Primal-dual algorithms and infinite-dimensional Jordan algebras of finite rank

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Primal-dual algorithms and infinite-dimensional Jordan algebras of finite rank

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Abstract

We consider primal-dual algorithms for certain types of infinite-dimensional optimization problems. Our approach is based on the generalization of the technique of finite-dimensional Euclidean Jordan algebras to the case of infinite-dimensional JB-algebras of finite rank. This generalization enables us to develop polynomial-time primal-dual algorithms for "infinite-dimensional second-order cone programs." We consider as an example a long-step primal-dual algorithm based on the Nesterov-Todd direction. It is shown that this algorithm can be generalized along with complexity estimates to the infinite-dimensional situation under consideration. An application is given to an important problem of control theory: multi-criteria analytic design of the linear regulator. The calculation of the Nesterov-Todd direction requires in this case solving one matrix differential Riccati equation plus solving a finite-dimensional system of linear algebraic equations on each iteration. The size of this algebraic system is $m+1$ by $m+1$, where m is a number of quadratic performance criteria.

Key words: Interior-point algorithms, primal-dual algorithms, second-order cone programming, infinite-dimensional problems, control theory

1 Introduction

Finite-dimensional Euclidean Jordan algebras proved to be very useful for the analysis of interior-point algorithms of optimization [2, 3, 4, 5, 11, 14]. In the present paper we analyze the possibility of using infinite-dimensional Jordan algebras of finite rank in a similar fashion for the analysis of an infinite-dimensional situation. In particular, we concentrate on primal-dual algorithms which constitute probably the most important class of interior-point algorithms though other classes of interior-point algorithms can be generalized following the pattern presented here.

Let $(V, \langle , \rangle)$ be a Hilbert space, $\Omega \subset V$ be an open convex cone in V, $a, b \in V$, $X \subset V$ be a closed vector subspace in V. Consider an optimization problem:

$$\langle a, z\rangle \to \min, \tag{1}$$

$$z \in (b + X) \cap \bar{\Omega} \tag{2}$$

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and its dual

$$\langle b, w\rangle \to \min, \tag{3}$$

$$w \in (a + X^{\perp}) \cap \bar{\Omega}^{*}. \tag{4}$$

Here $\bar{\Omega}$ is the closure of Ω (in the topology induced by norm $:\|z\| = \sqrt{\langle z, z\rangle}$),

$$\bar{\Omega}^{*} = \{w \in V : \langle w, z\rangle \geq 0, \forall z \in \Omega\}, \tag{5}$$

$X^{\perp}$ is the orthogonal complement of X in V with respect to the scalar product $\langle , \rangle$.

Let

$$\mathcal{F} = [(b + X) \cap \bar{\Omega}] \times [(a + X^{\perp}) \cap \bar{\Omega}^{*}].$$

We will assume throughout this paper that

$$\mathrm{int}(\mathcal{F}) = [(b + X) \cap \Omega] \times [(a + X^{\perp}) \cap \mathrm{int}(\bar{\Omega}^{*})] \neq \emptyset. \tag{6}$$

It is very easy to see that if the pair $\bar{z}, \bar{w}$ satisfy (2) and (4), respectively, and

$$\langle \bar{z}, \bar{w}\rangle = 0,$$

then $\bar{z}$ is an optimal solution to (1), (2) and $\bar{w}$ is an optimal solution to (3), (4), respectively. Given $(z, w) \in V \times V$, we introduce the so-called duality gap:

$$\mu(z, w) = \frac{\langle z, w\rangle}{r}, \tag{7}$$

where $r > 0$ is some positive constant which will be specified later.

A typical primal-dual algorithm generates a sequence $(z^{(k)}, w^{(k)}) \in \mathrm{int}(\mathcal{F})$, $k = 0, 1, ...$, such that:

$$\mu(z^{(k+1)}, w^{(k+1)}) \leq \left(1 - \frac{\delta}{r^{\omega}}\right) \mu(z^{(k)}, w^{(k)}), \tag{8}$$

for some positive constants δ and ω.

The following proposition is a direct consequence of (8).

Proposition 1.1 *Let $0 < \varepsilon < 1$ be given and a primal-dual algorithm generates a sequence satisfying (8). Then*

$$\mu(z^{(k)}, w^{(k)}) \leq \varepsilon$$

for

$$k \geq r^{\omega} \frac{\log\left(\dfrac{\mu(z^{(0)}, w^{(0)})}{\varepsilon}\right)}{\delta}$$

provided $\delta / r^{\omega} < 1$.

For a proof see e.g. [17]. Observe that the existence of a primal-dual sequence satisfying (8) for an arbitrary $0 < \varepsilon < 1$ is highly nontrivial in an infinite-dimensional situation and, in particular, implies that (1), (2) and (3), (4) have no duality gap.

In the present paper we consider a rather special but important situation where V is a JB-algebra of a finite rank, and Ω is the so-called "cone of squares." The classification of JB-algebras of finite rank is known (see e.g. [8]) and is briefly described in Section 2 of the paper. It turns out that each such an algebra is a direct sum of uniquely defined irreducible factors. Each factor is either an irreducible finite-dimensional Euclidean Jordan algebra or the so-called (infinite-dimensional) spin-factor. This enables us to reduce the analysis of interior-point algorithms to two cases: a) V is a finite-dimensional Euclidean Jordan algebra and b) V is a direct sum of a finite number of infinite-dimensional spin-factors. It is well-known that the cone of squares Ω for a) is the symmetric cones. The cone of squares for b) is infinite-dimensional second-order cones.

The case a) is very well understood by now (see e.g. [2, 3, 4, 5, 11, 14]). We analyze in detail the case b) and show that it has a lot of similarities with the second-order cone programming [9, 12, 16]. Specifically, we pick up the long-step path-following algorithm with the Nesterov-Todd direction as an example and show that the algorithm terminates in $O(r \log \mu^0/\varepsilon)$ iterations, where μ^0 is the initial duality gap and ε is the final duality gap, and r is the rank of the associated JB-algebra.

The crucial point in the implementation of primal-dual algorithms is the availability of an efficient procedure for the calculation of an appropriate "descent direction" which enables one to move from $(z^{(k)}, w^{(k)})$ to $(z^{(k+1)}, w^{(k+1)})$. In the infinite-dimensional setting this problem is reduced to solving an infinite-dimensional system of linear equations. In the present paper we consider a concrete example, a min-max optimization problem with linear constraints in a Hilbert space, and show that the corresponding infinite-dimensional system can be efficiently solved. This problem admits a natural control-theoretic interpretation as a multi-criteria problem of the analytic design of a linear regulator.

2 JB-algebras algebras of finite rank

The purpose of this section is to describe the classification of JB-algebras of finite rank. For further details see [8].

Let V be a real commutative algebra with the unit element e. Given $z \in V$, consider the multiplication operator $L(z) : V \to V$,

$$L(z)z_1 = z \circ z_1, \ z_1 \in V.$$

Definition 2.1 *We say that V is a Jordan algebra if the identity*

$$[L(z), L(z^2)] = L(z)L(z^2) - L(z^2)L(z) = 0 \tag{9}$$

holds for any $z \in V$.

We can introduce the so-called quadratic representation in an arbitrary Jordan algebra V. Given $z \in V$,

$$P(z) = 2L(z)^2 - L(z^2). \tag{10}$$

A direct computation shows:

Proposition 2.2 *Given $z_1, z_2 \in V$, we have:*

$$P(P(z_1)z_2) = P(z_1)P(z_2)P(z_1). \tag{11}$$

Let V be a Jordan algebra with the unit element e and the multiplication operator $\circ$.

