\upsilon(dx) < \infty$. The triplet $\left[\gamma,\ \eta^2,\ \upsilon(dx)\right]$ is called Lévy triplet and υ *Lévy measure* of X. From the Lévy-Khintchin formula, it is possible to see that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part.[1] The Lévy measure $\upsilon(dx)$ describes how the jumps occur. Jumps of sizes in the set $A \in B(\mathrm{R})$[2] occur according to a Poisson process with intensity parameter $\int_A \upsilon(dx)$. In other words, $\upsilon(A)$ is the expected number, per unit of time, of jumps whose size belongs to A.

If $\eta = 0$ and $\int_{-1}^{+1} |x|\, \upsilon(dx) < \infty$, it can be shown that a Lévy process is of *finite variation* and its Characteristic Exponent can be expressed as:

$$\zeta(u) = i\varsigma u + \int_R (\exp(iux) - 1)\upsilon(dx) \tag{4.4}$$

where $\varsigma = \gamma - \int_{-1}^{+1} x\upsilon(dx)$.

A Lévy process of finite variation can be decomposed into the difference of two increasing processes.

If $\eta = 0$ and $\int_{-1}^{+1} \upsilon(dx) < \infty$, there are finitely many jumps in any finite interval and the process is said to have *finite activity.*

If $\eta \neq 0$ the process is of *infinite variation* because the Brownian motion component is of infinite variation. A pure jump Lévy process is of infinite variation if and only if $\int_{-1}^{+1} |x|\, \upsilon(dx)$ is not finite. In this case the sum of small jumps does not converge. This leads to the necessity of the compensator term $iux1_{|x|\leq 1}$ in (4.3).

An important class of Lévy processes is represented by subordinators. A *subordinator* is a non negative and non decreasing Lévy process. Its characteristic triplet is such that $\eta = 0$, $\upsilon((-\infty, 0]) = 0$, and $\varsigma \geq 0$: the process has no diffusion component, only positive jumps of finite variation and a non negative drift.

4.1.1 Building Lévy processes and Brownian Motion Subordination

According to Cont and Tankov [33] there exist three convenient approaches to build a parametric Lévy process:

[1] See also Lévy-Ito decomposition [33] Proposition 3.7

[2] $B(\mathrm{R})$ is the Borel σ-algebra of R

- the direct specification of the Lévy measure;
- the specification of an infinitely divisible density as the density of increments at a given time scale;
- the subordination of a Brownian motion with an independent increasing Lévy process.

The first approach is described in Cont and Tankov [33] using the example of tempered stable processes. This way to construct Lévy processes provides a dynamic vision of the process because it allows to build directly the jump structure and to know, via the Levy-Khinchin formula, its distribution at any time. However, sometimes this distribution is not available in a very explicit form.
The second approach is to specify an infinitely divisible density as the density of increments at a given time scale. Generalized Hyperbolic processes can be constructed in this way. In this approach it is easy to simulate the increments of the process at the same time scale and to estimate parameters of the distribution if data are sampled with the same frequency, but in general the Lévy measure is not known. Therefore, unless this distribution belongs to some parametric class closed under convolution, we do not know the law of the increments at other time scales. In particular, given an infinitely divisible distribution it is not easy to infer from its density whether the corresponding Lévy process has a Gaussian component or whether it has finite or infinite activity.
The third way allows to compute the Characteristic Function of the resulting process immediately every t. However this approach does not always allow to find an explicit formula for the Lévy measure. Due to the conditionally Gaussian structure of the process, simulation and some computations can be considerably simplified. For example, call option price can be expressed as an integral involving Black-Scholes prices. The interpretation of the subordinator as a business time makes models of this type easier to understand and interpret (See Geman and Geman *et al.* [51, 52, 53, 54]). The first idea to use such stochastic clock for analysing financial processes was introduced by Clark (1973) [32]. Clark proposed to use Bochner (1950) [16] concept of subordinated process while he was analyzing cotton Futures price to address the non-normality of observed returns. In particular he time changed a Brownian motion with an independent process with log-normal

increments. The economic interpretation of the subordinator in his work was the cumulative volume of traded contracts. Five years later, Monroe [99] extended the Dubins-Schwarztz theorem [3] [36] to semimaringale and extablished that *any semimartingale can be written as a time changed Brownian motion.* Revisiting Clark's brillant conjecture, Ané and Geman (2000) [4] demonstrate that the structure of semimartingales necessarily prevails for stock prices by bringing together the No Arbitrage Assumption and Monroe's theorem to establish that any stock price may be written as a Brownian motion time changed with an almost surely increasing process. In particular, they show in a general non-parametric setting, that in order to recover a quasi perfect normality of returns, the transaction clock is better represented by the number of trades than the volume.
All Lévy processes are semimartingales because a Lévy process can be split into a sum of a square integrable martingale and a finite variation process: this is the Levy-Ito decomposition (See [33] Proposition 3.7). It follows from Monroe's theorem that every Lévy process can be expressed as a time changed Brownian Motion. The following is only a list of Lévy processes used in finance to describe the evolution of log-returns whose subordinators are known:

- Geman, Madan and Yor (2001) prove that the Poisson model with reflected normal jumps' intensity can be constructed by Poisson time changing a univarite Brownian motion (For details see [54]);

- the Normal Inverse Gaussian process can be obtained from a Brownian Motion with drift time changing the physical time with an Inverse Gaussian time (See for example Shoutens (2003) [112]);

- the Generalized Hyperbolic process can be obtained from a Brownian Motion with drift using a Generalized Inverse Gaussian process as a subordinator (See Prause (1999) [107]);

- Madan, Carr and Chang (1998) [82] introduce the asymmetric version of the Variance Gamma process by subordinating an arithmetic Brownian motion with a Gamma process ;

[3] Dubins-Schwarztz (1965) theorem: *Any continuous martingale is a time changed Brownian Motion.*

- Madan and Yor (2005) show that CGMY and Meixner processes are absolutely continuous with respect to One Sided Stable subordinators. For details on their subordinators see [83].

For a complete review of stochastic time change techniques from its origins in probability theory see Geman [51]. The relation among option pricing, stochastic volatility and time-changed Lévy processes is discussed by Carr and Wu (2004) [24].
We conclude this section with the first part of *THEOREM 4.2* [33] on *the subordination of a Lévy process*. In the next chapter we apply this part of the Theorem several times.
THEOREM: Subordination of a Lévy process
Fix a probability space $(\Omega,\ \Im,\ P)$. Let $\{X_t,\ t \geq 0\}$ be a Lévy process on $\Re^n$ with Characteristic Exponent $c(\mathbf{u})$ and triplet $(\gamma,\ A,\ \upsilon(d\mathbf{x}))$ and let $\{K_t,\ t \geq 0\}$ be a subordinator with Laplace Exponent $l(u)$ and triplet $(b,\ 0,\ \epsilon(dk))$. Then the process $\{Y_t,\ t \geq 0\}$ defined for each $\omega \in \Omega$ by $Y(t,\ \omega) = X(K(t,\omega),\omega)$ is a Lévy process. Its Characteristic Function is

$$E[\exp(iuY_t)] = \exp[tl(c(\mathbf{u}))]. \tag{4.5}$$

The Characteristic Exponent of $\mathbf{Y}$ is obtained by composition of the Laplace Exponent of K with the Characteristic Exponent of X[4]. The interested reader can find the complete version of this Theorem and a sketch of the proof in Cont and Tankov [33] (Section 4.2.2). For a detailed proof we refer the reader to Sato ([109] Theorem 30.1)

4.2 The Exponential Lévy Model and Equivalent Martingale Measures

4.2.1 The Exponential Lévy model

Consider the following financial market living on a stochastic basis $(\Omega,\ \Im,\ (\Im_t)_{t\geq 0},\ P)$ which satisfies the usual conditions. We model two assets. The first one is a risk

[4]This Theorem is valid for general Lévy process (not only Brownian motion) time changed with a subordinator.

free asset whose price at time t is

$$A_t = A_0 \exp(rt) \tag{4.6}$$

where r is the risk free rate. The second one is a risky asset with price process

$$S_t = S_0 \exp(X_t) \tag{4.7}$$

where X_t is a Lévy process. A price process of this type is called (ordinary) Exponential (or Geometric) Lévy process.

4.2.2 Absence of Arbitrage, Market Incompleteness and Equivalent Martingale Measures

The equivalent martingale measure method is one of the most powerful methods in the option pricing theory. The existence of a martingale measure is related to the absence of arbitrage, while the uniqueness of the equivalent martingale measure is related to market completeness, i.e. perfect hedging. The Exponential Lévy market model is arbitrage free if the log-price process satisfies one of the following (not mutually exclusive) conditions (See Cont and Tankov [33] section 9.5):

- $\eta > 0$;
- it has infinite variation;
- it has both positive and negative jumps;
- it has positive jumps and negative drift or negative jumps and positive drift.

However, the risk due to jumps cannot be hedged and therefore there is no more a unique risk-neutral measure. This is always the case when the price process is an arbitrage-free Geometric Lévy process, excepted of course in the Geomatric Brownian motion case. In an arbitrage-free Lévy market model, there are many different equivalent martingale measures under which the discounted asset price process is a martingale (See Cont and Tankov [33] chapters 9 and 10). Many candidates for suitable martingale measure equivalent to the real world probability measure have been suggested and from a theoretical point of view, different criteria can be chosen based on hedging arguments or distance minimization. Hellinger distance, L^2 distance, entropy or Kullback Leibler distance have frequently been

put forward (see [12, 57, 95]).
The following is only a list of possible equivalent martingale measures:

- Minimal Martingale Measure (MMM) (Follmer & Schweizer (1991) [42])
- Variance Optimal Martingale Measure (VOMM) (Schweizer (1995) [116], Jeanblanc & Miyahara (2005) [62])
- Mean Correcting Martingale Measure (MCMM)
- Utility Based Martingale Measure (UBMM)
- Minimal Entropy Martingale Measure (MEMM) (Miyahara (1996) [88], Frittelli (2000) [43] , Fujiwara & Miyahara (2003) [44])
- Esscher Martingale Measure (ESMM) (Gerber & Shiu (1994, 1996) [55, 56], Buhlmann *et al.* (1996) [22]).

The interested reader can find a good survey of martingale measures for Geometric Lévy models in Mihayara (2005) [95] (See also [41]). In the following section we describe the Esscher Equivalent Martingale Measure.

4.2.3 Esscher Equivalent Martingale Measure

The Esscher transform is very popular and thought to be very important in the actuary theory. Esscher in 1932 has introduced the risk function and the transformed risk function for the calculation of collective risk [39]. In a financial environment, Gerber and Shiu (1994, 1996) use the Esscher transform to construct equivalent martingale measures for processes with independent and stationary increments [55, 56]. Inspired by this, Buhlmann *et al.* (1996) more generally use conditional Esscher transforms to construct equivalent martingale measures for classes of semi-martingales [21, 22]. This measure sometimes is called the *compound return Esscher transformed martingale measure* to distinguish it from the *simple return Esscher transformed martingale measure*, which is known as the Minimal Entropy Martingale Measure (see for example [94, 96, 61]). From a mathematical point of view, ESMM is the nearest equivalent martingale measure to the historical probability P in the sense of power function metric [57, 95]. From an economic viewpoint, the risk-neutral universe being not unique, prices will rely on the attitude of economic agents toward risk. Using the expected utility or the neo-Bernoulli theory, a fair price can be obtained with the marginal utility principle and it can be shown that the fair price or indifference price given by a power utility or logarithmic utility function can be expressed via the compound return Esscher Measure [55, 56, 57]. From an operational perspective: most of the usual univariate Lévy processes used in financial modelling remain of the same kind in this particular risk-neutral universe. Schoutens ([112] Section 6.2.2) shows that the Normal Inverse Gaussian and the Meixner processes remain of the same type under the Esscher risk neutral probability measure. The same result is obtained in [73] for the Kou jump-diffusion process and in [107] for the Generalized Hyperbolic model. In the special case of the Brownian Motion with drift the ESMM coincides with the Mean Correcting Martingale Measure (this is obvious since in this special Lévy model the market is complete). Finally, the passage of one set of parameters of these processes in the historical universe to the set of parameters in the compound return Esscher risk-neutral universe is very simple.

Esscher Transform and Risk Neutral Dynamics

In this subsection we explain how to use the Esscher transform method to find an equivalent martingale measure for the process (4.7).
The Q_h Esscher transform associated with the risk process X_t is defined by the following Radon-Nicodym derivative:

$$\frac{dQ_h}{dP}|\Im_t = \frac{\exp(hX_t)}{E\left[\exp(hX_t)\right]}. \tag{4.8}$$

The parameter h must be fixed such that the discounted price process of the asset is a martingale under the probability measure Q_h equivalent to P:

$$E^{Q_h}\left[\exp\left(-rt\right)S_t\right] = \exp\left(-rt\right)S_0E^{Q_h}\left[\exp\left(X_t\right)\right] = S_0. \tag{4.9}$$

This implies

$$E^{Q_h}\left[\exp\left(X_t\right)\right] = E\left[\exp\left(X_t\right)\frac{dQ_h}{dP}|\Im_t\right] = \exp\left(rt\right). \tag{4.10}$$

This condition can be written in terms of the Moment Generating Function of X_t (if it exists)

$$\frac{M_{X_t}(h+1)}{M_{X_t}(h)} = \frac{E[\exp((h+1)X_t)]}{E\left[\exp(hX_t)\right]} = \exp\left(rt\right) \tag{4.11}$$

or equivalently in terms of its Characteristic Function

$$\frac{\Psi_{X_t}(-i(h+1))}{\Psi_{X_t}(-ih)} = \frac{E[\exp((h+1)X_t)]}{E\left[\exp(hX_t)\right]} = \exp\left(rt\right). \tag{4.12}$$

Since the distribution of X_t is infinitely divisible the above conditions are equivalent to the followings:

$$\frac{M_{X_1}(h+1)}{M_{X_1}(h)} = \frac{E[\exp((h+1)X_1)]}{E\left[\exp(hX_1)\right]} = \exp\left(r\right) \tag{4.13}$$

$$\frac{\Psi_{X_1}(-i(h+1))}{\Psi_{X_1}(-ih)} = \frac{E[\exp((h+1)X_1)]}{E\left[\exp(hX_1)\right]} = \exp\left(r\right). \tag{4.14}$$

The last two expressions show that if a solution $\hat{h}$ exists, it does not depend on t. It can be shown that if (4.13) has a solution $\hat{h}$ then the ESMM exists and is unique. The process X_t is also a Lévy process under the new probability measure Q_h. Lévy triplets of X_t under the measures Q_h and P are linked by the following relations:

$$\gamma^{Q_h} = \gamma + \hat{h}\eta^2 + \int_{|x|\leq 1} x[\exp(\hat{h}x - 1)]\upsilon(dx) \tag{4.15}$$

$$\eta^{Q_h 2} = \eta^2 \tag{4.16}$$

$$\upsilon^{Q_h}(dx) = \exp(\hat{h}x)\upsilon(dx) \tag{4.17}$$

where the P generating triplet is $\left[\gamma,\ \eta^2,\ \upsilon(dx)\right]$. However, in this case it should be noted that since $\eta \neq 0$ the easiest way to a get an equivalent martingale measure is represented by the Mean Correcting Martingale method.[5] In models with jumps, if the Gaussian component is absent ($\eta \neq 0$) we cannot change the drift to get an equivalent martingale measure, but we can obtain a greater variety of equivalent martingale measures by altering the distribution of the jumps.[6] If the Gaussian part is not present, the components of the Lévy triplet under the ESMM are given by:

$$\gamma^{Q_h} = \gamma + \int_{|x|\leq 1} x[\exp(\hat{h}x - 1)]\upsilon(dx) \tag{4.18}$$

---

[5] It is sufficient to change the drift like in the Black & Scholes model

[6] When option prices are available, a change of measure called *Mean Correcting Martingale measure* (See for example Schoutens [112]) is usually applied even if the process has zero Gaussian component. The model parameters are calibrated directly under the risk neutral measure *chosen by the market*. Since no explicit link among historical and risk neutral parameters exists, this method is *improperly* called Mean Correcting Martingale method. Under this new measure both the drift and the Lévy measure differ from the real world.

$$\eta^{Q_h 2} = 0 \tag{4.19}$$

$$\upsilon^{Q_h}(dx) = \exp(\hat{h}x)\upsilon(dx). \tag{4.20}$$

The Q_h Moment Generating Function of X_1 can be computed in the following way:

$$E^{Q_h}[\exp(uX_1)] = \frac{E[\exp(u+\hat{h})X_1]}{E[\exp(\hat{h}X_1)]} \tag{4.21}$$

$$M_{X_1}^{Q_h}(u) = \frac{M_{X_1}(u+\hat{h})}{M_{X_1}(\hat{h})}. \tag{4.22}$$

In a similar way, we get the Q_h Characteristic Function of X_1:

$$E^{Q_h}[\exp(iuX_1)] = \frac{E[\exp(iu+\hat{h})X_1]}{E[\exp(\hat{h}X_1)]} \tag{4.23}$$

$$\Psi_{X_1}^{Q_h}(u) = \frac{\Psi_{X_1}(u-i\hat{h})}{\Psi_{X_1}(-i\hat{h})}. \tag{4.24}$$

4.3 Jump-Diffusions vs Infinite Activity Lévy Processes

Financial models with jumps can be divided into two main categories:

- jump-diffusion models;
- infinite activity Lévy Processes.

In the first category, the evolution of prices is given by an exponential of a diffusion process, punctuated by jumps at random intervals. In these models the jumps represent rare events, crashes and large draw downs. Such an evolution can be represented by modelling the log-price as a Lévy process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. Examples of such models are the Merton jump-diffusion model with Gaussian jumps [86] and the Kou model with double exponential jumps [69].

The second category consists of models with infinite number of jumps in every interval. In these models, one does not need to introduce a Brownian component since the dynamics of jumps is already rich enough to generate nontrivial small time behaviour (See Carr *et al.* [23]) and it has been argued [23, 54] that such processes give a more realistic description of the price process at various time

scales. Examples of such models are the Variance Gamma [80, 81, 82], the Normal Inverse Gaussian [9, 10], the Hyperbolic [37], the Generalized Hyperbolic [38, 107], the CGMY [23] and the Meixner models [113, 114, 115]. However, the most important thing is that many models from this class can be constructed via Brownian subordination, which gives them additional analytical tractability compared to jump-diffusion models.
In the following subsections, we describe the Gamma and the Variance Gamma processes in detail because we will make a wide use of them in the next chapter.

4.3.1 The Gamma Process

The density function of a Gamma random variable G_1 with parameters $a > 0$ and $b > 0$ is given by

$$f_{G_1}(g) = \frac{b^a}{\Gamma(a)} g^{a-1} \exp(-gb) \quad g > 0 \tag{4.25}$$

where $\Gamma(a) = \int_0^{+\infty} z^{a-1} \exp(-z) dz$ is the Gamma function.
The Characteristic Function is given by

$$\psi_{G_1}(u) = (1 - \frac{iu}{b})^{-a} \tag{4.26}$$

This distribution is infinitely divisible since it easy to show that

$$\psi_{G_t}(u) = (1 - \frac{iu}{b})^{-at} \tag{4.27}$$

and so we the following relation hold

$$\psi_{G_t}(u) = [\psi_{G_1}(u)]^t . \tag{4.28}$$

The Gamma process $\{G_t,\ t \geq 0\}$ with parameters $a > 0$ and $b > 0$ is defined as the stochastic process which starts at 0 and has stationary and independent Gamma distributed increments. More precisely, G_t follows a Gamma$(at,\ b)$ distribution.
The Lévy triplet of the Gamma process is given by

$$[a(1 - \exp(1 - b))/b,\ 0,\ a \exp(-bg) g^{-1} 1_{g>0} dg)] \tag{4.29}$$

The following properties of the Gamma $(a,\ b)$ distribution can be derived through its Characteristic Function:

$$G_1 = \begin{cases} \text{Mean} & a/b \\ \text{Variance} & a/b^2 \\ \text{Skewness} & 2a^{-1/2} \\ \text{Kurtosis} & 3(1 + 2/a) \end{cases}$$

Notice also that G_1 has the following scaling property: cG follows a Gamma$(a,\ b/c)$ for $c > 0$.

