

## PRICING OF SPREAD OPTIONS ON A BIVARIATE JUMP MARKET AND STABILITY TO MODEL RISK

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ABSTRACT. We study the pricing of spread options. We consider a bivariate jump-diffusion model for the price process and we obtain a Margrabe type formula for the evaluation of the spread option. Moreover, we consider models in which we approximate the small jumps of the bivariate jump-diffusion by a two-dimensional Brownian motion scaled with the standard deviation of the small jumps. We prove the robustness of the spread option to such model risk. We illustrate our computations by several examples.

### 1. INTRODUCTION

Recent considerations in finance have led to an increasing interest in multidimensional models with jumps taking the dependence between components into account (see for instance Cont and Tankov [10]). In this context one is interested in finding closed-form formulas for option prices written in such models such as the spread options. A spread option is an option written on the difference of two underlying assets  $S^{(2)}(t) - S^{(1)}(t)$ ,  $t \geq 0$ .

In this paper we analyse the pricing and stability to model risk of spread options of European call type written in a bivariate jump-diffusion market. Thus, the pay-off function at maturity date  $T$  and with strike 0 takes the form

$$\max(S^{(2)}(T) - S^{(1)}(T), 0),$$

where  $(S^{(1)}(t), S^{(2)}(t))_{t \geq 0}$  is a bivariate jump-diffusion model for the price processes. We prove a Margrabe type formula for this spread option. The Margrabe formula is based on an appropriate change of measure which allows to move from pricing the spread option written on a bivariate process to pricing a European option written on a one-dimensional process (see Margrabe [20] and Carmona and Durrleman [11] for spread options in continuous models). In our computations we use the Girsanov theorem to derive formulas for the spread option price. Moreover, we effectively apply our approach to study robustness of the price towards model risk in the sense of small-jump approximations. We illustrate our findings with several examples. We first compute spread option prices written in models with stochastic volatility. Moreover, we derive formulas for the spread option prices in the case the bivariate Lévy process has a NIG distribution and in the case of Merton dynamics.

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Eberlein, Papapantoleon and Shiryaev [16] studied the problem of valuation of options depending on several assets using a duality approach. In particular, they derived a formula for the valuation of spread option written on exponential semimartingales in terms of the triplet of predictable characteristics of a one-dimensional semimartingale under the dual measure. In this paper we present a different approach for the valuation of spread option. Our approach is more direct and generalises their work to exponential jump diffusions with stochastic factors including stochastic volatility models.

From the modeling point of view, one can approximate the small jumps of the jump-diffusion by a continuous martingale appropriately scaled. This was introduced by Asmussen and Rosinski [1] in the case of Lévy processes. Benth, Di Nunno, and Khedher [6] [8] studied convergence results of option prices written in one-dimensional jump-diffusion models. They also studied the robustness of the option prices after a change of measure where the measure depends on the model choice. The main contribution of this paper is to apply our Margrabe type formula to prove the robustness of the spread option prices towards model risk using one dimensional Fourier techniques. By approximating the small jumps by a two-dimensional Brownian motion appropriately scaled, we prove the rate of convergence of the spread option prices to the correct. This rate turns out to be proportional to the variance of the small jumps. Gaussian approximations of multivariate Lévy processes are studied in Cohen and Rosinski [13].

The paper is organised as follows: in Section 2 we make a short introduction to Lévy processes and state a Margrabe type formula for the spread option written on a bivariate jump-diffusion. Moreover we present several examples to illustrate our findings on the pricing of spread options. In Section 3 we prove the robustness of the spread option prices and compute the convergence rate in the case the price process is driven by a bivariate Lévy process.

## 2. PRICING OF SPREAD OPTIONS IN A JUMP-DIFFUSION FRAMEWORK

Before we derive a formula of Margrabe type in order to price a European spread call option written on assets driven by a bivariate jump-diffusion, we first recall some basic results on Lévy processes and introduce the necessary notation. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space equipped with a filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$  ( $T > 0$ ) satisfying the *usual conditions* (see Karatzas and Shreve [19]). We introduce the generic notation  $L = (L^{(1)}(t), \dots, L^{(d)}(t))^*$ ,  $0 \leq t \leq T$ , for an  $\mathbb{R}^d$ -valued Lévy process on the given probability space. Here  $.^*$  denotes the transpose of a given vector or a given matrix. We work with the right continuous version with left limits of the Lévy process and we let  $\Delta L(t) := L(t) - L(t-)$ . Denote the Lévy measure of  $L$  by  $\nu(dz)$ , satisfying

$$\int_{\mathbb{R}_0^d} \min(1, |z|^2) \nu(dz) < \infty,$$

where  $|z| = \sqrt{\sum_{i=1}^d z_i^2}$  is the canonical norm in  $\mathbb{R}^d$ . Recall that  $\nu(dz)$  is a  $\sigma$ -finite Borel measure on  $\mathbb{R}_0^d := (\mathbb{R} - \{0\})^d$ . From the Lévy-Itô decomposition of a Lévy process (see

Sato [23]),  $L$  can be written as

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$$(2.1) \quad L(t) = at + \sigma^{\frac{1}{2}}B(t) + \int_0^t \int_{|z| \geq 1} z N(ds, dz) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon \leq |z| < 1} z \tilde{N}(ds, dz)$$

for a Brownian motion  $B = (B^{(1)}(t), \dots, B^{(d)}(t))^*$  in  $\mathbb{R}^d$ , a vector  $a \in \mathbb{R}^d$  and a symmetric non-negative definite matrix  $\sigma \in \mathbb{R}^{d \times d}$ .  $N(dt, dz) = N(dt, dz_1, \dots, dz_d)$  is the Poisson random measure of  $L$  and  $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$  its compensated version. Notice here that  $\int_0^t \int_{|z| \geq 1} z N(ds, dz) = \left( \int_0^t \int_{|z| \geq 1} z_1 N(ds, dz), \dots, \int_0^t \int_{|z| \geq 1} z_d N(ds, dz) \right)^*$ . The convergence in (2.1) is  $\mathbb{P}$ - a.s and uniform on bounded time intervals. The characteristic function of an  $\mathbb{R}^d$ -valued Lévy process of the form (2.1) has the following Lévy-Khintchine representation (see Sato [23])

$$\mathbb{E}[e^{i\langle z, L(t) \rangle}] = e^{t\psi(z)},$$

where

$$\psi(z) = i \langle a, z \rangle - \frac{1}{2} \langle z, \sigma z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i \langle z, x \rangle \mathbf{1}_{|x| < 1}) \nu(dx).$$

Here  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{R}^d$ . The triplet  $(a, \sigma, \nu)$  is called the characteristic triplet of the Lévy process  $L$ .

**2.1. The Margrabe formula in a bivariate jump-diffusion framework.** In the following, we consider a spread option of European type written on the difference of two underlying assets whose values are driven by a jump-diffusion. This is an extension of Margrabe [20] and Carmona and Durrleman [11] who priced spread options when the underlying assets are driven by a Brownian Motion. The dynamics we consider below are more general. In our framework we consider a two-dimensional price process  $S$  given by the following dynamics under the measure  $\mathbb{P}$ :

$$dS(t) = S(t) \left\{ a(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0^2} \gamma(t, z) \tilde{N}(dt, dz) \right\},$$

where  $a(t) = a(t, \omega) \in \mathbb{R}^2$ ,  $\sigma(t) = \sigma(t, \omega) \in \mathbb{R}^{2 \times 2}$ , and  $\gamma(t, z) = \gamma(t, z, \omega) \in \mathbb{R}^2$  are adapted processes. Note that the equation we consider for the price process is a stochastic differential equation using as integrators the Brownian motion  $B$  and the compensated compound Poisson process  $\tilde{N}$  of the Lévy process  $L$  defined in equation (2.1), where we choose  $d = 2$ .

When written out in detail, the dynamics of the price processes  $S^{(i)}$ ,  $i = 1, 2$  get the form

$$(2.2) \quad \begin{aligned} dS^{(i)}(t) = S^{(i)}(t) & \left\{ a_i(t)dt + \sigma_{i1}(t)dB^{(1)}(t) + \sigma_{i2}(t)dB^{(2)}(t) \right. \\ & \left. + \int_{\mathbb{R}_0^2} \gamma_i(t, z_1, z_2) \tilde{N}(dt, dz_1, dz_2) \right\}, \quad S^{(i)}(0) > 0. \end{aligned}$$

The coefficients of the equation (2.2) are such that  $\gamma_i(t, z_1, z_2) > -1$ ,  $i = 1, 2$ , for almost all  $\omega \in \Omega$ ,  $(t, z) \in [0, T] \times \mathbb{R}_0^2$ , and moreover, for all  $0 < t < T$ , and  $i = 1, 2$ , we assume

$$(2.3) \quad \mathbb{E} \left[ \int_0^t \left( |a_i(s)S^{(i)}(s)| + \sum_{j=1}^2 |\sigma_{ij}(s)S^{(i)}(s)|^2 + \int_{\mathbb{R}_0^2} |\gamma_i(s, z_1, z_2)S^{(i)}(s)|^2 ds \right) ds \right] < \infty, \quad \mathbb{P} - \text{a.s.}$$

The latter condition implies that the stochastic integrals are well defined and martingales.

The solution of (2.2) is the process  $(S^{(1)}(t), S^{(2)}(t))$ , explicitly given by  $S^{(i)}(t) = S^{(i)}(0) \exp(X^{(i)}(t))$ , for  $i = 1, 2$ , where  $X^{(i)}(t)$  is given by

$$\begin{aligned} dX^{(i)}(t) = & \left\{ a_i(t) - \frac{1}{2}(\sigma_{i1}^2(t) + \sigma_{i2}^2(t)) + \int_{\mathbb{R}_0^2} \ln(1 + \gamma_i(t, z_1, z_2)) - \gamma_i(t, z_1, z_2) \nu(dz_1, dz_2) \right\} dt \\ & + \sigma_{i1}(t) dB^{(1)}(t) + \sigma_{i2}(t) dB^{(2)}(t) + \int_{\mathbb{R}_0^2} \ln(1 + \gamma_i(t, z_1, z_2)) \tilde{N}(dt, dz_1, dz_2). \end{aligned}$$

Hereafter we detail the following Girsanov-type measure change, which will be useful in the sequel.

**Lemma 2.1.** *Define the measure  $\tilde{\mathbb{P}}$  by the Radon-Nikodym derivative with respect to  $\mathbb{P}$  given on the  $\sigma$ -algebra  $\mathcal{F}_T$  as follows*

$$(2.4) \quad \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp(Y(t)), \quad 0 \leq t \leq T,$$

where

$$\begin{aligned} Y(t) = & -\frac{1}{2} \int_0^t (\sigma_{11}^2(s) + \sigma_{12}^2(s)) ds + \int_0^t \sigma_{11}(s) dB^{(1)}(s) + \int_0^t \sigma_{12}(s) dB^{(2)}(s) \\ & + \int_0^t \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(s, z_1, z_2)) - \gamma_1(s, z_1, z_2) \nu(dz_1, dz_2) ds \\ (2.5) \quad & + \int_0^t \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(s, z_1, z_2)) \tilde{N}(ds, dz_1, dz_2), \end{aligned}$$

satisfying

$$(2.6) \quad \mathbb{E}[\exp(Y(T))] = 1.$$

Thus the processes  $B_{\tilde{\mathbb{P}}}^{(1)}$  and  $B_{\tilde{\mathbb{P}}}^{(2)}$  defined by

$$dB_{\tilde{\mathbb{P}}}^{(1)}(t) = -\sigma_{11}(t)dt + dB^{(1)}(t)$$

$$dB_{\tilde{\mathbb{P}}}^{(2)}(t) = -\sigma_{12}(t)dt + dB^{(2)}(t)$$

remain Brownian motions with respect to  $\tilde{\mathbb{P}}$  and

$$(2.7) \quad \tilde{N}_{\tilde{\mathbb{P}}}(dt, dz_1, dz_2) = -\gamma_1(t, z_1, z_2) \nu(dz_1, dz_2) dt + \tilde{N}(dt, dz_1, dz_2)$$

remains a compensated (time-inhomogeneous) Poisson random measure under  $\tilde{\mathbb{P}}$ . We denote

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$$\nu_{\tilde{\mathbb{P}}}(dt, dz_1, dz_2) := -\gamma_1(t, z_1, z_2)\nu(dz_1, dz_2)dt$$

*Proof.* Recall the expression of  $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$  and notice that

$$d(e^{Y(t)}) = e^{Y(t)} \left\{ \sigma_{11}(t)dB^{(1)}(t) + \sigma_{12}(t)dB^{(2)}(t) + \int_{\mathbb{R}_0^2} \gamma_1(t, z_1, z_2)\tilde{N}(dt, dz_1, dz_2) \right\}.$$

Since the condition (2.6) is fulfilled,  $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$ ,  $t \leq T$ , is a martingale and the lemma follows from the Girsanov theorem for Lévy processes (Theorem 1.35 in Øksendal and Sulem [21]).  $\square$

**Remark 2.2.** Notice that the price of  $S^{(2)}$  expressed in the numéraire  $S^{(1)}$  is a geometric jump diffusion. In fact, Itô's formula gives

$$\begin{aligned} d\left(\frac{S^{(2)}(t)}{S^{(1)}(t)}\right) &= \frac{S^{(2)}(t)}{S^{(1)}(t)} \left\{ (a_2(t) - a_1(t) + \sigma_{11}^2(t) + \sigma_{12}^2(t) - \sigma_{11}(t)\sigma_{21}(t) - \sigma_{12}(t)\sigma_{22}(t))dt \right. \\ &\quad + (\sigma_{21}(t) - \sigma_{11}(t))dB^{(1)}(t) + (\sigma_{22}(t) - \sigma_{12}(t))dB^{(2)}(t) \\ &\quad + \left( \int_{\mathbb{R}_0^2} \frac{1 + \gamma_2(t, z_1, z_2)}{1 + \gamma_1(t, z_1, z_2)} - 1 + (\gamma_1(t, z_1, z_2) - \gamma_2(t, z_1, z_2))\mathbf{1}_{|z| < 1} \nu(dz_1, dz_2) \right) dt \\ &\quad \left. + \int_{\mathbb{R}_0^2} \left( \frac{1 + \gamma_2(t, z_1, z_2)}{1 + \gamma_1(t, z_1, z_2)} - 1 \right) \tilde{N}(dt, dz_1, dz_2) \right\}. \end{aligned}$$