Definition 2.3 *An element $z \in V$ is called <u>invertible</u> in V if there exists $w \in V$ such that $z \circ w = e$, $z^2 \circ w = z$. We denote such an element w by z^{-1}.*

Proposition 2.4 *An element $z \in V$ is invertible if and only if $P(z)$ is an invertible linear operator. Moreover, in this case*

$$z^{-1} = P(z)^{-1}z.$$

Proposition 2.5 *Given an invertible element $z \in V$, a subalgebra generated by z, z^{-1}, e is associative.*

Definition 2.6 *A JB-algebra is a Jordan algebra V with the unit element e endowed with a complete norm $\|\cdot\|$ such that:*

$$\|z_1 \circ z_2\| \leq \|z_1\|\|z_2\|, \quad \|z_1^2\| = \|z_1\|^2, \quad \|z_1^2 + z_2^2\| \geq \|z_1^2\|, \ \forall z_1, z_2 \in V.$$

Proposition 2.7 *In every JB-algebra V the set*

$$\bar{\Omega} = \{z^2 : z \in V\} \tag{12}$$

is a closed convex cone.

Example 2.8 *Let K be a compact set and $Cont(K)$ is the vector space of continuous real-valued functions on K endowed with the norm:*

$$\|f\| = \sup\{|f(t)| : t \in K\}, \ f \in Cont(K).$$

It is quite obvious that $Cont(K)$ is a JB-algebra. A Jordan-algebraic multiplication in this example is the pointwise multiplication of functions. The cone $\bar{\Omega}$ is the cone of nonnegative functions from $Cont(K)$.

Lemma 2.9 *For every element z in a JB-algebra V, the closed subalgebra $C(z)$ generated by z and e is associative.*

Proposition 2.10 *Let V be a JB-algebra and $\bar{\Omega}$ defined in (12) be its cone of squares. The interior of $\bar{\Omega}$, which we denote by Ω, has the following properties:*

i) Ω is a nonempty open convex cone.

ii) Ω is the connected component of the unit element e in the set of invertible elements of V.

Let $\mathcal{L}(V)$ be the Banach space of bounded linear operators on V. Let, further,

$$GL(\Omega) = \{g \in \mathcal{L}(V) : \ g(\Omega) = \Omega, \ g \text{ is invertible in } \mathcal{L}(V)\}. \tag{13}$$

Proposition 2.11 *The cone Ω is linear homogeneous, i.e., for any $z \in \Omega$ there exists $g \in GL(\Omega)$ such that $ge = z$.*

Denote by $Aut(V)$ the group of Jordan algebra isomorphisms of a JB-algebra V, i.e., the group of invertible linear maps on V which preserve the Jordan-algebraic operations.

Proposition 2.12 *Given $g \in Aut(V), \|g(x)\| = \|x\|, \forall x \in V$. In particular, $Aut(V) \subset \mathcal{L}(V)$. Every $g \in GL(\Omega)$ admits a unique representation of the form (the polar decomposition):*

$$g = P(x)g_1, \ x \in \Omega, \ g_1 \in Aut(V).$$

We are now in position to introduce the "JB-algebras of finite rank" and its classification.

Let $(Y, (\cdot|\cdot))$ be a real Hilbert space. Introduce a multiplication operator on the vector space $V = \mathbf{R} \oplus Y$ as follows:

$$(s, x) \circ (t, y) = (st + (x|y), sy + tx).$$

If we denote $(1, 0) \in V$ by e, we immediately see that:

$$e \circ z = z \circ e = z, \ \forall z \in V.$$

It is easy to verify by a direct calculation that (9) holds.

Let $p = 1 + \dim Y$, where $\dim Y$ is the cardinality of an orthonormal basis in V. We call V the spin-factor (notation: V_p). It is known that spin-factors are JB-algebras with the norm defined as follows:

$$\|(t, y)\| = |t| + \sqrt{(y|y)}, \ (t, y) \in V.$$

Proposition 2.13 *Let V be a JB-algebra. The following conditions are equivalent:*

i) for every $z \in V$ the operator $L(z)$ satisfies a polynomial equation in $\mathcal{L}(V)$ over $\mathbf{R}$.

ii) there exists a natural number r such that every $z \in V$ admits a representation:

$$z = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_r e_r, \tag{14}$$

where $e_i \circ e_j = \delta_{ij} e_i, \ \lambda_i \in \mathbf{R}, \ i, j = 1, 2, ..., r.$

Proposition 2.13 singles out a subclass of JB-algebras of *finite rank*. The number r in Proposition 2.13 is called the *rank* of V (notation: $r = r(V)$).

Theorem 2.14 *Every JB-algebra of a finite rank admits a unique direct sum decomposition:*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_k, \ r(V) = r(V_1) + ... + r(V_k) \tag{15}$$

and each V_i is either a spin-factor of infinite cardinality or a finite-dimensional irreducible JB-algebra.

Remark: Since the class of finite-dimensional JB-algebras coincides with the class of Euclidean Jordan algebras, there is a complete classification of finite-dimensional JB-algebras (see e.g. [1]).

3 Some Jordan-algebraic properties of spin-factors

In what follows we restrict ourselves to the analysis of problems (1), (2) and (3), (4) for the case where V is a JB-algebra of a finite rank. In view of Theorem 2.14, the only new feature in the analysis of interior-point algorithms for solving (1), (2) and (3),(4) is a possible presence of infinite-dimensional spin-factors in the decomposition (15). In this section we describe some Jordan-algebraic aspects of a spin-factor $\mathbf{R} \times Y$ essential for future considerations.

Let $z = (s, y) \in \mathbf{R} \times Y$. We start with the description of the multiplication operator $L(z)$. It is convenient to introduce the following notation. We think of $(s, y) \in V$ as a column vector $\begin{pmatrix} s \\ y \end{pmatrix}$. Then each linear operator on $\mathbf{R} \times Y$ admits the following block partition:

$$\begin{pmatrix} \alpha & A \\ B & C \end{pmatrix},$$

where $\alpha \in \mathbf{R}$, $A : Y \to \mathbf{R}$, $B : \mathbf{R} \to Y$, $C : Y \to Y$. Then

$$\begin{pmatrix} \alpha & A \\ B & C \end{pmatrix} \begin{pmatrix} s \\ y \end{pmatrix} = \begin{pmatrix} \alpha s + Ay \\ Bs + Cy \end{pmatrix}.$$

Since Y is a Hilbert space, each continuous linear map $A : Y \to \mathbf{R}$ has the form:

$$Ay = (a|y)$$

for some $a \in Y$. Each map $B : \mathbf{R} \to Y$ has the form $Bs = sb, b = B1$. Given $y \in Y$, introduce notation:

$$l_y : Y \to \mathbf{R},\ l_y(y_1) = (y|y_1),\ y_1 \in Y.$$

Observe that $l_y^T : \mathbf{R} \to Y$ has the form:

$$l_y^T(s) = sy,\ s \in \mathbf{R}.$$

Here l_y^T is the transpose of l_y with respect to the given scalar product $(\cdot|\cdot)$ on Y and the standard scalar product on $\mathbf{R}$, i.e.,

$$sl_y(y_1) = (l_y^T(s)|y_1),\ s \in \mathbf{R},\ y_1 \in Y.$$

With this notation, we have

Proposition 3.1 *Let $z = (s, y) \in \mathbf{R} \times Y$. Then*

$$L(z) = \begin{pmatrix} s & l_y \\ l_y^T & sI_Y \end{pmatrix}. \tag{16}$$

Here I_Y is the identity operator on Y.

Proof. The result immediately follows from definitions. ■

Our next goal is to explicitly calculate the spectral decomposition (14) for the spin-factor $\mathbf{R} \times Y$.