4.3.2 The Variance Gamma Process

The symmetric version of this process was introduced by Madan and Seneta [80] as a model for share market returns and subsequently used for option pricing purpose by Madan and Milne [81]. The general model with skewness was introduced by Madan, Carr and Chang [82].
If X_1 follows a Variance Gamma law, that is if $X_1 \sim VG(\sigma,\ \nu,\ \theta)$, then its Characteristic Function is given by the well-known following expression

$$\Psi_{X_1}(u) = \left(1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu\right)^{-1/\nu}. \tag{4.30}$$

Since this distribution is infinitely divisible then the process $X = \{X_t,\ t \geq 0\}$ is a Lévy process. This process, called Variance Gamma, as every Lévy process, it starts at zero, has stationary and independent increments. In particular, $X_{s+t} - X_s \sim VG(\sigma\sqrt{t},\ \nu/t,\ \theta t)$ which is the same distribution of X_t.
The VG process can be obtained by evaluating Brownian motion with drift at a random time change given by a Gamma process:

$$X_t = \theta G_t + \sigma W_{G_t} \tag{4.31}$$

where $\theta,\ \nu > 0$ and $\sigma > 0$ are constants;
$G = \{G_t,\ t \geq 0\}$ is a Gamma process $G_t \sim Gamma(t/\nu,\ 1/\nu)$;
$W = \{W_t,\ t \geq 0\}$ is a Wiener process $W_t \sim Normal(0,\ t)$.
The Characteristic Function of X_t is easily obtained from the one of X_1

$$\Psi_{X_t}(u) = E\left[\exp\left(iuX_t\right)\right] = \left(1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu\right)^{-t/\nu}. \tag{4.32}$$

It can be shown that the VG process is a pure jump process with infinite activity but with paths of finite variation. In particular, Madan *et al.* showed that the VG process can be also expressed as the difference of two independent Gamma processes:

$$X_t = G_t^{(1)} - G_t^{(2)} \tag{4.33}$$

with $G_1^{(1)} \sim Gamma(C,\ M)$ and $G_1^{(2)} \sim Gamma(C,\ G)$, where

$$\begin{aligned} C &= 1/\nu > 0 \\ G &= (\sqrt{1/4\theta^2\nu^2 + 1/2\sigma^2\nu} - 1/2\theta\nu)^{-1} > 0 \\ M &= (\sqrt{1/4\theta^2\nu^2 + 1/2\sigma^2\nu} + 1/2\theta\nu)^{-1} > 0. \end{aligned}$$

This characterization allows to compute the Lévy measure:

$$\upsilon(dx) = \begin{cases} C\exp(Gx)\,|x|^{-1}\,dx & x < 0 \\ C\exp(-Mx)x^{-1}dx & x > 0 \end{cases}$$

The Lévy measure has infinite mass and consequently a Variance Gamma process has infinitely many jumps in any finite interval. Furthermore, it has paths of finite variation because $\int_{-1}^{+1} |x|\,\upsilon(dx)$ is finite. Its Lévy triplet is given by $[\gamma,\ 0,\ \upsilon(dx)]$, where

$$\gamma_X = \frac{-C(G(\exp(-M) - 1) - M(\exp(-G) - 1))}{MG} \tag{4.34}$$

The Characteristic Function of X_1 can be expressed in terms of C, M and G:

$$\Psi_{X_1}(u) = \left(\frac{GM}{GM + (M - G)iu + u^2}\right)^C \tag{4.35}$$

With this parameterization it easy to show that the Variance Gamma process is a special case of the CGMY model (see [23, 112]). Expressions (4.30) and (4.35) allow to compute the theoretical mean and central moments of X_1 under different parameterizations. We report only the moments obtained using (4.30) since we are mainly interested in the representation of the Variance Gamma process as a Brownian motion with drift time changed by a Gamma subordinator.

$$X_1 = \begin{cases} \text{Expected Value} & \theta \\ \text{Variance} & \sigma^2 + \nu\theta^2 \\ \text{Skewness} & \theta\nu\left(3\sigma^2 + 2\nu\theta^2\right) / \left(\sigma^2 + \nu\theta^2\right)^{3/2} \\ \text{Kurtosis} & 3\left(1 + 2\nu - \nu\sigma^4\left(\sigma^2 + \nu\theta^2\right)^{-2}\right) \end{cases}$$

Looking at this table, it is easy to see that the parameter θ controls the skewness and ν the kurtosis. Since $\nu > 0$ this distribution is leptokurtic. If $\nu \to 0$ skewness tends to 0 and kurtosis to 3. In particular, if $\nu \to 0$ the Variance Gamma distribution converges to the $Normal(\theta,\ \sigma^2)$ since

$$\lim_{\nu \to 0}\left[1 - (iu\theta - \frac{1}{2}u^2\sigma^2)\nu\right]^{-1/\nu} = \exp(iu\theta - \frac{1}{2}u^2\sigma^2). \tag{4.36}$$

If $\theta = 0$ the Variance Gamma distribution is symmetric and its expected value is 0 (in this case we get the model proposed by Madan and Seneta [80]). If the Variance Gamma distribution is negatively skewed ($\theta < 0$) /(or positively ($\theta > 0$)) then the expected value is negative /(positive). To get a more flexible model to describe the behaviour of log-returns a linear drift can be added:

$$Y_t = \mu t + \theta G_t + \sigma W_{G_t} \tag{4.37}$$

Given the Characteristic Function of X_1 (or X_t) is very easy to get

$$\Psi_{Y_1}(u) = \exp\left(iu\mu\right)\left(1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu\right)^{-1/\nu}, \tag{4.38}$$

$$\Psi_{Y_t}(u) = E\left[\exp\left(iuY_t\right)\right] = \exp\left(iu\mu t\right)\left(1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu\right)^{-t/\nu}. \tag{4.39}$$

This extension does not influence the infinite divisibility property. This Lévy process is a Variance Gamma process with a deterministic trend. In particular, $Y_{s+t} - Y_s \sim VG(\sigma\sqrt{t},\ \nu/t,\ \theta t,\ \mu t)$ which is the same distribution of Y_t. The additional term μ is only a location parameter. This change is reflected only on the first term of the Lévy triplet: $\gamma_Y = \gamma_X + \mu$. For completeness, we report the expressions for the expectation and other central moments up to order four of Y_1

$$Y_1 = \begin{cases} \text{Expected Value} & \theta + \mu \\ \text{Variance} & \sigma^2 + \nu\theta^2 \\ \text{Skewness} & \theta\nu\left(3\sigma^2 + 2\nu\theta^2\right)/\left(\sigma^2 + \nu\theta^2\right)^{3/2} \\ \text{Kurtosis} & 3\left(1 + 2\nu - \nu\sigma^4\left(\sigma^2 + \nu\theta^2\right)^{-2}\right). \end{cases}$$

The additional location parameter allows to get a distribution whose skewness does not determine automatically the sign and the magnitude of the mean. In this case, if $\nu \to 0$ the Characteristic Function of Y_1 tends to the one of the $Normal(\theta + \mu,\ \sigma^2)$ since

$$\lim_{\nu \to 0} \exp(iu\mu)\left[1 - (iu\theta - \frac{1}{2}u^2\sigma^2)\nu\right]^{-1/\nu} = \exp(iu(\theta + \mu) - \frac{1}{2}u^2\sigma^2). \tag{4.40}$$

4.3.3 Fitting Monthly Log-returns of Hedge Funds

In the previous chapter we have shown that the Normal distribution is a very poor model to fit monthly log-returns of hedge fund indices. In order to achieve

a better fit we can replace the Gaussian distribution by the more sophisticated Variance Gamma distribution with four parameters, taking into account skewness and excess kurtosis. We want to show that the underlying Variance Gamma distribution allows a much better fit to the data. In particular, we try to fit the Variance Gamma distribution to the monthly log-returns of two indices: Convertible Arbitrage and Fixed Income Arbitrage. To estimate the Variance Gamma (annual) parameters we assume (implicitly) independent observations[7] and use moments estimators. The analysis is performed using smoothed and unsmoothed time series for each hedge fund index.
We start considering the Convertible Arbitrage Index.
The results of the estimation procedure are given by:

- $\mu = 0,1380$ $\theta = -0,0681$ $\sigma = 0,0415$ $\nu = 0,1208$ for observed data
- $\mu = 0,1426$ $\theta = -0,0728$ $\sigma = 0,0770$ $\nu = 0,1255$ for unsmoothed data

Figure 4.1 shows two Normal kernel densities:

- the blue one is obtained using smoothed data (173 monthly log-returns)
- the red one is estimated using 25000 log-returns generated by the (smoothed) VG model.

Figure 4.2 reports the densities estimated with unsmoothed data:

- the blue density is got with smoothed log-return data (173 monthly log-returns)
- the red one is based on 25000 simulated log-returns through the (unsmoothed) VG model.

In both cases the Variance Gamma distribution seems to work very well for Convertible Arbitrage monthly log-returns. Further evidence is provided by Quantile-Quantile plots. The plotted plusses should not deviate too much from a straight line if the sample comes from the selected distribution. Pictures 4.3 and 4.5 report Q-Q plots based on the Normal distribution. In Figures 4.4 and 4.6 Q-Q plots

[7] See the distinction between observed (smoothed) and unsmoothed data of Chapter 3. Observed data are serial correlated. Unsmoothed data are not auto-correlated, but this does not necessarily imply independent observations.

are based on the Variance Gamma distribution. It can be seen that there is a severe problem in the tails if we try to fit the data with the Normal distribution. This feature of the data has been already evidenced in chapter 3. This problem almost completely disappears when we use the Variance Gamma distribution to fit observed and unsmoothed log-returns, as can be seen in Figures 4.4 and 4.6.
We repeat the same analysis for the Fixed Income Arbitrage Index.
The results of the estimation procedure are given by:

- $\mu = 0,1113$ $\theta = -0,0630$ $\sigma = 0,0258$ $\nu = 0,2464$ for observed data
- $\mu = 0,0984$ $\theta = -0,0512$ $\sigma = 0,0474$ $\nu = 0,2525$ for unsmoothed data

All Figures from 4.7 to 4.12 indicate that the Variance Gamma density fits smoothed and unsmoothed log-returns of the Fixed Income Arbitrage Index much better than the Normal.

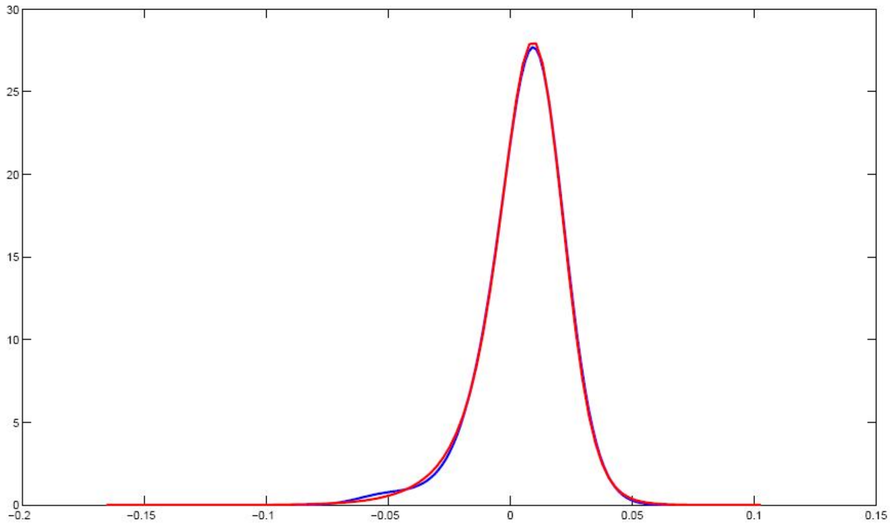

Figure 4.1: Kernel Densities for Convertible Arbitrage (smoothed data)

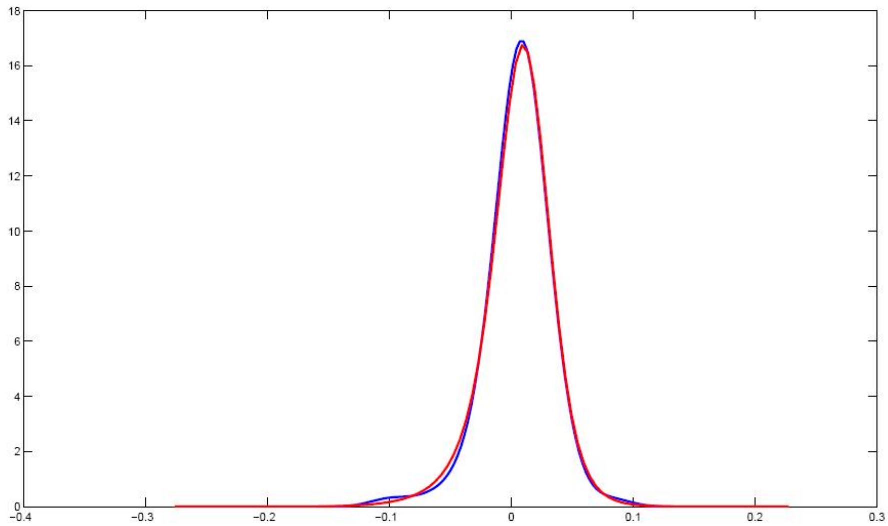

Figure 4.2: Kernel Densities for Convertible Arbitrage (unsmoothed data)

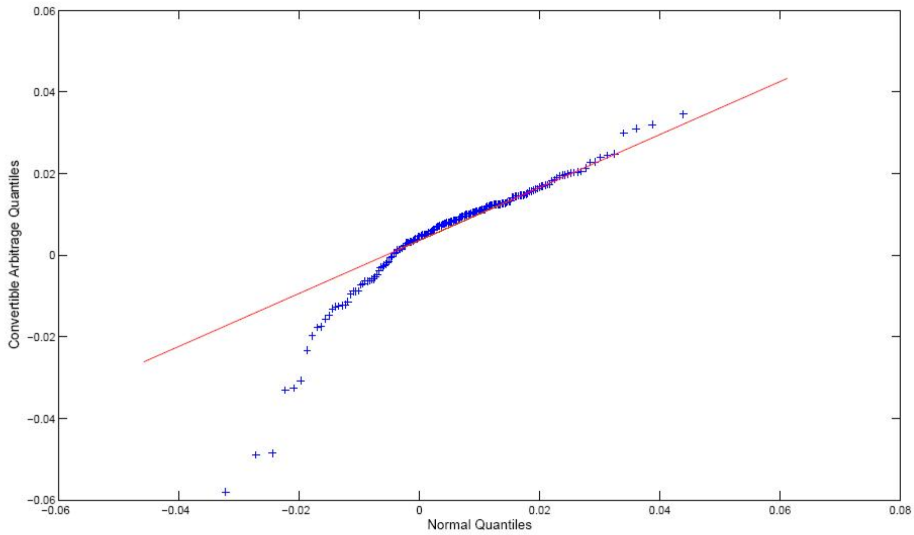

Figure 4.3: Normal Q-Q Plot for Convertible Arbitrage (smoothed data)

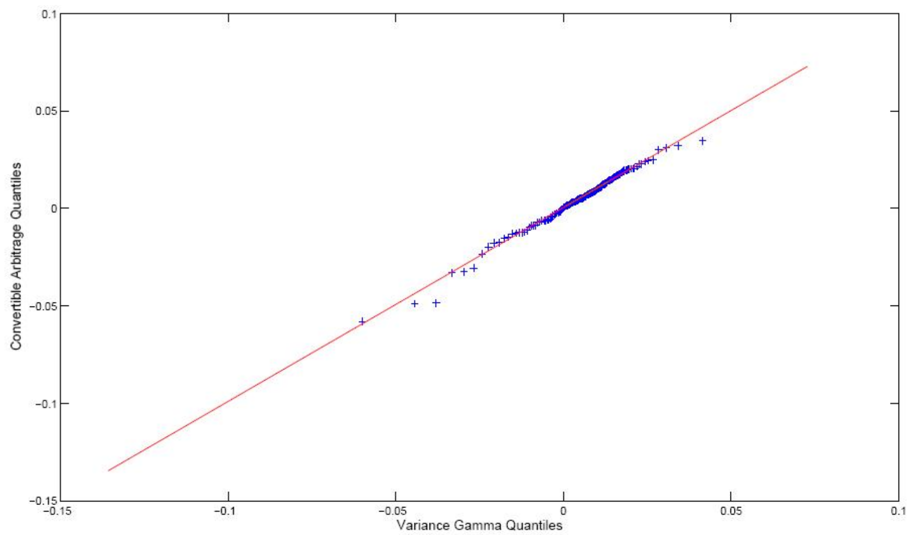

Figure 4.4: VG Q-Q Plot for Convertible Arbitrage (smoothed data)

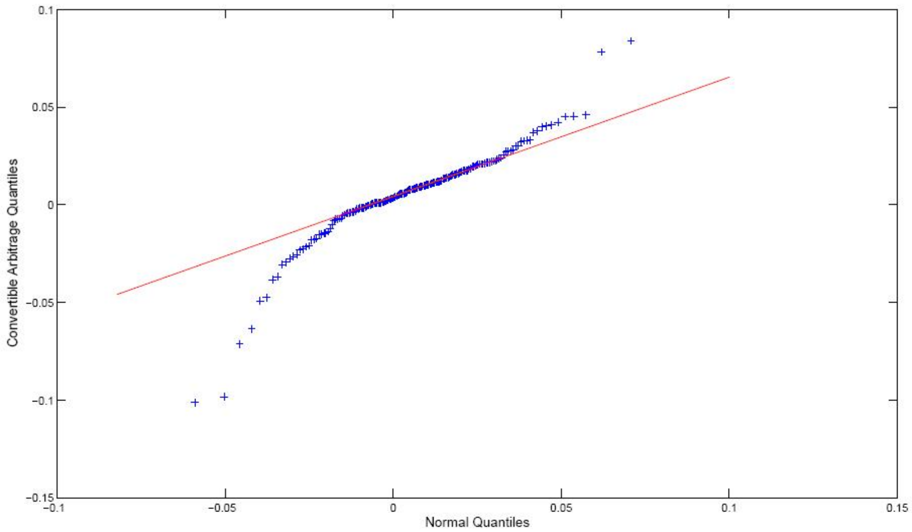

Figure 4.5: Normal Q-Q Plot for Convertible Arbitrage (unsmoothed data)

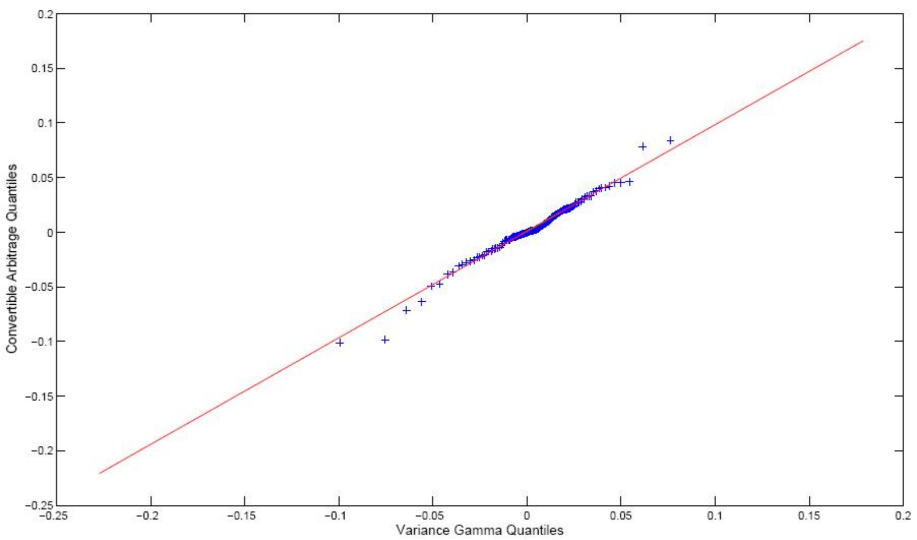

Figure 4.6: VG Q-Q Plot for Convertible Arbitrage (unsmoothed data)

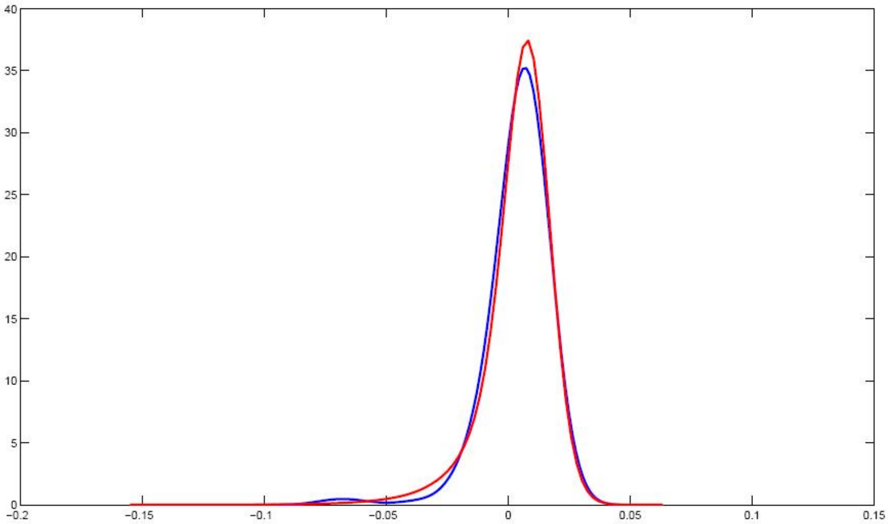

Figure 4.7: Kernel Densities for Fixed Income Arbitrage (smoothed data)

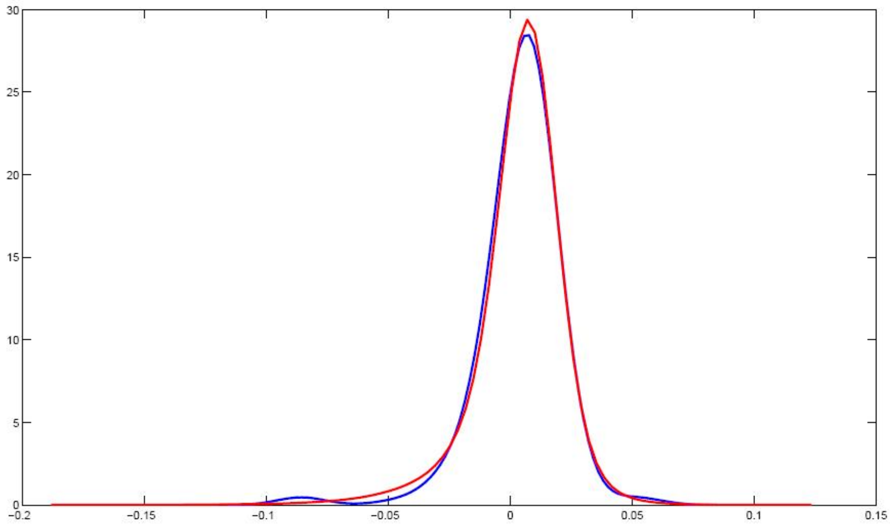

Figure 4.8: Kernel Densities for Fixed Income Arbitrage (unsmoothed data)

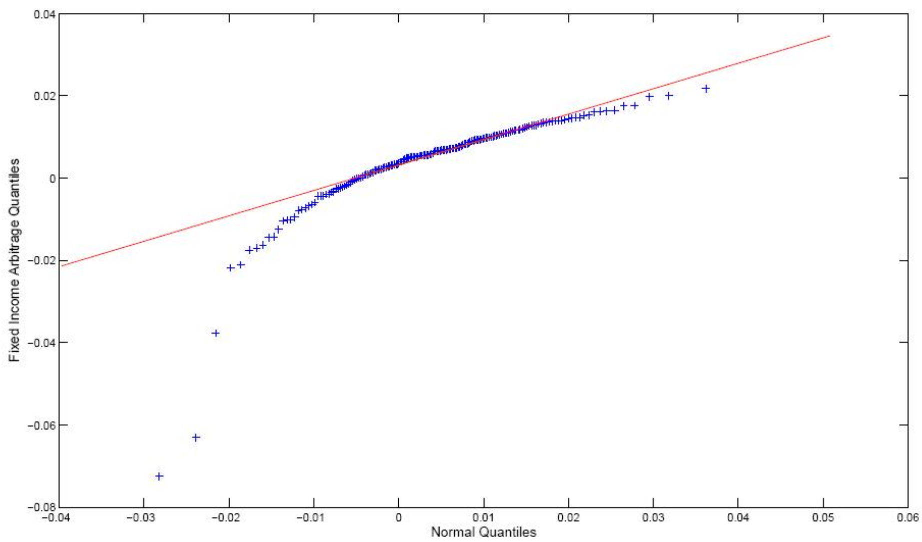

Figure 4.9: Normal Q-Q Plot for Fixed Income Arbitrage (smoothed data)