This remains a geometric jump diffusion also after applying the measure change (2.4). In fact, we have

$$\begin{aligned} d\left(\frac{S^{(2)}(t)}{S^{(1)}(t)}\right) &= \frac{S^{(2)}(t)}{S^{(1)}(t)} \left\{ (a_2(t) - a_1(t) + \sigma_{21}(t) - \sigma_{11}(t))dB_{\tilde{\mathbb{P}}}^{(1)}(t) + (\sigma_{22}(t) - \sigma_{12}(t))dB_{\tilde{\mathbb{P}}}^{(2)}(t) \right. \\ &\quad + \left( \int_{|z| \geq 1} (\gamma_2(t, z_1, z_2) - \gamma_1(t, z_1, z_2))\nu(dz_1, dz_2) \right) dt \\ &\quad \left. + \int_{\mathbb{R}_0^2} \left( \frac{1 + \gamma_2(t, z_1, z_2)}{1 + \gamma_1(t, z_1, z_2)} - 1 \right) \tilde{N}_{\tilde{\mathbb{P}}}(dt, dz_1, dz_2) \right\}. \end{aligned}$$

The solution of this equation is given by

$$(2.8) \quad \frac{S^{(2)}(t)}{S^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp(Z(t)),$$

where

$$\begin{aligned} Z(t) &= \int_0^t (a_2(s) - a_1(s))ds - \frac{1}{2} \int_0^t \{ (\sigma_{21}(s) - \sigma_{11}(s))^2 - (\sigma_{22}(s) - \sigma_{12}(s))^2 \} ds \\ &\quad + \int_0^t (\sigma_{21}(s) - \sigma_{11}(s))dB_{\tilde{\mathbb{P}}}^{(1)}(s) + \int_0^t (\sigma_{22}(s) - \sigma_{12}(s))dB_{\tilde{\mathbb{P}}}^{(2)}(s) \end{aligned}$$

$$\begin{aligned}
(2.9) \quad & + \int_0^t \int_{|z| \geq 1} \left( \gamma_2(s, z_1, z_2) - \gamma_1(s, z_1, z_2) \right) \nu(dz_1, dz_2) ds \\
& + \int_0^t \int_{\mathbb{R}_0^2} \left\{ \ln \left( \frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right) + \frac{\gamma_1(s, z_1, z_2) - \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right\} \nu_{\tilde{\mathbb{P}}}(ds, dz_1, dz_2) \\
& + \int_0^t \int_{\mathbb{R}_0^2} \ln \left( \frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right) \tilde{N}_{\tilde{\mathbb{P}}}(ds, dz_1, dz_2) \Big\}.
\end{aligned}$$

Here  $\nu_{\tilde{\mathbb{P}}}(dt, dz_1, dz_2)$  is the Lévy measure of the Poisson random measure  $\tilde{N}_{\tilde{\mathbb{P}}}$ .

The spread is defined by the difference of the two underlying asset prices  $S^{(2)}(t) - S^{(1)}(t)$ ,  $t \geq 0$ . Thus, the payout function of a European spread option with strike 0 at maturity date  $T$  is given by

$$(2.10) \quad \max(S^{(2)}(T) - S^{(1)}(T), 0).$$

This means that the buyer has the right to be paid at the maturity date  $T$  the difference  $S^{(2)}(T) - S^{(1)}(T)$  whenever it is positive and zero otherwise. The financial derivative (2.10) is sometimes called an exchange option in the literature.

In the following we state a Margrabe type formula for a spread option written on a bivariate jump-diffusion (see Section 5.2 in Carmona and Durrleman [11] for spread options written on continuous process prices). We choose the risk-free instantaneous interest rate  $r(t) = r(t, \omega)$  to be an  $\mathcal{F}_t$ -adapted stochastic process which is Lebesgue integrable on any compact.

**Proposition 2.3.** *Assume that*

$$\exp \left( \int_0^T \{a_1(s) - r(s)\} ds \right) \max \left( \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0 \right)$$

is  $\tilde{\mathbb{P}}$  integrable where the measure  $\tilde{\mathbb{P}}$  is defined in (2.4). Then the price  $C$  of a spread option with strike  $K = 0$  and maturity  $T$  is given by

$$(2.11) \quad C = S^{(1)}(0) \mathbb{E}_{\tilde{\mathbb{P}}} \left[ e^{\int_0^T \{a_1(s) - r(s)\} ds} \max \left( \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0 \right) \right].$$

*Proof.* The price of a zero-exercise spread option is given by

$$\begin{aligned}
(2.12) \quad C &= \mathbb{E}_{\mathbb{P}} \left[ e^{-\int_0^T r(s) ds} \max \left( S^{(2)}(T) - S^{(1)}(T), 0 \right) \right] \\
&= \mathbb{E}_{\mathbb{P}} \left[ e^{-\int_0^T r(s) ds} \max \left( \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0 \right) S^{(1)}(T) \right].
\end{aligned}$$

Writing the spread option price under the measure  $\tilde{\mathbb{P}}$ , we get

$$C = \mathbb{E}_{\tilde{\mathbb{P}}} \left[ e^{-\int_0^T r(s) ds} \max \left( \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0 \right) S^{(1)}(T) e^{-Y(T)} \right].$$

However we know that  $S^{(1)}(T) e^{-Y(T)} = S^{(1)}(0) e^{\int_0^T a_1(s) ds}$  and the result follows.  $\square$

**Remark 2.4.** In the framework we presented above we do not suppose that the process  $S$  is a martingale under the measure  $\mathbb{P}$ . Generally, in a financial setting, this would be the case as we are interested in the arbitrage-free price of the spread option and then  $S$  would be naturally set as a martingale under  $\mathbb{P}$ . However, the formula in Proposition 2.3 can be applied to other markets where the spot price  $S$  is not tradeable, like for example electricity and weather markets. In such cases  $S$  does not have to be a martingale under the pricing measure  $\mathbb{P}$  (see Benth, Šaltytė Benth and Koekebakker [4] for more on such markets). However, in the case we want to work under a risk neutral measure, say  $\mathbb{Q} \sim \mathbb{P}$ , the price process  $S$  will be a martingale under  $\mathbb{Q}$  and thus  $a_1(t) = a_2(t) = r(t)$ , a.s. In that case we apply Proposition 2.3 with  $S$  under  $\mathbb{Q}$  to find the price of the spread option as

$$C = S^{(1)}(0)\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\max\left(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0\right)\right],$$

where  $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\{Y(t)\}$  and the process  $Y$  is given by equation (2.5). Note that in fact  $\exp\{Y(t)\} = \frac{S^{(1)}(T)}{S^{(0)}(T)}e^{-\int_0^T r(s)ds}$ . The measure  $\tilde{\mathbb{Q}}$  with respect to the real world measure  $\mathbb{P}$  can be defined through  $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\frac{d\mathbb{Q}}{d\mathbb{P}}$ . We develop these arguments further in section 2.3 using the Esscher transform and we give an example where the logreturns follow a normal inverse Gaussian process in section 2.3.1.

**2.2. Application: The case of stochastic volatility.** We apply our result to a model for  $S$  with stochastic volatility. We specify the volatility model as a bivariate dynamics of the Barndorff-Nielsen and Shephard model (see Barndorff-Nielsen and Shephard [3]).

We consider the following stochastic process for  $S$ :

$$\begin{pmatrix} S^{(1)} \\ S^{(2)} \end{pmatrix} = \begin{pmatrix} a_1(t)S^{(1)}(t) \\ a_2(t)S^{(2)}(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1(t)S^{(1)}(t) & 0 \\ 0 & \sigma_2(t)S^{(2)}(t) \end{pmatrix} \begin{pmatrix} dB^{(1)}(t) \\ dB^{(2)}(t) \end{pmatrix},$$

$$(2.13) \quad d\sigma_i^2(t) = -\lambda_i\sigma_i^2(t)dt + dL^{(i)}(t), \quad \sigma_i^2(0) \geq 0, \quad i = 1, 2,$$

where  $\lambda_1$  and  $\lambda_2$  are positive constants and  $L = (L^{(1)}, L^{(2)})$  is a two dimensional subordinator process, that is, a two-dimensional Lévy process which is non decreasing in each of its coordinates. Note that  $L^{(1)}$  and  $L^{(2)}$  can be dependent. We assume for simplicity that  $B^{(1)}$  and  $B^{(2)}$  are independent. Note that a subordinator has paths of finite variation since it is monotonely increasing. It therefore has to be independent of  $B^{(1)}$  and  $B^{(2)}$ , which are processes with paths of infinite variation. Moreover, suppose that the Lévy process  $L$  has no deterministic drift and the Lévy measure has density  $\omega(z_1, z_2)$ , so that the cumulant functions  $\kappa_i(\theta) := \log \mathbb{E}[e^{\theta L^{(i)}(1)}]$ ,  $i = 1, 2$ , where they exist, take the form

$$\kappa_i(\theta) = \int_{\mathbb{R}_+^2} (e^{\theta z_i} - 1)\omega(z_1, z_2)dz_1dz_2.$$

The solution of (2.13) is given by

$$\sigma_i^2(t) = e^{-\lambda_i t}\sigma_i^2(0) + \int_0^t e^{-\lambda_i(t-s)}dL^{(i)}(s), \quad i = 1, 2.$$

We denote the integrated variance over the time period  $[0, T]$  by  $\sigma_i^{*2}(T) := \int_0^T \sigma_i^2(t)dt$ . A simple computation shows that

$$(2.14) \quad \sigma_i^{*2}(T) = \sigma_i^2(0)(1 - e^{-\lambda_i T})\lambda_i^{-1} + \int_0^T (1 - e^{-\lambda_i(T-u)})\lambda_i^{-1}dL_i(u), \quad i = 1, 2.$$

We assume that the price processes  $S^{(1)}$  and  $S^{(2)}$  have risk neutral dynamics. Thus we have  $a_1(t) = a_2(t) = r(t)$  (see Remark 2.4). The risk neutral valuation of the spread option price is given by

$$C = \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^T r(s)ds} \max(S^{(2)}(T) - S^{(1)}(T), 0) \right],$$

where  $\mathbb{Q}$  is the risk neutral probability density. We define the measure  $\tilde{\mathbb{Q}}$  by

$$(2.15) \quad \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} = \exp\{Y(t)\}, \quad t \leq T,$$

where

$$Y(t) = -\frac{1}{2} \int_0^t (\sigma_1^2(s) + \sigma_2^2(s))ds + \int_0^t \sigma_1(s)dB^{(1)}(s) + \int_0^t \sigma_2(s)dB^{(2)}(s).$$

From Lemma 2.1, we know that

$$\begin{aligned} dB_{\tilde{\mathbb{Q}}}^{(1)}(t) &= -\sigma_1(t)dt + dB^{(1)}(t), \\ dB_{\tilde{\mathbb{Q}}}^{(2)}(t) &= dB^{(2)}(t), \end{aligned}$$

remain Brownian motions under the measure  $\tilde{\mathbb{Q}}$ . Moreover, notice that the Lévy processes  $L^{(1)}$  and  $L^{(2)}$  remain Lévy processes under the new measure  $\tilde{\mathbb{Q}}$ . In fact  $\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{i\langle \theta, L(t) \rangle}] = \mathbb{E}[e^{i\langle \theta, L(t) \rangle}]$ . To explain, we have

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}[e^{i\langle \theta, L(t) \rangle}] &= \mathbb{E} \left[ e^{i\langle \theta, L(t) \rangle} \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} \right] \\ &= \mathbb{E} \left[ e^{i\langle \theta, L(t) \rangle} \exp\left\{-\frac{1}{2} \int_0^t (\sigma_1^2(s) + \sigma_2^2(s))ds + \int_0^t \sigma_1(s)dB^{(1)}(s) \right. \right. \\ &\quad \left. \left. + \int_0^t \sigma_2(s)dB^{(2)}(s)\right\} \right]. \end{aligned}$$

Denote by  $\sigma(L)$  the  $\sigma$ -algebra generated by  $L$  up to time  $T$ . Therefore conditioning on  $\sigma(L)$ , we get

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}[e^{i\langle \theta, L(t) \rangle}] &= \mathbb{E} \left[ \mathbb{E} \left[ e^{i\langle \theta, L(t) \rangle} \exp\left\{-\frac{1}{2} \int_0^t (\sigma_1^2(s) + \sigma_2^2(s))ds + \int_0^t \sigma_1(s)dB^{(1)}(s) \right. \right. \right. \\ &\quad \left. \left. + \int_0^t \sigma_2(s)dB^{(2)}(s)\right\} \Big| \sigma(L) \right] \right] \\ &= \mathbb{E} \left[ e^{i\langle \theta, L(t) \rangle} \exp\left\{-\frac{1}{2} \int_0^t (\sigma_1^2(s) + \sigma_2^2(s))ds\right\} \exp\left\{\frac{1}{2} \int_0^t (\sigma_1^2(s) + \sigma_2^2(s))ds\right\} \right] \end{aligned}$$



$$= \mathbb{E}[e^{i\langle \theta, L(t) \rangle}].$$

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We obtain the following lemma for the price of the spread option.

**Proposition 2.5.** *Let  $f(x) = \max((e^x - 1), 0)$  and  $\widehat{f}$  be the Fourier transform of  $f$ . Then for  $R \in \mathbb{R}$ , the price of the spread option written on  $S$  is given by*

$$C = \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} \widehat{f}(u + iR) \exp\left(-\frac{1}{2}(iu - R - (iu - R)^2)\sigma_1^2(0)(1 - e^{-\lambda_1 T})\lambda_1^{-1}\right) \\ \exp\left(\frac{1}{2}(iu - R + (iu - R)^2)\sigma_2^2(0)(1 - e^{-\lambda_2 T})\lambda_2^{-1}\right) \\ \exp\left(\int_0^T \{\kappa_1(g_1(u, s)) + \kappa_2(g_2(u, s))\} ds\right) du,$$

where  $g_1(u, s) = -\frac{1}{2}(iu - R - (iu - R)^2)(1 - e^{-\lambda_1(T-s)})\lambda_1^{-1}$ ,  $g_2(u, s) = -\frac{1}{2}(iu - R + (iu - R)^2)(1 - e^{-\lambda_2(T-s)})\lambda_2^{-1}$ , and  $\kappa_1$  and  $\kappa_2$  are the cumulant functions.