Proposition 3.2 *Let $(s, y) \in \mathbf{R} \times Y, y \neq 0$. Consider*

$$e_1 = \frac{1}{2}\left(1, \frac{y}{\|y\|}\right),\ e_2 = \frac{1}{2}\left(1, -\frac{y}{\|y\|}\right)$$
$$\lambda_1 = s + \|y\|,\ \lambda_2 = s - \|y\|,\ \|y\| = \sqrt{(y|y)}. \tag{17}$$

Then

$$(s, y) = \lambda_1 e_1 + \lambda_2 e_2, \tag{18}$$
$$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 \circ e_2 = 0. \tag{19}$$

Proof. A direct calculation. ■

Proposition 3.3 *Let $z \in (s, y) \in \mathbf{R} \times Y$. Then*

$$z^2 - 2sz + (s^2 - (y|y))e = 0.$$

Here $e = (1, 0)$ is the unit element in the Jordan algebra $\mathbf{R} \times Y$.

Proof. A direct computation. ∎

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Remark: Following the standard terminology (see e.g. [1]), we introduce the following notation:

$$\mathrm{tr}(z) = 2s, \ \det(z) = s^2 - (y|y). \tag{20}$$

Comparing (17) with (20), we see that

$$\mathrm{tr}(z) = \lambda_1(z) + \lambda_2(z), \ \det(z) = \lambda_1(z)\lambda_2(z). \tag{21}$$

The next proposition describes the inverse of an element $z = (s, y)$ in a spin-factor $\mathbf{R} \times Y$.

Proposition 3.4 *An element $z \in \mathbf{R} \times Y$ is invertible if and only if $\det(z) \neq 0$. In this case*

$$z^{-1} = \frac{1}{\det(z)}(s, -y) = \lambda_1(z)^{-1}e_1 + \lambda_2(z)^{-1}e_2,$$

(see (17), (18)).

Proof. A direct computation. ∎

We next describe the quadratic representation (see (10)) in a spin-factor $\mathbf{R} \times Y$. Given $y \in Y$, we introduce a linear operator $y \otimes y \in \mathcal{L}(Y)$ as follows:

$$y \otimes y(y_1) = (y|y_1)y, \ y_1 \in Y. \tag{22}$$

Proposition 3.5 *Let $z = (s, y) \in \mathbf{R} \times Y$. Then*

$$P(s, y) = \det(z)I_V + 2\begin{pmatrix} (y|y) & sl_y \\ sl_y^T & y \otimes y \end{pmatrix}.$$

Here I_V is the identity map on $V = \mathbf{R} \times Y$.

Proof. By Proposition 3.1

$$L(z) = sI_V + \begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix}.$$

Hence,

$$L(z)^2 = s^2 I_V + 2s\begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix} + \begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix}^2.$$

But

$$\begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix}^2 = \begin{pmatrix} l_y l_y^T & 0 \\ 0 & l_y^T l_y \end{pmatrix}.$$

Further,

$$l_y l_y^T(t) = l_y(ty) = t(y|y), \ t \in \mathbf{R}.$$

Hence, $l_y l_y^T = (y|y)$. On the other hand,

$$l_y^T l_y(y_1) = l_y^T((y|y_1)) = (y|y_1)y, \ y_1 \in Y,$$

i.e.,

$$L(z)^2 = s^2 I_V + 2s \begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix} + \begin{pmatrix} (y|y) & 0 \\ 0 & y \otimes y \end{pmatrix}.$$

Now,

$$z^2 = (s^2 + (y|y), 2sy).$$

Hence, using Proposition 3.1 again, we obtain:

$$L(z^2) = [s^2 + (y|y)] I_V + 2s \begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix}.$$

Finally, by (10),

$$P(z) = 2L(z)^2 - L(z^2) = \det(z) I_V + 2s \begin{pmatrix} 0 & l_y \\ l_y^T & 0 \end{pmatrix} + 2 \begin{pmatrix} (y|y) & 0 \\ 0 & y \otimes y \end{pmatrix}.$$

■

We now describe the cone of squares in the spin-factor $\mathbf{R} \times Y$.

Proposition 3.6 *We have:*

$$\begin{aligned} \Omega &= \{(s, y) \in \mathbf{R} \times Y : s > \|y\|\}, \\ \bar{\Omega} &= \{(s, y) \in \mathbf{R} \times Y : s \geq \|y\|\}, && (23) \\ \bar{\Omega}^* &= \bar{\Omega}, && (24) \end{aligned}$$

i.e., the cone $\bar{\Omega}$ is self-dual.

Proof. Let $z = (s, y)$ have the spectral decomposition (18). By (21) and Proposition 3.4, z is invertible if and only if $\lambda_1(z) \neq 0$, $\lambda_2(z) \neq 0$. Using (19), we immediately see that

$$z^2 = \lambda_1(z)^2 e_1 + \lambda_2(z)^2 e_2.$$

Hence, by Proposition 2.10 $w \in \Omega$ implies

$$\lambda_1(w) > 0, \lambda_2(w) > 0. \tag{25}$$

On the other hand, using (17), (25) is equivalent to $s > \|y\|$. Inversely, $\lambda_1(w) > 0, \lambda_2(w) > 0$ implies $w = u^2$,

$$u = \sqrt{\lambda_1(w)} e_1 + \sqrt{\lambda_2(w)} e_2.$$

It remains to prove (24). Let $(t, x) \in \bar{\Omega}^*$. Then (see (5))

$$st + (x|y) \geq 0, \ \forall (s, y) \in \bar{\Omega}. \tag{26}$$

Since by (23) $(s, 0) \in \bar{\Omega}$ for $s > 0$, we deduce from (26) that $t \geq 0$. Take $\tilde{y} = -x$, $\tilde{s} = \|x\| + \varepsilon$, $\varepsilon > 0$. Obviously, $(\tilde{s}, \tilde{y}) \in \Omega$ and (26) yields:

$$t(\|x\| + \varepsilon) - \|x\|^2 \geq 0 \text{ for } \varepsilon > 0.$$

Taking limit as $\varepsilon \to 0$, we conclude that $t\|x\| \geq \|x\|^2$, i.e., $t \geq \|x\|$ (in the case $\|x\| = 0$, we have already proven $t \geq 0$). Inversely, let $(t, x) \in \mathbf{R} \times Y$ and $t \geq \|x\|$. Given $(s, y) \in \bar{\Omega}$,

$$ts + (x|y) \geq t\|y\| + (x|y) \geq t\|y\| - \|x\|\|y\| = (t - \|x\|)\|y\| \geq 0.$$

Here we used the Cauchy-Schwarz inequality. ■

Proposition 3.7 *Given* $z_1, z_2 \in \mathbf{R} \times Y$,

$$\det(P(z_1)z_2) = [\det(z_1)]^2 \det z_2,$$

where $\det(z)$ *is defined in (20).*

Proof. A direct computation. ∎

We introduce now a canonical scalar product on $\mathbf{R} \times Y$:

$$\langle z_1, z_2 \rangle = \mathrm{tr}(z_1 \circ z_2)$$

If $z_i = (s_i, y_i)$, $i = 1, 2$, then by (20):

$$\langle z_1, z_2 \rangle = 2(s_1 s_2 + (y_1 | y_2)). \tag{27}$$

Proposition 3.8 *Given* $z \in \bar{\Omega}$, $L(z) \geq 0$, *i.e.,*

$$\langle L(z)z_1, z_1 \rangle \geq 0, \ \forall z_1 \in \mathbf{R} \times Y \tag{28}$$

Proof. Let $z = (s, y)$, $z_1 = (t, x)$. Since $(s, y) \in \bar{\Omega}$, We have $s \geq \sqrt{(y|y)}$ by (23). Evaluating (28), we see that we need to check that

$$st^2 + 2(x|y)t + s(x|x) \geq 0, \ \forall t \in \mathbf{R}, \ x \in Y.$$

We can assume without loss of generality that $s > 0$ (if $s = 0$, then $y = 0$). Thus, we need to check that the quadratic in t polynomial

$$t^2 + \frac{2(x|y)t}{s} + (x|x)$$

is everywhere nonnegative. But its discriminant has the form

$$\Delta = \frac{(x|y)^2}{s^2} - (x|x).$$

Using Cauchy-Schwarz inequality and $s^2 \geq (y|y)$, we obtain:

$$\Delta \leq \frac{(x|x)(y|y)}{s^2} - (x|x) \leq 0.$$

The result follows. ∎

In the next section, we will extend the polynomial-time convergence proof of primal-dual algorithms developed in [5] for finite-dimensional symmetric cone programs to the current infinite-dimensional setting. For this purpose, we need the following theorem which is an analogue of the result by Sturm [15] and plays a fundamental role in the analysis of finite-dimensional case.