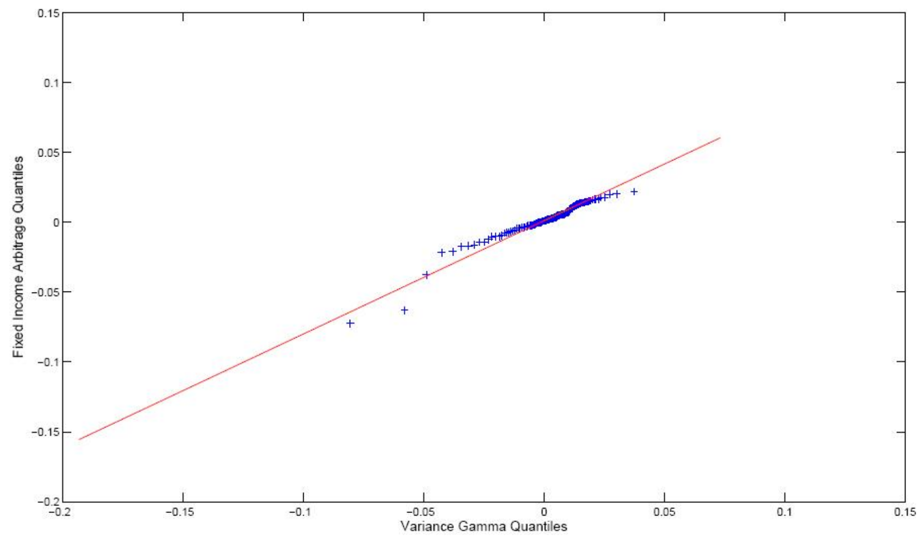

Figure 4.10: VG Q-Q Plot for Fixed Income Arbitrage (smoothed data)

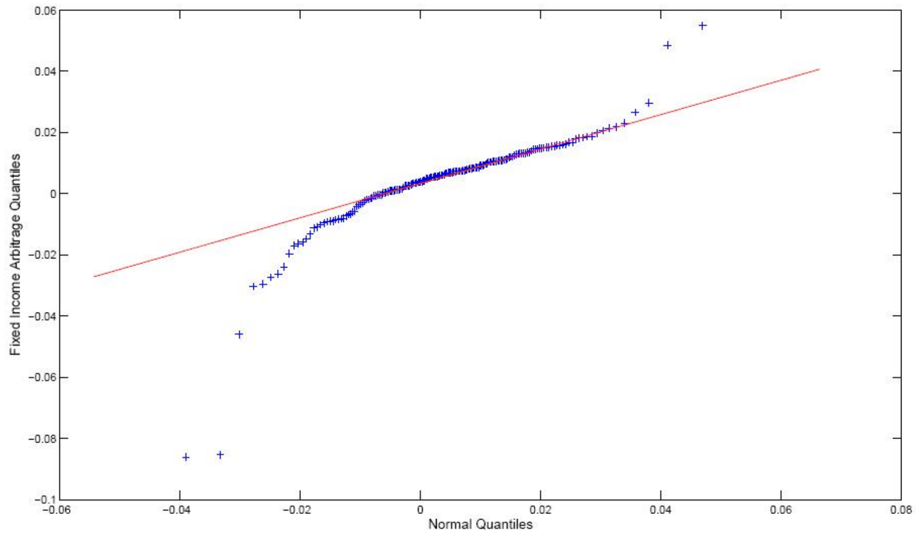

Figure 4.11: Normal Q-Q Plot for Fixed Income Arbitrage (unsmoothed data)

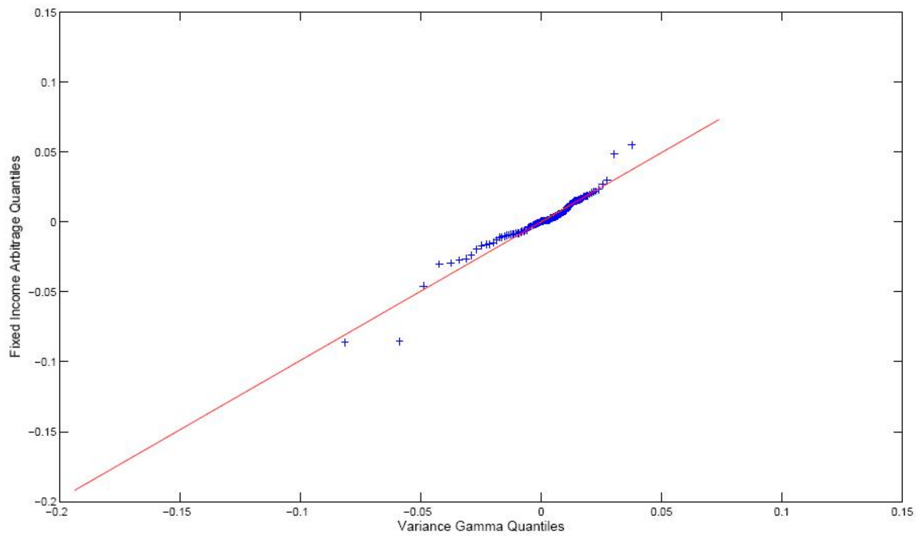

Figure 4.12: VG Q-Q Plot for Fixed Income Arbitrage (unsmoothed data)

Chapter 5

The Models

5.1 Multidimensional Lévy Processes and Dependence Structure

An increasing number of financial applications requires a multivariate model with dependence between components. Two typical examples are represented by the pricing of basket options and portfolio optimization. In most of these applications, jumps in the price process must be taken into account. However, multidimensional models with jumps are more difficult to construct than one-dimensional ones. As observed in [33] this has led to an important imbalance between the range of possible applications and the number of available models in the multidimensional and one-dimensional cases: a wide variety of one-dimensional models have been developed for relatively few applications, while multidimensional applications continue to be dominated by Brownian motion. The main reason for the abuse of multivariate Gaussian models is that dependence can be parameterized in a simple way, in terms of correlation matrices. In this case marginal properties given by volatilities are easy to separate from dependence properties described by correlations. Another reason is that it is very easy to simulate Gaussian time series with arbitrary correlation matrices. However, some methods have been developed to build multidimensional processes with jumps.

Lévy copulas

This concept parallels the notion of ordinary copula on the level of Lévy measure.

Ordinary copulas allow to separate the dependence structure of a random vector from its univariate margins. They provide a complete characterization of the possible dependence structures of a random vector with fixed margins. They can be used to construct multivariate distributions with specified dependence structure from a collection of univariate laws. The interested reader can find an introduction to copulas in [120, 63, 106] and for their multiple applications in finance can see Cherubini *et al.* (2004) [29]. However, ordinary copulas are used to study dependence in a static context. In principle, the whole distribution of a n-dimensional Lévy process $X = \{X_t,\ t \geq 0\}$ is determined by the law of X_t for one fixed t. Therefore, one can describe the dependence structure among components of X by the ordinary copula C_t of X_t. However, this copula generally depends on t and C_s for some $t \neq s$ cannot be computed from C_t alone; to compute it one also needs to know the marginal distributions at time t and at time s. Tankov (2004) [123] construct an explicit example of a Lévy process with non constant copula. Moreover, given n one-dimensional Lévy processes, it is not clear which copulas C_t lead to a n-dimensional Lévy process. Since every (n-dimensional) Lévy process $X = \{X_t,\ t \geq 0\}$ is described in a time-dependent fashion by its characteristic triplet $[\gamma,\ A,\ \upsilon(d\mathbf{x})]$, it seems therefore natural to describe the dependence between components of X also in terms of its characteristic triplet. Since the continuous martingale component of X is completely described by the covariance matrix A and is independent from the jump part, it is sufficient to focus on the dependence of the jump part of X. For such a process, separate modelling of margins and dependence is achieved by introducing Lévy copulas, which play the same role for Lévy measures as ordinary copulas for probability measures. Lévy copulas, introduced by Tankov (2003)[122] and further developed by Tankov and Kallsen (2004)[65, 123], are functions that completely characterize the law of a multidimensional Lévy process given the laws of its components, described by their Lévy measures. If a Brownian motion component is also present, since the Gaussian and the jump part are independent, one can use at the same time a correlation matrix and a Lévy copula to model separately both sources of dependence. It should be emphasized that a Lévy copula allows to describe in a *time-dependent fashion* the dependence structure of a Lévy process without Gaussian component. Lévy copulas completely characterize the possible dependence patterns of Lévy processes in the sense that for every Lévy process, there exists a Lévy copula that describes its dependence structure and for every Lévy copula and every n one-dimensional Lévy processes, there exists an n-dimensional Lévy process with this Lévy copula and with margins given by these one-dimensional processes. Multidimensional Lévy process models for applications can thus be constructed by taking any n one-dimensional processes and a Lévy copula from a parametric family (See for example Paragraph 5 [65]). Tankov in [123, 124] discuss the simulation of multivariate Lévy processes with dependence structure given by a Lévy copula. Financial applications include the pricing of basket option (with barriers) and portfolio management. Lévy copula models are also useful in insurance and in risk management, to model the dependence between loss processes of different business lines, and more generally, in all multivariate problems where dependence between jumps needs to be taken into account. From a theoretical point of view they have very interesting property that make them unique, that is *flexibility*. They allows to build multivariate Lévy processes with margins of different kind and to model their dependence in a dynamical way. However, their practical use is limited because are quite complex. In fact, from the point of view of applications they exhibit two main problems:

- Lévy copula models are really hard to estimate from data;
- The simulation procedure is quite complex and becomes intractable when the number of asset is greater than two or three.

For a detailed description of Lévy copula one can read the references mentioned above or Chapter 5 of [33] on Multidimensional Lévy processes[1].

Building Multivariate Lévy Process with Finite Activity

The simplest way to build an n-dimensional jump-diffusion model is to assume the existence of a unique Poisson process, common to all price processes. This essentially means that there is only one source of jump risk in the market affecting all assets and no specific sources of jump risk exist. All assets jump at the same

[1]For a financial application of Lévy copulas see [33, 123, 124]. In these works a bivariate Lévy process is built using a two parameters Clayton Lévy copula with Variance Gamma margins. Copula parameters are computed in different combination to reproduce the observed correlation between the log-returns. Different combinations produce quite different tail dependence structures

time but the size of the jumps may be different (See [75]). In this model, the dates of market crashes are modelled as jump times of a standard Poisson process $N = \{N_t,\ t \geq 0\}$.
This leads us to the following model for the log-price processes of n assets :

$$Y_t^j = \mu_j t + B_t^j + \sum_{k=1}^{N_t} L_k^j \quad j = 1, \dots, n \tag{5.1}$$

where B_t is an n-dimensional Brownian motion with covariance matrix Σ, and $\{L_k\}_{k=1}^{\infty}$ are i.i.d. n-dimensional random vectors which determine the sizes of jumps in individual assets during a market crash. To complete the model, we need to specify the distribution of the size of the jump in individual assets and the dependence between jumps in assets. If we assume that $\left\{L_k^j\right\}_{j=1}^n$ are Gaussian random vectors, then we have to define their covariance matrix Λ and the mean vector $\mathbf{m}$, thus obtaining a multivariate version of Merton's model [85]. To construct this multivariate model from real data we have to estimate $n^2 + 3n + 1$ parameters simultaneously. Even in a simple bivariate Merton's model we have to estimate jointly 11 parameters! As the number of assets increases the estimation becomes clearly unfeasible. If the jumps are not Gaussian, we must specify the distribution of jumps in each component and the copula describing their dependence. In particular, if the jumps are double exponential using different copulas we get different multivariate extensions of Kou's model [68, 69]. The model is thus completely specified by a covariance matrix Σ, a drift vector with n components, n jump size distributions, one n-dimensional copula C and a jump intensity parameter λ. Also in this case estimation is a real problem even with two assets.
It seems reasonable to model several independent shocks to account for events that affect individual assets or individual sectors rather than the entire market. To reach this task several driving Poisson processes have to be introduced into the model. This leads to the following model for the log-price processes of n assets :

$$Y_t^j = \mu_j t + B_t^j + \sum_{f=1}^{D} \sum_{k=1}^{N_t^f} L_{k,f}^j \quad j = 1, \dots, n \tag{5.2}$$

where $N_t^1, \dots,\ N_t^D$ are Poisson processes driving D independent shocks and $L_{k,f}^j$ is the size of jump in j-th component after k-th shock of type f. The vectors $\left\{L_{k,f}^j\right\}_{j=1}^n$ for different k and/or f are independent. To define a parametric model completely, one must specify a one-dimensional distribution for each component for each shock type (different shocks influence the same stock in different ways), and one n-dimensional copula for each shock type (that is D n-dimensional copulas). As the dimension of the problem grows, this kind of modelling quickly becomes unfeasible.[2]

Building Multivariate Lévy Processes through Subordination of Multivariate Brownian motion with drift

Another method to introduce jumps into a multidimensional model is to take a Multivariate Brownian motion with drift and time change it with an independent one-dimensional subordinator [33, 123, 76, 74].[3] This approach allows constructing multidimensional versions of many popular one-dimensional models, including Variance Gamma, Normal Inverse Gaussian, Generalized Hyperbolic, Meixner and Carr-Geman-Madan-Yor process. The principal advantage of this method is its simplicity and analytic tractability, especially if compared with the previous methods. In particular:

- the computation of the Characteristic Function of the process is simple;
- the knowledge of the Characteristic Function allows to find expressions for joint and marginal moments;
- conditional Normality of log-returns simplifies the simulation procedure;
- the knowledge of the joint Characteristic Function allows to estimate simultaneously (if the number of margins is not too high) all the parameters using the Spectral GMM estimator [25]. The GMM estimator based on the Moment Generating Function or other *ad hoc* technique based on Method of Moments can be used. Estimation of Lévy copula parameters is an open question. Estimation of parameters in a multidimensional jump-diffusion process is very difficult even in the bivariate case;
- a parsimonious description of dependence is especially important because one typically does not have enough information about the dependence structure to estimate many parameters;

[2] For an application of a multivariate jump-diffusion model in option pricing see[79].

[3] See Section 4.1.1

- since no traded option on hedge funds are available we cannot calibrate the model parameters directly in the *risk neutral measure chosen by the market.* The method we use to find an equivalent martingale measure requires the knowledge of the Characteristic Function of the multivariate process.

This method has also some drawbacks. The range of dependence patterns that one can obtain using this approach is limited (for example, independence is not included), and all components must follow the same parametric model. Finally, building a Multivariate Lévy process time changing a Multivariate Brownian Motion with a one-dimensional subordinator imposes some constraints in the parameters of the marginal processes. Therefore, the greater the number of parameters describing the distribution of the subordinator is, the more similar the moments of the margins are. In the Multivariate Variance Gamma process with linear drift we have only one constrained parameter for every margins.
Semeraro and Luciano (2006-2007)[78, 77, 118] suggest the use of a Multivariate subordinator in order to get a more flexible and realistic multidimensional model. In particular, they get some Multivariate Lévy processes with well-known margins (NIG, VG, CGMY).
After an accurate analysis of advantages and disadvantages of all the exposed methods, we decide to build multivariate Lévy processes using the technique of the stochastic time change. In the following three section we provide three different models. All of them have Variance Gamma margins but present different dependence structures.

5.1.1 Model 1: Multivariate Variance Gamma (Independent Brownian Motions)

The evolution of hedge funds' log-returns is described through a Multivariate Variance Gamma process with linear drift. The Multivariate Variance Gamma process is obtained time changing a Multivariate Brownian motion with drift, with independent components, with an independent one-dimensional Gamma process. As in [76] modelling dependence in this way is like starting from an independent Gaussian World in which all assets are driven by independent Geometric Brownian motions. Then, in order to introduce dependence, we time-change all the asset price processes by a common Gamma time-change. The economic interpretation is that all firms operate in the same economic environment. A jump in the time-change leads to a jump in the price processes and hence all jumps happen simultaneously. However, the jump sizes are caused by the individual Brownian motions. We thus introduce a new business time in which the general market operates. This new business time can also be interpreted as a model for the information arrival. Taking into account that the market does not forget information, the amount of information only cannot decrease. Moreover, it seems reasonable that the amount of new information released should not be affected by the amount already released. In other words, the information process should have independent increments. Finally, one can also require that the increment only depends on the length of that period and hence is stationary. The Gamma process satisfies these properties. In fact, it is a positive Lévy process, i.e. a subordinator (See subsection (4.3.1)).

The NAV at time t of each hedge fund is given by the product of the initial NAV times the exponential of a Variance Gamma process with linear drift:

$$F_t^j = F_0^j \exp(Y_t^j) \tag{5.3}$$

where F_t^j and F_0^j is the NAV of the hedge fund j at times t and 0, while Y_t^j is the log-return of the j-th hedge fund over the period $[0;\ t]$ for every $j = 1, \dots, n$. The log-return of the j-th hedge fund is

$$\begin{aligned} Y_t^j &= \mu_j t + X_t^j &(5.4)\\ &= \mu_j t + \theta_j G_t + \sigma_j W_{G_t}^j &(5.5) \end{aligned}$$

where $G = \{G_t,\ t \geq 0\}$ is the common Gamma stochastic time change process such that $G_t \sim Gamma(t/\nu,\ 1/\nu)$ and $\nu > 0$;

$W^j = \{W_t^j, t \geq 0\}$ and $W^k = \{W_t^k,\ t \geq 0\}$ are independent Wiener processes for all $j \neq k$, W_t^j and W_t^k are $Gaussian(0,\ t)$;

θ_j, μ_j and $\sigma_j > 0$ are costants.

If we set $t = 1$, we get the yearly log-return for asset j, that is

$$\begin{aligned} Y_1^j &= \mu_j + X_1^j &(5.6)\\ &= \mu_j + \theta_j G_1 + \sigma_j W_{G_1}^j &(5.7) \end{aligned}$$

The above assumptions leads to the following simple expression for hedge funds' j and k yearly log-returns covariance:

$$\sigma\left(Y_1^j; Y_1^k\right) = \theta_j \theta_k \nu \tag{5.8}$$

and for correlation:

$$\rho\left(Y_1^j; Y_1^k\right) = \frac{\theta_j \theta_k \nu}{\sqrt{\sigma_j^2 + \nu\theta_j^2}\sqrt{\sigma_k^2 + \nu\theta_k^2}} \tag{5.9}$$

Since ν is strictly positive, the j-th and k-th hedge funds are positively correlated if and only if θ_j and θ_k have the same sign. In other worlds, this model implies a positive correlation for all the assets having the same sign of skewness. Pairs of negatively skewed or pairs of positively skewed hedge funds have a positive correlation coefficient. If a hedge fund has a symmetric VG distribution then it will be uncorrelated with all other hedge funds. However, by construction, it is clear that this hedge fund cannot be independent with the others. Negative correlation between pairs of hedge funds is only possible if their distributions exhibit skewness of opposite sign.
The Characteristic Function of Y_1^j for $j = 1, \ldots, n$ is

$$\Psi_{Y_1^j}(u) = \exp\left(iu\mu_j\right)\left(1 - iu\theta_j\nu + \frac{1}{2}u^2\sigma_j^2\nu\right)^{-1/\nu}. \tag{5.10}$$

Conditional normality allows to compute the joint Characteristic Function of the Multivariate Variance Gamma distribution (with $t = 1$). The computations required are easy but tedious. Yet, we can compute this function in a more efficient way by using *THEOREM 4.2* [33]. In order to apply this theorem we need to know:

- the Laplace Exponent $l(u)$ of the Gamma subordinator;
- the Charactheristic Exponent $c(\mathbf{u})$ of a Multivariate Brownian motion, with independent components.

The Laplace Exponent of a generic subordinator is defined in the following way:

$$E\left[\exp\left(uG_t\right)\right] = \exp\left[tl\left(u\right)\right] \quad \forall u \leq 0 \tag{5.11}$$

$$l\left(u\right) = \ln\frac{E\left[\exp\left(uG_t\right)\right]}{t} \tag{5.12}$$

The Moment Generating Function of $G_t \sim Gamma(t/\nu, 1/\nu)$ is

$$E\left[\exp\left(uG_t\right)\right] = \left(1 - u\nu\right)^{-t/\nu} \quad u < \frac{1}{\nu} \tag{5.13}$$

and consequently its Laplace Exponent is simply

$$l\left(u\right) = -\frac{\ln\left(1 - u\nu\right)}{\nu} \tag{5.14}$$

The joint Characteristic Function of the Multivariate Brownian motion with independent components ($t = 1$)

$$\begin{aligned}\Psi(\mathbf{W})_1(\mathbf{u}) &= E\left[\exp\left(i\sum_{j=1}^{n}(u_j\theta_j + u_j\sigma_j W_1^j)\right)\right] \\ &= \exp\left[\sum_{j=1}^{n}\left(iu_j\theta_j - \frac{1}{2}u_j^2\sigma_j^2\right)\right]\end{aligned} \tag{5.15}$$

and therefore its Characteristic Exponent is

$$c\left(\mathbf{u}\right) = \sum_{j=1}^{n}\left(iu_j\theta_j - \frac{1}{2}u_j^2\sigma_j^2\right) \tag{5.16}$$

Now we have all the elements necessary to compute the Characteristic Function of the Multivariate Variance Gamma distribution. Using *THEOREM 4.2* [33] we get

$$\begin{aligned}\Psi(\mathbf{Y})_1\left(\mathbf{u}\right) = \exp\left(i\sum_{j=1}^{n}u_j\mu_j\right) \times \\ \left[1 - \nu\sum_{j=1}^{n}(iu_j\theta_j - \frac{1}{2}u_j^2\sigma_j^2)\right]^{(-1/\nu)}\end{aligned} \tag{5.17}$$

From this function it is immediate to derive the joint Moment Generating Function[4] of $\mathbf{Y}_1$, whose existence requires that the argument between the square brack-

[4]Contrarily to the Characteristic Function, which is always well-defined (as the Fourier transform of a probability measure), the Moment Generating Function is not always defined: the integral (5.18) may not converge for some values of $\mathbf{u}$. When it is well-defined, it can be formally related to the Characteristic Function (5.17) by: $M(\mathbf{Y})_1\left(\mathbf{u}\right) = \Psi(\mathbf{Y})_1\left(-i\mathbf{u}\right)$. However, we can use this relation to find the formal expression for the Moment Generating Function for the set of values of $\mathbf{u}$ such that the expectation (5.18) is finite. See Cont and Tankov [33] Paragraph 2.2.4.

ets is positive:

$$M(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j \mu_j\right) \times \left[1 - \nu \sum_{j=1}^{n}(u_j\theta_j + \frac{1}{2}u_j^2\sigma_j^2)\right]^{(-1/\nu)} \tag{5.18}$$

This function will play a crucial role in the section devoted to the change of measure.