We recall the following theorem in which the price of an option is written in terms of the Fourier transform of the pay-off function. For the proof we refer to Eberlein, Glau, and Papapantoleon [15]. We use this theorem in our computations hereafter.

**Theorem 2.6.** *Let  $X$  be a jump-diffusion in  $\mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a payoff function. We denote by  $P_{X(T)}(dx)$  the probability of  $X(T)$  and by  $\widehat{f}$  the Fourier transform of  $f$ . Assume that for  $R \in \mathbb{R}$  we have*

- (1)  $e^{-Rx} f(x) \in L^1(\mathbb{R})$ ,
- (2)  $e^{-Rx} \widehat{f}(x) \in L^1(\mathbb{R})$ ,
- (3)  $e^{Rx} P_{X(T)}(dx) \in L^1(\mathbb{R})$ .

Thus we have

$$\mathbb{E}[f(X(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{E}[e^{-i(u+iR)X(T)}] \widehat{f}(u + iR) du.$$

*Proof of Proposition 2.5.* From Proposition 2.3 and Remark 2.4, the risk neutral formula for the spread option price is given by

$$C = S^{(1)}(0) \mathbb{E}_{\widetilde{\mathbb{Q}}} \left[ \max\left(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0\right) \right].$$

Here  $\widetilde{\mathbb{Q}}$  is defined in (2.15) and  $\frac{S^{(2)}(t)}{S^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp(Z(t))$ , where

$$Z(t) = \frac{1}{2} \int_0^t (\sigma_1^2(s) - \sigma_2^2(s)) ds - \int_0^t \sigma_1(s) dB_{\widetilde{\mathbb{Q}}}^{(1)}(s) + \int_0^t \sigma_2(s) dB_{\widetilde{\mathbb{Q}}}^{(2)}(s).$$

Notice that the option price takes the form  $C = S^{(1)}(0) \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(Z(T))]$ . Thus from Theorem 2.6, we have

$$C = \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} \widehat{f}(iR + u) \mathbb{E}_{\widetilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}] du.$$

Therefore to compute the option price  $C$  we need to compute  $\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}]$ . To this end we see that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}] = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{-i(u+iR)\left\{\frac{1}{2}\int_0^T(\sigma_1^2(s)-\sigma_2^2(s))ds - \int_0^T\sigma_1(s)dB_{\tilde{\mathbb{Q}}}^{(1)}(s) + \int_0^T\sigma_2(s)dB_{\tilde{\mathbb{Q}}}^{(2)}(s)\right\}}\right].$$

Conditioning on  $\sigma(L)$ , and recalling the expressions of  $\sigma_1^{*2}(T)$  and  $\sigma_2^{*2}(T)$  in (2.14) we get

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}] &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{\frac{iu-R}{2}(-\sigma_1^{*2}(T)+\sigma_2^{*2}(T))}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{(iu-R)\left\{\int_0^T\sigma_1(s)dB_{\tilde{\mathbb{Q}}}^{(1)}(s) - \int_0^T\sigma_2(s)dB_{\tilde{\mathbb{Q}}}^{(2)}(s)\right\}}\middle|\sigma(L)\right]\right] \\ &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{\frac{iu-R}{2}(-\sigma_1^{*2}(T)+\sigma_2^{*2}(T))}e^{\frac{(iu-R)^2}{2}(\sigma_1^{*2}(T)+\sigma_2^{*2}(T))}\right] \\ &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{-\frac{1}{2}\sigma_1^{*2}(T)(iu-R-(iu-R)^2) + \frac{1}{2}\sigma_2^{*2}(T)((iu-R)+(iu-R)^2)}\right].\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}] &= e^{-\frac{1}{2}(iu-R-(iu-R)^2)\sigma_1^2(0)(1-e^{-\lambda_1 T})\lambda_1^{-1}}e^{\frac{1}{2}(iu-R+(iu-R)^2)\sigma_2^2(0)(1-e^{-\lambda_2 T})\lambda_2^{-1}} \\ &\quad \mathbb{E}_{\tilde{\mathbb{Q}}}\left[e^{\int_0^Tg_1(u,s)dL^{(1)}(s) + \int_0^Tg_2(u,s)dL^{(2)}(s)}\right].\end{aligned}$$

Using an extension of the key formula in Eberlein and Raible [14], we have

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{Q}}}[e^{-i(u+iR)Z(T)}] &= e^{-\frac{1}{2}(iu-R-(iu-R)^2)\sigma_1^2(0)(1-e^{-\lambda_1 T})\lambda_1^{-1}}e^{\frac{1}{2}(iu-R+(iu-R)^2)\sigma_2^2(0)(1-e^{-\lambda_2 T})\lambda_2^{-1}} \\ &\quad \exp\left(\int_0^T\{\kappa_1(g_1(u,s)) + \kappa_2(g_2(u,s))\}ds\right),\end{aligned}$$

where  $\kappa_1$  and  $\kappa_2$  are the cumulant functions and the result follows.  $\square$

The computations we did in this section are based on a change of measure which allows to move from pricing a spread option written on a bivariate jump-diffusion to pricing a European option written on a one dimensional jump-diffusion dynamics. To derive such a formula, we used the Girsanov theorem. In some situations it is more convenient to consider a special type of measure transform known as the Esscher transform. We next specialize our results to the case of spread options on exponential bivariate Lévy process.

**2.3. Application: Exponential Lévy processes and Esscher transforms.** Our computations will be based on the Esscher transform of Gerber and Shiu [17] for options on several risky assets. The Esscher probability  $\mathbb{P}_\theta$  is defined by means of the Esscher transform as follows (see Gerber and Shiu [17])

$$(2.16) \quad \frac{d\mathbb{P}_\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{e^{\langle\theta, L(t)\rangle}}{\mathbb{E}_{\mathbb{P}}[e^{\langle\theta, L(t)\rangle}]}.$$

The transform depends on the parameter  $\theta \in \mathbb{R}^2$ . First, we apply an Esscher transform with parameter  $\theta$ , such that the corresponding measure  $\mathbb{P}_\theta$  is risk neutral for the price dynamics and the spread option price  $C$  can be written as expectation under  $\mathbb{P}_\theta$ . Afterwards, we apply Magrabe's formula as in Proposition 2.3 and state  $C$  as expectation under Magrabe's measure  $\tilde{\mathbb{P}}_\theta$ . Furthermore, we explore the relations between the real world measure  $\mathbb{P}$ , the risk-neutral measure  $\mathbb{P}_\theta$  and Magrabe's pricing measure  $\tilde{\mathbb{P}}_\theta$  in terms of Esscher transforms.

In fact, Margrabe's pricing measure can be specified with respect to  $\mathbb{P}$  directly through a single Esscher transform with parameter  $\theta + \mathbf{1}_1$ , where  $\mathbf{1}_1$  denotes the first unit vector.

We suppose here that the risk-free rate of return is constant, that is,  $r(t) = r$  for a positive constant  $r$  and consider a spread option written on  $S^{(1)}(t) = S^{(1)}(0)e^{L^{(1)}(t)}$  and  $S^{(2)}(t) = S^{(2)}(0)e^{L^{(2)}(t)}$ , where  $L = (L^{(1)}(t), L^{(2)}(t))$  is a bivariate Lévy process with characteristic triplet  $(a, 0, \nu)$ . Let  $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ . The moment generating function of  $L$  is given by

$$\begin{aligned} M_t(\theta) &= \mathbb{E}_{\mathbb{P}}[e^{\langle \theta, L(t) \rangle}] \\ &= \exp \left\{ t(a_1\theta_1 + a_2\theta_2 + \int_{\mathbb{R}_0^2} (e^{\langle \theta, z \rangle} - 1 - \langle \theta, z \rangle \mathbf{1}_{|z| < 1}) \nu(dz_1, dz_2)) \right\}, \end{aligned}$$

for  $\theta$  such that this exists. In order for (2.16) to be well-defined, we must assume exponential integrability conditions on  $L(1)$ . Hence, suppose that there exists a constant  $c > 0$  such that

$$(2.17) \quad \int_{\mathbb{R}^2} e^{\langle x, z \rangle} \nu(dz) < \infty,$$

for all  $|x| \leq c$ . This ensures finite exponential moments for  $L(1)$  up to order  $c$ . To get a risk neutral probability measure, the parameter  $\theta$  is determined such that, for  $i = 1, 2$ , the discounted price process  $e^{-rt}S^{(i)}(t)$  is a martingale. Hence

$$S^{(i)}(0) = \mathbb{E}_{\theta}[e^{-rt}S^{(i)}(t)]$$

which is equivalent to

$$(2.18) \quad e^{rt} = \mathbb{E}_{\theta}[e^{L^{(i)}(t)}] = \mathbb{E}_{\mathbb{P}} \left[ \frac{e^{L^{(i)}(t) + \theta L(t)}}{M_t(\theta)} \right] = \frac{M_t(\mathbf{1}_i + \theta)}{M_t(\theta)},$$

where  $\mathbf{1}_i$  denotes the  $i$ th unit vector and  $\mathbb{E}_{\theta}$  denotes the expectation under the new measure  $\mathbb{P}_{\theta}$ . The existence and uniqueness of the parameter  $\theta = (\theta_1, \theta_2)$  which verifies (2.18) is proved in Gerber and Shiu [18]. By the risk neutral valuation rule, the price of the spread option is then given by

$$C = e^{-rT} \mathbb{E}_{\theta} \left[ \max \left( \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0 \right) S^{(1)}(T) \right].$$

In order to apply Proposition 2.3, define

$$(2.19) \quad \left. \frac{d\tilde{\mathbb{P}}_{\theta}}{d\mathbb{P}_{\theta}} \right|_{\mathcal{F}_t} = e^{rt - L^{(1)}(t)}$$

according to Lemma 2.1 and Remark 2.4. Note that (2.19) corresponds to an Esscher transform with parameter  $\mathbf{1}_1$ . Furthermore, it is

$$(2.20) \quad \left. \frac{d\tilde{\mathbb{P}}_{\theta}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \left. \frac{d\tilde{\mathbb{P}}_{\theta}}{d\mathbb{P}_{\theta}} \frac{d\mathbb{P}_{\theta}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{rt - L^{(1)}(t)} \frac{e^{\langle \theta, L(t) \rangle}}{M_t(\theta)} = \frac{e^{\langle \theta + \mathbf{1}_1, L(t) \rangle}}{M_t(\theta + \mathbf{1}_1)} =: \left. \frac{d\mathbb{P}_{\theta + \mathbf{1}_1}}{d\mathbb{P}} \right|_{\mathcal{F}_t},$$

using (2.18). Thus,  $\tilde{\mathbb{P}}_\theta$  corresponds to the measure  $\mathbb{P}_{\theta+\mathbf{1}_1}$ , defined through an Esscher transform with parameter  $\theta + \mathbf{1}_1$ , with respect to  $\mathbb{P}$ . Applying Proposition 2.3 it follows therefore

$$(2.21) \quad C = S^{(1)}(0)\mathbb{E}_{\theta+\mathbf{1}_1} \left[ \max\left(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0\right) S^{(1)}(T) \right].$$

This is in accordance with the result in Gerber and Shiu [17] for options on several risky assets. By Theorems 33.1 and 33.2 in Sato [23], the new characteristic triplet of the Lévy process  $L$  under the new martingale measure  $\mathbb{P}_\theta$  is given by  $(\tilde{a}, 0, \tilde{\nu})$ , where

$$\tilde{\nu}(dz_1, dz_2) = e^{\langle \theta, z \rangle} \nu(dz_1, dz_2),$$

and

$$\tilde{a}_i = a_i + \int_{|z|<1} z_i e^{\langle \theta, z \rangle} \nu(dz_1, dz_2), \quad \text{for } i = 1, 2.$$

The characteristic triplets of the Lévy process  $L$  under the new measure  $\mathbb{P}_{\theta+\mathbf{1}_1}$  is given by  $(\hat{a}, 0, \hat{\nu})$ , where

$$\hat{\nu}(dz_1, dz_2) = e^{(\theta_1+1)z_1 + \theta_2 z_2} \nu(dz_1, dz_2)$$

and

$$\hat{a}_i = a_i + \int_{|z|<1} z_i e^{(\theta_1+1)z_1 + \theta_2 z_2} \nu(dz_1, dz_2), \quad i = 1, 2.$$

Therefore the process  $\frac{S^{(2)}(t)}{S^{(1)}(t)}$  is given by

$$(2.22) \quad \begin{aligned} \frac{S^{(2)}(t)}{S^{(1)}(t)} &= \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp \left\{ (\hat{a}_2 - \hat{a}_1)t + \int_0^t \int_{|z|<1} (z_2 - z_1) \tilde{N}_{\theta+\mathbf{1}_1}(ds, dz_1, dz_2) \right. \\ &\quad \left. + \int_0^t \int_{|z|\geq 1} (z_2 - z_1) N_{\theta+\mathbf{1}_1}(ds, dz_1, dz_2) \right\}, \end{aligned}$$

where  $N_{\theta+\mathbf{1}_1}(dt, dz_1, dz_2)$  is a Poisson random measure with Lévy measure  $\hat{\nu}(dz_1, dz_2)$ . Note that, under  $\mathbb{P}_{\theta+\mathbf{1}_1}$ , (2.22) can be written as

$$(2.23) \quad \frac{S^{(2)}(t)}{S^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp \left\{ L^{(2)}(t) - L^{(1)}(t) \right\},$$

where

$$L^{(i)}(t) = \hat{a}_i t + \int_0^t \int_{|z|<1} z_i \tilde{N}_{\theta+\mathbf{1}_1}(ds, dz_1, dz_2) + \int_0^t \int_{|z|\geq 1} z_i N_{\theta+\mathbf{1}_1}(ds, dz_1, dz_2), \quad i = 1, 2,$$

are the coordinates of a bivariate Lévy process. Hence

$$C = S^{(1)}(0)\mathbb{E}_{\theta+\mathbf{1}_1} \left[ \max \left( \frac{S^{(2)}(0)}{S^{(1)}(0)} e^{L^{(2)}(T) - L^{(1)}(T)} - 1, 0 \right) \right].$$