Theorem 3.9 *Let* $z \in \Omega$. *Then* $L(z)$ *is invertible in* $\mathcal{L}(\mathbf{R} \times Y)$ *(i.e.,* $L(z)^{-1}$ *is a bounded linear operator from* $\mathbf{R} \times Y$ *onto itself) and, moreover,*

$$L(z)^{-1}\Omega \subset \Omega.$$

Proof. Let $z = (s, y) \in \Omega$ and $(t, x) \in \mathbf{R} \times Y$. We claim that

$$\begin{aligned} L(z)^{-1} \begin{pmatrix} t \\ x \end{pmatrix} &= \begin{pmatrix} r \\ u \end{pmatrix}, \\ r &= \frac{st - (x|y)}{\det(z)}, & (29) \\ u &= \frac{1}{s}\left(x + \frac{(x|y) - st}{\det(z)} y\right). & (30) \end{aligned}$$

It suffices to check that

$$L(z) \begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} t \\ x \end{pmatrix},$$

which is a direct computation by using (16).

In order to prove the theorem, given $s > \|y\|$ and $t > \|x\|$, we need to check that $r > \|u\|$ (see Proposition 3.6). Observe that $(s, -y) \in \Omega$. Hence, (29) and Proposition 3.6 imply that $r \geq 0$. Observe that by (29), (30):

$$u = \frac{1}{s}(x - ry)$$

and consequently

$$(u|u) = \frac{1}{s^2}((x|x) + r^2(y|y) - 2r(x|y)).$$

Thus, $r^2 > (u|u)$ is equivalent to:

$$r^2(s^2 - (y|y)) + 2r(x|y) > (x|x). \tag{31}$$

Using (29), we can rewrite (31) in the form

$$(st - (x|y))^2 + 2(x|y)(st - (x|y)) > (x|x)\det(z)$$

or

$$s^2t^2 - 2st(x|y) + (x|y)^2 + 2st(x|y) - 2(x|y)^2 > (x|x)(s^2 - (y|y))$$

(Recall that $\det(z) = s^2 - (y|y)$). This can be further simplified to:

$$s^2(t^2 - (x|x)) > (x|y)^2 - (x|x)(y|y).$$

But the last inequality is obvious, since $t^2 > (x|x)$ and $|(x|y)|^2 \leq (x|x)(y|y)$ by Cauchy-Schwarz inequality. ■

4 Primal-dual algorithms

We now return to our pair of dual problems (1), (2) and (3), (4). In the remaining part of the paper we will assume that V is a JB-algebra of a finite rank, Ω is the cone of squares in V and $r = rank(V)$ in the definition of the duality gap (7). We continue to assume that the condition (6) is satisfied. The vector space V is endowed with the canonical Hilbert space structure. First of all there exists a canonical linear form $\mathrm{tr} : V \to \mathbf{R}$. It is defined through the direct sum decomposition (15). If $\dim V_i < \infty$, then there is a standard way to define the trace operator [1]. Otherwise V_i is an infinite-dimensional spin-factor and we use (20).

The scalar product is then defined as:

$$\langle z, w\rangle = \mathrm{tr}(z \circ w),\ \ z, w \in V.$$

Proposition 3.6 (along with the standard properties of finite-dimensional Euclidean Jordan Algebras) enables us to conclude that

$$\bar{\Omega}^* = \bar{\Omega}.$$

The advantage of the Jordan-algebraic framework suggested in the present paper is that we can easily carry over literally all interior-point algorithms along with their complexity estimates to the infinite-dimensional situation. Let us illustrate this point by considering a long-step primal-dual algorithm based on the Nesterov-Todd direction [13].

The main ingredient in the construction of primal-dual algorithms is the choice of a "descent" direction which drives the duality gap μ to zero. The class of scaling-invariant "descent" directions is obtained by solving the following system of linear equations. Given $(z, w) \in \Omega\times\Omega$ and $g \in GL(\Omega)$ (see (13)), observe first of all that $g^{-T} \in GL(\Omega)$, since $\bar{\Omega}^* = \bar{\Omega}$. The system of linear equations has the form:

$$L(\tilde{z})\tilde{\xi} + L(\tilde{w})\tilde{\eta} = \gamma\mu(z, w)e - \tilde{z} \circ \tilde{w}, \tag{32}$$

$$\tilde{\xi} \in g(X),\ \ \tilde{\eta} \in g^{-T}(X^{\perp}), \tag{33}$$

$$\tilde{z} = g(z),\ \ \tilde{w} = g^{-T}(w). \tag{34}$$

Here $0 < \gamma < 1$ is a real parameter and $(\tilde{\xi}, \tilde{\eta})$ is a scaled "descent direction." For a motivation of this construction see e.g. [4, 10, 16]. We consider a special choice of the cone automorphism g.

Proposition 4.1 *Given $(z_1, z_2) \in \Omega \times \Omega$, there exists a unique $z_3 \in \Omega$ such that*

$$P(z_3)z_1 = z_2. \tag{35}$$

Proof. The decomposition (15) leads to the corresponding decomposition of the cone of squares Ω:

$$\Omega = \Omega_1 \oplus \Omega_2 \oplus ... \oplus \Omega_k, \tag{36}$$

where Ω_i is the cone of squares in V_i, $i = 1, 2, ..., k$.

Hence, to prove (35) it suffices to consider two cases: a) dim $V < \infty$ and b) $V = \mathbf{R} \times Y$ is a spin-factor. For the case a) we refer to [4]. We derive an explicit formula for the case b). The derivation below is a simplified modification of the one given in [16] for the analysis of the Nesterov-Todd direction for the finite-dimensional second-order cone programming.

Let $z_1 = (s, y)$, $z_2 = (t, x)$, $z_3 = (r, u)$. Consider, first, the case $\det(z_1) = \det(z_2) = 1$. By Proposition 3.7 we should have $\det(z_3) = 1$ or

$$r^2 - (u|u) = 1. \tag{37}$$

Using Proposition 3.5, we can rewrite (35) in the form:

$$y + u\langle(r, u), (s, y)\rangle = x, \tag{38}$$

$$s + 2(u|u)s + 2r(u|y) = t, \tag{39}$$

where

$$\langle(r, u), (s, y)\rangle = 2rs + 2(u|y)$$

(Compare with (27)). We can eliminate $(u|u)$ from (39), using (37). We obtain:

$$r\langle (r,u),(s,y)\rangle = t+s \tag{40}$$

Now (38) can be rewritten in the form:

$$u\langle (r,u),(s,y)\rangle = x-y. \tag{41}$$

From (40) and (41), we obtain:

$$r = \frac{s+t}{\delta},\ u = \frac{x-y}{\delta},\ \delta = \langle (r,u),(s,y)\rangle. \tag{42}$$

Substituting (41), (42) into (37), we obtain:

$$\delta^2 = (t+s)^2 - (x-y|x-y) = 2 + \langle (t,x),(s,y)\rangle, \tag{43}$$

where we used $\det(z_1) = \det(z_2) = 1$. The formulas (42), (43) give explicit expressions for (r,u), proving the uniqueness of z_3 in (35).

