Czechoslovak Mathematical Journal

E. W. Johnson

Primary factorization in semigroups

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 2, 180–184

Persistent URL: http://dml.cz/dmlcz/102082

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

PRIMARY FACTORIZATION IN SEMIGROUPS

E. W. JOHNSON, Iowa City

(Received February 6, 1984)

Throughout, S will denote a commutative, multiplicative semigroup with 0 and 1.

Factorization theory, in one form or another, has been a topic of ongoing interest in algebra since the beginnings of the subject. In this paper, we consider the implications of factorizations, of various types, of ideals as products of primary ideals.

By a *prime ideal*, we shall mean an ideal P $(\neq S)$ which has the property that if it contains the product of two elements then it must contain one of them. The set M of all nonunits of S is a prime ideal, in fact the unique maximal ideal of S. By a *primary ideal*, we shall mean an ideal Q $(\neq S)$ which has the property that if it contains the product xy of two elements and fails to contain x, then it must contain a power of y. Any power of the maximal ideal M is easily seen to be primary. The *radical* of an ideal I, denoted $\mathrm{rad}(I)$, is the set of elements having a power in I. It is easy to see that an ideal P is prime iff whenever P contains the product of two ideals, it must contain one of them. Similarly, an ideal Q is primary iff whenever Q contains the product AB of two ideals and fails to contain A, $\mathrm{rad}(Q)$ must contain B. The radical of a primary ideal is prime, and any ideal having radical M is primary, as is easily seen. If Q is a primary ideal and $\mathrm{rad}(Q) = P$, then we will say that Q is *P-primary* or that Q is *primary with associated prime P*. We shall say that a semigroup has a *primary decomposition theory* if every ideal has a representation as a finite intersection of primary ideals (i.e., a primary decomposition). If S is Noetherian (i.e., satisfies A.C.C. on ideals) then every ideal has a primary decomposition. Any primary decomposition can be refined to a normal decomposition (i.e., one which is as short as possible and in which distinct primary terms have distinct radicals). If S is Noetherian then every primary ideal contains a power of its associated prime. We shall say that S has a *strong primary decomposition theory* if S has a primary decomposition theory and every primary contains a power of its radical.

By an irreducible ideal, we shall mean a nonzero ideal which cannot be properly factored (i.e., $A = BC$ implies $B = S$ or $C = S$). If A and B are subsets of S then we shall use $A : B$ to denote the set of all elements x such that $xB \subset A$. If A is an ideal of S, then $A : B$ is an ideal of S. If x is any element of S and A is an ideal of S, then $A \cap (x) = (A : (x))(x)$, as is easily seen. Hence a principal ideal is a factor of any ideal which it contains. By a principally reduced semigroup we shall mean a semigroup S in which no principal ideal $(x) \neq 0$ is a proper factor of itself. By a factor reduced semigroup we shall mean a semigroup in which no ideal $A \neq 0$ is a proper factor of itself.

Note 1. A semigroup S is principally reduced if, and only if, $M(x) = (x)$ implies $x = 0$. S is factor reduced if, and only if, $MA = A$ implies $A = 0$.

We begin by considering what is easily the simplest possible setting for factoring as products of primaries, namely that in which every ideal is already primary. We note that semigroups in which every ideal is primary were considered by Satyananarayana [2], but under different cancellative assumptions.

Theorem 1. *Let S be a principally reduced semigroup in which every ideal is primary. If P is a prime different from M, then $P = 0$. Conversely, if S is a semigroup in which M is the only nonzero prime, then every ideal of S is primary.*

Proof. Assume P is a prime ideal different from M. Choose x in P. Then $M(x)$ is primary. If $x \neq 0$, then $x \notin M(x)$, so $M \subset \operatorname{rad}(M(x)) \subset \operatorname{rad}((x)) \subset P$, a contradiction. Hence $x = 0$, and M is the only nonzero prime ideal of S.

It was shown in [2] that the radical of an ideal is the intersection of the primes containing it. Hence, if M is the only nonzero prime ideal of S, and if A is any non-prime ideal of S, then $M = \operatorname{rad}(A)$, and therefore A is primary.

Note 2. If S satisfies a strong primary decomposition theory, then S is principally reduced if, and only if, $\bigcap_n M^n = 0$. If $\bigcap_n M^n = 0$, then S is factor reduced.

Proof. Assume S is principally reduced and $y \in \bigcap_n M^n$. If Q is any term from a primary decomposition of $M(y)$ and $y \notin Q$, then $M \subset \operatorname{rad}(Q)$, so $y \in M^n \subset Q$, for some n. Hence $M(y) = y$. Since S is principally reduced, it follows that $y = 0$, and hence that $\bigcap_n M^n = 0$. Since $(y) = (y) M$ implies $(y) = (y) M^n$ for all n, the converse is clear. Since $A = AB$ implies $A = AB^n \subset M^n$ for all n, the last statement follows.

We now consider the case in which every ideal is a product of primaries. Noetherian rings satisfying this condition have also attracted some interest as generalizations of Dedekind domains [1].

Theorem 2. *Let S be a semigroup satisfying $\bigcap_n M^n = 0$ in which every ideal is a product of primaries. Then S has at most three primes different from M, each of which is principal. If P_1 is a principal prime and P_0 is a prime properly contained in P_1, then $P_0 = 0$. If P_1 and P_2 are noncomparable primes, then $M = P_1 \cup P_2$.*

Proof. Let P be any prime different from 0 and M. Then the quotient P/MP is one-dimensional. To see this, note that if I is any ideal strictly between MP and P, then one of the primary factors of I, say Q, is contained in P. Since I is not contained in MP, P is not a factor of Q, so Q is properly contained in P. But then from $MP \subset Q$

we get $M \subset \operatorname{rad}(Q) \subset P$, a contradiction. On the other hand, MP is properly contained in P by Note 2. Hence P/MP has dimension 1. It now follows that if x is any element of $P \setminus MP$, then necessarily $P = (x) \cup MP$. But then $P = \bigcap_n [(x) \cup PM^n] = (x) \cup \bigcap PM^n = (x)$. Hence, every prime different from M is principal.