We now consider two examples of the application of this Esscher transform-based pricing of a spread option. First we study the case of a bivariate normal inverse Gaussian Lévy process, and afterwards we consider the so-called Merton dynamics. In both cases we can

relate the process under the pricing measure in our Margrabe formula to explicit processes which are possible to apply for analytical pricing. ØPT. OF MATH./CMA UNIV. OF OSLO

2.3.1. *Example: Normal inverse Gaussian Lévy process.* Given the parameters of the distribution of a normal inverse Gaussian (NIG) Lévy process under the real world measure  $\mathbb{P}$ , one can derive parameters under a risk neutral measure  $\mathbb{P}_\theta$  after an Esscher transform as in Benth and Henriksen [5]. The bivariate NIG distribution has parameters  $\alpha > 0$ ,  $\beta \in \mathbb{R}^2$ ,  $\mu \in \mathbb{R}^2$ ,  $\delta > 0$  and  $\Delta \in \mathbb{R}^{2 \times 2}$ , where  $\Delta$  is a positive definite matrix with determinant 1 (see Barndorff-Nielsen [2] and Rydberg [22] for more about the bivariate NIG distribution). Let  $L$  be a Lévy process such that  $L(1) \sim NIG(\alpha, \beta, \mu, \delta, \Delta)$  under  $\mathbb{P}$ . Then the density function of  $L(1)$  takes the form

$$(2.24) \quad f(z) = \frac{\delta}{\sqrt{2}} \left( \frac{\alpha}{\pi q(z)} \right)^{\frac{3}{2}} \exp(p(z)) K_{\frac{3}{2}}(\alpha q(z)),$$

where  $K_{\frac{3}{2}}$  is the modified Bessel function of second kind of order  $\frac{3}{2}$  and

$$\begin{aligned} p(z) &= \delta \sqrt{\alpha^2 - \beta^* \Delta \beta} + \beta^* (z - \mu), \\ q(z) &= \sqrt{\delta^2 + (z - \mu)^* \Delta^{-1} (z - \mu)}. \end{aligned}$$

The parameters have the following interpretation:  $\alpha$  corresponds to the tail heaviness of the marginals and  $\delta$  is the scaling of the distribution. The centering is described by  $\mu$  and  $\beta$  controls the skewness. The dependency structure between the marginals is modelled by  $\Delta$ . The cumulant function is explicitly given by

$$(2.25) \quad \Psi_L(s) = \delta \sqrt{\alpha^2 - \beta^* \Delta \beta} - \delta \sqrt{\alpha^2 - (\beta + is)^* \Delta (\beta + is)} + is^* \mu.$$

One recalls the cumulant function to be the logarithm of the characteristic function.

The price dynamics for the stocks are given by  $S^{(1)}(t) = S^{(1)}(0) \exp\{L^{(1)}(t)\}$  and  $S^{(2)}(t) = S^{(2)}(0) \exp\{L^{(2)}(t)\}$  with  $S^{(i)}(0) > 0$ ,  $i = 1, 2$ . Define a probability measure  $\mathbb{P}_\theta \sim \mathbb{P}$  for  $\theta \in \mathbb{R}^2$  through an Esscher transform as in (2.16). Calculating the characteristic function, it follows that under  $\mathbb{P}_\theta$ ,

$$L(1) \sim NIG(\alpha, \beta + \theta, \mu, \delta, \Delta).$$

We choose the parameter  $\theta$  such that we have risk neutral dynamics. This is the case when the discounted price process is a  $\mathbb{P}_\theta$  martingale, where discounting is done using the risk-free interest rate  $r > 0$ . Hence

$$\mathbb{E}_\theta[e^{-rt} S(t)] = S(0),$$

or equivalently

$$\Psi_L(-i\mathbf{1}_i; \theta) = r$$

for  $i = 1, 2$ , see (2.18). This condition turns into a system of two equations for  $\theta$ ,

$$r = \mu_1 - \delta \sqrt{\alpha^2 - [\beta_1 + 1 + \theta_1, \beta_2 + \theta_2] \Delta \begin{bmatrix} \beta_1 + 1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}}$$

$$\begin{aligned}
& + \delta \sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + \theta_2] \Delta} \begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}, \\
r = \mu_2 - \delta \sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + 1 + \theta_2] \Delta} & \begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + 1 + \theta_2 \end{bmatrix} \\
& + \delta \sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + \theta_2] \Delta} \begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}.
\end{aligned}$$

The probability measure  $\tilde{\mathbb{P}}$  defined in Lemma 2.1 and used in the Margrabe's formula in Proposition 2.3 corresponds here to the pricing measure  $\mathbb{P}_{\theta+\mathbf{1}_1}$  as in (2.20). It follows that under  $\mathbb{P}_{\theta+\mathbf{1}_1}$

$$(L^{(1)}(1), L^{(2)}(1)) \sim NIG(\alpha, \beta_{\theta+\mathbf{1}_1}, \mu, \delta, \Delta),$$

with  $\beta_{\theta+\mathbf{1}_1} = \beta + \theta + \mathbf{1}_1$ . Under  $\mathbb{P}_{\theta+\mathbf{1}_1}$ , it holds that

$$\frac{S^{(2)}(t)}{S^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp \left\{ L^{(2)}(t) - L^{(1)}(t) \right\}.$$

Observe that the cumulant of  $L^{(2)}(t) - L^{(1)}(t)$  is given as

$$\Psi_{L_2-L_1}(s) = \ln \mathbb{E}[e^{is(L^{(2)}(1)-L^{(1)}(1))}] = \Psi_{(L^{(1)}, L^{(2)})}(-s, s)$$

where  $\Psi_L(s_1, s_2)$  is given by (2.25) with  $\beta = \beta_{\theta+\mathbf{1}_1}$ . Then we have that

$$\Psi_{L_1-L_2}(s) = \tilde{\delta} \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2} - \tilde{\delta} \sqrt{\tilde{\alpha}^2 - (\tilde{\beta} + is)^2} + is\tilde{\mu}$$

with  $\tilde{\delta} = \delta \sqrt{z_1}$ ,  $\tilde{\alpha}^2 = \frac{1}{z_1}(\alpha^2 - \beta^* \Delta \beta + \tilde{\beta}^2)$ ,  $\tilde{\beta} = \frac{z_2}{2z_1}$ ,  $\tilde{\mu} = \mu_2 - \mu_1$ , and  $z_1 = \vec{\Gamma}^* \Delta \vec{\Gamma}$ ,  $z_2 = \vec{\Gamma}^* \Delta \beta + \beta^* \Delta \vec{\Gamma}$  and  $\vec{\Gamma}^* = (-1, 1)$ . This is the cumulant of a one-dimensional NIG-distribution with parameters  $\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}, \tilde{\delta}$ . Hence,  $L^{(2)}(t) - L^{(1)}(t)$  is a NIG Lévy process under  $\mathbb{P}_{\theta+\mathbf{1}_1}$  and the pricing of the European spread is computable by means of Fourier transform, say. We can follow the same approach as in Lemma 2.5, however, with a different characteristic function of course.

**2.3.2. Example: Merton-Dynamics.** Now we apply the results to the case when the logarithm of the stock prices follows a compound Poisson process with normally distributed jump sizes, the so called Merton dynamics. In this case it is possible to get an infinite sum, where each summand can be evaluated as in the classical Black and Scholes framework. This case has been analysed by Cheang and Chiarella [12], who also investigated the American-type spread options.

Assume now that the stock prices are given as in the Merton dynamics by  $S^{(i)}(t) = S^{(i)}(0) \exp\{L^{(i)}(t)\}$ ,  $S^{(i)}(0) > 0$ ,  $i = 1, 2$ , where  $L(t) = (L^{(1)}(t), L^{(2)}(t))$  is a Lévy process

of jump diffusion type

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$$(2.26) \quad \begin{aligned} L^{(1)}(t) &= (a_1 - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2))t + \sigma_{11}B^{(1)}(t) + \sigma_{12}B^{(2)}(t) + \sum_{k=0}^{N(t)} Y_k^{(1)} \\ L^{(2)}(t) &= (a_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2))t + \sigma_{21}B^{(1)}(t) + \sigma_{22}B^{(2)}(t) + \sum_{k=0}^{N(t)} Y_k^{(2)} \end{aligned}$$

where  $Y_k = (Y_k^{(1)}, Y_k^{(2)})$ ,  $k \in \mathbb{N}$ , is a sequence of iid bivariate random variables and  $N(t)$  is a Poisson process with jump intensity  $\lambda$  independent of  $Y_k$ ,  $k \in \mathbb{N}$  and  $B(t)$ . The compound Poisson processes in (2.26) can be written in integral form

$$\begin{aligned} \sum_{k=1}^{N(t)} Y_k^{(1)} &= \int_0^t \int_{\mathbb{R}_0^2} z_1 N(ds, dz_1, dz_2) \\ \sum_{k=1}^{N(t)} Y_k^{(2)} &= \int_0^t \int_{\mathbb{R}_0^2} z_2 N(ds, dz_1, dz_2) \end{aligned}$$

where  $N(dt, dz_1, dz_2)$  is a Poisson random measure with Lévy measure

$$(2.27) \quad \nu(dz_1, dz_2) = \lambda f_{\mu, \Sigma}(z_1, z_2) dz_1 dz_2,$$

and

$$f_{\mu, \Sigma}(z) = \frac{1}{2\pi |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(z - \mu)^* \Sigma^{-1}(z - \mu)\right\}$$

is the density function of the normal distribution with parameters  $\mu = (\mu_1, \mu_2)$  and  $\Sigma = \begin{pmatrix} \Sigma_1^2 & \Sigma_{12} \\ \Sigma_{12} & \Sigma_2^2 \end{pmatrix}$ . The stock price dynamics has then the following form:

$$\begin{aligned} dS^{(1)}(t) &= S_1(t) \left\{ a_1 dt + \sigma_{11} dB^{(1)}(t) + \sigma_{12} dB^{(2)}(t) \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_0^2} (e^{z_1} - 1) N(ds, dz_1, dz_2) \right\} \\ dS^{(2)}(t) &= S_2(t) \left\{ a_2 dt + \sigma_{21} dB^{(1)}(t) + \sigma_{22} dB^{(2)}(t) \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_0^2} (e^{z_2} - 1) N(ds, dz_1, dz_2) \right\}. \end{aligned}$$

In the previous example with the NIG dynamics, we showed how to use the Esscher transform twice to go from the physical measure  $\mathbb{P}$  to the pricing measure  $\mathbb{P}_{\theta+1_1}$  in the Margrabe formula. We can do the same two-step measure change procedure for the Merton model, but to reduce technicalities, we simply assume that the dynamics is already in the risk-neutral setting, which means that

$$a_1 = a_2 = r,$$

where  $r$  is the interest rate. Note that our dynamics are in the form (2.2) with  $\gamma_j(s, z_1, z_2) = e^{z_j} - 1$ ,  $j = 1, 2$ . The dynamics of  $\frac{S^{(2)}(t)}{S^{(1)}(t)}$  are then given by

$$\begin{aligned} d\left(\frac{S^{(2)}(t)}{S^{(1)}(t)}\right) &= \frac{S^{(2)}(t)}{S^{(1)}(t)} \left\{ (\sigma_{11}^2 + \sigma_{12}^2 - \sigma_{11}\sigma_{21} - \sigma_{12}\sigma_{22})dt + (\sigma_{21} - \sigma_{11})dB^{(1)}(t) \right. \\ &\quad \left. + (\sigma_{22} - \sigma_{12})dB^{(2)}(t) + \int_0^t \int_{\mathbb{R}_0^2} (e^{z_2 - z_1} - 1)N(ds, dz_1, dz_2) \right\} \end{aligned}$$

For the risk-neutral measure  $\mathbb{P}$ , define the measure  $\tilde{\mathbb{P}}$  as in Proposition 2.3 through

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \frac{S^{(1)}(T)}{S^{(1)}(0)} e^{-rT}.$$

Additionally, introduce the measure  $\hat{\mathbb{P}}$  by the density:

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \frac{S^{(2)}(T)}{S^{(2)}(0)} e^{-rT}.$$

Note that these measure changes of Girsanov type correspond to Esscher transforms as in (2.16) with parameters  $\theta = (1, 0)$  and  $\theta = (0, 1)$ , respectively, as long as we neglect the Gaussian component. Therefore we find

$$\begin{aligned} \nu_{\tilde{\mathbb{P}}}(dz) &= e^{z_1} \nu(dz) \\ \nu_{\hat{\mathbb{P}}}(dz) &= e^{z_2} \nu(dz). \end{aligned}$$

Using (2.27), we can conclude that the jumps are still compound Poisson processes with jump intensities  $\tilde{\lambda} = \lambda M_1((1, 0))$  and  $\hat{\lambda} = \lambda M_1((0, 1))$ , where  $M_t(\theta) = \exp\{\mu^* \theta + \frac{1}{2} \theta^* \Sigma \theta\}$  is the moment generating function of  $Y$ . The jump sizes are again normally distributed with expectations  $\tilde{\mu} = (\mu_1 + \Sigma_1^2, \mu_2 + \Sigma_{12})$  and  $\hat{\mu} = (\mu_1 + \Sigma_{12}, \mu_2 + \Sigma_2^2)$ , respectively, and an unchanged volatility  $\Sigma$ .