5.1.2 Model 2: Multivariate Variance Gamma (Correlated Brownian Motions)

The dynamic of hedge funds is modelled through a Multivariate Variance Gamma process with a linear drift. The pure Multivariate jump part of the process is got by time changing a Multivariate Brownian motion, with correlated components, with an independent one-dimensional Gamma process. Modelling dependence in this way is like starting from a Gaussian World in which all assets are driven by correlated Geometric Brownian motions and then introducing another source of dependence through a stochastic time-change represented by a common Gamma stochastic clock. The main difference with respect to the previous model is that now there are two sources of co-movement among NAV of different hedge funds:

- cross-correlations between the underlying Brownian motions;
- dependence produced by the same stochastic clock.

The log-return of the j-th hedge fund over the period $[0,\ t]$ is

$$Y_t^j = \mu_j t + X_t^j \tag{5.19}$$

$$= \mu_j t + \theta_j G_t + \sigma_j W_{G_t}^j \tag{5.20}$$

where $G = \{G_t,\ t \geq 0\}$ is the Gamma stochastic time-change process with $G_t \sim Gamma(t/\nu,\ 1/\nu)$ and $\nu > 0$;
$W^j = \{W_t^j,\ t \geq 0\}$ and $W^k = \{W_t^k,\ t \geq 0\}$ are correlated Wiener processes with correlation coefficient ρ_{jk} for all $j \neq k$;

θ_j, μ_j and $\sigma_j > 0$ are constants.
If we consider $t = 1$, we get the annual log-return for asset j, that is

$$Y_1^j = \mu_j + X_1^j \tag{5.21}$$
$$= \mu_j + \theta_j G_1 + \sigma_j W_{G_1}^j \tag{5.22}$$
$$= \mu_j + \theta_j G_1 + \sigma_j \sqrt{G_1} W_1^j \tag{5.23}$$

The covariance between yearly log-returns of hedge funds j and k is given by:

$$\sigma\left(Y_1^j; Y_1^k\right) = \theta_j \theta_k E(G_1^2) + \sigma_j \sigma_k E(G_1) E(W_1^j W_1^K) \tag{5.24}$$
$$= \theta_j \theta_k \nu + \sigma_j \sigma_k \rho_{jk} \tag{5.25}$$

The correlation is:

$$\rho\left(Y_1^j; Y_1^k\right) = \frac{\theta_j \theta_k \nu + \sigma_j \sigma_k \rho_{jk}}{\sqrt{\sigma_j^2 + \nu \theta_j^2}\sqrt{\sigma_k^2 + \nu \theta_k^2}} \tag{5.26}$$

These expressions show some important differences with respect to the same ones of the previous model:

- pairs of assets with skewness of the same sign could be negatively correlated;
- pairs of assets with skewness of opposite sign could be positively correlated;
- a hedge fund with a symmetric distribution could be correlated with other assets;
- pairs of assets have null correlation if and only if at least one of them has a symmetric distribution and their underlying Brownian Motions are uncorrelated.

However, both models do not contemplate independence. Null correlation does not imply independence. Hedge funds' Navs are always dependent due to the existence of a common stochastic clock. As we mentioned before, Semeraro and Luciano developed some models, using multivariate subordinators, able to produce also independence [77, 78, 118].
In order to compute the Characteristic Function of this more general Multivariate Variance Gamma process we follow the same procedure described in the previous

case. Therefore, we report only the different formulas. The joint Characteristic Function of the Multivariate Brownian motion with correlated components ($t = 1$)

$$\begin{aligned}\Psi(\mathbf{W})_1(\mathbf{u}) &= E\left[\exp\left(i\sum_{j=1}^{n}(u_j\theta_j + u_j\sigma_j W_1^j)\right)\right] \\ &= \exp\left[\sum_{j=1}^{n} iu_j\theta_j - \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j \sigma_k \rho_{jk}\right]\end{aligned} \tag{5.27}$$

Therefore, the Characteristic Exponent of the Multivariate Brownian motion with dependent components is simply

$$c\left(\mathbf{u}\right) = \sum_{j=1}^{n} iu_j\theta_j - \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j \sigma_k \rho_{jk} \tag{5.28}$$

Using *THEOREM* 4.2 [33] we get the Characteristic Function of the Multivariate Variance Gamma process with linear drift :

$$\begin{aligned}\Psi(\mathbf{Y})_1\left(\mathbf{u}\right) &= \exp\left(i\sum_{j=1}^{n} u_j\mu_j\right) \times \\ &\left[1 - \nu\left(\sum_{j=1}^{n} iu_j\theta_j - \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j \sigma_k \rho_{jk}\right)\right]^{-1/\nu}\end{aligned} \tag{5.29}$$

From this expression we can derive the joint Moment Generating Function[5] of $\mathbf{Y}_1$, which is defined when the argument between the square brackets is positive:

$$\begin{aligned}M(\mathbf{Y})_1\left(\mathbf{u}\right) &= \exp\left(\sum_{j=1}^{n} u_j\mu_j\right) \times \\ &\left[1 - \nu\left(\sum_{j=1}^{n} u_j\theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j \sigma_k \rho_{jk}\right)\right]^{-1/\nu}\end{aligned} \tag{5.30}$$

5.1.3 Model 3: Multivariate ν-Variance Gamma Model (Independent Brownian Motions)

The traditional Multivariate Lévy process constructed by subordinating a Brownian motion with drift through a univariate subordinator presents a number of

[5] See footnote 4 of section (5.1.1)

drawbacks, including the lack of independence and a limited range of dependence. In order to face these problems, Semeraro and Luciano [77, 78, 118] investigate multivariate subordination, with a common and an idiosyncratic component. From the intuitive point of view, the main feature of such multivariate subordination is that it allows to incorporate both a common time transform, which can be interpreted in financial applications as a measure of the overall market activity, and an idiosyncratic time shift. The evolution of hedge funds' log-returns under the real world probability measure now is described through a Multivariate Brownian motion with drift and independent Gaussian components time changed by a Multivariate Gamma process with correlated marginal processes. Finally, we add to each marginal process a linear drift to get a more flexible model for hedge funds' log-returns.

The log-return of the j-th hedge fund over the period $[0,\ t]$ is

$$
\begin{aligned}
Y_t^j &= \mu_j t + X_t^j && (5.31)\\
&= \mu_j t + \theta_j G_t^j + \sigma_j W^j_{G_t^j} && (5.32)\\
&= \mu_j t + \theta_j (U_t^j + \nu_j Z_t)\sigma_j W^j_{U_t^j+\nu_j Z_t} && (5.33)
\end{aligned}
$$

where $G^j = \{G_t^j,\ t \geq 0\}$ is the Gamma stochastic time change for the j-th asset process with $G_t^j \sim Gamma(t/\nu_j,\ 1/\nu_j)$ and $\nu_j > 0$ for all $j = 1, \dots,\ n$;

$U^j = \{U_t^j,\ t \geq 0\}$ is the idiosyncratic part of the stochastic clock such that $U_t^j \sim Gamma(t/\nu_j - at,\ 1/\nu_j)$ where $U^j = \{U_t^j,\ t \geq 0\}$ and $U^k = \{U_t^k,\ t \geq 0\}$ are independent Gamma processes for all $j \neq k$;

$Z = \{Z_t,\ t \geq 0\}$ is the part of the stochastic clock common to all asset where $Z_t \sim Gamma(at,\ 1)$ and is independent with respect to the specific parts of each time-change;

$W^j = \{W_t^j, t \geq 0\}$ and $W^k = \{W_t^k, t \geq 0\}$ are uncorrelated Wiener processes for all $j \neq k$;

$\theta_j,\ \mu_j$ and $\sigma_j > 0$ are constants. The process $\mathbf{G} = \{\mathbf{G}_t,\ t \geq 0\}$ is a multivariate ν-Gamma subordinator with dependent components and it satisfies the condition $0 < \nu_j < 1/a$ for $j = 1, \dots, n$.

The covariance between the j-th and the k-th components of the Gamma subordinator for $t = 1$ is:

$$\sigma(G_1^j; G_1^k) = a\nu_j\nu_k \tag{5.34}$$

The covariance and the correlation between the log-returns of hedge funds j and k are

$$\sigma(Y_1^j; Y_1^k) = \theta_j \theta_k \sigma(G_1^j; G_1^k) = \theta_j \theta_k a \nu_j \nu_k \tag{5.35}$$

$$\rho(Y_1^j; Y_1^k) = \frac{\theta_j \theta_k a \nu_j \nu_k}{\sqrt{(\sigma_j^2 + \theta_j^2 \nu_j)(\sigma_k^2 + \theta_k^2 \nu_k)}} \tag{5.36}$$

Changing the parameter a it is possible to modify all the correlations without modifying the marginal distributions.

- The marginal processes are independent if and only if $a = 0$.
- Two margins are uncorrelated but dependent if at least one of them is symmetric and $a \neq 0$.

To compute the joint Characteristic Function we need to use the *generalized* version of *THEOREM 4.2* [33] in the case of a *multidimensional subordinator* (See [11] Theorem 3.3). The Laplace Exponent $l(\mathbf{u})$ of the subordinator is

$$l(\mathbf{u}) = \ln \left\{ [\prod_j^n (1 - u_j \nu_j)^{-(\frac{1}{\nu_j} - a)}][1 - \sum_j^n u_j \nu_j]^{-a} \right\} \tag{5.37}$$

The Charactheristic Exponent $c(\mathbf{u})$ of our Multivariate Brownian motion, with independent components is given by:

$$c(\mathbf{u}) = \sum_j^n (i\theta_j u_j - \frac{1}{2}\sigma^2 u_j^2) \tag{5.38}$$

We compute the Characteristc Function of the process for $t = 1$ using the following formula

$$\Psi(\mathbf{Y})_1(\mathbf{u}) = \exp(i \sum_j^n u_j \mu_j) \exp(l(c(\mathbf{u}))) \tag{5.39}$$

and we obtain

$$\begin{aligned} \Psi(\mathbf{Y})_1(\mathbf{u}) = \exp(i \sum_j^n u_j \mu_j) \prod_j^n (1 - \nu_j (i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-(\frac{1}{\nu_j} - a)} \times \\ (1 - \sum_j^n \nu_j (i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-a} \end{aligned} \tag{5.40}$$

To compute the j-th marginal Characteristic Function it is sufficient to evaluate formula (5.40) in $(0, \dots, u_j, \dots, 0)$

$$\Psi_{Y_1^j}(u_j) = \exp(iu_j \mu_j)(1 - \nu_j (i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-\frac{1}{\nu_j}} \tag{5.41}$$

Of course this is the Characteristic Function of a univariate Variance Gamma process with linear trend. The joint Moment Generating Function is simply:

$$M(\mathbf{Y})_1(\mathbf{u}) = \exp(\sum_j^n u_j\mu_j)\prod_j^n(1-\nu_j(\theta_j u_j+\frac{1}{2}\sigma_j^2u_j^2))^{-(\frac{1}{\nu_j}-a)}\times \\ (1-\sum_j^n \nu_j(\theta_j u_j+\frac{1}{2}\sigma_j^2u_j^2))^{-a} \tag{5.42}$$

5.2 The Change of Measure

The evolution of the Net Asset Value of each hedge fund is described as an Exponential Variance Gamma process under the real world probability measure in all our models. According to Cont and Tankov ([33] Section 9.5) these models are arbitrage free since the price process of every asset have both positive and negative jumps. Consequently, there exists for each of them an equivalent martingale measure. However, they belong to the class of incomplete market models: the equivalent martingale measure is not unique (See section (4.2.2)). Among the possible candidates we select the Esscher Equivalent Martingale Measure[6].

5.2.1 Multivariate Esscher Transform

In this section we explain how to use the Esscher Transform in a multivariate contest in order to find the Esscher Equivalent Martingale Measure.

Consider a market with n risky assets and a bank account which provides a risk free interest rate r constant over the time period $[0, T]$. The value of the bank account at time t is $A_t = A_0 \exp(rt)$. Suppose that the price of every risky asset at time $t \in [0, T]$ can be described by a Geometric Lévy model, say $F_t^j = F_0^j \exp(Y_t^j)$ for $j = 1, \ldots, n$.

Let $\mathbf{Y} = \{\mathbf{Y}_t,\ t \geq 0\}$ be the n-dimensional Lévy process describing the multivariate log-returns process, then the $Q_\mathbf{h}$ Esscher measure associated with the risk process $\mathbf{Y}$ is defined by the following Radon-Nikodym derivative

$$\frac{dQ_\mathbf{h}}{dP}|\Im_t = \frac{\exp(\sum_{j=1}^n h_j Y_t^j)}{E\left[\exp(\sum_{j=1}^n h_j Y_t^j)\right]} \tag{5.43}$$

[6] Actually we cannot be sure about its existence.

In order to find the Esscher risk neutral dynamic of $\mathbf{Y} = \{\mathbf{Y}_t,\ t \geq 0\}$ two steps are necessary:

- find a vector $\hat{\mathbf{h}}$ such that the discounted price process of every asset is a martingale under the new probability measure $Q_{\mathbf{h}}$;
- find the joint Characteristic Function of the multivariate process $\mathbf{Y} = \{\mathbf{Y}_t,\ t \geq 0\}$ under $Q_{\mathbf{h}}$.

Any transformation of the Lévy measure respecting some integrability constraints (See [33] Section 4.2.3) leads to a new Lévy process. In particular, the Esscher Transform corresponds to an *exponential tilting* of the P Lévy measure. If there exists a vector $\hat{\mathbf{h}}$ such that

$$\int_{|\mathbf{y}|\geq 1} \upsilon^{Q_h}(d\mathbf{y}) = \int_{|\mathbf{y}|\geq 1} \exp(\mathbf{h}^T\mathbf{y})\upsilon(d\mathbf{y}) < \infty \tag{5.44}$$

then the process is a Lévy process under this new probability measure. However, if it is possible to find a vector $\hat{\mathbf{h}}$ such that the discounted price process of each asset is a martingale under the measure $Q_{\mathbf{h}}$, then the existence of the Esscher Martingale Measure is ensured.

In the following sections we apply these steps to our multivariate models. Actually, we cannot be sure that such an equivalent martingale measure exists.

Notice that in our case the choice of this risk neutral measure (if it exists) seems to be the best as possible for at least to reasons[7]:

- Each step requires the knowledge of the joint P Characteristic Function. Usually, it is not easy to find this function explicitly. However, building a multidimensional Lévy process by a stochastic time change of a multivariate Brownian motion makes easy the computation of the Characteristic Function of the process (See Cont and Tankov [33] Theorem 4.3).
- If this equivalent martingale measure exists usually it possible to find the link between the physical and the risk neutral parameters. This is very useful when no option prices are available to calibrate the model. Since no traded option on hedge funds are available we cannot apply the *improperly* called Mean Correcting Martingale method, which is the easiest and most frequently change of measure encountered in financial applications.

[7]See also the previous discussion of section 4.2.3

Finally, it should be emphasized that even if the ESMM exists, we cannot be sure that marginal and joint processes remain of the same type.

5.2.2 Model 1 and ESMM

The first step of the procedure described in the previous section requires the solution of the following system of n equations[8]

$$\begin{cases} E\left[\exp(\sum_{j=1}^{n} h_j Y_t^j + Y_t^1)\right] / E\left[\exp(\sum_{j=1}^{n} h_j Y_t^j)\right] & = \exp(rt) \\ & \vdots \\ E\left[\exp(\sum_{j=1}^{n} h_j Y_t^j + Y_t^n)\right] / E\left[\exp(\sum_{j=1}^{n} h_j Y_t^j)\right] & = \exp(rt) \end{cases} \tag{5.45}$$

To solve this system we need the P Moment Generating function of the model introduced in section 5.1.1:

$$M(\mathbf{Y})_t\,(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j \mu_j t\right)\left[1 - \nu \sum_{j=1}^{n}(u_j\theta_j + \frac{1}{2}u_j^2\sigma_j^2)\right]^{(-t/\nu)}$$

Thanks to the infinitely divisibility property of hedge funds' log-returns distributions, the solution of the *Esscher* system does not depend on t. The previous system after some computation leads to the next one:

$$\begin{cases} \ln\left[1 - (\nu(\theta_1 + h_1\sigma_1^2 + 0.5\sigma_1^2))/(1 - \nu\sum_j^n(h_j\theta_j + 0.5h_j^2\sigma_j^2))\right] & = (\mu_1 - r)\nu \\ & \vdots \\ \ln\left[1 - (\nu(\theta_n + h_n\sigma_n^2 + 0.5\sigma_n^2))/(1 - \nu\sum_j^n(h_j\theta_j + 0.5h_j^2\sigma_j^2))\right] & = (\mu_n - r)\nu \end{cases} \tag{5.46}$$

with the following constraints

$$[1 - \nu\sum_{j}^{n}(h_j\theta_j + 0.5h_j^2\sigma_j^2)] > 0 \tag{5.47}$$

and

$$[1-\nu(\sum_{j\neq k}^{n}(h_j\theta_j+0.5h_j^2\sigma_j^2)+((h_k+1)\theta_k+0.5(h_k+1)^2\sigma_k^2))] > 0 \;\; k = 1, \ldots, n. \tag{5.48}$$

[8] One equation for each hedge fund

It is not possible to find a closed formula for the solution of this system. However, numerically the solution is obtained very quickly[9]. The existence of the vector $\hat{\mathbf{h}}$ is sufficient to ensure the existence of the Esscher Equivalent Martingale Measure. The joint Moment Generating Function for $t = 1$ under the ESMM can be computed as follows:

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = E^{Q_{\mathbf{h}}}\left[\exp\sum_{j=1}^{n} u_j Y_1^j\right] \tag{5.49}$$

$$E^{Q_{\mathbf{h}}}\left[\exp\sum_{j=1}^{n} u_j Y_1^j\right] = \frac{E\left[\exp\sum_{j=1}^{n}(\hat{h}_j + u_j)Y_1^j\right]}{E\left[\exp\sum_{j=1}^{n}\hat{h}_j Y_1^j\right]} \tag{5.50}$$

where $\hat{h}_j$ is the j-th component of the vector $\hat{\mathbf{h}}$. The computation of the Esscher risk neutral joint Moment Generating Function requires the knowledge of the Moment Generating Function of the Multivariate Variance Gamma process for $t = 1$ under the statistical measure (See (5.18)). We substitute the following expressions into equation (5.50)

$$E\left[\exp\sum_{j=1}^{n}(\hat{h}_j + u_j)Y_1^j\right] = \exp\left(\sum_{j=1}^{n}(\hat{h}_j + u_j)\mu_j\right) \times \left[1 - \nu\sum_{j=1}^{n}((\hat{h}_j + u_j)\theta_j + \frac{1}{2}(\hat{h}_j + u_j)^2\sigma_j^2)\right]^{(-1/\nu)} \tag{5.51}$$

$$E\left[\exp\sum_{j=1}^{n}\hat{h}_j Y_1^j\right] = \exp\left(\sum_{j=1}^{n}\hat{h}_j\mu_j\right) \times \left[1 - \nu\sum_{j=1}^{n}(\hat{h}_j\theta_j + \frac{1}{2}\hat{h}_j^2\sigma_j^2)\right]^{(-1/\nu)} \tag{5.52}$$

and after tedious computations and rearrangements we get

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j\mu_j\right) \times \left[1 - \nu\sum_{j=1}^{n}\frac{(u_j(\theta_j + \hat{h}_j\sigma_j^2) + \frac{1}{2}u_j^2\sigma_j^2)}{1 - \nu\sum_{j=1}^{n}(\hat{h}_j\theta_j + \frac{1}{2}\hat{h}_j^2\sigma_j^2)}\right]^{-1/\nu}. \tag{5.53}$$

[9] See next chapter

The joint $Q_{\mathbf{h}}$ Moment Generating Function can be written in the following more compact form:

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j \mu_j^{Q_{\mathbf{h}}}\right)\left[1 - \nu^{Q_{\mathbf{h}}} \sum_{j=1}^{n}(u_j \theta_j^{Q_{\mathbf{h}}} + \frac{1}{2} u_j^2 \sigma_j^{Q_{\mathbf{h}}2})\right]^{\left(-1/\nu^{Q_{\mathbf{h}}}\right)} \tag{5.54}$$

where relations among physical and Esscher risk neutral parameters are

$$\mu_j^{Q_{\mathbf{h}}} = \mu_j \tag{5.55}$$

$$\nu_j^{Q_{\mathbf{h}}} = \nu \tag{5.56}$$

$$\theta_j^{Q_{\mathbf{h}}} = \frac{\theta_j + \hat{h}_j \sigma_j^2}{1 - \nu \sum_{j=1}^{n}(\hat{h}_j \theta_j + \frac{1}{2}\hat{h}_j^2 \sigma_j^2)} \tag{5.57}$$

$$\sigma_j^{Q_{\mathbf{h}}2} = \frac{\sigma_j^2}{1 - \nu \sum_{j=1}^{n}(\hat{h}_j \theta_j + \frac{1}{2}\hat{h}_j^2 \sigma_j^2)} \tag{5.58}$$

$$\rho_{jk}^{Q_{\mathbf{h}}} = \rho_{jk} = 0 \tag{5.59}$$

From (5.54) it is easy to get the joint $Q_{\mathbf{h}}$ Characteristic Function

$$\Psi^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(i \sum_{j=1}^{n} u_j \mu_j\right)\left[1 - \nu \sum_{j=1}^{n}(i u_j \theta_j^{Q_{\mathbf{h}}} - \frac{1}{2} u_j^2 \sigma_j^{Q_{\mathbf{h}}2})\right]^{(-1/\nu)} \tag{5.60}$$

The j-th $Q_{\mathbf{h}}$ marginal Characteristic Function (for $j = 1, \ldots, n$) is given by:

$$\Psi_{Y_1{}^j}^{Q_{\mathbf{h}}}(u_j) = \exp(i u_j \mu_j)\left[1 - \nu(i u_j \theta_j^{Q_{\mathbf{h}}} - \frac{1}{2} u_j^2 \sigma_j^{Q_{\mathbf{h}}2})\right]^{(-1/\nu)} \tag{5.61}$$

Comparing (5.60) and (5.61) with (5.17) and (5.10) it results clear that the joint and marginal Characteristic Functions under the P and $Q_{\mathbf{h}}$ measures are of the same type. For each marginal process, only two parameters changes. Under the ESMM the multivariate log-returns process can be expressed again as a Multivariate Brownian motion with independent components, time-changed by an independent Gamma process, identical to the physical one (plus a linear drift). In other worlds, the underlying dependence structure remains unchanged. However, covariances, correlations, and marginal moments change. In particular, the log-return of the j-th hedge fund over the period $[0;\ t]$ under $Q_{\mathbf{h}}$ is

$$Y_t^j = \mu_j t + \theta_j^{Q_{\mathbf{h}}} G_t + \sigma_j^{Q_{\mathbf{h}}} W_{G_t}^j \tag{5.62}$$

where $G = \{G_t,\ t \geq 0\}$ is the common Gamma stochastic process with $G_t \sim Gamma(t/\nu,\ 1/\nu)$, $\nu > 0$;
$W^j = \{W_t^j,\ t \geq 0\}$ and $W^k = \{W_t^k,\ t \geq 0\}$ are independent Wiener processes for all $j \neq k$;
$\theta_j^{Q_\mathbf{h}}$, μ_j and $\sigma_j^{Q_\mathbf{h}} > 0$ are constants.