A direct substitution of (42), (43) into (38), (39) shows that $z_3 = (r,u)$ solves (35). The general case can be reduced to the considered case as follows. Let

$$\mu_i = \frac{1}{\sqrt{\det(z_i)}},\ i = 1,2.$$

Then $\det(\mu_i z_i) = 1$. Let $\tilde{z}_3 \in \Omega$ be such that $P(\tilde{z}_3)(\mu_1 z_1) = \mu_2 z_2$. Then

$$P\left(\sqrt{\frac{\mu_1}{\mu_2}}\tilde{z}_3\right) z_1 = z_2,$$

i.e,

$$z_3 = \sqrt{\frac{\mu_1}{\mu_2}}\tilde{z}_3. \tag{44}$$

This completes the proof ■

Combining (42)–(44) we obtain

Corollary 4.2 *Let $z_1, z_2 \in \Omega$, $z_1 = (s,y), z_2 = (t,x)$. Consider $z_3 = (r,u)$ with*

$$r = \sqrt{\frac{\mu_1}{\mu_2}}\frac{\mu_1 s + \mu_2 t}{\sqrt{2 + \mu_1\mu_2\langle (s,y),(t,x)\rangle}},$$
$$u = \sqrt{\frac{\mu_1}{\mu_2}}\frac{\mu_2 x - \mu_1 y}{\sqrt{2 + \mu_1\mu_2\langle (s,y),(t,x)\rangle}},$$
$$\mu_i = \frac{1}{\sqrt{\det(z_i)}},\ i = 1,2.$$

Then $z_3 \in \Omega$ is a unique solution to (35) for the case $V = \mathbf{R} \times Y$.

Proposition 4.3 *Let Ω be the cone of squares in the spin-factor $\mathbf{R} \times Y$ and $(s,y) \in \Omega$. Consider*

$$z = \left(\frac{\mu}{2}, \frac{y}{\mu}\right),\ \mu = \sqrt{s + \|y\|} + \sqrt{s - \|y\|}.$$

Then $z \in \Omega$ and $z^2 = (s,y)$. Moreover, if

$$(s,y) = \lambda_1 e_1 + \lambda_2 e_2$$

be the spectral decomposition of (s,y), then

$$z = \sqrt{\lambda_1}e_1 + \sqrt{\lambda_2}e_2.$$

Proof. A direct computation. ■

Remark: We denote z by $(s, y)^{1/2}$. Given $z \in \Omega$, we have $P(z^{1/2})^2 = P(z)$. It easily follows from (11). Thus

$$P(z^{1/2}) = P(z)^{1/2}. \tag{45}$$

Observe that (45) holds for an arbitrary JB-algebra V of a finite rank. It follows from decomposition (15) and the validity of (45) in the case $\dim V < \infty$ (see [1]).

We use Proposition 4.1 to introduce the so-called Nesterov-Todd direction in the infinite-dimensional setting. Given $z_1, z_2 \in \Omega$, let $z_3 \in \Omega$ be such that (35) holds. Take $g = P(z_3^{1/2}) \in GL(\Omega)$. Then

$$g z_1 = g^{-T} z_2 = v$$

and equations (32)–(34) takes the form

$$\tilde{\xi} + \tilde{\eta} = \gamma\mu(v, v)v^{-1} - v, \tag{46}$$

$$\tilde{\xi} \in P(z_3^{1/2})X, \ \tilde{\eta} \in P(z_3^{-1/2})(X^{\perp}), \tag{47}$$

$$v = P(z_3^{1/2})z_1 = P(z_3^{-1/2})z_2. \tag{48}$$

Observe that in the original variables, (46)–(48) has the form:

$$\xi + P(z_3)^{-1}\eta = \gamma\mu(z_1, z_2)z_2^{-1} - z_1. \tag{49}$$

$$\xi \in X, \ \eta \in X^{\perp} \tag{50}$$

It is obvious from (46)–(48) that the Nesterov-Todd direction exists and unique. Indeed, (46) and (47) show that $\tilde{\xi}$ is the orthogonal projection of the vector $\gamma\mu(v, v)v^{-1} - v$ onto the closed vector subspace $P(z_3^{1/2})X$. The existence and uniqueness of other popular directions (e.g., HRVW/KSH/M direction [16]) can be shown in a similar fashion.

As an example, consider a long-step primal-dual algorithm based on the Nesterov-Todd direction. Given $(z_1, z_2) \in \Omega \times \Omega$, let $z_4 \in \Omega$ be such that

$$v = v(z_1, z_2) = P(z_4^{1/2})z_2 = P(z_4)^{-1/2}z_1.$$

(Observe that $z_4 = z_3^{-1}$ in our previous notation.) Given $0 < \beta < 1$, introduce the so-called wide-neighborhood in $\Omega \times \Omega$:

$$N_{-\infty}(\beta) = \{(z_1, z_2) \in \Omega \times \Omega : \lambda_{\min}(v(z_1, z_2)^2) \geq (1 - \beta)\mu(z_1, z_2)\}.$$

Here $\lambda_{\min}(z) = \min\{\lambda_i : i = 1, 2, ..., r\}$ in the decomposition (14). We can show that this neighborhood is scaling invariant in exactly the same way as in the case of finite-dimensional Euclidean Jordan algebra [5]. Note that the duality gap μ is also scaling invariant.

Fix $\varepsilon > 0$. Suppose that $(z_1^{(0)}, z_2^{(0)}) \in \mathrm{int}(\mathcal{F}) \cap N_{-\infty}(\beta)$ (see (6)). Let (ξ_k, η_k) be the Nesterov-Todd direction at the point $(z_1^{(k)}, z_2^{(k)})$ defined as in (49), (50). Let $\bar{t}$ be the largest value of $t \in [0, 1]$ such that $z_1^{(k)} + t\xi^{(k)}, z_2^{(k)} + t\eta^{(k)}) \in N_{-\infty}(\beta)$. Set $(z_1^{(k+1)}, z_2^{(k+1)}) = (z_1^{(k)} + \bar{t}\xi_k, z_2^{(k)} + \bar{t}\eta_k)$. We stop the iteration when $\mu(z_1^{(k)}, z_2^{(k)}) \leq \varepsilon$.

Theorem 4.4 *For the primal-dual algorithm described above, we have:*

$$\mu(z_1^{(k)}, z_2^{(k)}) \leq \varepsilon$$

for

$$k \geq r \frac{\log \left(\dfrac{\mu(z_1^{(0)}, z_2^{(0)})}{\varepsilon} \right)}{(1-\gamma)\delta}$$

provided $\sqrt{\beta(1-\beta)} \leq 1-\gamma$ *and*

$$\delta(\beta, \gamma) = \frac{2\beta\gamma}{\beta\gamma^2/(1-\beta) + (1-\gamma)^2}.$$

The proof of this theorem is exactly as [5] where the case of general symmetric finite-dimensional cone programming have been considered. Observe that it is essential that we have Theorem 3.9 at our disposal. A direct proof of the analogous theorem for finite-dimensional second-order cone programs developed in [16] is also extended in a straightforward way to prove the theorem under the restriction that Ω is the direct sum of several finite/infinite-dimensional second-order cones.

Corollary 4.5 *There exists a sequence* $(z_1^{(k)}, z_2^{(k)}) \in \mathrm{int}(\mathcal{F})$ *such that*

$$\mu(z_1^{(k)}, z_2^{(k)}) \rightarrow 0$$

where $k \rightarrow \infty$.