We know from Proposition 2.3 the price of a spread option to be

$$C = S^{(1)}(0) \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \max\left(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0\right) \right].$$

This can be rewritten as

$$C = S^{(2)}(0) \hat{\mathbb{P}}(A) - S^{(1)}(0) \tilde{\mathbb{P}}(A)$$

with  $A = \{\omega \in \Omega : \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1 > 0\}$  (see for example Corollary 6.13 in Bingham and Kiesel [9]). Conditioning on the number of jumps we get

$$\begin{aligned} \tilde{\mathbb{P}}(A) &= \sum_{n=0}^{\infty} \tilde{p}_n \tilde{\mathbb{P}}(A_n) \\ \hat{\mathbb{P}}(A) &= \sum_{n=0}^{\infty} \hat{p}_n \hat{\mathbb{P}}(A_n) \end{aligned}$$



with

$$\tilde{p}_n = \tilde{\mathbb{P}}(N(T) = n) = \frac{e^{-\tilde{\lambda}T} (\tilde{\lambda}T)^n}{n!}, \quad \hat{p}_n = \hat{\mathbb{P}}(N(T) = n) = \frac{e^{-\hat{\lambda}T} (\hat{\lambda}T)^n}{n!}$$

and  $A_n = \{\omega : \frac{S^{(2)}(T)}{S^{(1)}(T)} - 1 > 0 | N(T) = n\}$ . Under  $\tilde{\mathbb{P}}$  and given  $N(T) = n$ , it is

$$\begin{aligned} \frac{S^{(2)}(t)}{S^{(1)}(t)} &= \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp \left\{ -\frac{1}{2}(\sigma_{21} - \sigma_{11})^2 t - \frac{1}{2}(\sigma_{22} - \sigma_{12})^2 t \right. \\ &\quad \left. + (\sigma_{21} - \sigma_{11})B_{\tilde{\mathbb{P}}}^{(1)}(t) + (\sigma_{22} - \sigma_{12})B_{\tilde{\mathbb{P}}}^{(2)}(t) \right. \\ &\quad \left. + \sum_{k=1}^n \left( Y_k^{(2)} - Y_k^{(1)} \right) \right\}, \end{aligned}$$

and analogously for  $\hat{\mathbb{P}}$ . Therefore, one can see that

$$\tilde{\mathbb{P}}(A_n) = 1 - \Phi(\tilde{d}_n)$$

$$\hat{\mathbb{P}}(A_n) = 1 - \Phi(\hat{d}_n)$$

where  $\Phi$  denotes the standard normal distribution,

$$\tilde{d}_n = \frac{\ln\left(\frac{S^{(2)}(0)}{S^{(1)}(0)}\right) - \tilde{\mu}_n}{\tilde{\sigma}_n}, \quad \hat{d}_n = \frac{\ln\left(\frac{S^{(2)}(0)}{S^{(1)}(0)}\right) - \hat{\mu}_n}{\hat{\sigma}_n}$$

and

$$\tilde{\mu}_n = -\frac{1}{2}T((\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2) + n(\mu_2 + \Sigma_{12} - \mu_1 - \Sigma_1^2)$$

$$\tilde{\sigma}_n^2 = T((\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2) + n(\Sigma_2^2 + \Sigma_1^2 - 2\Sigma_{12})$$

$$\hat{\mu}_n = \frac{1}{2}T((\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2) + n(\mu_2 + \Sigma_2^2 - \mu_1 - \Sigma_{12})$$

$$\hat{\sigma}_n^2 = T((\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2) + n(\Sigma_2^2 + \Sigma_1^2 - 2\Sigma_{12}).$$

In conclusion, we find that the spread option price is expressible as an infinite sum of Black-Scholes prices, scaled by the Poisson probabilities. Summarizing, we have

$$C = S^{(2)}(0) \sum_{n=0}^{\infty} \hat{p}_n (1 - \Phi(\hat{d}_n)) - S^{(1)}(0) \sum_{n=0}^{\infty} \tilde{p}_n (1 - \Phi(\tilde{d}_n)).$$

One may truncate the sums to obtain efficient pricing procedures for the spread option.

### 3. ROBUSTNESS OF SPREAD OPTIONS

In this section we consider dynamics with no explicit Brownian component, namely we consider dynamics driven by a pure Lévy process or by centered Poisson random measures. This is done for simplicity in the exposition. We refer to Benth, Di Nunno, and Khedher [7] Section 3.2 for a discussion on the relationship between the Brownian motion in the price dynamics and the Brownian motion of the approximating dynamics from the point of view of an efficient computation of option prices.

**3.1. Robustness of a bivariate Lévy process.** In this section, we first consider a pure-jump bivariate Lévy process  $(L^{(1)}, L^{(2)})$  with Lévy measure  $\nu$ . We assume that the Lévy measure satisfies

$$\int_{\mathbb{R}_0^2} z_i^2 \nu(dz_1, dz_2) < \infty, \quad i = 1, 2.$$

Let  $D_1^\varepsilon = \{(x, y) \in \mathbb{R}^2; \sqrt{x^2 + y^2} \leq \varepsilon\}$  and  $D_2^\varepsilon$  be the complementary of  $D_1^\varepsilon$ . We define  $\nu_1^\varepsilon = \mathbf{1}_{D_1^\varepsilon} \nu$  and  $\nu_2^\varepsilon = \mathbf{1}_{D_2^\varepsilon} \nu$ . We can write  $(L^{(1)}, L^{(2)})$  as the sum of the two following Lévy processes

$$(3.1) \quad \begin{pmatrix} L^{(1)}(t) \\ L^{(2)}(t) \end{pmatrix} = \begin{pmatrix} L_{1,\varepsilon}^{(1)}(t) \\ L_{1,\varepsilon}^{(2)}(t) \end{pmatrix} + \begin{pmatrix} L_{2,\varepsilon}^{(1)}(t) \\ L_{2,\varepsilon}^{(2)}(t) \end{pmatrix},$$

where  $(L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)})$  and  $(L_{2,\varepsilon}^{(1)}, L_{2,\varepsilon}^{(2)})$  have the following two characteristic functions

$$\phi_{(L_{j,\varepsilon}^{(1)}, L_{j,\varepsilon}^{(2)})}(z_1, z_2) = \exp\left\{t \int_{\mathbb{R}_0^2} (e^{iz_1 x + iz_2 y} - 1 - iz_1 x - iz_2 y) \nu_j^\varepsilon(dx, dy)\right\}, \quad j = 1, 2.$$

We allow the processes  $L^{(1)}, L^{(2)}$  to be dependent. Thus the processes  $L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}$  are dependent and  $L_{2,\varepsilon}^{(1)}, L_{2,\varepsilon}^{(2)}$  are dependent, too. It is easy to see that the characteristic function of the process  $(L^{(1)}, L^{(2)})$  is given by

$$(3.2) \quad \phi_{(L^{(1)}, L^{(2)})}(z_1, z_2) = \phi_{(L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)})}(z_1, z_2) \phi_{(L_{2,\varepsilon}^{(1)}, L_{2,\varepsilon}^{(2)})}(z_1, z_2),$$

from which we conclude that the couples  $(L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)})$  and  $(L_{2,\varepsilon}^{(1)}, L_{2,\varepsilon}^{(2)})$  are independent. The covariance matrix is given by

$$(3.3) \quad \Sigma(\varepsilon) = \int_{\sqrt{x^2 + y^2} \leq \varepsilon} (x, y)(x, y)^T \nu(dx, dy) = \begin{pmatrix} \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) & \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}(\varepsilon) \\ \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}(\varepsilon) & \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) \end{pmatrix}.$$

The Lévy process given by equation (3.1) can be approximated by a two-dimensional Lévy process  $(L_\varepsilon^{(1)}, L_\varepsilon^{(2)})$  given by

$$(3.4) \quad \begin{pmatrix} L_\varepsilon^{(1)}(t) \\ L_\varepsilon^{(2)}(t) \end{pmatrix} = \alpha(\varepsilon) \begin{pmatrix} W^{(1)}(t) \\ W^{(2)}(t) \end{pmatrix} + \begin{pmatrix} L_{2,\varepsilon}^{(1)}(t) \\ L_{2,\varepsilon}^{(2)}(t) \end{pmatrix},$$

where  $\alpha(\varepsilon)$  is a  $2 \times 2$  matrix such that  $\alpha(\varepsilon) = \Sigma^{\frac{1}{2}}(\varepsilon)$  and  $W^{(1)}$  and  $W^{(2)}$  are two independent standard Brownian motions. In this context we mention a paper by Asmussen and Rosinski [1] in which an approximation of the small jumps of one dimensional Lévy processes by a scaled Brownian motion was studied. In this paper the authors choose to scale the Brownian motion by the standard deviation of the jumps smaller than  $\varepsilon$  so that the original process and the approximation have the same variance. In the multivariate case, this approximation was studied by Cohen and Rosinski [13]. Also Benth, Di Nunno, and Khedher [6], [7] worked with this method. That is what explains our choice of the matrix  $\alpha(\varepsilon)$ . Indeed our aim is to keep unchanged the variance-covariance matrix of the original process  $(L^{(1)}, L^{(2)})$ .

We need to study the convergence of the coefficients of the matrix  $\alpha(\varepsilon)$  when  $\varepsilon$  goes to 0. We have

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$$(3.5) \quad \text{Var} \left( \begin{pmatrix} \alpha_1(\varepsilon) & \alpha_2(\varepsilon) \\ \alpha_2(\varepsilon) & \alpha_3(\varepsilon) \end{pmatrix} \begin{pmatrix} W^{(1)}(t) \\ W^{(2)}(t) \end{pmatrix} \right) = t\Sigma(\varepsilon).$$

Since  $W^{(1)}$  and  $W^{(2)}$  are two independent Brownian motions, (3.5) is equivalent to

$$\text{Var} \left( \begin{pmatrix} \alpha_1(\varepsilon)W^{(1)} \\ \alpha_2(\varepsilon)W^{(1)} \end{pmatrix} \right) + \text{Var} \left( \begin{pmatrix} \alpha_2(\varepsilon)W^{(2)} \\ \alpha_3(\varepsilon)W^{(2)} \end{pmatrix} \right) = t\Sigma(\varepsilon)$$

and we get the following set of equations for the coefficients  $\alpha_1(\varepsilon)$ ,  $\alpha_2(\varepsilon)$ , and  $\alpha_3(\varepsilon)$

$$(3.6) \quad \begin{aligned} \alpha_1^2(\varepsilon) + \alpha_2^2(\varepsilon) &= \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon), \\ \alpha_1(\varepsilon)\alpha_2(\varepsilon) + \alpha_2(\varepsilon)\alpha_3(\varepsilon) &= \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}(\varepsilon), \\ \alpha_2^2(\varepsilon) + \alpha_3^2(\varepsilon) &= \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon). \end{aligned}$$

We know that  $\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon)$  and  $\sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon)$  vanish when  $\varepsilon$  goes to 0. Therefore also  $\alpha_1(\varepsilon)$ ,  $\alpha_2(\varepsilon)$ , and  $\alpha_3(\varepsilon)$  converge to 0 when  $\varepsilon$  goes to 0. We use this to prove the following convergence result.

**Proposition 3.1.** *Let the process  $(L^{(1)}, L^{(2)})$  respectively  $(L_\varepsilon^{(1)}, L_\varepsilon^{(2)})$  be defined as in equation (3.1), respectively (3.4). Then, for every  $t \geq 0$ ,*

$$\lim_{\varepsilon \rightarrow 0} (L_\varepsilon^{(1)}(t), L_\varepsilon^{(2)}(t)) = (L^{(1)}(t), L^{(2)}(t)) \quad \mathbb{P} - a.s.$$

In fact, the limit above also holds in  $L^1(\Omega, \mathcal{F}, \mathbb{P})$  with

$$\mathbb{E} [|L_\varepsilon^{(1)}(t) - L^{(1)}(t)|] \leq (\alpha_1(\varepsilon) + \alpha_2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(1)}}(\varepsilon))\sqrt{t}$$

and

$$\mathbb{E} [|L_\varepsilon^{(2)}(t) - L^{(2)}(t)|] \leq (\alpha_2(\varepsilon) + \alpha_3(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}(\varepsilon))\sqrt{t}.$$

*Proof.* The  $\mathbb{P}$ -a.s. convergence follows directly from the proof of the Lévy-Kintchine formula (See Thm. 19.2 in Sato [23]). Concerning the  $L^1$ -convergence, we argue as follows. The combined application of the triangle and Cauchy-Schwarz inequalities give

$$\begin{aligned} \mathbb{E} [|L^{(1)}(t) - L_\varepsilon^{(1)}(t)|] &= \mathbb{E} \left[ \left| \alpha_1(\varepsilon)W^{(1)}(t) + \alpha_2(\varepsilon)W^{(2)}(t) - \int_0^t \int_{|z| \leq \varepsilon} x \tilde{N}(ds, dz_1, dz_2) \right| \right] \\ &\leq \alpha_1(\varepsilon)\mathbb{E} [|W^{(1)}(t)|] + \alpha_2(\varepsilon)\mathbb{E} [|W^{(2)}(t)|] \\ &\quad + \mathbb{E} \left[ \left| \int_0^t \int_{|z| \leq \varepsilon} x \tilde{N}(ds, dz_1, dz_2) \right| \right] \\ &\leq \alpha_1(\varepsilon)\mathbb{E} [|(W^{(1)}(t))^2|]^{\frac{1}{2}} + \alpha_2(\varepsilon)\mathbb{E} [|(W^{(2)}(t))^2|]^{\frac{1}{2}} \\ &\quad + \mathbb{E} \left[ \left| \int_0^t \int_{|z| \leq \varepsilon} x \tilde{N}(ds, dz_1, dz_2) \right|^2 \right]^{\frac{1}{2}} \end{aligned}$$

$$\leq (\alpha_1(\varepsilon) + \alpha_2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(1)}}(\varepsilon))\sqrt{t}.$$

The coefficients  $\sigma_{L_{1,\varepsilon}^{(1)}}$ ,  $\sigma_{L_{2,\varepsilon}^{(1)}}$ , and  $\sigma_{L_{1,\varepsilon}^{(1)}, L_{2,\varepsilon}^{(1)}}$  converge to 0 when  $\varepsilon$  goes to 0. Therefore, from equation (3.6), we deduce that the coefficients  $\alpha_1(\varepsilon)$ ,  $\alpha_2(\varepsilon)$ , and  $\alpha_3(\varepsilon)$  go to 0 when  $\varepsilon$  goes to 0. In the same manner, we can prove that  $\mathbb{E}[|L^{(2)} - L_\varepsilon^{(2)}|] \leq (\alpha_2(\varepsilon) + \alpha_3(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}(\varepsilon))\sqrt{t}$  and the result follows.  $\square$

Notice that in the proof of the Proposition 3.1, the convergence of the coefficients of the matrix  $\alpha(\varepsilon)$  to 0 when  $\varepsilon$  goes to 0 is enough to prove the convergence in  $L^1$  of the process  $(L_\varepsilon^{(1)}, L_\varepsilon^{(2)})$  to  $(L^{(1)}, L^{(2)})$ .