5.2.3 Model 2 and ESMM

In the first step we have to solve the system (5.45) in order to find a vector $\hat{\mathbf{h}}$ such that the discounted Net Asset Value of each hedge fund is a martingale under the new measure. The first ingredient we need is the P Moment Generating Function of the Multivariate Variance Gamma process ($t = 1$) (5.30):

$$M(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j\mu_j\right)\left[1-\nu\left(\sum_{j=1}^{n} u_j\theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k\sigma_j\sigma_k\rho_{jk}\right)\right]^{-1/\nu}$$

After some computations and rearrangements the system to solve may be written as:

$$\begin{cases} \ln\left[1-\frac{\nu(\theta_1+0.5\sigma_1^2+\sum_{j=1}^{n} h_j\sigma_1\sigma_j\rho_{1j})}{1-\nu\left(\sum_{j=1}^{n} h_j\theta_j+\frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} h_j h_k\sigma_j\sigma_k\rho_{jk}\right)}\right] & = (\mu_1 - r)\nu \\ & \vdots \\ \ln\left[1-\frac{\nu(\theta_n+0.5\sigma_n^2+\sum_{j=1}^{n} h_j\sigma_n\sigma_j\rho_{nj})}{1-\nu\left(\sum_{j=1}^{n} h_j\theta_j+\frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} h_j h_k\sigma_j\sigma_k\rho_{jk}\right)}\right] & = (\mu_n - r)\nu \end{cases} \tag{5.63}$$

with the following constraints

$$\left[1-\nu\left(\sum_{j=1}^{n} h_j\theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} h_j h_k\sigma_j\sigma_k\rho_{jk}\right)\right] > 0 \tag{5.64}$$

and

$$\begin{aligned} &1-\nu\Big(\sum_{j\neq q}^{n} h_j\theta_j + \frac{1}{2}\sum_{j\neq q}^{n}\sum_{k\neq q}^{n} h_j h_k\sigma_j\sigma_k\rho_{jk}\Big) \\ &-\nu\Big((h_q+1)\theta_q + \frac{1}{2}\sum_{j\neq q}^{n} h_j(h_q+1)\sigma_j\sigma_q\rho_{jq} + \frac{1}{2}(h_q+1)^2\sigma_q^2\Big) > 0 \end{aligned} \tag{5.65}$$

for $q = 1, \ldots, n$.
It is possible to find again the vector $\hat{\mathbf{h}}$ only numerically[10]. The solution $\hat{\mathbf{h}}$ is

[10] See next chapter

sufficient to guarantee the existence of the Esscher Equivalent Martingale Measure. The joint Moment Generating Function of the process for $t = 1$ under the ESMM can be computed following the procedure described in section (5.2.2). After tedious computations and some rearrangements we get the following expression:

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j \mu_j\right) \times \left[1 - \frac{\nu(\sum_{j=1}^{n} u_j(\theta_j + \sum_{k=1}^{n} \hat{h}_j \sigma_j \sigma_k \rho_{jk}) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} u_j u_k \sigma_j \sigma_k \rho_{jk}))}{1 - \nu\left(\sum_{j=1}^{n} \hat{h}_j \theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} \hat{h}_j \hat{h}_k \sigma_j \sigma_k \rho_{jk}\right)}\right]^{-1/\nu} \tag{5.66}$$

The joint $Q_{\mathbf{h}}$ Moment Generating Function can be written in the following form

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^{n} u_j \mu_j^{Q_{\mathbf{h}}}\right) \times \left[1 - \nu^{Q_{\mathbf{h}}}\left(\sum_{j=1}^{n} u_j \theta_j^{Q_{\mathbf{h}}} + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j^{Q_{\mathbf{h}}} \sigma_k^{Q_{\mathbf{h}}} \rho_{jk}^{Q_{\mathbf{h}}}\right)\right]^{-1/\nu^{Q_{\mathbf{h}}}} \tag{5.67}$$

where the relations among statistical and Esscher risk neutral parameters are

$$\mu_j^{Q_{\mathbf{h}}} = \mu_j \tag{5.68}$$

$$\nu^{Q_{\mathbf{h}}} = \nu \tag{5.69}$$

$$\theta_j^{Q_{\mathbf{h}}} = \frac{\theta_j + \sum_{k=1}^{n} \hat{h}_k \sigma_j \sigma_k \rho_{jk}}{1 - \nu\left(\sum_{j=1}^{n} \hat{h}_j \theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} \hat{h}_j \hat{h}_k \sigma_j \sigma_k \rho_{jk}\right)} \tag{5.70}$$

$$\sigma_j^{Q_h 2} = \frac{\sigma_j^2}{1 - \nu\left(\sum_{j=1}^{n} \hat{h}_j \theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} \hat{h}_j \hat{h}_k \sigma_j \sigma_k \rho_{jk}\right)} \tag{5.71}$$

$$\sigma_{jk}^{Q_{\mathbf{h}}} = \frac{\sigma_{jk}}{1 - \nu\left(\sum_{j=1}^{n} \hat{h}_j \theta_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} \hat{h}_j \hat{h}_k \sigma_j \sigma_k \rho_{jk}\right)} \tag{5.72}$$

$$\rho_{jk}^{Q_{\mathbf{h}}} = \rho_{jk} \tag{5.73}$$

From (5.67) we can easily obtain the joint $Q_{\mathbf{h}}$ Characteristic Function

$$\Psi^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(i\sum_{j=1}^{n} u_j\mu_j\right) \times \left[1-\nu\left(i\sum_{j=1}^{n} u_j\theta_j^{Q_{\mathbf{h}}} - \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n} u_j u_k \sigma_j^{Q_{\mathbf{h}}}\sigma_k^{Q_{\mathbf{h}}}\rho_{jk}\right)\right]^{-1/\nu} \tag{5.74}$$

The j-th $Q_{\mathbf{h}}$ marginal Characteristic Function (for $j = 1, \ldots, n$) is given by:

$$\Psi^{Q_{\mathbf{h}}}_{Y_1{}^j}(u_j) = \exp(iu_j\mu_j)\left[1-\nu(iu_j\theta_j^{Q_{\mathbf{h}}} - \frac{1}{2}u_j^2\sigma_j^{Q_{\mathbf{h}}2})\right]^{(-1/\nu)} \tag{5.75}$$

Marginal and joint Characteristic Functions under both probability measures (real and risk neutral) are of the same type (Compare (5.74) and (5.75) to (5.29) and (5.10)). For each marginal process, only two parameters changes. Under the ESMM the log-returns process is still obtained time-changing a Multivariate Brownian motion with correlated components, with an independent Gamma process. In particular, the underlying dependence structures is not affected by the Esscher change of measure. To be more precise, the Brownian motions have the same correlation matrix and the Gamma process has the same parameters under both *Worlds*. However, covariances, correlations, and marginal moments are different. In particular, the log-return of the j-th hedge fund over the period $[0, t]$ under $Q_{\mathbf{h}}$ is

$$Y_t^j = \mu_j t + \theta_j^{Q_{\mathbf{h}}} G_t + \sigma_j^{Q_{\mathbf{h}}} W_{G_t}^j \tag{5.76}$$

where $G = \{G_t,\ t \geq 0\}$ is the common stochastic clock process with $G_t \sim Gamma(t/\nu,\ 1/\nu)$ and $\nu > 0$;
$W^j = \{W_t^j,\ t \geq 0\}$ and $W^k = \{W_t^k,\ t \geq 0\}$ are correlated Wiener processes for all $j \neq k$;
$\theta_j^{Q_{\mathbf{h}}}$, μ_j and $\sigma_j^{Q_{\mathbf{h}}} > 0$ are constants.

5.2.4 Model 3 and ESMM

First off all, to solve the system (5.45) we need the P Moment Generating Function of the Multivariate ν-Variance Gamma process ($t = 1$) (5.42):

$$M(\mathbf{Y})_1(\mathbf{u}) = \exp(\sum_j^n u_j\mu_j)\prod_j^n(1-\nu_j(\theta_j u_j+\frac{1}{2}\sigma_j^2 u_j^2))^{-(\frac{1}{\nu_j}-a)}\times$$
$$(1-\sum_j^n \nu_j(\theta_j u_j+\frac{1}{2}\sigma_j^2 u_j^2))^{-a}$$

After some computations and rearrangements the system (5.45) may be written as :

$$(\frac{1}{\nu_k}-a)\ln[1-\frac{\nu_k(\theta_k+h_k\sigma_k^2+0.5\sigma_k^2)}{1-\nu_k(\theta_k h_k+0.5\sigma_k^2 h_k^2)}]+a\ln[1-\frac{\nu_k(\theta_k+h_k\sigma_k^2+0.5\sigma_k^2)}{1-\sum_j^n\nu_j(\theta_j h_j+0.5\sigma_j^2 h_j^2)}]$$
$$=\mu_k-r\quad\text{for } k=1,\dots,n. \tag{5.77}$$

with the following constraints

$$[1-\nu\sum_j^n(h_j\theta_j+0.5h_j^2\sigma_j^2)]>0, \tag{5.78}$$

$$[1-\nu(\sum_{j\neq k}^n(h_j\theta_j+0.5h_j^2\sigma_1^2)+((h_k+1)\theta_k+0.5(h_k+1)^2\sigma_k^2))]>0\;\; k=1,\dots,n, \tag{5.79}$$

$$[1-\nu(h_k\theta_k+0.5h_k^2\sigma_k^2)]>0,\quad k=1,\dots,n, \tag{5.80}$$

$$[1-\nu((h_k+1)\theta_k+0.5(h_k+1)^2\sigma_k^2))]>0\quad k=1,\dots,n. \tag{5.81}$$

We cannot find the elements of the vector $\hat{\mathbf{h}}$ in closed form. However, we are able to solve numerically this system[11]. The joint Moment Generating Function of the process for $t = 1$ under this new probability measure can be computed following the procedure described in section (5.2.2). Long and tedious calculations lead to the following expression:

$$M^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(\sum_{j=1}^n u_j\mu_j\right)\prod_j^n\left\{1-\frac{\nu_j[u_j(\theta_j+\hat{h}_j\sigma_j^2)+\frac{1}{2}u_j^2\sigma_j^2]}{1-\nu_k(\theta_j\hat{h}_j+\frac{1}{2}\hat{h}_j^2\sigma_j^2)}\right\}^{-(\frac{1}{\nu_j}-a)}\times$$
$$\left\{1-\frac{\sum_j^n\nu_j[u_j(\theta_j+\hat{h}_j\sigma_j^2)+\frac{1}{2}u_j^2\sigma_j^2]}{1-\sum_j^n\nu_j(\theta_j\hat{h}_j+\frac{1}{2}\sigma_j^2\hat{h}_j^2)}\right\}^{-a} \tag{5.82}$$

[11] See next chapter

The joint Characteristic Funtion is easily obtained from (5.82)

$$\Psi^{Q_{\mathbf{h}}}(\mathbf{Y})_1(\mathbf{u}) = \exp\left(i\sum_{j=1}^{n} u_j\mu_j\right)\prod_j^n \left\{1 - \frac{\nu_j[iu_j(\theta_j + \hat{h}_j\sigma_j^2) - \frac{1}{2}u_j^2\sigma_j^2]}{1-\nu_k(\theta_j\hat{h}_j + \frac{1}{2}\hat{h}_j^2\sigma_j^2)}\right\}^{-(\frac{1}{\nu_j}-a)} \times \left\{1 - \frac{\sum_j^n \nu_j[(iu_j(\theta_j + \hat{h}_j\sigma_j^2) - \frac{1}{2}u_j^2\sigma_j^2]}{1-\sum_j^n \nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma_j^2\hat{h}_j^2)}\right\}^{-a} \tag{5.83}$$

The j-th marginal Characteristic Function under the ESMM is obtained substituting $(0, \ldots, u_j, \ldots,\ 0)$ in (5.83)

$$\Psi^{Q_{\mathbf{h}}}(Y_1^j)(u_j) = \exp(iu_j\mu_j)\left\{1 - \frac{\nu_j[iu_j(\theta_j + \hat{h}_j\sigma_j^2) - \frac{1}{2}u_j^2\sigma_j^2]}{1-\nu_k(\theta_j\hat{h}_j + \frac{1}{2}\hat{h}_j^2\sigma_j^2)}\right\}^{-(\frac{1}{\nu_j}-a)} \times \left\{1 - \frac{\nu_j[(iu_j(\theta_j + \hat{h}_j\sigma_j^2) - \frac{1}{2}u_j^2\sigma_j^2]}{1-\sum_j^n \nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma_j^2\hat{h}_j^2)}\right\}^{-a} \tag{5.84}$$

Under the selected risk neutral measure the multivariate process of log returns is not a Multivariate ν-Variance Gamma (Compare (5.83) to (5.40)). Furthermore, the marginal processes are no longer Variance Gamma (Compare (5.84) to (5.41)).

Proposition

Under the ESMM the log-return of the j-th hedge fund over the period $[0,\ t]$ for $j = 1, \ldots,\ n$ can be expressed in the following way

$$Y_t^j = \mu_j t + (\theta_j + \hat{h}_j\sigma_j^2)\left\{\frac{U_t^j}{1-\nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma^2\hat{h}_j^2)} + \frac{\nu_j Z_t}{1-\sum_j^n \nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma_j^2\hat{h}_j^2)}\right\} + + \sigma_j\sqrt{\frac{U_t^j}{1-\nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma^2\hat{h}_j^2)} + \frac{\nu_j Z_t}{1-\sum_j^n \nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma_j^2\hat{h}_j^2)}}W_1^j \tag{5.85}$$

The j-th stochastic time change process is given by

$$T^j = \frac{U^j}{1-\nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma^2\hat{h}_j^2)} + \frac{\nu_j Z}{1-\sum_j^n \nu_j(\theta_j\hat{h}_j + \frac{1}{2}\sigma_j^2\hat{h}_j^2)} = \{T_t^j,\ t \geq 0\}. \tag{5.86}$$

$U^j = \{U_t^j,\ t \geq 0\}$ drives the idiosyncratic part of the stochastic clock T^j. $U_t^j \sim Gamma(t/\nu_j - at,\ 1/\nu_j)$ and the processes U^j and U^k are independent for $j \neq k$. $Z = \{Z_t,\ t \geq 0\}$ drives the common part of each stochastic time-change. $Z_t \sim$

$Gamma(at,\ 1)$ and is independent with respect to U^j for all j. $W^j = \{W_t^j,\ t \geq 0\}$ and $W^k = \{W_t^k,\ t \geq 0\}$ are uncorrelated Wiener processes for all $j \neq k$. W^j is independent with respect to all previous Gamma processes for all j. Finally, θ_j, μ_j and $\sigma_j > 0$ are constants for $j = 1, \ldots,\ n$.
The process $\mathbf{T} = \{\mathbf{T}_t,\ t \geq 0\}$ is a multivariate subordinator with dependent components.

Proof

The proof is divided into two parts:

- in the first part of the proof we show that under the assumptions of the proposition the process $\mathbf{T} = \{\mathbf{T}_t,\ t \geq 0\}$ is a subordinator;
- in the second part of proof we need to show that time changing a Multivariate Brownian motion with uncorrelated components with this subordinator we get the Characteristic Function (5.83)

First part

Consider the process $\mathbf{T} = \{\mathbf{T}_t,\ t \geq 0\}$. Its j-th component at time t is given by the following expression:

$$T_t^j = \frac{U_t^j}{1 - \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)} + \frac{\nu_j Z_t}{1 - \sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)} \tag{5.87}$$

where

$$\frac{U_t^j}{1 - \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)} \sim Gamma((\frac{1}{\nu_j} - a)t,\ \frac{1 - \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)}{\nu_j}) \tag{5.88}$$

$$\frac{\nu_j Z_t}{1 - \sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)} \sim Gamma(at,\ \frac{1 - \sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)}{\nu_j}) \tag{5.89}$$

T_t^j for $t \in [0,\ t]$ it does not follow a Gamma distribution. However, under the assumptions of this proposition we can compute its Characteristic Function:

$$\begin{aligned} E[\exp(iu_j T_t^j)] = &\left[1 - iu_j \frac{\nu_j}{1 - \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)}\right]^{-(\frac{1}{\nu_j} - a)t} \times \\ &\left[1 - iu_j \frac{\nu_j}{1 - \sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)}\right]^{-at} \end{aligned} \tag{5.90}$$

From equation 5.90 clearly the distribution of T_t^j for all j is infinitely divisible since

$$E[\exp(iu_j T_t^j)] = E[\exp(iu_j T_1^j)]^t \tag{5.91}$$

The multivariate distribution of $\mathbf{T}_t$ is also infinitely divisible since its Characteristic Function can be obtained considering the Characteristic Function of $\mathbf{T}_1$ to the power t:

$$\begin{aligned}\Psi^{Q_{\mathbf{h}}}(\mathbf{T})_t(\mathbf{u}) = \prod_j^n \left[1 - iu_j \frac{\nu_j}{1-\nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)}\right]^{-(\frac{1}{\nu_j}-a)t} \times \\ \left[1 - \frac{i\sum_j^n u_j \nu_j}{1-\sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)}\right]^{-at}\end{aligned} \tag{5.92}$$

Finally, since all marginal processes are expressed as the sum of two independent Gamma processes they are non-negative and non decreasing. This consideration concludes the first part of the proof. The process $\mathbf{T} = \{\mathbf{T}_t,\ t \geq 0\}$ is a multivarite subordinator since for every t its distribution is infinitely divisible and each component is a non-negative and non decreasing Lévy process.

Second part

To compute the joint Characteristic Function of (5.85) we use the generalized version of *THEOREM 4.2* [33] for the case of *multivariate subordinators* [11].

The Laplace Exponent $l(\mathbf{u})$ of $\mathbf{T}_1$

$$l(\mathbf{u}) = \ln(\prod_j^n \left[1 - \frac{u_j \nu_j}{1-\nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma^2 \hat{h}_j^2)}\right]^{-(\frac{1}{\nu_j}-a)} \left[1 - \frac{\sum_j^n u_j \nu_j}{1-\sum_j^n \nu_j(\theta_j \hat{h}_j + \frac{1}{2}\sigma_j^2 \hat{h}_j^2)}\right]^{-a}) \tag{5.93}$$

The Characteristic Exponent $c(\mathbf{u})$ of the Multivariate Brownian motion is

$$c(\mathbf{u}) = \sum_j^n (i(\theta_j + \hat{h}_j \sigma_j^2) - \frac{1}{2}\sigma^2 u_j^2)) \tag{5.94}$$

The Characteristic Function of the vector $\mathbf{Y}_1$ can be computed from the following expression:

$$\Psi(\mathbf{Y})_1(\mathbf{u}) = \exp(i\sum_j^n u_j \mu_j)\exp(l(c(\mathbf{u}))) \tag{5.95}$$

$$\Psi(\mathbf{Y})_1(\mathbf{u}) = \exp(i\sum_j^n u_j\mu_j)\prod_j^n\left[1-\frac{\nu_j(i(\theta_j+\hat{h}_j\sigma_j^2)-\frac{1}{2}\sigma^2u_j^2))}{1-\nu_j(\theta_j\hat{h}_j+\frac{1}{2}\sigma^2\hat{h}_j^2)}\right]^{-(\frac{1}{\nu_j}-a)}\times$$
$$\left[1-\frac{\sum_j^n\nu_j(i(\theta_j+\hat{h}_j\sigma_j^2)-\frac{1}{2}\sigma^2u_j^2))}{1-\sum_j^n\nu_j(\theta_j\hat{h}_j+\frac{1}{2}\sigma_j^2\hat{h}_j^2)}\right]^{-a} \tag{5.96}$$

A comparison between (5.96) and (5.83) concludes the proof. This result is very useful because it allows to express the Esscher risk neutral hedge funds' log-returns process through a Multivariate Brownian motion time-changed by a Multivariate subordinator, whose components are linear combination of two Gamma processes. In particular, all risk neutral parameters are explicitly linked to the physical ones. It should be emphasized that Model 3 is an example of a Lévy process that do not remain of the same kind under the Esscher Equivalent Martingale Measure. It is important to note that not only the multivariate but also the marginal processes change. In particular, the margins are no longer Variance Gamma.
To simplify the notation for the sequel, we rewrite expression (5.85) in the following way:

$$Y_t^j = \mu_j t + \lambda_j(\alpha_j U_t^j + \beta_j Z_t) + \sigma_j\sqrt{\alpha_j U_t^j + \beta_j Z_t}W_1^j \tag{5.97}$$

where

$$\lambda_j = \theta_j + \hat{h}_j\sigma_j^2, \tag{5.98}$$

$$\alpha_j = \frac{1}{1-\nu_j(\theta_j\hat{h}_j+\frac{1}{2}\sigma^2\hat{h}_j^2)}, \tag{5.99}$$

$$\beta_j = \frac{\nu_j}{1-\sum_j^n\nu_j(\theta_j\hat{h}_j+\frac{1}{2}\sigma_j^2\hat{h}_j^2)} \tag{5.100}$$

for $j = 1, \ldots, n$.