The next theorem provides an infinite-dimensional generalization of the optimality criterion for (1), (2), and (3), (4) (see e.g. [2]).

Theorem 4.6 *Suppose that* V *is a JB-algebra of a finite rank,* Ω *is a cone of squares in* V *and (6) is satisfied. Then problems (1), (2), and (3), (4) both have optimal solutions. The sets of optimal solutions for both problems are bounded closed convex sets. If* z^* *(respectively,* w^**) is an optimal solution to (1), (2), (respectively, (3), (4)), then*

$$\langle z^*, w^* \rangle = 0. \tag{51}$$

Inversely, if z^* *satisfies (2),* w^* *satisfies (4) and (51) holds, then* z^* *is an optimal solution to (1), (2), and* w^* *is an optimal solution to (3), (4).*

Proof. Consider the sequence $(z^{(k)}, w^{(k)}) \in \mathrm{int}(\mathcal{F})$ such that $\langle z^{(k)}, w^{(k)} \rangle \rightarrow 0$, $k \rightarrow +\infty$. Without loss of generality, we can assume that

$$\langle z^{(k)}, w^{(k)} \rangle \leq \langle z^{(0)}, w^{(0)} \rangle, \;\; k = 0, 1, ...$$

Since $z^{(k)} - z^{(0)} \in X$, $w^{(k)} - w^{(0)} \in X^{\perp}$, we have:

$$\langle z^{(k)} - z^{(0)}, w^{(k)} - w^{(0)} \rangle = 0, \;\; k = 0, 1, ...$$

Hence,

$$\langle z^{(k)}, w^{(0)} \rangle + \langle z^{(0)}, w^{(k)} \rangle = \langle z^{(0)}, w^{(0)} \rangle + \langle z^{(k)}, w^{(k)} \rangle \leq 2\langle z^{(0)}, w^{(0)} \rangle, \;\; k = 0, 1, ... \tag{52}$$

Observe that (52) implies that $(z^{(k)}, w^{(k)}), k = 0, 1, ...$, is bounded. Indeed, due to decomposition (36), it suffices to consider the case where V is irreducible. If $\dim V < +\infty$, the result is well-known (see e.g. [1]). Let $V = \mathbf{R} \times Y$ be a spin-factor. Let $(t, x) \in \mathbf{R} \times Y$, $t > \|x\|$. Given $\alpha > 0$, consider the set

$$B_\alpha = \{(s, y) \in \mathbf{R} \times Y : s \geq \|y\|,\ st + (y|x) \leq \alpha\}.$$

If $(s, y) \in B_\alpha$, then by Cauchy-Schwarz inequality:

$$st + (y|x) \geq st - \|y\|\|x\| = s(t - \|x\|) + \|x\|(s - \|y\|) \geq s(t - \|x\|).$$

Hence,

$$\|y\| \leq s \leq \frac{\alpha}{t - \|x\|}.$$

Thus, the set B_α is bounded.

Since $(z^{(k)}, w^{(k)})$ is bounded, it follows that there is a subsequence $(z^{(k_l)}, w^{(k_l)})$, $l = 0, 1, \ldots$ which converges weakly to a feasible point (z^*, w^*). Observe that the feasible region $\mathcal{F}$ is convex and closed and, hence, weakly closed. Let us show that

$$\langle z^*, w^* \rangle = 0. \tag{53}$$

To simplify the notation, assume that $(z^{(k)}, w^{(k)})$ weakly converges to (z^*, w^*) when $k \to \infty$. We have:

$$\langle b - z^{(k)}, a - w^{(k)} \rangle = 0 \text{ or } \langle a, b \rangle + \langle z^{(k)}, w^{(k)} \rangle = \langle a, z^{(k)} \rangle + \langle b, w^{(k)} \rangle. \tag{54}$$

Taking limit in (54), when $k \to \infty$ and using $\langle z^{(k)}, w^{(k)} \rangle \to 0$, $z^{(k)} \to z^*$ (weakly), $w^{(k)} \to w^*$ (weakly), we obtain:

$$\langle a, b \rangle = \langle a, z^* \rangle + \langle b, w^* \rangle. \tag{55}$$

On the other hand,

$$\langle a - w^*, b - z^* \rangle = 0.$$

Comparing this with (55), we conclude that (53) holds. Let us show that each $(\tilde{z}, \tilde{w}) \in \mathcal{F}$ such that $\langle \tilde{z}, \tilde{w} \rangle = 0$ is a pair of optimal solutions for (1), (2), and (3), (4), respectively. Let z_1 be feasible for (1), (2). Then

$$\begin{aligned} \langle a, b \rangle &= \langle b, \tilde{w} \rangle + \langle a, \tilde{z} \rangle, \\ \langle a, b \rangle + \langle z_1, \tilde{w} \rangle &= \langle b, \tilde{w} \rangle + \langle a, z_1 \rangle, \end{aligned}$$

Using $\langle z, \tilde{w} \rangle \geq 0$, we obtain:

$$\langle b, \tilde{w} \rangle + \langle a, z_1 \rangle \geq \langle b, \tilde{w} \rangle + \langle a, \tilde{z} \rangle,$$

i.e., $\langle a, z_1 \rangle \geq \langle a, \tilde{z} \rangle$. Thus $\tilde{z}$ is an optimal solution to (1), (2). Similarly, we show that $\tilde{w}$ is an optimal solution to (3), (4). In particular, (z^*, w^*) constructed above is the pair of optimal solutions to (1), (2) and (3), (4), respectively. Besides, $\langle z^*, w^* \rangle = 0$. We then immediately see as above that if $\langle z, w \rangle > 0$ for a feasible pair (z, w),then (z, w) is not a pair of optimal solutions. Take any $(z, w) \in \mathrm{int}(\mathcal{F})$. Then for any optimal pair (z^*, w^*), the condition (53) implies:

$$\langle w, z^* \rangle + \langle z, w^* \rangle = \langle z, w \rangle.$$

Reasoning as in the proof of boundedness of the sequence $(z^{(k)}, w^{(k)})$ above, we conclude that the set of optimal pairs is bounded. ∎

5 Example

Consider the following optimization problem:

$$\max_{i\leq i\leq m} \|W_i y\| \to \min, \tag{56}$$
$$y \in c + Z. \tag{57}$$

Here $W_i : Y \to Y, i = 1, 2, ..., m$, are bounded linear operators on Y, Z is a closed vector subspace in the Hilbert space Y. Recall that $(\cdot|\cdot)$ the inner product associated with Y.

We can rewrite (56) and (57) in the form:

$$t \to \min, \tag{58}$$
$$\|W_i y\| \leq t,\ \ i = 1, ..., m, \tag{59}$$
$$y \in c + Z. \tag{60}$$

Our immediate goal is to rewrite (58)–(60) in the form (1), (2).