For completeness, we add that to compute  $\alpha(\varepsilon) = \Sigma^{\frac{1}{2}}(\varepsilon)$  we first compute the eigenvalues and the corresponding eigenvectors of the covariance matrix  $\Sigma(\varepsilon)$ . That is we have

$$\Sigma(\varepsilon) = \begin{pmatrix} \cos \theta(\varepsilon) & -\sin \theta(\varepsilon) \\ \sin \theta(\varepsilon) & \cos \theta(\varepsilon) \end{pmatrix} \begin{pmatrix} \lambda_1(\varepsilon) & 0 \\ 0 & \lambda_2(\varepsilon) \end{pmatrix} \begin{pmatrix} \cos \theta(\varepsilon) & -\sin \theta(\varepsilon) \\ \sin \theta(\varepsilon) & \cos \theta(\varepsilon) \end{pmatrix}^*.$$

$$\lambda_1(\varepsilon) = \frac{1}{2} \left( \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) + \sqrt{(\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon))^2 - 4(\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon)\sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) - \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}^2(\varepsilon))} \right),$$

$$\lambda_2(\varepsilon) = \frac{1}{2} \left( \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) - \sqrt{(\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon))^2 - 4(\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon)\sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) - \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}^2(\varepsilon))} \right),$$

$$\theta(\varepsilon) = \arctan \left( \frac{\lambda_1(\varepsilon) - \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon)}{\sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)}}(\varepsilon)} \right).$$

Thus

$$\alpha_1(\varepsilon) = \lambda_1^{\frac{1}{2}}(\varepsilon) \cos^2 \theta(\varepsilon) + \lambda_2^{\frac{1}{2}}(\varepsilon) \sin^2 \theta(\varepsilon),$$

$$\alpha_2(\varepsilon) = \lambda_1^{\frac{1}{2}}(\varepsilon) \cos \theta(\varepsilon) \sin \theta(\varepsilon) - \lambda_2^{\frac{1}{2}}(\varepsilon) \cos \theta(\varepsilon) \sin \theta(\varepsilon),$$

$$\alpha_3(\varepsilon) = \lambda_1^{\frac{1}{2}}(\varepsilon) \cos^2 \theta(\varepsilon) + \lambda_2^{\frac{1}{2}}(\varepsilon) \sin^2 \theta(\varepsilon).$$

This specifies  $\alpha(\varepsilon)$  in terms of the coefficients of the matrix  $\Sigma(\varepsilon)$ .

**3.2. Robustness of the price process.** Now we assume that the price process  $S = (S^{(1)}, S^{(2)})$  is given by the following dynamics

$$(3.7) \quad S(t) = x + \int_0^t a(s)S(s)ds + \int_0^t \int_{\mathbb{R}_0^2} S(s)\gamma(s, z)\tilde{N}(ds, dz),$$

where  $S(0) = x \in \mathbb{R}^2$ . We assume that the solution of the latter equation exists and that for  $i = 1, 2$ ,

$$\gamma_i(s, z) = g_i(z)\widehat{\gamma}_i(s),$$

where  $\int_{|z| \leq \varepsilon} g_i^2(z)\nu(dz) < \infty$ . Moreover we assume that the stochastic factors  $a_i(s)$  and  $\widehat{\gamma}_i(s)$  are such that

$$|a_i(s)|, |\widehat{\gamma}_i(s)| \leq C, \quad i = 1, 2,$$

where  $C$  is a positive constant (not depending on  $\omega$ ).

We define the matrix  $G(\varepsilon) = \left( G_{ij}(\varepsilon) \right)_{1 \leq i, j \leq 2}$ , by

$$G_{ij}(\varepsilon) = \int_{|z| \leq \varepsilon} g_i(z) g_j(z) \nu(dz), \quad \text{for } 1 \leq i, j \leq 2$$

and the matrix  $\beta(\varepsilon)$  by the square root of  $G(\varepsilon)$ , namely

$$(3.8) \quad \beta(\varepsilon) = \begin{pmatrix} \beta_1(\varepsilon) & \beta_2(\varepsilon) \\ \beta_2(\varepsilon) & \beta_3(\varepsilon) \end{pmatrix} = G^{\frac{1}{2}}(\varepsilon).$$

We approximate the price process  $S$  by

$$(3.9) \quad \begin{aligned} S_\varepsilon^{(1)}(t) &= x_1 + \int_0^t a_1(s) S_\varepsilon^{(1)}(s) ds + \beta_1(\varepsilon) \int_0^t S_\varepsilon^{(1)}(s) \widehat{\gamma}_1(s) dW^{(1)}(s) \\ &\quad + \beta_2(\varepsilon) \int_0^t S_\varepsilon^{(1)}(s) \widehat{\gamma}_1(s) dW^{(2)}(s) + \int_0^t \int_{|z| \geq \varepsilon} S_\varepsilon^{(1)}(s) \gamma_1(s, z) \widetilde{N}(ds, dz), \\ S_\varepsilon^{(2)}(t) &= x_2 + \int_0^t a_2(s) S_\varepsilon^{(2)}(s) ds + \beta_2(\varepsilon) \int_0^t S_\varepsilon^{(2)}(s) \widehat{\gamma}_2(s) dW^{(1)}(s) \\ &\quad + \beta_3(\varepsilon) \int_0^t S_\varepsilon^{(2)}(s) \widehat{\gamma}_2(s) dW^{(2)}(s) + \int_0^t \int_{|z| \geq \varepsilon} S_\varepsilon^{(2)}(s) \gamma_2(s, z) \widetilde{N}(ds, dz), \end{aligned}$$

where  $S_\varepsilon(0) = (x_1, x_2)$  and  $W = (W^{(1)}, W^{(2)})$  is a two dimensional Brownian motion. Notice here that the variance-covariance matrix of the process  $S_\varepsilon$  is given by  $\widetilde{\Sigma}(\varepsilon, t) = \left( \widetilde{\Sigma}_{i,j}(\varepsilon, t) \right)_{1 \leq i, j \leq 2}$ , where

$$\begin{aligned} \widetilde{\Sigma}_{1,1}(\varepsilon, t) &= (\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon)) \mathbb{E} \left[ \int_0^t (S_\varepsilon^{(1)}(s))^2 \widehat{\gamma}_1^2(s) ds \right], \\ \widetilde{\Sigma}_{1,2}(\varepsilon, t) &= \widetilde{\Sigma}_{2,1}(\varepsilon, t) = (\beta_1(\varepsilon) \beta_2(\varepsilon) + \beta_2(\varepsilon) \beta_3(\varepsilon)) \mathbb{E} \left[ \int_0^t S_\varepsilon^{(1)}(s) S_\varepsilon^{(2)}(s) \widehat{\gamma}_1(s) \widehat{\gamma}_2(s) ds \right] \\ \widetilde{\Sigma}_{2,2}(\varepsilon, t) &= (\beta_2^2(\varepsilon) + \beta_3^2(\varepsilon)) \mathbb{E} \left[ \int_0^t (S_\varepsilon^{(2)}(s))^2 \widehat{\gamma}_2^2(s) ds \right]. \end{aligned}$$

Since the matrix  $\beta(\varepsilon)$  is given by equation (3.8), the matrix  $\widetilde{\Sigma}(\varepsilon)$  is the same as the variance-covariance matrix of the small jumps of the process  $S$ .

We state the following lemma which shows the boundedness of  $S$  and  $S_\varepsilon$ . The proof is similar to the proof of Lemma 3.2 in Benth, Di Nunno and Khedher [6].

**Lemma 3.2.** *Let  $S$  and  $S_\varepsilon$ , be the unique solutions of (3.7) and (3.9), respectively. For every  $0 \leq t \leq T < \infty$ , we have the following type of estimate for the respective norms*

$$\|S^{(i)}(t)\|_2^2, \|S_\varepsilon^{(i)}(t)\|_2^2 \leq ae^{bt}, \quad i = 1, 2,$$

where  $a$  and  $b$  are positive constants depending on  $T$  but independent of  $\varepsilon$ .

With the same arguments as in equations (3.5) and (3.6) and using the fact that  $S$  is bounded we can show that the coefficients  $\beta_1(\varepsilon)$ ,  $\beta_2(\varepsilon)$ , and  $\beta_3(\varepsilon)$  converge to 0 when  $\varepsilon$  goes to 0. We use the latter arguments to prove the following robustness result of the price process.

**Proposition 3.3.** *For every  $0 \leq t \leq T < \infty$ , we have*

$$\begin{aligned} \|S^{(1)}(t) - S_\varepsilon^{(1)}(t)\|_2^2 &\leq CG_{11}(\varepsilon), \\ \|S^{(2)}(t) - S_\varepsilon^{(2)}(t)\|_2^2 &\leq CG_{22}(\varepsilon), \end{aligned}$$

where  $S$  and  $S_\varepsilon$  are solutions of (3.7) and (3.9), respectively and  $C$  is a positive constant depending on  $T$ , but independent of  $\varepsilon$ .

*Proof.* We prove the result for the process  $S^{(1)}$ . The proof for  $S^{(2)}$  follows the same lines. We have

$$\begin{aligned} S^{(1)}(t) - S_\varepsilon^{(1)}(t) &= \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\} a_1(s) ds - \beta_1(\varepsilon) \int_0^t S_\varepsilon^{(1)}(s) \widehat{\gamma}_1(s) dW^{(1)}(s) \\ &\quad - \beta_2(\varepsilon) \int_0^t S_\varepsilon^{(1)}(s) \widehat{\gamma}_1(s) dW^{(2)}(s) \\ &\quad + \int_0^t \int_{|z|>\varepsilon} \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\} \gamma_1(s, z) \widetilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{|z|\leq\varepsilon} S^{(1)}(s) \gamma_1(s, z) \widetilde{N}(ds, dz). \end{aligned}$$

Applying Hölder inequality and Itô isometry we get

$$\begin{aligned} \|S^{(1)}(t) - S_\varepsilon^{(1)}(t)\|_2^2 &\leq T \mathbb{E} \left[ \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\}^2 a_1^2(s) ds \right] \\ &\quad + \{\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon)\} \mathbb{E} \left[ \int_0^t (S_\varepsilon^{(1)}(s))^2 \widehat{\gamma}_1^2(s) ds \right] \\ &\quad + \int_{|z|>\varepsilon} g_1^2(z) \nu(dz) \mathbb{E} \left[ \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\}^2 \widehat{\gamma}_1^2(s) ds \right] \\ &\quad + G_{11}(\varepsilon) \mathbb{E} \left[ \int_0^t (S^{(1)}(s))^2 \widehat{\gamma}_1^2(s) ds \right]. \end{aligned}$$

Since  $a_1(s)$ ,  $\widehat{\gamma}_1(s)$  are bounded, we get

$$\begin{aligned} \|S^{(1)}(t) - S_\varepsilon^{(1)}(t)\|_2^2 &\leq C \mathbb{E} \int_0^t \|S^{(1)}(s) - S_\varepsilon^{(1)}(s)\|^2 ds + (\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon)) \int_0^t \|S_\varepsilon^{(1)}(s)\|^2 ds \\ &\quad + G_{11}(\varepsilon) \int_0^t \|S^{(1)}(s)\|^2 ds, \end{aligned}$$

where  $C$  is a constant depending on  $T$ . Since  $\beta_1^2(\varepsilon) + \beta_2^2(\varepsilon) = G_{11}(\varepsilon)$  and applying Lemma 3.2 and Gronwall's inequality, we prove the statement.  $\square$

**3.3. Robustness of the Margrabe formula.** In the following we study the robustness of the spread option written on a bivariate geometric Lévy process under the considerations of Remark 2.4. We suppose that the dynamics of the price processes  $S$  and  $S^\varepsilon$  are given by equations (3.7) and (3.9), resp. Applying Proposition 2.3, the price of the spread option written in the underlying process  $S$  is given by

$$(3.10) \quad C = S^{(1)}(0) \mathbb{E}_{\tilde{\mathbb{P}}} \left[ e^{\int_0^T \{a_1(s) - r(s)\} ds} \max\left(\frac{S^{(2)}(T)}{S^{(1)}(T)} - 1, 0\right) \right],$$

where the measure  $\tilde{\mathbb{P}}$  is defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \exp(Y(T)).$$

Here above

$$\begin{aligned} Y(T) = & \left( \int_0^T \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(t, z_1, z_2)) - \gamma_1(t, z_1, z_2) \nu(dz_1, dz_2) dt \right. \\ & \left. + \int_0^T \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(t, z_1, z_2)) \tilde{N}(dt, dz_1, dz_2) \right). \end{aligned}$$

Under  $\tilde{\mathbb{P}}$ , the process  $\frac{S^{(2)}}{S^{(1)}}$  is given by

$$\frac{S^{(2)}(t)}{S^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp(Z(T)),$$

where

$$\begin{aligned} Z(t) = & \int_0^t (a_2(s) - a_1(s)) ds + \int_0^t \int_{|z| \geq 1} \gamma_2(s, z_1, z_2) - \gamma_1(s, z_1, z_2) \nu(dz_1, dz_2) ds \\ & + \int_0^t \int_{\mathbb{R}_0^2} \left\{ \log\left(\frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)}\right) + \frac{\gamma_1(s, z_1, z_2) - \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right\} \nu_{\tilde{\mathbb{P}}}(dz_1, dz_2) ds \\ & + \int_0^t \int_{\mathbb{R}_0^2} \log\left(\frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)}\right) \tilde{N}_{\tilde{\mathbb{P}}}(ds, dz_1, dz_2) \Big\}. \end{aligned}$$

Here  $\nu_{\tilde{\mathbb{P}}}$  is the Lévy measure associated with the Poisson random measure  $\tilde{N}_{\tilde{\mathbb{P}}}$  defined by equation (2.7).