Chapter 6

Estimation and Simulation

In order to use the models developed in chapter 5 to price equity and debt tranches it is necessary to be able to simulate trajectories of the Net Asset Value of the collateral fund of hedge funds under the selected risk neutral probability measure. To reach this task the following steps are necessary:

- the estimation of the real world parameters;
- the computation of the risk neutral parameters;
- the simulation of the NAV of every hedge fund under the selected Equivalent Martingale Measure, with the desired frequency;
- the computation of the NAV of the fund of hedge funds by summing the value of each hedge fund in the collateral portfolio.

In this chapter we briefly describe the estimation procedure and report the estimated real world parameters for each model, making a distinction between estimates obtained using smoothed or unsmoothed data. We report also the risk neutral parameters computed following the procedures described in the previous chapter. Finally, we explain how to simulate paths of hedge funds' NAV for each model.

6.1 Estimation Procedure and Results

To estimate real world parameters we use method of moments.
We start with Model 1 and Model 2. First, we impose a value for the common pa-

rameter ν. Then, we estimate marginal parameters requiring the equality among the first three empirical moments of log-returns and their theoretical VG counterparts. By so doing, we get Variance Gamma distributions able to replicate empirical means, variances and skewnesses. Then, we compute the mean of the resulting kurtoses and we compare this value with the empirical one of hedge funds in the collateral portfolio[1]. Varying the value of parameter ν we estimate again the model replicating the first three moments. The resulting mean kurtosis depends of course on the value of ν. After several trials we choose $\nu = 0,33333$. This value leads to a mean fitted kurtosis similar to the mean empirical one. Annual marginal parameter estimates are reported in table A.1 and table A.4. Model 2 requires also to estimate the correlation matrix of the underlying Brownian Motions. Formula (5.26) allows us to find the implied correlation ρ_{jk} for each pair of Brownian Motions. However, this matrix is not positive definite. Using Matlab it is easy to find a positive definite approximation of this implied correlation matrix (See table A.11 and table A.12).

We want to underline that our interest is mainly theoretical. We know that this estimation procedure is far from being rigorous. A more accurate way to proceed requires the joint estimation of all parameters at the same time. Since we know in explicit form the joint Characteristic Function, a multivariate extension of the Spectral GMM estimator of continuous-time processes, introduced in the univariate case by Chacko and Viceira (2003) [25], is possible at list theoretically. However, also for practical reasons we decided to use the method described above.

We estimate Model 3 using unconstrained method of moments for each time series of monthly log-returns. In this way, we obtain Variance Gamma distributions that replicate perfectly empirical means, standard deviations, skewnesses and kurtosis. Then, we choose a value for the common parameter a which satisfies the constraint $0 < a < \min(\frac{1}{\nu_j})$ for $j = 1, \ldots,\ n$. Theoretically, Model 3 is more flexible than Model 1 in modelling dependence. If $a = 0$ hedge funds are independent. As a increases all correlations increase in absolute value. However, when this model is applied to real data, the existence of an asset j with a big ν_j can reduce the domain of a considerably with a huge impact on maximal correlations attainable. When we estimate Model 3 using observed log-returns we get $\min(\frac{1}{\nu_j}) = 1,98$.

[1]See next chapter for the composition of the fund of hedge funds

When we use unsmoothed data we get $\min(\frac{1}{\nu_j}) = 1,79$. These values impose very strict limits on the maximal magnitude of correlations that can be reached choosing a big value for the common parameter a. This is caused by the presence of a hedge fund with a very high kurtosis in the collateral portfolio. This is the main disadvantage of Model 3. Since maximal correlations produced by this model are low in absolute value, we decide to use a high value of the parameter a inside its domain. In particular, we fix $a = 1,6$.
Tables A.7 and A.8 report annual estimates for Model 3.

6.1.1 Risk Neutral Parameters Computation

Given the estimates of real world parameters, we solve numerically the *Esscher systems* described in chapter 5. We find a solution $\hat{\mathbf{h}}$ for each model. This ensures the existence of the Esscher Risk Neutral Probability Measure for all our processes. Then, using the vectors $\hat{\mathbf{h}}$, the estimates of physical parameters and their relations with the risk neutral ones, we get the following tables reporting the Esscher risk neutral parameters for each process:

- table A.2 for Model 1 (Smoothed data)
- table A.5 for Model 1 (Unsmoothed data)
- table A.3 for Model 2 (Smoothed data)
- table A.6 for Model 2 (Unsmoothed data)
- table A.9 for Model 3 (Smoothed data)
- table A.10 for Model 3 (Unsmoothed data)

Furthermore, in chapter 5 we showed that this change of measure does not modify the underlying dependence structure among Brownian motions. Consequently, the underlying Brownian motions are still independent in the case of Model 1 and Model 3. In the case of Model 2 they have the same correlation matrix as under the statistical probability measure (See table A.11 and table A.12). Figures 6.1, 6.2, and 6.4 display comparisons between real world and risk neutral Kernel densities of Convertible Arbitrage for each model. Figure 6.3 show a comparison between risk neutral densities under Models 1 and 2 for the same index.

6.2 Simulation

To simulate the paths of n dependent hedge fund NAVs under the Esscher Equivalent Martingale Measure we can proceed as follows. Discretize the time-interval $[0,\ T]$ into N equally spaced intervals:
Let $\Delta t = T/N$ and set $t_k = k\Delta t$, for $k = 0, ..., N$.
$F^j_{t_0}$ is the NAV of hedge fund j at time 0 for $j = 1, \ldots,\ n$.

6.2.1 Model 1

To simulate $F^j_{t_k}$ for every hedge fund repeat the following steps for k from 1 to N:

- sample a random number g_k out of the Gamma$(\Delta t/\nu,\ 1/\nu)$ distribution;
- sample for each $j = 1, \ldots, n$ an independent standard Normal random number $w^j_{t_k}$.
- Set

$$F^j_{t_k} = F^j_{t_{k-1}} \exp\left[\mu_j \Delta t + \theta_j^{Q_h} g_k + \sigma_j^{Q_h} \sqrt{g_k} w^j_{t_k}\right] \tag{6.1}$$

6.2.2 Model 2

To simulate $F^j_{t_k}$ for every hedge fund repeat the following steps for k from 1 to N:

- sample a random number g_k out of the Gamma$(\Delta t/\nu,\ 1/\nu)$ distribution;
- generate for each $j = 1, \ldots, n$ an independent standard Normal random number $w^j_{t_k}$;
- convert these random numbers $w^j_{t_k}$ into correlated random numbers $v^j_{t_k}$ by using the Cholesky decomposition of the implied correlation matrix of the underlying Brownian Motions.
- Set

$$F^j_{t_k} = F^j_{t_{k-1}} \exp\left[\mu_j \Delta t + \theta_j^{Q_h} g_k + \sigma_j^{Q_h} \sqrt{g_k} v^j_{t_k}\right] \tag{6.2}$$

6.2.3 Model 3

To simulate $F^j_{t_k}$ for every hedge fund repeat the following steps for k from 1 to N:

- sample a random number z_k out of the Gamma($a\Delta t,\ 1$) distribution;
- sample for each $j = 1, \ldots, n$ an independent random number u^j_k from a Gamma($(\frac{1}{\nu_j} - a)\Delta t,\ \frac{1}{\nu_j}$);
- sample for each $j = 1, \ldots, n$ an independent standard Normal random number $w^j_{t_k}$.
- Set

$$F^j_{t_k} = F^j_{t_{k-1}} \exp\left[\mu_j \Delta t + \lambda_j(\alpha_j u^j_k + \beta_j z_k) + \sigma_j \sqrt{\alpha_j u^j_k + \beta_j z_k} w^j_{t_k}\right] \tag{6.3}$$

Figures 6.5, 6.6 and 6.7 show simulated paths of the log-return process for the pair Convertible Arbitrage and Event Driven over a period of 5 years. Finally, pictures 6.8, 6.8, 6.8 report scatter plots of risk neutral monthly log-returns for the same indices.

6.2.4 Simulation of Fund of Hedge Funds NAV

Chosen a model, to simulate a simple trajectory of the NAV of the collateral fund of hedge funds it is sufficient to compute for k from 1 to N

$$F_{t_k} = \sum_{j=1}^{n} F^j_{t_k}. \tag{6.4}$$

In the applications of chapter 7, we will also take into account the impact of CFO structural features such as coupon payments, equity distribution rules, Over Collateralization tests, liquidity profile and management fees to describe the temporal evolution of the NAV of the collateral portfolio.

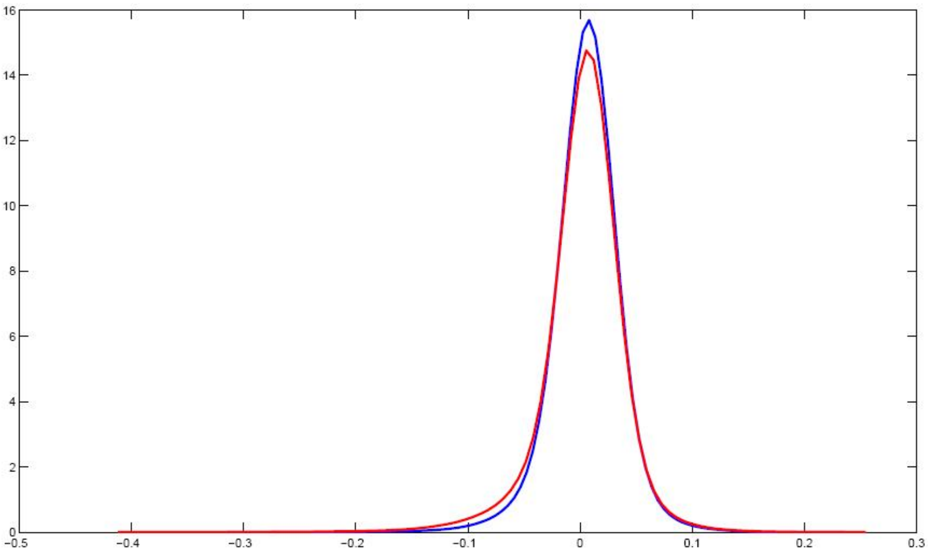

Figure 6.1: Risk Neutral (red) and Real World (blue) Kernel Densities for Convertible Arbitrage (Model 1 Unsmoothed case)

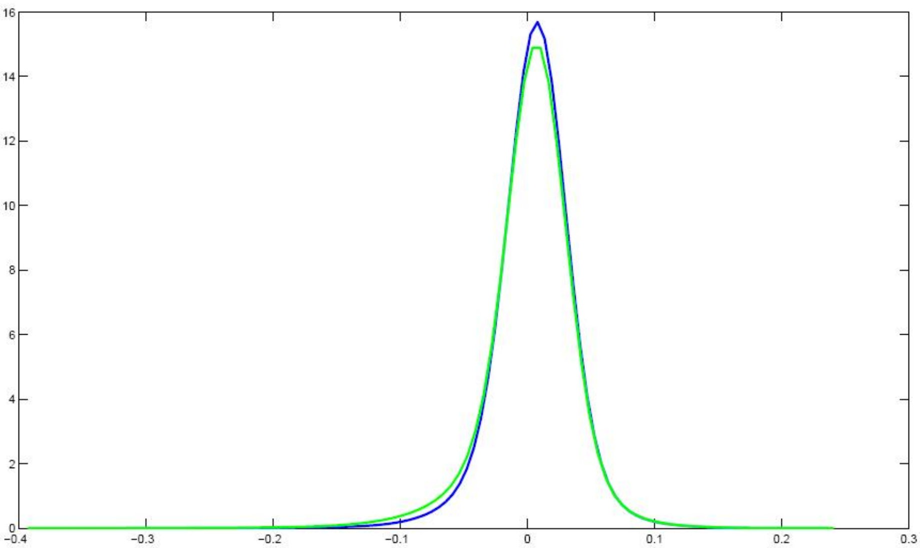

Figure 6.2: Risk Neutral (green) and Real World (blue) Kernel Densities for Convertible Arbitrage (Model 2 Unsmoothed case)

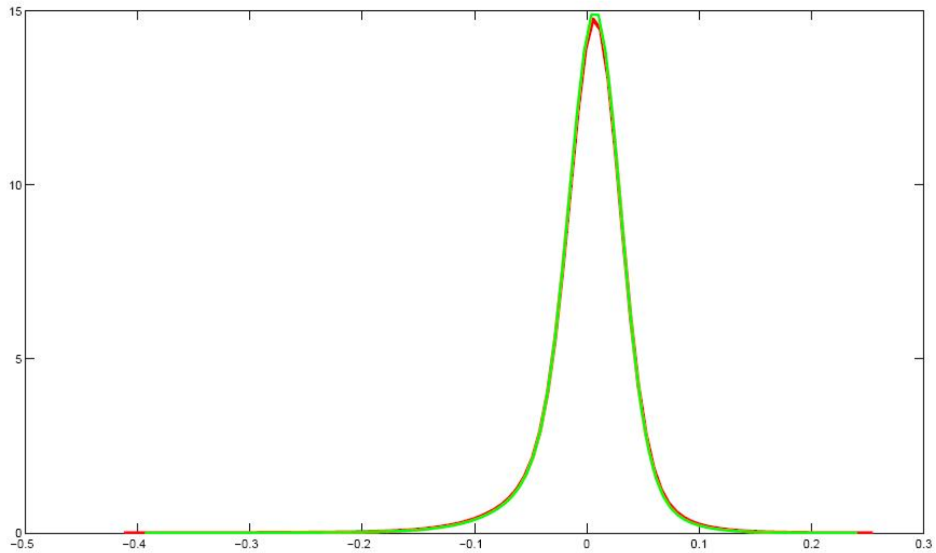

Figure 6.3: Risk Neutral Kernel Densities for Convertible Arbitrage (Models 1 and 2 Unsmoothed case)

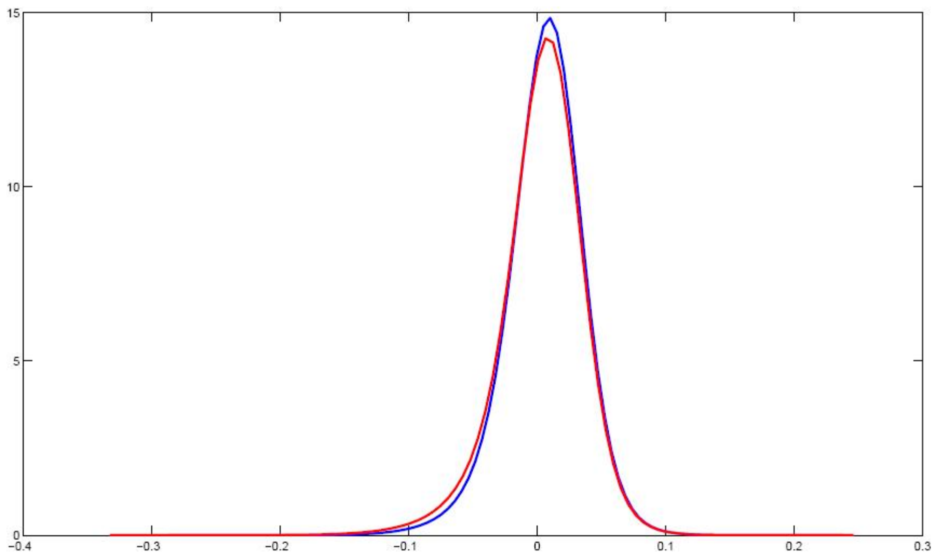

Figure 6.4: Risk Neutral (red) and Real World (blue) Kernel Densities for Convertible Arbitrage (Model 3 Unsmoothed case)

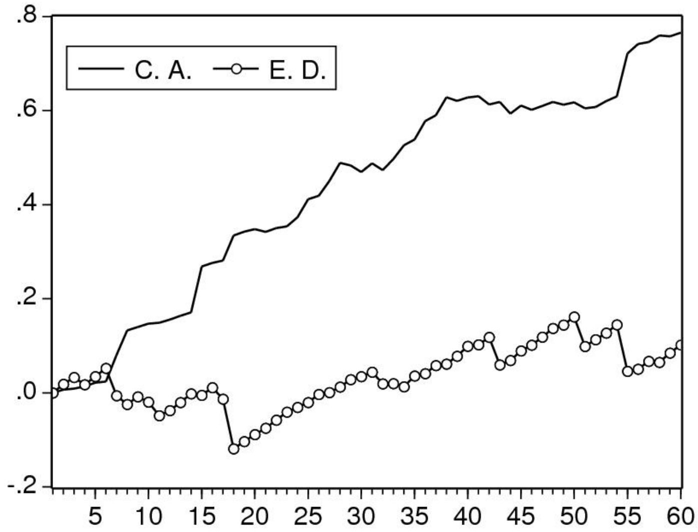

Figure 6.5: Model 1 (Unsmoothed case)

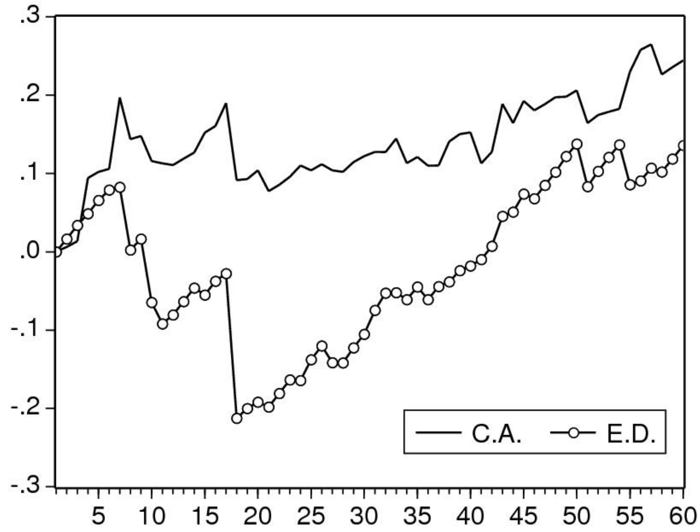

Figure 6.6: Model 2 (Unsmoothed case)

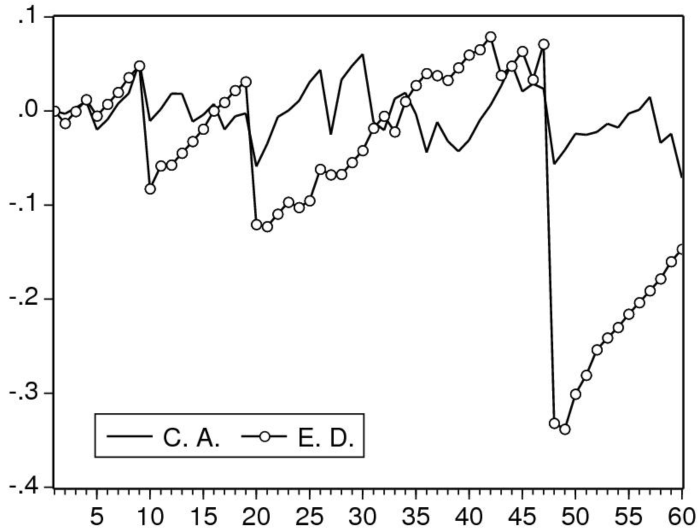

Figure 6.7: Model 3 (Unsmoothed case)

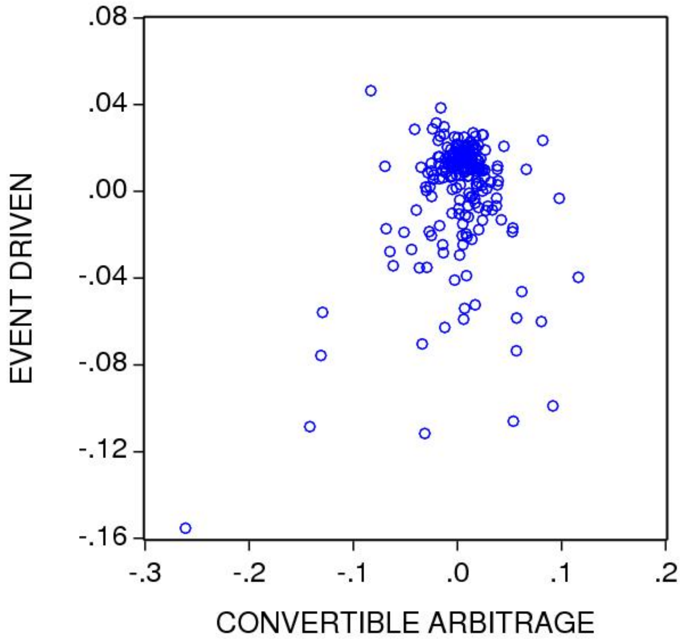

Figure 6.8: Model 1 (Unsmoothed case)

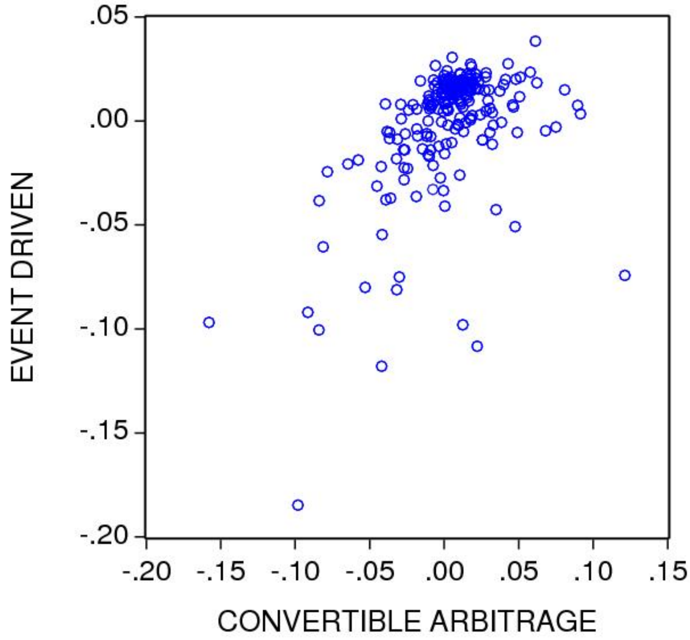

Figure 6.9: Model 2 (Unsmoothed case)

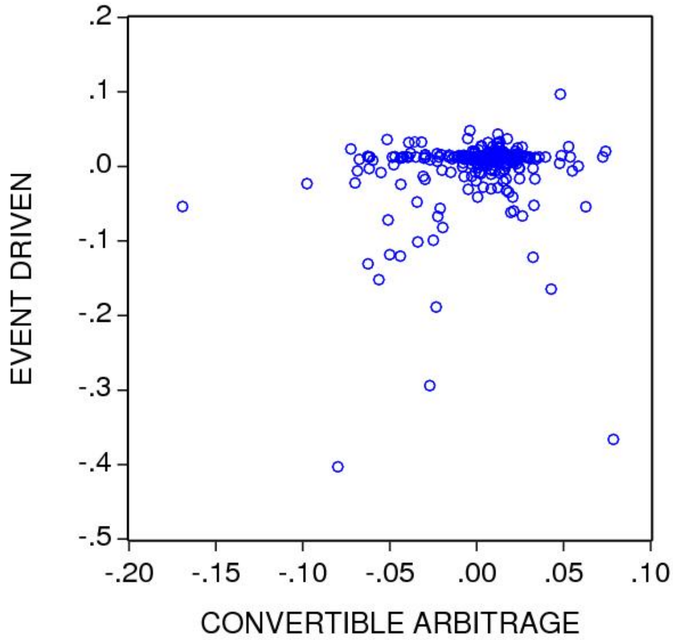

Figure 6.10: Model 3 (Unsmoothed case)

Chapter 7

Pricing CFOs equity and debt tranches

As we have already mentioned, CFOs equity securities and notes are different types of investment in the underlying pool of hedge funds. In this chapter we price debt and equity securities of a CFO as options written on a basket of hedge funds. In particular, we price equity and debt tranches for a theoretical CFO using a sort of *structural firm value approach*. In fact, CFOs can be seen as firms with a fixed maturity (if we do not consider default). We use a Merton-type model and a Black-Cox-type model, where we assume that the hedge fund NAV processes are described by dependent Geometric Variance Gamma processes under the real world probability measure[1]. Default can be triggered either by the fact that the CFO Net Asset Value at maturity is too low to cover promised debt payments, as in the traditional Merton's model, or by the violation of an Over Collateralization Test, which represent a barrier, as in the traditional Black-Cox's model. In the second case default before the scheduled maturity is possible.
While the CFO collateral is the same in all our applications, the covenants of the CFO constitutive document change. First of all, we price a very simple CFO, in which its liability side is represented only by zero coupon bonds with different priorities and an equity tranche. Secondly, we consider a CFO structure in which liabilities are represented by different coupon bonds and a paying dividend equity tranche. In both cases, we assume that default can happen only at maturity. Fi-

[1]See the three models introduced in Chapter 5

nally, we introduce the possibility of default before the CFO maturity and consider the CFO liquidity profile.

