PRIMARY FACTORIZATION IN A WEAK BEZOUT DOMAIN

R. A. BEAUREGARD AND R. E. JOHNSON

ABSTRACT. It is well known that in a weak Bezout domain each prime factorization of an element is unique up to similarity. In this paper, a corresponding extension to primary factorizations is obtained.

An integral domain with unity is called a weak Bezout domain[1] by P. M. Cohn in [2] iff the sum and intersection of any two principal right ideals that have nonzero intersection are again principal. There it is shown that in such a ring any prime factorization of an element is unique up to order of factors and similarity. This generalizes the familiar result for commutative weak Bezout domains (called Bezout rings) that any prime factorization of an element is unique up to order of factors and associates. Just as in the commutative case one also considers primary decompositions, and when they exist the question of uniqueness arises. In this note we study primary factorizations in a weak Bezout domain, and we show that any primary factorization of an element is unique up to order of factors and similarity (associates for the commutative case).

In what follows R denotes a weak Bezout domain, $R^* = R\backslash\{0\}$, and U denotes the group of units of R. We assume the reader is familiar with the gcld $(a, b)_l$, lcrm $[a, b]_r$, etc., and similarity $a \sim b$ of $a, b \in R^*$. In particular, if $ab' = ba'$ in R^* then $(a, b)_l = (a', b')_r = 1$ iff $[a, b]_r = [a', b']_l = ab' = ba'$, in which case $a \sim a'$ and $b \sim b'$. We call c an *S-factor* of b iff $c \sim c'$ and c' is a factor of b. Left and right S-factors are defined in the obvious way. If $p \in R^*\backslash U$, p is called prime iff in every factorization $p = p_1p_2$ either $p_1 \in U$ or $p_2 \in U$. An element $b \in R^*\backslash U$ is called *p-primary* for some prime p iff every prime factor of b is similar to p and every nonunit r- or l-factor of b has at least one prime factor. We call $c = c_1c_2 \cdots c_n$ a *primary factorization* of c iff each c_i is p_i-primary for some prime p_i and $p_i \not\sim p_j$ whenever $i \neq j$. An element of $R^*\backslash U$ need not have a primary factorization even if it can be

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[1] The definition of weak Bezout domain is left-right symmetric [2], and can be weakened by omitting the requirement that the intersection be principal [4].

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factored into primes. This is obviously the case in a power series ring in noncommuting indeterminates over a field.

The main result is the following theorem whose proof will follow from the lemmas below.

THEOREM. *If R is a weak Bezout domain and $c\in R^*\backslash U$ has two primary factorizations $c=a_1a_2\cdots a_m=b_1b_2\cdots b_n$ then $m=n$ and $a_i\sim b_{i\sigma}$ for some permutation σ of $\{1, 2, \cdots, n\}$.*

LEMMA 1. *If $ab'=ba'$, $(a, b)_l=c$, and $(a', b')_r=c'$ in R^*, then there exist $x, x', y, y'\in R^*$ such that $a=cx$, $b=cy$, $a'=x'c'$, $b'=y'c'$, $xy'=yx'$, and $x\sim x'$, $y\sim y'$.*

PROOF. Obvious.

LEMMA 2. *If $ab'=ba'$ and $(a, b)_l=1$, then every l-factor of a is a l-S-factor of a'. If $ab'=ba'$ and a is prime, then a is either a l-factor of b or a l-S-factor of a'.*

PROOF. If $a_1\in R^*\backslash U$ is a l-factor of a, then $(a_1, b)_l=1$ and $[a_1, b]_r=a'b''=ba_1'$, $a_1\sim a_1'$, for some b'', $a_1'\in R^x$. Since $ba'\in[a_1, b]_rR$, $ba'=ba'_1c$ for some $c\in R^*$ and $a'=a_1'c$. The second part is now obvious.

LEMMA 3. *If $ab'=ba'$ and no nonunit l-factor of b' is a r-S-factor of b, then $b=au$ for some unit u.*

PROOF. (See [1, Theorem 1] for a special case.) By Lemma 1, y and y' must be units.

LEMMA 4. *If p is a prime factor of ab' then p is an S-factor of either a or b'.*

PROOF. Assume $ab'=bpe$ and $a'=pe$. In the notation of Lemma 1, $pe=x'c'$ and p is either a l-factor of x' or a l-S-factor of c' by Lemma 2. Hence, p is either a l-S-factor of x by Lemma 2 and thus an S-factor of a or an S-factor of b'.