Let $V_1 = \mathbf{R} \times Y$, $V = V_1 \times .... \times V_1$ (m times), $\Omega_1 = \{(s, y) \in \mathbf{R} \times Y : s > \|y\|\}$, $\Omega = \Omega_1 \times ... \times \Omega_1$. Consider a linear operator

$$\Lambda : V_1 \to V,$$
$$\Lambda(\mu, \zeta) = ((\mu, W_1\zeta), (\mu, W_2\zeta), ..., (\mu, W_m\zeta)).$$

Let, further, $a = ((1, 0), (0, 0), ..., (0, 0)) \in V$, $b = ((0, W_1c), (0, W_2c), ..., (0, W_mc)) \in V$, $z = (z_1, ..., z_m)$, $z_i = (t_i, x_i) \in V_1, i = 1, ..., m$. The scalar product in V is defined as follows:

$$\langle((t_1^{(1)}, x_1^{(1)}), ..., (t_m^{(1)}, x_m^{(1)})), ((t_1^{(2)}, x_1^{(2)}), ..., (t_m^{(2)}, x_m^{(2)}))\rangle = \sum_{i=1}^{m}[t_i^{(1)}t_i^{(2)} + (x_i^{(1)}|x_i^{(2)})].$$

We now can rewrite (58)–(60) in the form:

$$\langle a, z\rangle \to \min,$$
$$z \in (b + X) \cap \bar{\Omega},$$

where

$$X = \Lambda(\mathbf{R} \times Z). \tag{61}$$

An easy calculation shows that the orthogonal complement $X^\perp$ of X in V has the form:

$$X^\perp = \{((r_1, u_1), ..., (r_m, u_m)) \in V : r_1 + r_2 + ...r_m = 0,\ \sum_{i=1}^{m} W_i^* u_i \in Z^\perp\},$$

where $Z^\perp$ is the orthogonal complement of Z in Y and W_i^* is the adjoint of W_i for each i. According to (3), (4), its dual will be of the form

$$\sum_{i=1}^{m}(W_i c|u_i) \to \min$$
$$\sum_{i=1}^{m} r_i = m,\ \ \|u_i\| \leq r_i,\ \ i = 1, 2, ..., m,$$
$$W_1^* u_1 + W_2^* u_2 + ... + W_m^* u_m \in Z^\perp.$$

It is easy to see that the condition (6) is satisfied. We can apply Theorem 4.6 in this example. Consider the Nesterov-Todd direction for our problem. Let $(m_1, m_2) \in \Omega \times \Omega$. According to (49) and (50) we need to find $(\xi, \eta) \in X \times X^{\perp}$ such that

$$P(z)\xi + \eta = \Delta. \tag{62}$$

Here $z \in \Omega$ is the scaling point uniquely determined from the equation $P(z)m_1 = m_2$ and $\Delta \in V$ is a known vector, depending on m_1, m_2.

We can rewrite (62) in the form:

$$P(z)\xi - \Delta \in X^{\perp},\ \xi \in X, \tag{63}$$

which is equivalent to:

$$\frac{\langle P(z)\xi, \xi\rangle}{2} - \langle \xi, \Delta\rangle \to \ \min, \tag{64}$$

$$\xi \in X. \tag{65}$$

Using the parameterization (61), we can write (64), (65) in the form:

$$\begin{aligned} &\rho(\mu, \zeta) = \frac{\langle P(z)\xi, \xi\rangle}{2} - \langle \xi, \Delta\rangle \to \ \min, \\ &(\mu, \zeta) \in \mathbf{R} \times Z. \end{aligned}$$

Observe that ρ is a convex quadratic function in variables (μ, ζ). Let $z = (z_1, ..., z_m)$, $z_i = (t_i, x_i) \in \Omega_1$, $\Delta = ((r_1^*, u_1^*), ..., (r_m^*, u_m^*)) \in V$, $\xi = (\xi_1, ..., \xi_m) \in X$. We obviously have:

$$P(z)\xi = (P(z_1)\xi_1, ..., P(z_m)\xi_m).$$

Using Proposition 3.5, we can easily calculate that

$$\begin{aligned} \rho(\mu, \zeta) &= \frac{1}{2}\sum_{i=1}^{m}(t_i^2 - \|x_i\|^2)\|W_i\zeta\|^2 + \sum_{i=1}^{m}(x_i|W_i\zeta)^2 - \sum_{i=1}^{m}(u_i^*|W_i\zeta) + \frac{\nu_1\mu^2}{2} + \nu_2\mu, \\ \nu_1 &= \sum_{i=1}^{m}(t_i^2 + \|x_i\|^2),\ \nu_2 = 2\sum_{i=1}^{m} t_i(x_i|W_i\zeta) - \sum_{i=1}^{m} r_i^*. \end{aligned}$$

Hence,

$$\begin{aligned} \phi(\zeta) &= \min\{\rho(\mu, \zeta) : \mu \in \mathbf{R}\} \\ &= \frac{(\zeta, M\zeta)}{2} + \frac{1}{2}(\zeta|(\sum_{i=1}^{m+1} \varepsilon_i(v_i \otimes v_i))\zeta) + (v_0|\zeta) - \frac{(\sum_{i=1}^{m} r_i^*)^2}{2\nu_1} \\ &= \frac{(\zeta, M\zeta)}{2} + \frac{1}{2}\sum_{i=1}^{m+1} \varepsilon_i(v_i|\zeta)^2 + (v_0|\zeta) - \frac{(\sum_{i=1}^{m} r_i^*)^2}{2\nu_1}, \end{aligned} \tag{66}$$

where

$$\begin{aligned} M &= \sum_{i=1}^{m}(t_i^2 - \|x_i\|^2)W_i^*W_i, \\ v_i &= \sqrt{2}W_i^*x_i,\ i = 1, 2, ..., m, \\ v_{m+1} &= \frac{2}{\sqrt{\nu_1}}\sum_{i=1}^{m} t_iW_i^*x_i, \\ \varepsilon_i &= 1,\ i = 1, ..., m,\ \varepsilon_{m+1} = -1, \\ v_0 &= \sqrt{\frac{1}{\nu_1}}(\sum_{i=1}^{m} r_i^*)v_{m+1} - \sum_{i=1}^{m} W_i^*u_i^*, \end{aligned}$$

and $v_i \otimes v_i$ is defined as in (22).

Assume that

$$(M\zeta|\zeta) \geq \delta\|\zeta\|^2, \quad \forall \zeta \in Z \tag{67}$$

for some $\delta > 0$. Under this condition, we can show that the problem

$$\phi(\zeta) \to \min, \ \zeta \in Z, \tag{68}$$

where ϕ is described in (66) can be reduced to solving $(m+1)$ problems of the form

$$\frac{1}{2}(M\zeta|\zeta) + (v|\zeta) \to \min, \ \zeta \in Z \tag{69}$$

for appropriate choices of $v \in Y$, and the system of $(m+1) \times (m+1)$ linear algebraic equations. This observation makes sense because in some applications we have nice efficient algorithms to solve (69). Below we describe the procedure.

Let $\zeta_0 \in Z$ be the optimal solution to the problem

$$\frac{(\zeta|M\zeta)}{2} + (v_0|\zeta) \to \min, \ \zeta \in Z, \tag{70}$$

and $\zeta_i \in Z$, $i = 1, .., m+1$ be the optimal solutions to the problems

$$\frac{(\zeta|M\zeta)}{2} + (\varepsilon_i v_i|\zeta) \to \min, \ \zeta \in Z. \tag{71}$$

Let $S = [s_{ij}]$, $s_{ij} = (\zeta_i|v_j)$, $i, j = 1, 2, ..., m+1$, and

$$(I - S)\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_{m+1} \end{pmatrix} = \begin{pmatrix} (v_0|\zeta_1) \\ \vdots \\ (v_0|\zeta_{m+1}) \end{pmatrix}, \tag{72}$$

then

$$\zeta(\delta) = \zeta_0 + \sum_{i=1}^{m+1} \delta_i \zeta_i$$

is an optimal solution to the problem (68). The procedure is a simple modification of the argument in [7] which deals with a version of the Sherman-Morrison-Woodbury formula in the infinite-dimensional setting. We give a derivation of (72) in Appendix.

Remark: It is easy to see that if (67) holds at an interior feasible solution z, then the linear operator $W : Y \to Y \times Y \times ... \times Y$ (m times) defined as $Wu \equiv (W_1 u, ..., W_m u)$ is invertible. Then y ((56) and (57)) is determined uniquely from z.