7.1 Applications and Results

Consider the following theoretical Collateralized Fund of Hedge Funds Obligations structure with a scheduled maturity $T = 5$ years.

- Asset Side: fund of hedge funds with a current price of 1000 monetary units;
- Liability Side: debt and equity with total initial investment of 1000 monetary units;
    1. Debt tranche A with nominal value of 570 monetary units;
    2. Debt tranche B with nominal value of 150 monetary units;
    3. Debt tranche C with nominal value of 100 monetary units;
    4. Equity tranche with nominal value of 180 monetary units.

The most senior tranche in this CFO is tranche A and is credit enhanced due to the subordination of the lower tranches. This means that the lowest tranche, the equity tranche, absorbs losses first. When this tranche is exhausted, the next lowest tranche, i.e tranche C, begins absorbing losses. If tranche C is consumed, tranche B starts absorbing losses. Finally, only if tranche B is completely dissipated then tranche A is exposed to losses.
Now, we report the collateral portfolio composition and the amounts invested at time 0 in each underlying hedge funds:

1. Convertible Arbitrage: 175 monetary units;
2. Dedicated Short Bias: 50 monetary units;
3. Emerging Markets: 50 monetary units;
4. Equity Market Neutral: 250 monetary units;
5. Event Driven: 100 monetary units;
6. ED Distressed: 50 monetary units

7. ED Multi-Strategy: 100 monetary units;

8. ED Risk Arbitrage: 225 monetary units.

Finally, assume the existence of a risk free asset with a constant annual log-return $r = 4\%$.
These features are common to all the CFOs we price.

7.1.1 First CFO: pricing and sensitivity analyses

The distinctive characteristics of this CFO are the followings:

1. Debt tranche A is a zero coupon bond with a promised maturity payment $D_T^A = 696,20$, with an implicit promised annual log-return of $r = 4\%$;

2. Debt tranche B is a zero coupon bond with a promised final payment of $D_T^B = 183,67$, with $r = 4,05\%$;

3. Debt tranche C is a zero coupon bond with a promised payment $D_T^C = 125,23$, with $r = 4,5\%$;

4. Equity tranche is a stock that pays no dividends.

Default is only possible at the CFO scheduled maturity date if the value of the collateral pool of hedge funds is not sufficient to repay the liabilities. Note that it is not very precise to talk of default at the maturity for a CFO. In fact, even if CFOs have a capital structure similar to firms, at the fixed maturity date, they always cease to exist. Instead, it makes sense to talk of CFOs default prior to their scheduled maturity date.
In this simple case, each tranche can be expressed as a European option on the collateral portfolio. In particular, the equity tranche is a European call option on the pool of hedge funds with strike price $D_T = D_T^A + D_T^B + D_T^C$ and maturity T. Its intrinsic value is given by:

$$E_0 = \exp(-rT)E^{Q_h}\left[\max(F_T - D_T;\ 0)\right] \tag{7.1}$$

where $F_T = \sum_{j=1}^{8} F_T^j$ is the value of the collateral portfolio at time T.
Current fair prices of tranches $A,\ B,\ C$ are given by the following expressions:

$$A_0 = \exp(-rT)E^{Q_h}\left[D_T^A - \max(D_T^A - F_T;\ 0)\right] \tag{7.2}$$

$$B_0 = \exp(-rT)E^{Q_h}\left[\max(F_T - D_T^A;\ 0) - \max(F_T - (D_T^A + D_T^B);\ 0)\right] \quad (7.3)$$

$$C_0 = \exp(-rT)E^{Q_h}\left[\max(F_T - (D_T^A + D_T^B);\ 0) - \max(F_T - D_T;\ 0)\right] \quad (7.4)$$

To compute fair prices we perform the following steps:

- we simulate several times (50000) the Nav of the collateral portfolio at the maturity $T = 5$, under the Esscher risk neutral probability measure;
- we compute the mean payoff of every tranche;
- we discount these values with the risk free rate.

Tables B.1, B.2, B.3 and B.4 report CFOs notes and equity fair prices obtained using Model 1 and Model 2. These tables also display some sensitivity analyses. All the results exposed in the first two are based on a value of the common parameter ν equal to 0,3333, while those in the last two on $\nu = 0,5833$. In the Benchmark case of tables B.1 and B.2 prices are based on the risk neutral parameters reported respectively in tables A.2 for Model 1, and A.3 and A.11 for Model 2. Half variances is the scenario under which, preserving the common Gamma parameter value $\nu = 0,3333$ (or $\nu = 0,5833$), all other real parameters are estimated using method of moments, with all the empirical variances divided by two. In the case called Double Variances all empirical variances are multiplied by two, other empirical moments and the common parameter ν are unchanged. Then real world parameters are again estimated by constrained method of moments. Similar considerations hold for the other cases reported on the tables. In each scenario, risk neutral parameters are then computed as explained in Chapter 5.
Finally, all tables report the minimum value of each thanche and the number of losses based on a simulated sample of 50000 values. These details are not very useful, but we report them only to show how the risk can change under different hypotheses concerning empirical marginal moments.
An analysis of the results let us to make some observations.
Ceteris paribus:

- if variance increases the equity tranche becomes a more attractive investment opportunity, while the debt becomes riskier and its valuation diminishes. On the contrary, a reduction of the variance results in a decline of the equity fair price, while the debt tranches become more appreciated;

- if negative skewness increase in absolute value the equity tranche is more valued while the price of the debt tranches decreases;
- building a collateral portfolio with a positive skewness is the best thing a CFO manager can try to do for debt holders but the worst for equity investors;
- if kurtosis increases the theoretical value of the equity tranche increases, while the prices of the notes decrease;
- the consideration of implied correlations among Brownian Motions increase the equity value; this increase is particularly relevant in situation of high risk (double variances);
- tranche A is very protected by the structure and only in some extreme and rare scenarios (trajectories) can suffer medium losses. Its fair price is almost always equal by the amount invested. A triple AAA rating for this tranche seems very plausible;
- tranche A has a fair price less than its initial invested amount if and only if two risks are high at the same time. Examples are high kurtosis and big variance, high variance and high negative skewness, or high negative skewness and big kurtosis;
- tranche B has also a good protection, but not at the same level of tranche A;
- tranche C is the most risky among debt tranches. Its fair price is often less then the initial investment;
- notice that the model can be used to infer final promised payments, i.e. the promised rates of return, to make the price of each debt tranche fair;
- to sum up: as risk increases equity holders take advantage over debt investors;

Table 7.2 summarize our main results.

7.1.2 Second CFO: pricing and sensitivity analysis

The distinctive features of this CFO are the followings:

1. Debt tranche A is a coupon bond with an annual cash flow of $23,26$, i.e. the coupon rate is $c = 4\%$
2. Debt tranche B is a coupon bond with an annual cash flow of $6,20$, i.e. the coupon rate is $c = 4,05\%$
3. Debt tranche C is a coupon bond with an annual cash flow of $4,60$, i.e. the coupon rate is $c = 4,5\%$
4. Equity tranche is a stock that pays dividends computed as a given percentage of the annual net profit. Notice that the dividend payment at the end of a year is not sure. Only if the NAV of the collateral portfolio at the end of a year is greater than 1000 after the payment of coupons to bondholders, a portion of the profits is distributed. In particular, we consider three different hypotheses concerning the equity distribution rule: 0%, 50%, 100% of annual net profit.

These differences influence the simulation procedure. In the previous case, it was sufficient to simulate directly the value of the collateral portfolio at the CFO maturity. Now, we have to simulate the NAV at the end of every year until time T, to take into account jumps due to coupon payments and possible dividend payments. It is assumed that every payment is made through the liquidation of a part of the collateral portfolio. In particular, we suppose that a part of each hedge fund, proportional to its NAV at the payment date, is sold. Implicitly, we presume that the CFO has enough liquidity to pay coupons and dividends.
Tables B.5, B.6, B.7, B.8 show equity and fair prices obtained using Model 1 and Model 2 and some sensitivity analyses. All previous observations are still valid. However, in this case we can analyse the impact on fair prices of different equity distribution rules. In particular, the following observations can be made:

- if the dividend increases then equity fair price increases of an amount approximately equal to the value lost by lower debt tranches. Especially, the dividend policy has a direct impact on the price of equity and C tranches.

Tranche A is unaffected by a change in the equity distribution rule. Tranche B is only marginally influenced in extremely risky situations;

- dividend policy relevance is strictly linked to the degree of risk of the collateral portfolio. Specifically, the greater the risk is more relevant the impact of a change on the portion of net profits distributed on fair prices is. When variances and correlations are high, the distribution of a high portion of profits really creates value for stockholders. On the contrary, when the collateral pool is made up by positively skewed hedge funds, the dividend policy seems to be irrelevant.

Table 7.1 summarize these results.

7.1.3 Third CFO: pricing and sensitivity analysis

The third and the second CFO have the same liability structure. However, now we take into account the possibility of default prior to maturity and CFO liquidity profile. Tables B.9 and B.10 show the price of each CFO tranche computed under different equity distribution rules and using four different models to describe the physical evolution of hedge funds log-returns:

1. Multivariate Brownian Motion;
2. Multivariate Variance Gamma Process with independent underlying Brownian Motions (Model 1);
3. Multivariate Variance Gamma Process with dependent underlying Brownian Motions (Model 2);
4. Multivariate ν - Variance Gamma Process (Model 3).

Tables B.9 and B.10 report prices based respectively on observed and unsmoothed data. Only in the case of Model 3 the change of measure modifies joint and marginal processes as we have proved in Chapter 5. Under the risk neutral probability measure a Multivariate Brownian Motion changes only its drift. In the first case, to simulate this process we use the Cholesky decomposition of the empirical log-returns correlation matrix. In the second case, we employ the same decomposition of the correlation matrix obtained from unsmoothed time series[2].

[2]These matrices are not reported in this work.

In the simulation procedure we consider a barrier equal to 1,05 times the total nominal value of the debt tranches. If the NAV of the fund of hedge funds falls below this level, when its value is checked by the CFO manager, then the collateral portfolio will be sold in order to redeem the rated notes. In the event of default, we model the sale of the assets by assuming this simple liquidity profile:

- 30% after three months;
- 30% after six months;
- all the residual collateral portfolio value after nine months.

If default happens six months before CFO legal maturity the liquidity profile will be the following:

- 30% after three months;
- all the residual collateral portfolio value at the maturity.

If default occurs three months before CFO legal maturity, the liquidity profile will be 100% of the NAV at maturity. For simplicity, we assume that hedge funds are liquidated proportionally to their NAV. In the default event, tranche A is redeemed first. In particular, we assume that both capital and current coupon have to be paid. Then, tranche B has to be repaid in the same way and so on. The CFO manager usually makes the Over Collateralization test on a monthly basis. However, it can happen that to do all the necessary operations, more time is needed. For pratical reasons, we simulate portfolio NAV and make Over collateralization test every three months. Finally, we assume the existence of an initial lock out period of two years. This implies that redemptions before two years are not admitted.

Fair prices reported in these tables allow us to make the following observations:

- barriers destroy value for all tranches;
- the equity price is only marginally affected by the choice of the model;
- the intrinsic value of the equity tranche is slightly influenced by CFO dividend policy. Notice that the introduction of a barrier modifies the sign of the relation between equity price and the percentage of net profits distributed;

- debt tranche prices are strongly affected by the choice of the model. In particular, tranche C is the most influenced by this choice;
- when the percentage of net profits distributed increases debt tranches values decrease; this effect is especially relevant for tranche C.

Tables B.11, B.12, B.13 and B.14 report some sensitivity analyses. The main results can be summarized as follows:

- the higher the barrier is, the greater the value destroyed is in terms of fair prices;
- the higher the risk is, the bigger the negative impact is on debt tranches theoretical price ;
- as the barrier increases equity price tends to become independent with respect to risk. Without barrier, the value of equity tranche clearly increases when risk increases;
- the introduction of a barrier can protect apparently the capital invested by debt holders. Early redemptions, force to sell the assets when the price is low and bondholders loses one ore more promised coupon payments;
- as the level of the barrier decreases all tranches becomes more valued and converge to the prices obtained in the case of the second CFO;
- the less risky the collateral portfolio is, the faster the speed of convergence of prices towards prices without barrier is. As an example, look at the case opposite skewnesses in all these tables. The barrier seems to be irrelevant.

We perform further analyses[3] increasing the incidence of the equity tranche from 18% of the collateral to 30%. This increment of course gives more protection to debt tranches. The level of the barrier decreases and its negative effect on fair prices is reduced. In particular, tranche C take the biggest advantage by an increase of the equity capital. Other results, previously exposed are still valid.
Finally, tables B.15 and B.16 display fair prices computed considering also an annual management fee of 0,5% of the total nominal amount of CFO tranches. If

[3]These analyse are not reported in Appendix B

there are no barriers, at the end of each year the CFO manager receive 5. In this case the discounted value of the management fees is 22,20. If there are barriers the manager is paid until default. It is clear that the present value of this fee diminishes. However, if we look at tables B.15 and B.16 it is easy to see that the global impact on fair prices of management fees is even greater than 22,20. In presence of a barrier management fees have a double impact on the value:

- direct, i.e the fees paid. This cost is essentially faced by equity ;
- indirect, i.e they make easier to default. This cost mainly affect debt tranches because they become more risky.

Table 7.2 summarize the main results of this section:

7.2 Conclusion and Future Developments

Our models were applied to evaluate the equity and the debt tranches of a CFO. The analysis was performed starting from a simple CFO structure, which was then progressively complicated with the introduction of the structural features we encounter in typical CFOs. In this way, at each step of the evolution of the structure, the reader can understand the impact on the value, measured with respect to the first four moments of the distribution of log-returns, and how this value is divided among the different tranches. The result is a useful schema that can provide some help in designing a CFO transaction. In particular, we believe these models can be useful for rating agencies as well as for deal structurers, to efficiently evaluate various capital structures, test levels, liquidity profiles, coupons and equity distribution rules. The analysis is also helpful for the CFO manager who usually invests in the equity tranche, because gives him some suggestions on how to increase the value of his investment.
In this work we built a multivariate Lévy process to describe by time change a Multivariate Brownian motion with a univariate and multivariate subordinator. The main limit of this method is the lack of flexibility:

- marginal processes are of the same type;
- the pattern of dependence structure is quite limited. On the other hand we explained that Lévy copulas are very flexible, but are difficult to use in real

applications.

A possible future line of research tries to combine both methodologies in order to get the advantages of both methods and to mitigate their drawbacks. The idea is to build multidimensional Lévy processes time-changing a Multivariate Brownian motion with a multivariate subordinator whose components are linked by a *positive* Lévy copula. This gives more flexibility to the first method because we may have different margins and more patterns of dependence are possible. At the same time the simulation procedure of the multivariate process should be simpler. To simulate from a positive L'evy copula is less complicated than for general copulas. If we are able to simulate the multivariate subordinator then all simulation problems are solved thanks to the conditional Gaussianity of the process.

Results for CFO 1

Hedge Funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Variance ↑	↑	↓	↓	↓
Variance ↓	↓	↑	↑	↑
Skewness ↑	↓	↑	↑	↑
Skewness ↓	↑	↓	↓	↓
Kurtosis ↑	↑	↓	↓	↓
Kurtosis ↓	↓	↑	↑	↑
Correlation ↑	↑	↓	↓	↓
Correlation ↓	↓	↑	↑	↑

Table 7.1:Results for CFO 2

Hedge Funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Variance ↑	↑	↓	↓	↓
Variance ↓	↓	↑	↑	↑
Skewness ↑	↓	↑	↑	↑
Skewness ↓	↑	↓	↓	↓
Kurtosis ↑	↑	↓	↓	↓
Kurtosis ↓	↓	↑	↑	↑
Correlation ↑	↑	↓	↓	↓
Correlation ↓	↓	↑	↑	↑
Dividend ↑	↑	$\downarrow or \perp$	↓	↓
Dividend ↓	↓	$\uparrow or \perp$	↑	↑

Table 7.2:Results for CFO 3

Hedge Funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Variance ↑	↑	↓	↓	↓
Variance ↓	↓	↑	↑	↑
Skewness ↑	↑	↑	↑	↑
Skewness ↓	↓	↓	↓	↓
Kurtosis ↑	↑	↓	↓	↓
Kurtosis ↓	↓	↑	↑	↑
Correlation ↑	↑	↓	↓	↓
Correlation ↓	↓	↑	↑	↑
Barrier ↑	↓	↓	↓	↓
Barrier ↓	↑	↑	↑	↑
Dividend ↑	↓	↓	↓	↓
Dividend ↓	↑	↑	↑	↑
Fees ↑	↓	↓	↓	↓

Appendix A

Tables: Annual Real World and Risk Neutral Parameters

Table A.1:(Smoothed) Real World Parameters-Models 1 and 2

Index	μ_j	θ_j	σ_j	ν
Convertible Arbitrage	0,09318	-0,02330	0,04590	0,33333
Dedicated Short Bias	-0,05208	0,02691	0,16397	0,33333
Emerging Markets	0,13886	-0,05419	0,15268	0,33333
Equity Market Neutral	0,08316	0,00281	0,02647	0,33333
Event Driven	0,17030	-0,07013	0,03866	0,33333
ED Distressed	0,17588	-0,06401	0,04969	0,33333
ED Multi-Strategy	0,14482	-0,05025	0,05321	0,33333
ED Risk Arbitrage	0,08215	-0,01534	0,03925	0,33333

Table A.2:(Smoothed) Risk Neutral Parameters-Model 1

Index	μ_j	$\theta_j^{Q_h}$	$\sigma_j^{Q_h}$	ν
Convertible Arbitrage	0,09318	-0,05559	0,06214	0,33333
Dedicated Short Bias	-0,05208	0,06605	0,22197	0,33333
Emerging Markets	0,13886	-0,12187	0,20668	0,33333
Equity Market Neutral	0,08316	-0,04412	0,03584	0,33333
Event Driven	0,17030	-0,13454	0,05233	0,33333
ED Distressed	0,17588	-0,14126	0,06726	0,33333
ED Multi-Strategy	0,14482	-0,10927	0,07204	0,33333
ED Risk Arbitrage	0,08215	-0,04386	0,05313	0,33333

Table A.3:(Smoothed) Risk Neutral Parameters-Model 2

Index	μ_j	$\theta_j^{Q_h}$	$\sigma_j^{Q_h}$	ν
Convertible Arbitrage	0,09318	-0,05524	0,05619	0,33333
Dedicated Short Bias	-0,05208	0,07054	0,20072	0,33333
Emerging Markets	0,13886	-0,11797	0,18690	0,33333
Equity Market Neutral	0,08316	-0,04400	0,03241	0,33333
Event Driven	0,17030	-0,13429	0,04732	0,33333
ED Distressed	0,17588	-0,14085	0,06083	0,33333
ED Multi-Strategy	0,14482	-0,10879	0,06514	0,33333
ED Risk Arbitrage	0,08215	-0,04360	0,04804	0,33333

Table A.4:(Unsmoothed) Real World Parameters-Models 1 and 2

Index	μ_j	θ_j	σ_j	ν
Convertible Arbitrage	0,09668	-0,02685	0,07974	0,33333
Dedicated Short Bias	-0,05341	0,02913	0,18126	0,33333
Emerging Markets	0,16393	-0,08836	0,19764	0,33333
Equity Market Neutral	0,08424	0,00257	0,03326	0,33333
Event Driven	0,20534	-0,10811	0,03994	0,33333
ED Distressed	0,20328	-0,09448	0,06129	0,33333
ED Multi-Strategy	0,15701	-0,06522	0,06800	0,33333
ED Risk Arbitrage	0,08382	-0,01723	0,04935	0,33333

Table A.5:(Unsmoothed) Risk Neutral Parameters-Models 1

Index	μ_j	$\theta_j^{Q_h}$	$\sigma_j^{Q_h}$	ν
Convertible Arbitrage	0,09668	-0,06227	0,10046	0,33333
Dedicated Short Bias	-0,05341	0,06589	0,22837	0,33333
Emerging Markets	0,16393	-0,15753	0,24900	0,33333
Equity Market Neutral	0,08424	-0,04544	0,04190	0,33333
Event Driven	0,20534	-0,17125	0,05032	0,33333
ED Distressed	0,20328	-0,17079	0,07722	0,33333
ED Multi-Strategy	0,15701	-0,12300	0,08567	0,33333
ED Risk Arbitrage	0,08382	-0,04608	0,06218	0,33333