LEMMA 5. *If $c=b_1a_1d_1=b_2a_2d_2$ and no nonunit l- (r-) factor of a_1 or a_2 is an S-factor of b_1 or b_2 (d_1 or d_2), then $a_1\sim a_2$.*

PROOF. We might as well assume $(b_1, b_2)_l=1$. If $m=[b_1, b_2]_r$, then $m=b_1b_2'=b_2b_1'$, $c=mc'$, and $a_1d_1=b_2'c'$, $a_2d_2=b_1'c'$ for some b_i', $c'\in R^*$ with $b_i'\sim b_i$. By assumption, $(a_1, b_2')_l=(a_2, b_1')=1$. If $(d_i, c')_r=e_i$ with $d_i=d_i'e_i$ and $c'=c_i'e_i$, then $a_1d_1'=b_2'c_1'$, $a_2d_2'=b_1'c_2'$ and $a_i\sim c_i'$. Therefore, no nonunit r-factor of c_i' is a l-S-factor of e_i and $c_1'=c_2'u$ for some unit u by Lemma 3. Hence, $c_1'\sim c_2'$ and $a_1\sim a_2$.

PROOF OF THE THEOREM. If a_i is p_i-primary and b_j is q_j-primary,

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then for each i there exists a unique j such that $p_i \sim q_j$ by Lemma 4. Thus, $m=n$ and $p_i \sim q_{i\sigma}$ for some permutation σ of $\{1, 2, \cdots, n\}$. In turn, $a_i \sim b_{i\sigma}$ by Lemma 5.

Since similar elements are associates in commutative rings, we have the following.

COROLLARY. *Let R be a commutative Bezout ring and let $c \in R^*$ have a primary decomposition. Then this is unique up to order of factors and associates.*

We conclude with an example[2] of a commutative Bezout ring with elements having a primary decomposition but no prime factorization.

For $i=1, 2, \cdots, n$, let p_i be the ith prime integer and consider the subring R_i of $Q[x]$ (polynomials in x with rational coefficients) consisting of all polynomials f such that $f(i)$ has a denominator prime to p_i. On changing the variable to $x-i$, we can verify as in [3] that R_i is a Bezout ring. We claim that $R=\bigcap_{i=1}^n R_i$ is again a Bezout ring. For let $a', b' \in R$, and let d_i be the gcd of a', b' in R_i, then d_i is determined up to unit factor in R_i, i.e. a rational number prime to p_i. Hence we can write $d_i = d' p_i^{\alpha_i}$ for some $d' \in Q[x]$, and it follows that $d = d' p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ is a gcd of a', b' in R. Thus $a' = da$, $b' = db$, and $av_i - bu_i = 1$ $(u_i, v_i \in R_i)$. Suppose we have found $u, v \in R_1 \cap \cdots \cap R_{i-1}$ to satisfy

$$av - bu = 1, \tag{1}$$

then $f = (u-u_i)/a = (v-v_i)/b$ lies in $Q[x]$. Hence we can find a power product of $p_1, \cdots, p_{i-1}$, say γ, such that $\gamma f \in R_1 \cap \cdots \cap R_{i-1}$ and a power of p_i, say δ, such that $\delta f \in R_i$. Since γ, δ are coprime, there are integers λ, μ such that $\lambda\gamma - \mu\delta = 1$. Then $f = \lambda\gamma f - \mu\delta f$ and writing

$$\begin{aligned} u' &= u - \lambda\gamma f a = u_i - \mu\delta f a, \\ v' &= v - \lambda\gamma f b = v_i - \mu\delta f b, \end{aligned} \tag{2}$$

we see that (1) holds with u, v replaced by u', v', and by (2) the latter are in $R_1 \cap \cdots \cap R_i$. By induction we find $u, v \in R$ to satisfy (1) and this shows R to be a Bezout ring.

In R, $x-i$ is p_i-primary but not a product of primes, and e.g. $x(x-1) \cdots (x-n)$ has a primary factorization.

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[2] We wish to thank the referee for communicating this example to us.

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University of Rhode Island, Kingston, Rhode Island 02881 and
University of New Hampshire, Durham, New Hampshire 03824

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