Consider now a more concrete situation in control theory which is similar to [6]. Denote by $L_2^n[0, T]$ the vector space of square integrable functions $f : [0, T] \to \mathbf{R}^n$. Let

$$Y = L_2^n[0, T] \times L_2^l[0, T], \ T > 0,$$

and

$$\begin{aligned} Z \ = \ & \{(\alpha, \beta) \in Y : \alpha \text{ is absolutely continuous on } [0, T], \\ & \alpha(0) = 0, \dot{\alpha}(t) = A(t)\alpha(t) + B(t)\beta(t), \ t \in [0, T]\}. \end{aligned}$$

Here $A(t)$ (respectively $B(t)$) is an n by n (respectively, n by l) continuous matrix-valued function. Observe that

$$((\alpha_1, \beta_1)|(\alpha_2, \beta_2)) = \int_0^T [\alpha_1^T(t)\alpha_2(t) + \beta_1^T(t)\beta_2^T(t)]dt,\ (\alpha_i, \beta_i) \in Y,\ i = 1, 2.$$

In this case, $Z^\perp$ is easily calculated:

$$Z^\perp = \{(\dot{p} + A^T p, B^T p) : p \text{ is absolutely continous on } [0,T], \dot{p} \in L_2^n[0,T], p(T) = 0\}.$$

In the following, we deal with the following min-max optimization problem:

$$\max_i \int_0^T [(\alpha(t) - \bar{\alpha}_i(t))^T Q_i(\alpha(t) - \bar{\alpha}_i(t)) + (\beta(t) - \bar{\beta}_i)^T R_i(\beta(t) - \bar{\beta}_i(t))]dt, \to \min \tag{73}$$

where $Q_i(t)$ (respectively $R_i(t)$) is a continuous matrix-valued function such that $Q_i(t) = Q_i^T(t)$, $R_i(t) = R_i^T(t)$ are positive-definite symmetric matrices for any $t \in (0, T]$ and $(\bar{\alpha}(t), \bar{\beta}(t)) \in Y$. This problem is a very important problem in control theory, namely, a problem of multi-criteria design of the analytic regulator. This problem can be solved with our algorithm as follows.

For $i = 1, ..., m$, let $L_{Q_i}(t)$ and $L_{R_i}(t)$ be the lower triangular matrices obtained with the Cholesky factorizations of $Q_i(t)$ and $R_i(t)$, respectively. Letting

$$W_i \equiv \begin{pmatrix} L_{Q_i}^T & 0 \\ 0 & L_{R_i}^T \end{pmatrix},\ i = 1, 2, ..., m$$

in (58)–(60), we obtain the problem equivalent to (73). In this case, we have

$$M = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix},$$

where

$$Q = \sum_{i=1}^m (t_i^2 - \|x_i\|^2) Q_i, \quad R = \sum_{i=1}^m (t_i^2 - \|x_i\|^2) R_i.$$

It is readily seen that (67) is satisfied here. The major part in computation of the Nesterov-Todd search direction is solution of (69) (with different v) to obtain ζ_i ($i = 0, ..., m+1$). Interestingly, this can be done as follows just by solving a matrix differential Riccati equation.

Let $v = (\gamma, \theta)$ in (69). Observe that the optimality condition is

$$M\zeta + v \in Z^\perp.$$

we are done if we can find $\zeta = (\alpha, \beta)$ satisfying the following condition:

$$\begin{aligned} Q\alpha + \gamma &= -\dot{p} - A^T p,\ \alpha(0) = 0, && (74) \\ R\beta + \theta &= -B^T p,\ p(T) = 0, && (75) \\ \dot{\alpha} &= A\alpha + B\beta. && (76) \end{aligned}$$

Let us try to find p in the form:

$$p = K\alpha + \rho,\ K(T) = 0,\ \rho(T) = 0. \tag{77}$$

where $K = K(t)$ is $n \times n$ matrix-valued function. Then

$$\dot{p} = \dot{K}\alpha + K\dot{\alpha} + \dot{\rho}. \tag{78}$$

Substituting this into (74), (75), we arrive at the following system of equations.

$$\dot{K} + A^T K + KA - KBR^{-1}B^T K + Q = 0,\ K(T) = 0, \tag{79}$$

$$\dot{\rho} + (A^T - KBR^{-1}B^T)\rho = -\gamma + KB\theta,\ \rho(T) = 0. \tag{80}$$

The system (79) is a matrix differential Riccati equation which admits a unique solution on the interval $[0, T]$ under natural control-theoretic constraints on the pair (A, B). To find $\zeta = (\alpha, \beta)$, we need to solve a linear system (80) and then find α and β using (74)–(78). Observe that the matrix differential Riccati equation (79) does not depend on $v = (\gamma, \theta)$, which is $\varepsilon_i v_i$ $(i = 0, 1, .., m+1)$ in our case. This means that (79) needs to be solved just once in one computation of the Nesterov-Todd direction, and (80) needs to be integrated $m + 2$ times.

6 Concluding Remarks

In the present paper we have considered infinite-dimensional generalization of interior-point algorithms using the framework of infinite-dimensional Jordan algebras of finite rank. Specifically, we developed a framework for primal-dual interior-point algorithms associated with the infinite-dimensional spin-factors and established a polynomial convergence result using the Nesterov-Todd direction. Though we have analyzed in detail only one primal-dual algorithm based on the Nesterov-Todd direction, it is pretty clear how to generalize other interior-point algorithms analyzed earlier in the finite-dimensional setting of Euclidean Jordan algebras.

We showed by considering an important example of a control problem that Nesterov-Todd direction can be calculated in an efficient way. Other popular directions (e.g., HRVW/KSH/M direction) can be analyzed in a similar fashion.

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Appendix: Derivation of (72)

First we observe that the functional which gives the optimal solution of (69) is linear with respect to v. Let M_Z be the restriction of superposition of the orthogonal projection to Z with M, to Z. Since M is positive definite on Z, there exists inverse of M_Z. We denote the inverse by M_Z^{-1}. Furthermore, let Π_Z be the orthogonal projection from X onto Z. Then the optimal solution of (69) is given as

$$\zeta = -M_Z^{-1}\Pi_Z v. \tag{81}$$

The optimality condition of (68) is

$$M\zeta + v_0 + \sum_{i=1}^{m+1} (v_i|\zeta)\varepsilon_i v_i \in Z^{\perp}.$$

Now, $(v_i|\zeta)$ is not yet known, but let δ_i be $(v_i|\zeta)$, and we continue as if we know δ. Then, we see that ζ is an optimal solution to (69) with

$$v = v_0 + \sum_{i=1}^{m+1} \delta_i \varepsilon_i v_i,$$

Due to (81), we see that the optimal solution of (69) is written as linear combination of the optimal solutions ζ_0 of (70) and ζ_i, $i = 1, ..., m+1$ of (71). Substituting $\zeta(\delta)$ into $(v_i|\zeta)$, we obtain

$$\delta_i = (v_i|\zeta(\delta)) \quad i = 1, ..., m+1. \tag{82}$$

This relation is obviously equivalent to (72), and is a necessary condition for $\zeta(\delta)$ to be the optimal solution of (68). Observe that such δ_i is ensured to exist due to solvability of (68). In the following, we show that (82) is sufficient for $\zeta(\delta)$ to be an optimal solution of (68). Let δ be the solution of (72) (and, equivalently, (82)). Due to the definition of ζ_i, we have

$$M\zeta(\delta) + v_0 + \sum_{i=1}^{m+1} \varepsilon_i v_i (v_i|\zeta(\delta)) = M\zeta_0 + v_0 + \sum_{i=1}^{m+1} \delta_i (M\zeta_i + \varepsilon_i v_i) \in Z^\perp.$$

This yields that $\zeta(\delta)$ is indeed the solution of (68).

Therefore, δ_i, $i = 1, ..., m+1$ determines the optimal solution of (68) if and only if (72) is satisfied. Since (68) has a unique optimal solution, (72) has a unique solution.

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