For the approximating processes, the spread option price is analogously given by

$$(3.11) \quad C_\varepsilon = S_\varepsilon^{(1)}(0) \mathbb{E}_{\tilde{\mathbb{P}}_\varepsilon} \left[ e^{\int_0^T \{a_1(s) - r(s)\} ds} \max\left(\frac{S_\varepsilon^{(2)}(T)}{S_\varepsilon^{(1)}(T)} - 1, 0\right) \right],$$

where  $\tilde{\mathbb{P}}_\varepsilon$  is defined by

$$\frac{d\tilde{\mathbb{P}}_\varepsilon}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \exp(Y_\varepsilon(T)).$$

Here above

$$\begin{aligned}
Y_\varepsilon(T) &= -\frac{1}{2} \left( \beta_1^2(\varepsilon) + \beta_2^2(\varepsilon) \right) \int_0^T \widehat{\gamma}_1^2(t) dt + \beta_1(\varepsilon) \int_0^T \widehat{\gamma}_1(t) dW^{(1)}(t) \\
&\quad + \beta_2(\varepsilon) \int_0^T \widehat{\gamma}_1^2(t) dW^{(2)}(t) \\
&\quad + \int_0^T \int_{|z| \geq \varepsilon} \ln(1 + \gamma_1(t, z_1, z_2)) - \gamma_1(t, z_1, z_2) \nu(dz_1, dz_2) dt \\
&\quad + \int_0^T \int_{|z| \geq \varepsilon} \ln(1 + \gamma_1(t, z_1, z_2)) \widetilde{N}(dt, dz_1, dz_2).
\end{aligned}$$

The price  $\frac{S_\varepsilon^{(2)}}{S_\varepsilon^{(1)}}$  is given by

$$\frac{S_\varepsilon^{(2)}(t)}{S_\varepsilon^{(1)}(t)} = \frac{S^{(2)}(0)}{S^{(1)}(0)} \exp(Z_\varepsilon(T)),$$

where

$$\begin{aligned}
Z_\varepsilon(t) &= \int_0^t (a_2(s) - a_1(s)) ds - \frac{1}{2} \int_0^t \left\{ \left( \beta_2(\varepsilon) \widehat{\gamma}_1(s) - \beta_1(\varepsilon) \widehat{\gamma}_1(s) \right)^2 \right. \\
&\quad \left. - \left( \beta_3(\varepsilon) \widehat{\gamma}_2(s) - \beta_2(\varepsilon) \widehat{\gamma}_2(s) \right)^2 \right\} ds \\
&\quad + \int_0^t (\beta_2(\varepsilon) - \beta_1(\varepsilon)) \widehat{\gamma}_1(s) dW_{\mathbb{P}_\varepsilon}^{(1)}(s) + \int_0^t (\beta_3(\varepsilon) - \beta_2(\varepsilon)) \widehat{\gamma}_2(s) dW_{\mathbb{P}_\varepsilon}^{(2)}(s) \\
&\quad + \int_0^t \int_{|z| \geq 1} \left( \gamma_2(s, z_1, z_2) - \gamma_1(s, z_1, z_2) \right) \nu_{\mathbb{P}_\varepsilon}(dz_1, dz_2) ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0^2} \left\{ \log \left( \frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right) + \frac{\gamma_1(s, z_1, z_2) - \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right\} \nu_{\mathbb{P}_\varepsilon}(dz_1, dz_2) ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0^2} \log \left( \frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)} \right) \widetilde{N}_{\mathbb{P}_\varepsilon}(ds, dz_1, dz_2) \}.
\end{aligned}$$

Here

$$\widetilde{N}_{\mathbb{P}_\varepsilon}(dt, dz) = \left( -\gamma_1(t, z) \nu(dz)(dz) dt + \widetilde{N}(dt, dz) \right) \mathbf{1}_{|z| \geq \varepsilon},$$

$\nu_{\mathbb{P}_\varepsilon}$  is the Lévy measure associated with  $\widetilde{N}_{\mathbb{P}_\varepsilon}$ ,  $dW_{\mathbb{P}_\varepsilon}^{(1)}(t) = -\beta_1(\varepsilon) \widehat{\gamma}_1(t) dt + dW^{(1)}(t)$ , and  $dW_{\mathbb{P}_\varepsilon}^{(2)}(t) = -\beta_2(\varepsilon) \widehat{\gamma}_2(t) dt + dW^{(2)}(t)$ .

We have the following technical lemma which is used in the forthcoming convergence result on spread prices.

**Lemma 3.4.** *For  $t \in [0, T]$ ,  $u, R \in \mathbb{R}$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{P}_\varepsilon} \left[ e^{-i(u+iR)Z_\varepsilon(t)} \right] = \mathbb{E}_{\mathbb{P}} \left[ e^{-i(u+iR)Z(t)} \right]$$



*Proof.* We have

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$$\begin{aligned}
& \left| \mathbb{E}_{\tilde{\mathbb{P}}} \left[ e^{-i(u+iR)Z(t)} \right] - \mathbb{E}_{\tilde{\mathbb{P}}_\varepsilon} \left[ e^{-i(u+iR)Z_\varepsilon(t)} \right] \right| \\
& \leq \mathbb{E} \left[ \left| \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} e^{-i(u+iR)Z(t)} - \frac{d\tilde{\mathbb{P}}_\varepsilon}{d\mathbb{P}} e^{-i(u+iR)Z_\varepsilon(t)} \right| \right] \\
& = \mathbb{E} \left[ \left| \exp \left\{ \int_0^T \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \nu(dz) dt \right. \right. \right. \\
& \quad + \int_0^T \int_{\mathbb{R}_0^2} \ln(1 + \gamma_1(t, z)) \tilde{N}(dt, dz) \\
& \quad + (-iu + R) \left( \int_0^t (a_2(s) - a_1(s)) ds + \int_0^t \int_{|z| \geq 1} \gamma_2(s, z) - \gamma_1(s, z) \nu(dz) ds \right. \\
& \quad + \int_0^t \int_{\mathbb{R}_0^2} \left\{ \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right\} \nu_{\tilde{\mathbb{P}}}(dz) ds \\
& \quad \left. \left. \left. + \int_0^t \int_{\mathbb{R}_0^2} \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) \tilde{N}_{\tilde{\mathbb{P}}}(ds, dz) \right\} \right. \right. \\
& \quad - \exp \left\{ -\frac{1}{2} G_{11}(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dt + \beta_1(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(1)}(t) \right. \\
& \quad + \beta_2(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(2)}(t) + \int_0^T \int_{|z| \geq \varepsilon} \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \nu(dz) dt \\
& \quad + \int_0^T \int_{|z| \geq \varepsilon} \ln(1 + \gamma_1(t, z)) \tilde{N}(dt, dz) + (-iu + R) \left( \int_0^t (a_2(s) - a_1(s)) ds \right. \\
& \quad - \frac{1}{2} \int_0^t \left\{ \left( \beta_2(\varepsilon) \hat{\gamma}_1(s) - \beta_1(\varepsilon) \hat{\gamma}_1(s) \right)^2 - \left( \beta_3(\varepsilon) \hat{\gamma}_2(s) - \beta_2(\varepsilon) \hat{\gamma}_2(s) \right)^2 \right\} ds \\
& \quad + \int_0^t (\beta_2(\varepsilon) - \beta_1(\varepsilon)) \hat{\gamma}_1(s) dW_{\tilde{\mathbb{P}}_\varepsilon}^{(1)}(s) + \int_0^t (\beta_3(\varepsilon) - \beta_2(\varepsilon)) \hat{\gamma}_2(s) dW_{\tilde{\mathbb{P}}_\varepsilon}^{(2)}(s) \\
& \quad + \int_0^t \int_{|z| \geq 1} \left( \gamma_2(s, z) - \gamma_1(s, z) \right) \nu_{\tilde{\mathbb{P}}_\varepsilon}(dz) ds \\
& \quad + \int_0^t \int_{\mathbb{R}_0^2} \left\{ \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right\} \nu_{\tilde{\mathbb{P}}_\varepsilon}(dz) ds \\
& \quad \left. \left. \left. + \int_0^t \int_{\mathbb{R}_0^2} \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) \tilde{N}_{\tilde{\mathbb{P}}_\varepsilon}(ds, dz) \right\} \right] \right|. \\
& = \mathbb{E} \left[ \left| \exp \left\{ \int_0^t (a_2(s) - a_1(s)) ds + \int_0^T \int_{|z| > \varepsilon} \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \nu(dz) dt \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_0^T \int_{|z|>\varepsilon} \ln(1 + \gamma_1(t, z)) \tilde{N}(dt, dz) \\
& + (-iu + R) \left( \int_0^t \int_{|z|\geq 1} \gamma_2(s, z) - \gamma_1(s, z) \nu(dz) ds \right. \\
& + \int_0^t \int_{|z|>\varepsilon} \left\{ \log\left(\frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)}\right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right\} \nu_{\tilde{\mathbb{P}}}(dz) ds \\
& \left. + \int_0^t \int_{|z|>\varepsilon} \log\left(\frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)}\right) \tilde{N}_{\tilde{\mathbb{P}}}(ds, dz) \right\} \\
& \left( \exp \left\{ \int_0^T \int_{|z|\leq\varepsilon} \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \nu(dz) dt \right. \right. \\
& + \int_0^T \int_{|z|\leq\varepsilon} \ln(1 + \gamma_1(t, z)) \tilde{N}(dt, dz) \\
& + (-iu + R) \left( \int_0^t \int_{|z|\leq\varepsilon} \left\{ \log\left(\frac{1 + \gamma_2(s, z_1, z_2)}{1 + \gamma_1(s, z_1, z_2)}\right) \right. \right. \\
& + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \left. \right\} \nu_{\tilde{\mathbb{P}}}(dz) ds \\
& \left. + \int_0^t \int_{|z|\leq\varepsilon} \log\left(\frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)}\right) \tilde{N}_{\tilde{\mathbb{P}}}(ds, dz) \right\} \\
& - \exp \left\{ -\frac{1}{2} G_{11}(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dt + \beta_1(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(1)}(t) \right. \\
& + \beta_2(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(2)}(t) \\
& + (-iu + R) \left( \frac{1}{2} \int_0^t \left\{ \left( \beta_2(\varepsilon) - \beta_1(\varepsilon) \right)^2 \hat{\gamma}_1^2(s) - \left( \beta_3(\varepsilon) - \beta_2(\varepsilon) \right)^2 \hat{\gamma}_2^2(s) \right\} ds \right. \\
& \left. \left. + \int_0^t (\beta_2(\varepsilon) - \beta_1(\varepsilon)) \hat{\gamma}_1(s) dW_{\tilde{\mathbb{P}}_\varepsilon}^{(1)}(s) + \int_0^t (\beta_3(\varepsilon) - \beta_2(\varepsilon)) \hat{\gamma}_2(s) dW_{\tilde{\mathbb{P}}_\varepsilon}^{(2)}(s) \right\} \right) \Big].
\end{aligned}$$

Using Hölder inequality and the triangle inequality, we get

$$\begin{aligned}
& |\mathbb{E}_{\tilde{\mathbb{P}}} [e^{-i(u+iR)Z(t)}] - \mathbb{E}_{\tilde{\mathbb{P}}_\varepsilon} [e^{-i(u+iR)Z_\varepsilon(t)}]| \\
& \leq C \left( \mathbb{E} \left[ \exp 2 \left\{ \int_0^T \int_{|z|\leq\varepsilon} \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \nu(dz) dt \right. \right. \right. \\
& \quad + \int_0^T \int_{|z|\leq\varepsilon} \ln(1 + \gamma_1(t, z)) \tilde{N}(dt, dz) \\
& \quad \left. \left. + R \int_0^t \int_{|z|\leq\varepsilon} \left\{ \log\left(\frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)}\right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right\} \nu_{\tilde{\mathbb{P}}}(dz) ds \right. \right. \\
& \quad \left. \left. \right\} \right] \Big).
\end{aligned}$$

$$\begin{aligned}
& + R \int_0^t \int_{|z| \leq \varepsilon} \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) \tilde{N}_{\mathbb{P}_\varepsilon}(ds, dz) \Big] \Big]_{\text{MATH./CMA UNIV. OF OSLO}}^{\frac{1}{2}} \\
& + \mathbb{E} \left[ \exp 2 \left\{ -\frac{1}{2} G_{11}(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dt + \beta_1(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(1)}(t) \right. \right. \\
& + \beta_2(\varepsilon) \int_0^T \hat{\gamma}_1^2(t) dW^{(2)}(t) \\
& \left. \left. + R \left( \int_0^t (\beta_2(\varepsilon) - \beta_1(\varepsilon)) \hat{\gamma}_1(s) dW_{\mathbb{P}_\varepsilon}^{(1)}(s) + \int_0^t (\beta_3(\varepsilon) - \beta_2(\varepsilon)) \hat{\gamma}_2(s) dW_{\mathbb{P}_\varepsilon}^{(2)}(s) \right) \right\} \right]^{\frac{1}{2}}.
\end{aligned}$$

We have

$$\begin{aligned}
& \int_0^T \int_{|z| \leq \varepsilon} \left| \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \right| \nu(dz) dt + \int_0^T \int_{|z| \leq \varepsilon} \left| \ln(1 + \gamma_1(t, z)) \right| \tilde{N}(dt, dz) \\
& + R \int_0^t \int_{|z| \leq \varepsilon} \left| \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right| \nu_{\mathbb{P}}(dz) ds \\
& + R \int_0^t \int_{|z| \leq \varepsilon} \left| \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) \right| \tilde{N}_{\mathbb{P}}(ds, dz) \\
& \leq \int_0^T \int_{|z| \leq 1} \left| \ln(1 + \gamma_1(t, z)) - \gamma_1(t, z) \right| \nu(dz) dt + \int_0^T \int_{|z| \leq 1} \left| \ln(1 + \gamma_1(t, z)) \right| \tilde{N}(dt, dz) \\
& + R \int_0^t \int_{|z| \leq 1} \left| \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) + \frac{\gamma_1(s, z) - \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right| \nu_{\mathbb{P}}(dz) ds \\
& + R \int_0^t \int_{|z| \leq 1} \left| \log \left( \frac{1 + \gamma_2(s, z)}{1 + \gamma_1(s, z)} \right) \right| \tilde{N}_{\mathbb{P}}(ds, dz).
\end{aligned}$$

We also have  $\beta_1(\varepsilon), \beta_2(\varepsilon) \leq 1$ . Thus by dominated convergence, we can take the limit inside the expectation and the result follows.  $\square$

We can now conclude the following convergence result.