Table A.6:(Unsmoothed) Risk Neutral Parameters-Models 2

Index	μ_j	$\theta_j^{Q_h}$	$\sigma_j^{Q_h}$	ν
Convertible Arbitrage	0,09668	-0,06165	0,09415	0,33333
Dedicated Short Bias	-0,05341	0,06907	0,21402	0,33333
Emerging Markets	0,16393	-0,15375	0,23336	0,33333
Equity Market Neutral	0,08424	-0,04534	0,03927	0,33333
Event Driven	0,20534	-0,17109	0,04716	0,33333
ED Distressed	0,20328	-0,17042	0,07237	0,33333
ED Multi-Strategy	0,15701	-0,12255	0,08029	0,33333
ED Risk Arbitrage	0,08382	-0,04584	0,05827	0,33333

Table A.7:(Smoothed) Real World Parameters-Models 3 $a = 1,6$

Index	μ_j	θ_j	σ_j	ν_j
Convertible Arbitrage	0,13801	-0,06812	0,04156	0,12084
Dedicated Short Bias	-0,40369	0,37852	0,15358	0,02470
Emerging Markets	0,18611	-0,10144	0,14978	0,18022
Equity Market Neutral	0,04379	0,04219	0,02575	0,02263
Event Driven	0,14267	-0,04250	0,04718	0,50277
ED Distressed	0,15713	-0,04527	0,05393	0,45167
ED Multi-Strategy	0,14018	-0,04562	0,05399	0,36426
ED Risk Arbitrage	0,09662	-0,02981	0,03826	0,17449

Table A.8:(Unsmoothed) Real World Parameters-Models 3 $a = 1,6$

Index	μ_j	θ_j	σ_j	ν_j
Convertible Arbitrage	0,14268	-0,07284	0,07701	0,12560
Dedicated Short Bias	-0,45876	0,43452	0,16953	0,02331
Emerging Markets	0,22752	-0,15192	0,19267	0,19703
Equity Market Neutral	0,06288	0,02388	0,03298	0,03599
Event Driven	0,15216	-0,05484	0,06173	0,55842
ED Distressed	0,16524	-0,05652	0,07126	0,51766
ED Multi-Strategy	0,14772	-0,05592	0,06959	0,38403
ED Risk Arbitrage	0,09912	-0,03252	0,04843	0,17883

Table A.9:(Smoothed) Risk Neutral Parameters-Models 3 $a = 1,6$

Index	μ_j	λ_j	σ_j	α_j	β_j
Convertible Arbitrage	0,1380	-0,0788	0,0416	1,0580	0,2578
Dedicated Short Bias	-0,4037	0,4061	0,1536	1,0114	0,0527
Emerging Markets	0,1861	-0,1219	0,1498	1,0187	0,3845
Equity Market Neutral	0,0438	-0,0041	0,0258	0,9708	0,0483
Event Driven	0,1427	-0,0554	0,0472	1,1658	1,0726
ED Distressed	0,1571	-0,0656	0,0539	1,2120	0,9636
ED Multi-Strategy	0,1402	-0,0612	0,0540	1,1165	0,7771
ED Risk Arbitrage	0,0966	-0,0426	0,0383	1,0586	0,3723

Table A.10:(Unsmoothed) Risk Neutral Parameters-Models 3 $a = 1,6$

Index	μ_j	λ_j	σ_j	α_j	β_j
Convertible Arbitrage	0,1427	-0,0898	0,0770	1,0301	0,2299
Dedicated Short Bias	-0,4588	0,4620	0,1695	1,0101	0,0427
Emerging Markets	0,2275	-0,1690	0,1927	1,0148	0,3606
Equity Market Neutral	0,0629	-0,0224	0,0330	0,9989	0,0659
Event Driven	0,1522	-0,0678	0,0617	1,1324	1,0221
ED Distressed	0,1652	-0,0779	0,0713	1,1720	0,9476
ED Multi-Strategy	0,1477	-0,0735	0,0696	1,0993	0,7029
ED Risk Arbitrage	0,0991	-0,0479	0,0484	1,0494	0,3273

Table A.11:(Smoothed) Brownian Motions Implied Correlations

ρ_{jk}	CA	DSB	EM	EMN	ED	D	MS	RA
CA	1,00	-0,27	0,29	0,35	0,52	0,43	0,52	0,31
DSB	-0,27	1,00	-0,54	-0,34	-0,72	-0,68	-0,57	-0,49
EM	0,29	-0,54	1,00	0,28	0,74	0,63	0,69	0,42
EMN	0,35	-0,34	0,28	1,00	0,53	0,47	0,41	0,30
ED	0,52	-0,72	0,74	0,53	1,00	0,82	0,86	0,66
D	0,43	-0,68	0,63	0,47	0,82	1,00	0,67	0,53
MS	0,52	-0,57	0,69	0,41	0,86	0,67	1,00	0,60
RA	0,31	-0,49	0,42	0,30	0,66	0,53	0,60	1,00

Table A.12:(Unsmoothed) Brownian Motions Implied Correlations

ρ_{jk}	CA	DSB	EM	EMN	ED	D	MS	RA
CA	1,00	-0,38	0,31	0,30	0,50	0,47	0,56	0,35
DSB	-0,38	1,00	-0,59	-0,37	-0,64	-0,73	-0,63	-0,51
EM	0,31	-0,59	1,00	0,35	0,63	0,61	0,69	0,45
EMN	0,30	-0,37	0,35	1,00	0,48	0,50	0,44	0,28
ED	0,50	-0,64	0,63	0,48	1,00	0,72	0,72	0,58
D	0,47	-0,73	0,61	0,50	0,72	1,00	0,60	0,51
MS	0,56	-0,63	0,69	0,44	0,72	0,60	1,00	0,61
RA	0,35	-0,51	0,45	0,28	0,58	0,51	0,61	1,00

Appendix B

Tables: Equity and Debt Tranches Fair Prices

Table B.1:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Equity and Three Zero Coupon Bonds

Model 1: Independent Brownian Motions (Smoothed Data)

Fund of Hedge funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Benchmark $\nu = 0,3333$				
Prices	178,641	570	150,281	101,078
Minimum	0	551,879	0	0
Num. Losses	25106	1	138	1749
Half Variances				
Prices	177,754	570	150,353	101,894
Minimum	0	570	33,488	0
Num. Losses	25044	0	43	905
Double Variances				
Prices	180,778	569,995	149,973	99,254
Minimum	0	478,205	0	0
Num. Losses	25427	7	509	3348
Double Skewnesses				
Prices	179,373	569,999	150,196	100,433
Minimum	0	532,191	0	0
Num. Losses	25206	3	240	2361
Opposite Skewnesses				
Prices	177,093	570	150,375	102,531
Minimum	0	570	150,375	68,888
Num. Losses	25958	0	0	2

Table B.2:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Equity and Three Zero Coupon Bonds

Mode 2: Dependent Brownian Motions (Smoothed Data)

Fund of Hedge funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Benchmark $\nu = 0,3333$				
Prices	179,036	570	150,278	100,686
Minimum	0	560,854	0	0
Num. Losses	25860	1	150	2237
Half Variances				
Prices	177,871	570	150,354	101,775
Minimum	0	570	38,482	0
Num. Losses	25579	0	42	1126
Double Variances				
Prices	182,092	569,992	149,845	98,071
Minimum	0	491,691	0	0
Num. Losses	26227	12	722	4386
Double Skewnesses				
Prices	179,661	570	150,219	100,113
Minimum	0	554,721	0	0
Num. Losses	25921	1	249	2768
Opposite Skewnesses				
Prices	177,106	570	150,375	102,515
Minimum	0	570	150,375	18,874
Num. Losses	26308	0	0	36

Table B.3:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Equity and Three Zero Coupon Bonds

Model 1: Independent Brownian Motions (**Smoothed Data**)

Fund of Hedge funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Benchmark $\nu = 0,5833$				
Prices	179,968	569,998	150,102	99,932
Minimum	0	507,777	0	0
Num. Losses	24659	6	356	2734
Half Variances				
Prices	178,668	570	150,273	101,059
Minimum	0	570	30,955	0
Num. Losses	24544	0	157	1709
Double Variances				
Prices	182,514	569,984	149,556	97,945
Minimum	0	496,739	0	0
Num. Losses	24930	32	854	4237
Double Skewnesses				
Prices	181,261	569,993	149,851	98,895
Minimum	0	527,943	0	0
Num. Losses	24737	15	600	3545
Opposite Skewnesses				
Prices	177,336	570	150,374	102,29
Minimum	0	570	106,936	0
Num. Losses	24830	0	4	386

Table B.4:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Equity and Three Zero Coupon Bonds

Model 2:Dependent Brownian Motions (Smoothed Data)

Fund of Hedge funds	EQUITY TRANCHE	ZCB A TRANCHE	ZCB B TRANCHE	ZCB C TRANCHE
Benchmark $\nu = 0,5833$				
Prices	180,444	570	150,072	99,484
Minimum	0	569,995	0	0
Num. Losses	25515	1	391	3208
Half Variances				
Prices	178,831	570	150,283	100,886
Minimum	0	570	26,2	0
Num. Losses	25146	0	149	1946
Double Variances				
Prices	184,051	569,981	149,334	96,633
Minimum	0	484,994	0	0
Num. Losses	25944	34	1121	5357
Double Skewnesses				
Prices	181,497	569,996	149,872	98,634
Minimum	0	530,132	0	0
Num. Losses	25364	11	609	3862
Opposite Skewnesses				
Prices	177,557	570	150,367	102,077
Minimum	0	570	73,996	0
Num. Losses	25546	0	26	696

Table B.5:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

Model 1: Independent Brownian Motions (Smoothed Data)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,3333$				
Prices (0% Div.)	178,339	570	150,284	101,304
Prices (50% Div.)	178,439	570	150,276	101,165
Prices (100% Div.)	178,623	570	150,264	100,978
Half Variances				
Prices (0% Div.)	177,685	570	150,336	101,914
Prices (50% Div.)	177,707	570	150,334	101,86
Prices (100% Div.)	177,773	570	150,332	101,785
Double Variances				
Prices (0% Div.)	180,058	569,997	150,008	99,852
Prices (50% Div.)	180,395	569,997	149,969	99,491
Prices (100% Div.)	180,914	569,997	149,915	99,004
Double Skewnesses				
Prices (0% Div.)	178,906	570	150,212	100,799
Prices (50% Div.)	179,078	570	150,196	100,587
Prices (100% Div.)	179,386	570	150,173	100,285
Opposite Skewnesses				
Prices (0% Div.)	177,308	570	150,347	102,317
Prices (50% Div.)	177,298	570	150,347	102,317
Prices (100% Div.)	177,295	570	150,347	102,317

Table B.6:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

Model 2: Dependent Brownian Motions (Smoothed Data)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,3333$				
Prices (0% Div.)	178,539	570	150,268	101,134
Prices (50% Div.)	178,714	570	150,258	100,954
Prices (100% Div.)	179,009	570	150,244	100,668
Half Variances				
Prices (0% Div.)	177,749	570	150,336	101,863
Prices (50% Div.)	177,793	570	150,333	101,808
Prices (100% Div.)	177,884	570	150,33	101,716
Double Variances				
Prices (0% Div.)	180,788	569,998	149,891	99,253
Prices (50% Div.)	181,381	569,997	149,842	98,693
Prices (100% Div.)	182,267	569,994	149,762	97,884
Double Skewnesses				
Prices (0% Div.)	178,990	570	150,213	100,727
Prices (50% Div.)	179,245	570	150,197	100,462
Prices (100% Div.)	179,659	570	150,173	100,064
Opposite Skewnesses				
Prices (0% Div.)	177,329	570	150,347	102,312
Prices (50% Div.)	177,332	570	150,347	102,312
Prices (100% Div.)	177,333	570	150,347	102,310

Table B.7:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

Model 1: Independent Brownian Motions (**Smoothed Data**)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,5833$				
Prices (0% Div.)	179,249	569,998	150,117	100,537
Prices (50% Div.)	179,545	569,998	150,095	100,239
Prices (100% Div.)	179,927	569,998	150,061	99,881
Half Variances				
Prices (0% Div.)	178,296	570	150,265	101,352
Prices (50% Div.)	178,439	570	150,259	101,196
Prices (100% Div.)	178,644	570	150,250	100,993
Double Variances				
Prices (0% Div.)	181,294	569,980	149,691	98,921
Prices (50% Div.)	181,924	569,978	149,609	98,345
Prices (100% Div.)	182,729	569,977	149,498	97,637
Double Skewnesses				
Prices (0% Div.)	180,328	569,994	149,907	99,655
Prices (50% Div.)	180,821	569,994	149,856	99,187
Prices (100% Div.)	181,433	569,994	149,786	98,634
Opposite Skewnesses				
Prices (0% Div.)	177,413	570	150,345	102,188
Prices (50% Div.)	177,415	570	150,345	102,174
Prices (100% Div.)	177,432	570	150,345	102,153

Table B.8:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

Model 2:Dependent Brownian Motions (**Smoothed Data**)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,5833$				
Prices (0% Div.)	179,786	569,998	150,086	100,049
Prices (50% Div.)	180,215	569,997	150,044	99,632
Prices (100% Div.)	180,816	569,997	149,987	99,075
Half Variances				
Prices (0% Div.)	178,488	570	150,273	101,164
Prices (50% Div.)	178,677	570	150,261	100,964
Prices (100% Div.)	178,966	570	150,244	100,682
Double Variances				
Prices (0% Div.)	182,804	569,979	149,390	97,737
Prices (50% Div.)	183,784	569,975	149,245	96,865
Prices (100% Div.)	185,095	569,970	149,032	95,757
Double Skewnesses				
Prices (0% Div.)	180,661	569,995	149,900	99,346
Prices (50% Div.)	181,255	569,994	149,833	98,786
Prices (100% Div.)	182,050	569,992	149,743	98,068
Opposite Skewnesses				
Prices (0% Div.)	177,569	570	150,340	102,047
Prices (50% Div.)	177,601	570	150,339	102,003
Prices (100% Div.)	177,668	570	150,337	101,933

Table B.9:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

4 Models with barrier 105% (**Smoothed Data**)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
M.G.B.M.				
Prices (0% Div.)	177,265	569,974	150,229	102,022
Prices (50% Div.)	177,282	569,974	150,226	102,014
Prices (100% Div.)	177,274	569,972	150,212	101,979
Model 1				
Prices (0% Div.)	176,547	568,242	148,111	96,218
Prices (50% Div.)	176,443	568,073	147,725	95,291
Prices (100% Div.)	176,309	567,800	147,131	93,927
Model 2				
Prices (0% Div.)	176,354	567,715	147,503	94,611
Prices (50% Div.)	176,199	567,475	146,921	93,294
Prices (100% Div.)	175,950	567,046	146,111	91,319
Model 3				
Prices (0% Div.)	176,907	569,472	149,463	99,945
Prices (50% Div.)	176,841	569,416	149,281	99,521
Prices (100% Div.)	176,793	569,322	149,069	99,014

Table B.10:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying Dividend Equity and Three Coupon Bonds

4 Models with barrier 105% (**Unsmoothed Data**)

Fund of Hedge funds	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
M.G.B.M.				
Prices (0% Div.)	176,700	569,529	149,266	99,487
Prices (50% Div.)	176,576	569,475	148,910	98,769
Prices (100% Div.)	176,504	569,069	148,877	98,459
Model 1				
Prices (0% Div.)	176,673	566,882	146,735	92,738
Prices (50% Div.)	176,557	565,982	146,010	91,083
Prices (100% Div.)	176,395	565,982	145,053	88,835
Model 2				
Prices (0% Div.)	176,943	565,868	145,499	90,076
Prices (50% Div.)	176,843	565,405	144,507	87,884
Prices (100% Div.)	176,625	564,645	142,999	84,520
Model 3				
Prices (0% Div.)	176,631	568,700	148,608	97,458
Prices (50% Div.)	176,495	568,579	148,263	96,637
Prices (100% Div.)	176,371	568,346	147,805	95,503

Table B.11:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds

CFO tranche prices with barriers and liquidity profile

Model 1: Independent Brownian Motions (**Smoothed Data**)

Collateral NAV and Debt Ratio	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,3333$				
(105%)	176,443	568,073	147,725	95,291
(100%)	178,141	569,357	148,410	99,280
(95%)	178,338	569,806	149,351	100,854
Half Variances				
(105%)	176,564	569,054	148,952	98,461
(100%)	177,581	569,769	149,523	101,052
(95%)	177,654	569,943	149,996	101,774
Double Variances				
(105%)	176,637	566,035	145,380	89,697
(100%)	179,654	568,250	145,899	95,418
(95%)	180,201	569,318	147,379	98,400
Double Skewnesses				
(105%)	176,364	567,251	146,836	93,037
(100%)	178,572	568,967	147,479	97,770
(95%)	178,920	569,650	148,690	100,004
Opposite Skewnesses				
(105%)	177,289	569,996	150,330	102,279
(100%)	177,296	570,000	150,347	102,317
(95%)	177,296	570,000	150,347	102,317

Table B.12:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds

CFO tranche prices with barriers and liquidity profile

Model 2: Dependent Brownian Motions (Smoothed Data)

Collateral NAV and Debt Ratio	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,3333$				
(105%)	176,199	567,475	146,921	93,294
(100%)	178,420	569,099	147,808	98,332
(95%)	178,682	569,733	148,951	100,337
Half Variances				
(105%)	176,495	568,797	148,650	97,638
(100%)	177,649	569,676	149,312	100,664
(95%)	177,735	569,924	149,954	101,646
Double Variances				
(105%)	176,626	564,732	143,618	85,940
(100%)	180,496	567,556	144,120	92,884
(95%)	181,358	568,950	146,050	96,869
Double Skewnesses				
(105%)	176,081	566,754	146,118	91,290
(100%)	178,825	568,760	147,005	97,114
(95%)	179,181	569,567	148,381	99,595
Opposite Skewnesses				
(105%)	177,241	569,963	150,268	102,099
(100%)	177,305	569,999	150,332	102,294
(95%)	177,306	570,000	150,347	102,313

Table B.13:

Asset Side 1000: Fund of Hedge Funds

Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds

CFO tranche prices with barriers and liquidity profile

Model 1: Independent Brownian Motions (**Smoothed Data**)

Collateral NAV and Debt Ratio	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,5833$				
(105%)	177,099	567,153	146,950	93,704
(100%)	179,021	568,746	147,292	97,667
(95%)	179,352	569,481	148,270	99,699
Half Variances				
(105%)	176,785	568,004	147,641	95,388
(100%)	178,376	569,270	148,302	99,175
(95%)	178,557	569,747	149,153	100,676
Double Variances				
(105%)	178,000	564,900	143,755	87,171
(100%)	181,243	567,363	144,095	93,077
(95%)	182,021	568,679	145,582	96,579
Double Skewnesses				
(105%)	177,462	565,697	144,801	89,175
(100%)	180,335	567,921	145,285	94,742
(95%)	180,905	569,035	146,700	97,787
Opposite Skewnesses				
(105%)	177,007	569,584	149,671	100,506
(100%)	177,442	569,920	149,987	101,792
(95%)	177,460	569,984	150,220	102,104

Table B.14:
Asset Side 1000: Fund of Hedge Funds
Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds
CFO tranche prices with barriers and liquidity profile
Model 1: Dependent Brownian Motions (Smoothed Data)

Collateral NAV and Debt Ratio	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
Benchmark $\nu = 0,5833$				
(105%)	177,111	566,021	145,182	89,681
(100%)	179,861	568,182	145,857	95,451
(95%)	180,327	569,237	147,301	98,419
Half Variances				
(105%)	176,812	567,498	147,130	93,929
(100%)	178,605	569,047	147,841	98,422
(95%)	178,805	569,693	148,926	100,313
Double Variances				
(105%)	178,732	563,285	141,718	83,014
(100%)	182,797	566,363	142,037	89,987
(95%)	183,823	568,115	143,725	94,408
Double Skewnesses				
(105%)	177,716	565,138	144,150	87,659
(100%)	180,774	567,602	144,607	93,675
(95%)	181,379	568,894	146,204	97,212
Opposite Skewnesses				
(105%)	176,763	569,106	149,120	98,864
(100%)	177,625	569,784	149,636	101,204
(95%)	177,677	569,953	150,056	101,852

Table B.15:
Asset Side 1000: Fund of Hedge Funds
Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds
CFO tranche prices with barrier (105%) and management fees
(**Smoothed Data**)

MODEL	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
M.G.B.Motion				
Prices with fees	154,977	569,912	149,994	101,439
(Prices with no fees)	(177,282)	(569,974)	(150,226)	(102,014)
Model 1 $\nu = 0,333$				
Prices with fees	154,894	567,517	146,788	92,873
(Prices with no fees)	(176,443)	(568,073)	(147,725)	(95,291)
Model 2 $\nu = 0,333$				
Prices with fees	154,837	566,788	145,762	90,356
(Prices with no fees)	(176,199)	(567,475)	(146,921)	(93,294)
Model 3				
Prices with fees	154,768	569,101	148,693	97,973
(Prices with no fees)	(176,841)	(569,416)	(149,281)	(99,521)

Table B.16:
Asset Side 1000: Fund of Hedge Funds
Liability Side 1000: Paying (50%) Dividend Equity and Three Coupon Bonds
CFO tranche prices with barrier (105%) and management fees
(**Unsmoothed Data**)

MODEL	EQUITY TRANCHE	CB A TRANCHE	CB B TRANCHE	CB C TRANCHE
M.G.B.Motion				
Prices with fees	154,317	569,185	148,217	97,010
(Prices with no fees)	(176,700)	(569,529)	(149,266)	(99,487)
Model 1 $\nu = 0,333$				
Prices with fees	155,638	565,833	144,838	88,181
(Prices with no fees)	(176,557)	(566,552)	(146,010)	(91,083)
Model 2 $\nu = 0,333$				
Prices with fees	156,393	564,549	143,148	84,570
(Prices with no fees)	(176,843)	(565,405)	(144,507)	(87,883)
Model 3				
Prices with fees	154,647	568,066	147,367	94,273
(Prices with no fees)	(176,495)	(568,578)	(148,263)	(96,637)

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