**Proposition 3.5.** *Let  $C$  and  $C_\varepsilon$  be defined in equations (3.10) and (3.11). It holds that*

$$\lim_{\varepsilon \rightarrow 0} C_\varepsilon = C.$$

*Proof.* From Theorem 2.6, we have for  $d \in \mathbb{R}$  and  $f = \max((e^x - 1), 0)$ ,

$$\begin{aligned}
C &= \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} \hat{f}(u + iR) \mathbb{E}_{\mathbb{P}} \left[ e^{\int_0^T \{a_1(s) - r_1(s)\} ds} e^{-i(u+iR)Z(T)} \right] du, \\
C_\varepsilon &= \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} \hat{f}(u + iR) \mathbb{E}_{\mathbb{P}_\varepsilon} \left[ e^{\int_0^T \{a_1(s) - r_1(s)\} ds} e^{-i(u+iR)Z_\varepsilon(T)} \right] du
\end{aligned}$$

and the result follows easily from Lemma 3.4.  $\square$

Note that we only used a one dimensional Fourier transform to analyse a two dimensional problem.

In the latter proposition we proved the robustness of the Margrabe formula in the case where the price processes are modeled under the real world measure  $\mathbb{P}$ , see Remark 2.4. In the following we prove the robustness of the Margrabe formula when the price processes are modeled under a risk neutral measure. For that we consider a two dimensional exponential Lévy process, we use the Esscher transform to compute the Margrabe formula, and we give a convergence rate for the robustness result. We refer to Benth, Di Nunno, and Khedher [8] for robustness studies of options under martingale measures.

**3.4. Convergence rate for the Margrabe formula.** In this section, we restrict ourselves to the case where the price processes  $S$  and  $S_\varepsilon$  are given by  $S(t) = (S^{(1)}(0)e^{L^{(1)}(t)}, S^{(2)}(0)e^{L^{(2)}(t)})$ , and  $S_\varepsilon(t) = (S^{(1)}(0)e^{L_\varepsilon^{(1)}(t)}, S^{(2)}(0)e^{L_\varepsilon^{(2)}(t)})$ , where  $L = (L^{(1)}, L^{(2)})$  and  $L_\varepsilon = (L_\varepsilon^{(1)}, L_\varepsilon^{(2)})$  are bivariate Lévy processes given by (3.1) and (3.4). Recall from Section 2.3 that the risk-neutral price for the spread option written in  $S$  is given by

$$C = S^{(2)}(0)\mathbb{E}_{\theta+\mathbf{1}_1}[f(Z(T))],$$

where  $f = \max((e^x - 1), 0)$  and

$$Z(T) = (\widehat{a}_2 - \widehat{a}_1)T + \int_0^T \int_{|z| \leq 1} (z_2 - z_1) \widetilde{N}_{\theta+\mathbf{1}_1}(ds, dz) + \int_0^T \int_{|z| > 1} (z_2 - z_1) N_{\theta+\mathbf{1}_1}(ds, dz).$$

Here  $\widehat{a}_1$ ,  $\widehat{a}_2$ , and  $\widetilde{N}_{\theta+\mathbf{1}_1}$  are all defined in Section 2.3.

Now we define the parameter  $\theta_\varepsilon$  such that for  $i = 1, 2$ , the discounted price process  $e^{-rt}S_\varepsilon^{(i)}$  is a martingale. Thus  $\theta_\varepsilon = (\theta_{1,\varepsilon}, \theta_{2,\varepsilon})$  has to fulfill the following equation

$$e^{rt} = \frac{M_t(\mathbf{1}_i + \theta_\varepsilon)}{M_t(\theta_\varepsilon)}.$$

This is similar to the computations done in Section 2.3. Benth, Di Nunno, and Khedher [8] proved the existence and uniqueness of the parameter  $\theta_\varepsilon$  when the price process is modeled by one dimensional Lévy process. With similar computations we prove the existence and uniqueness of the parameter  $\theta_\varepsilon$  in our case. We define a new measure  $\mathbb{P}_{\theta_\varepsilon+\mathbf{1}_1}$  as follows

$$(3.12) \quad \frac{d\mathbb{P}_{\theta_\varepsilon+\mathbf{1}_1}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{e^{\langle \theta_\varepsilon + \mathbf{1}_1, L(t) \rangle}}{M_t(\theta_\varepsilon + \mathbf{1}_1)}.$$

Similar computation as in Section 2.3 leads to the following risk-neutral price for the spread option written on  $S_\varepsilon$

$$C_\varepsilon = S^{(2)}(0)\mathbb{E}_{\theta_\varepsilon+\mathbf{1}_1}[f(Z_\varepsilon(T))],$$

where

$$\begin{aligned} Z_\varepsilon(T) &= (\widehat{a}_2^\varepsilon - \widehat{a}_1^\varepsilon)T - \alpha_1(\varepsilon)W^{(1)}(T) - \alpha_2(\varepsilon)W^{(2)}(T) + \alpha_2(\varepsilon)W^{(1)}(T) + \alpha_3(\varepsilon)W^{(2)}(T) \\ &\quad + \int_0^T \int_{\varepsilon < |z| \leq 1} (z_2 - z_1) \widetilde{N}_{\theta_\varepsilon+\mathbf{1}_1}(ds, dz) + \int_0^T \int_{|z| > 1} (z_2 - z_1) N_{\theta_\varepsilon+\mathbf{1}_1}(ds, dz). \end{aligned}$$

Here  $\alpha_i$ ,  $1 \leq i \leq 3$  are defined by equation (3.6),  $\tilde{N}_{\theta_\varepsilon+1_1}(ds, dz)$  is a Poisson random measure with Lévy measure

$$\widehat{\nu}_\varepsilon(dz_1, dz_2) = e^{(\theta_1^\varepsilon+1)z_1+\theta_2^\varepsilon z_2} \nu(dz_1, dz_2),$$

and

$$\widehat{a}_i^\varepsilon = a_i + \int_{\varepsilon \leq |z| \leq 1} z_i e^{(\theta_1^\varepsilon+1)z_1+\theta_2^\varepsilon z_2} \nu(dz_1, dz_2), \quad i = 1, 2.$$

Notice that in the paper by Benth, Di Nunno, and Khedher [8] it is proved that the parameter  $\theta_\varepsilon$  is bounded uniformly in  $\varepsilon$  in the case of a one-dimensional Lévy process. In our case we can also prove with the same arguments that  $\theta_\varepsilon$  is bounded uniformly in  $\varepsilon$  and that

$$(3.13) \quad |\theta_i^\varepsilon - \theta_i| \leq C_\theta \sigma_{L_{1,\varepsilon}^{(i)}}^2(\varepsilon), \quad i = 1, 2,$$

where  $C_\theta$  is a constant depending on  $\theta$  and  $\sigma_{L_{1,\varepsilon}^{(i)}}^2(\varepsilon)$ ,  $i = 1, 2$ , is given by (3.3). We use this result to prove the following convergence rate.

**Proposition 3.6.** *It follows that*

$$|\mathbb{E}_{\theta+1_1}[e^{-i(u+iR)Z(T)}] - \mathbb{E}_{\theta_\varepsilon+1_1}[e^{-i(u+iR)Z_\varepsilon(T)}]| \leq K_1(u, R, \theta) \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + K_2(u, R, \theta) \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon),$$

where  $K_1(u, R, \theta)$  and  $K_2(u, R, \theta)$  are constants depending on  $u$ ,  $R$ , and  $\theta$ .

*Proof.* Define

$$\begin{aligned} \psi(u) &= \exp \left\{ -i(u+iR)(\widehat{a}_2 - \widehat{a}_1)T + T \int_{\mathbb{R}_0^2} e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1) \widehat{\nu}(dz) \right\} \\ \psi_\varepsilon(u) &= \exp \left\{ -i(u+iR)(\widehat{a}_2^\varepsilon - \widehat{a}_1^\varepsilon)T - \frac{1}{2}(u+iR)^2 b_\varepsilon^2 T \right. \\ &\quad \left. + T \int_{|z|>\varepsilon} e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1) \widehat{\nu}_\varepsilon(dz) \right\}, \end{aligned}$$

where  $b_\varepsilon = \int_{|z| \leq \varepsilon} (z_2 - z_1)^2 \widehat{\nu}_\varepsilon(dz)$ . Let  $\tau(u, \varepsilon)$  be defined as follows

$$\begin{aligned} \tau(u, \varepsilon) &= \left\{ -i(u+iR)(\widehat{a}_2^\varepsilon - \widehat{a}_1^\varepsilon)T - \frac{1}{2}(u+iR)^2 b_\varepsilon^2 T \right. \\ &\quad \left. + T \int_{|z|>\varepsilon} e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1) \widehat{\nu}_\varepsilon(dz) \right\} \\ &\quad + \left\{ i(u+iR)(\widehat{a}_2 - \widehat{a}_1)T \right. \\ &\quad \left. - T \int_{\mathbb{R}_0^2} e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1) \widehat{\nu}(dz) \right\}. \end{aligned}$$

Thus we have

$$|\mathbb{E}[e^{-i(u+iR)Z(T)}] - \mathbb{E}[e^{-i(u+iR)Z_\varepsilon(T)}]| \leq |\psi(u)| |1 - \exp(\tau(u, \varepsilon))|$$

$$\leq \exp(\tau(u, 1)|\psi(u)||\tau(u, \varepsilon)|).$$

We have

$$\begin{aligned} |\tau(u, \varepsilon)| &\leq \left| \left\{ i(u + iR)T \left( \int_{\varepsilon \leq |z| \leq 1} (z_2 - z_1) (e^{(\theta_1+1)z_1 + \theta_2 z_2} - e^{(\theta_1^\varepsilon+1)z_1 + \theta_2^\varepsilon z_2}) \nu(dz) \right. \right. \right. \\ &\quad \left. \left. + i(u + iR)T \left( \int_{|z| < \varepsilon} (z_2 - z_1) e^{(\theta_1+1)z_1 + \theta_2 z_2} \nu(dz) \right) \right\} \right. \\ &\quad \left. + \left\{ T \int_{|z| > \varepsilon} (e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1)) \right. \right. \\ &\quad \left. \left. (-e^{(\theta_1+1)z_1 + \theta_2 z_2} + e^{(\theta_1^\varepsilon+1)z_1 + \theta_2^\varepsilon z_2}) \nu(dz) \right\} - \frac{1}{2}(u+iR)^2 b_\varepsilon^2 T \right. \\ &\quad \left. + \left\{ -T \int_{|z| \leq \varepsilon} (e^{i(u+iR)(z_2-z_1)} - 1 - i(u+iR)(z_2-z_1)) \right. \right. \\ &\quad \left. \left. e^{(\theta_1+1)z_1 + \theta_2 z_2} \nu(dz) \right\} \right| \\ &\leq K_1(u, R, \theta) \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + K_2(u, R, \theta) \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon), \end{aligned}$$

where  $K_1(u, R, \theta)$  and  $K_2(u, R, \theta)$  are constants depending on  $u$ ,  $R$ , and  $\theta$ . To prove the latter, we used the fact that  $\theta_1^\varepsilon$  and  $\theta_2^\varepsilon$  are bounded, the equation (3.13), and the Taylor expansion of the function  $e^{(\theta_1+1)z_1 + \theta_2 z_2} - e^{(\theta_1^\varepsilon+1)z_1 + \theta_2^\varepsilon z_2}$ . Thus we proved the result.  $\square$

From Theorem 2.6 and assuming that  $\int_{\mathbb{R}} |\widehat{f}(u + iR)K_1(u, R, \theta)| du < \infty$  and  $\int_{\mathbb{R}} |\widehat{f}(u + iR)K_2(u, R, \theta)| du < \infty$ , we deduce the following

$$|C_\varepsilon - C| \leq C_1 \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon) + C_2 \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon),$$

where  $C_1 = \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} |\widehat{f}(u + iR)K_1(u, R, \theta)| du$  and  $C_2 = \frac{S^{(1)}(0)}{2\pi} \int_{\mathbb{R}} |\widehat{f}(u + iR)K_2(u, R, \theta)| du$ .

We are now interested in the case where  $(L^{(1)}, L^{(2)})$  follows a bivariate normal inverse Gaussian Lévy process as in subsection 2.3.1. For this case we would like to investigate the behavior of  $\sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon)$  and  $\sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon)$ , the entries of the covariance matrix (3.3) of the small jumps. Using Proposition 8.9 in Sato [23] we find the Lévy density corresponding to the probability density (2.24) to be

$$g(z) = \frac{\delta}{\sqrt{2}} \left( \frac{\alpha}{\pi \sqrt{z' \Delta^{-1} z}} \right)^{\frac{3}{2}} \exp(\beta z) K_{\frac{3}{2}}(\alpha \sqrt{z' \Delta^{-1} z}).$$

Using that for  $z \rightarrow 0$

$$K_{\frac{3}{2}}(z) \sim z^{-\frac{3}{2}}$$

we find for small  $z$

$$(3.14) \quad g(z) \sim \sqrt{z' \Delta^{-1} z}^{-3}.$$

It follows for the variance and covariance of the small jumps

$$\begin{aligned} \sigma_{L_{1,\varepsilon}^{(1)}}^2(\varepsilon), \sigma_{L_{1,\varepsilon}^{(2)}}^2(\varepsilon) &\sim \varepsilon \ln(\varepsilon) + \varepsilon \\ \sigma_{L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(1)}}^2(\varepsilon) &\sim \varepsilon \end{aligned}$$

giving the speed of convergence in terms of the truncation level  $\varepsilon$ . With L'Hôpital's rule it is easy to see that  $\varepsilon \ln(\varepsilon)$  converges to 0 for  $\varepsilon \rightarrow 0$ .

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