If P_0 and P_1 are distinct principal primes with P_0 contained in P_1, then $P_0 = P_0 \cap P_1 = (P_0 : P_1) P_1 = P_0 P_1$, from which it follows (Note 2) that $P_0 = 0$. Now assume that P_1 and P_2 are noncomparable principal primes. It is easily seen that $(P_1 \cup P_2) = P$ is another prime, and since P_1 is nonzero, it follows that $P = M$. If P_0 is a third principal prime, then since P_0 is principal and contained in $M = P_1 \cup P_2$, it follows that P_0 is contained in either P_1 or P_2. But then $P_0 = 0$. Hence S has at most three primes different from M and they are all principal.

In the special case of Theorem 2 where S has exactly three primes different from M, there is more to be said.

Theorem 3. *Let S be a semigroup satisfying $\bigcap_n M^n = 0$ in which every ideal is a product of primaries. If S has exactly three primes different from M, then S is Noetherian of (Krull) dimension 2.*

Proof. Assume that S has three primes, P_0, P_1 and P_2, different from M. By Theorem 2, we may assume that $P_0 = 0$ and that $M = P_1 \cup P_2$. It is clear that S has (Krull) dimension 2, since $0 < P_1 < M$ and $0 < P_2 < M$ are the only maximal prime chains.

There are a variety of ways to see that S is Noetherian. We choose one which is fairly unique to this situation:

Let F be a maximal nonfinitely generated ideal. Since the product of finitely generated ideals is finitely generated, and since F is the product of primary ideals, it must be that F is itself primary. Hence F is primary for one of P_1, P_2 and M. If F is primary for, say, P_1, then we can choose n so that F is contained in P_1^n but not in P_1^{n+1}. Then $F = F \cap P_1^n = (F : P_1^n) P_1^n$, with $F : P_1^n$ not contained in P_1. Since F is primary, it follows that P_1^n is contained in F, and hence that $P_1 = F$. But then F is principal, a contradiction. Hence F must be primary for $M = P_1 \cup P_2$. But then $F = F \cap M = (F \cap P_1) \cup (F \cap P_2) = (F : P_1) P_1 \cup (F : P_2) P_2$, and both of $F : P_1$ and $F : P_2$ must be greater than F, and hence finitely generated, since F has radical M. But then F is again finitely generated. Therefore every ideal of S is finitely generated and S is Noetherian.

Note 3. In the final paragraph of the proof of Theorem 3 it is shown that if F is a primary ideal with a principal associated prime, then F is a power of its associated prime, and hence principal. It is easy to see that if $A = Q \cap Q_1 \cap \ldots \cap Q_n$ is a normal decomposition in which the Q_i are principal and have noncomparable associated primes, then $A = [(Q : Q_n) \cap Q_1 \cap \ldots \cap Q_{n-1}] Q_n$. It follows that if S is a principally reduced semigroup satisfying a strong primary decomposition theory in which every prime ideal $P \neq M$ is principal, then every ideal is a product of primaries.

We now proceed to consider situations in which we have some sort of uniqueness of factorization.

The case in which every nonzero ideal of S has a unique factorization as a product of primaries is trivial: M^2 automatically has two different factorizations, since it is itself primary, so it follows that $M^2 = 0$. Hence, it is clear that the most we should ask for is that every nonzero ideal be a unique product of *irreducible* primaries. On the other hand, it is easily seen that every ideal A of such a semigroup satisfies the cancellative property $AB = AC = 0$ implies $B = C$. Semigroups satisfying this condition are Noetherian with $\bigcap_n M^n = 0$ [3]. We obtain a characterization under weaker hypotheses.

Theorem 4. *Let S be a semigroup satisfying $\bigcap_n M^n = 0$ in which every M-primary ideal contains a power of its radical. Assume that the prime ideals P of S satisfy the property $PA = PB \neq 0$ implies $A = B$, for all ideals A and B. Then either $M^2 = 0$, or S is Noetherian, every ideal of S is principal and every nonzero ideal of S is a power of M.*

Proof. We observe that if P is prime and $PA \subset PB \neq 0$, then $PB = P(A \cup B) \neq 0$, so $B = A \cup B$ and hence $A \subset B$.

First consider the case in which S has dimension 0 and $M^2 \neq 0$. Choose $x \in M \setminus M^2$ such that $M(x) \neq 0$ (this is clearly possible since M is generated by the elements of $M \setminus M^2$). Since the radical of (x) is the intersection of the primes containing it, (x) has radical M, and hence is M-primary. Choose n least such that M^n is contained in (x). Then $M^n = M^n \cap (x) = (M^n : (x))(x)$ and M^n is not contained in $M(x)$, so $M^n : (x) = S$. It follows that $M^n = (x)$. By the choice of $x \in M \setminus M^2$, we get that $n = 0$ and that $M = (x)$.

Now assume S has dimension greater than 0. Let F be the family of all subsets B of $M \setminus M^2$ such that $x, y \in B$ and $(x) = (y)$ imply $x = y$. Let G be a maximal element of F. If z is any element of $M \setminus M^2$, then $z \notin G$ implies $(z) = (g)$, for some element $g \in G$. Since $M = M \setminus M^2 \cup M^2$ and $\bigcap_n M^n = 0$, it follows that the ideal generated by G is M.

Fix $g \in G$ and let $H = H_g = G \setminus \{g\}$. Let J_g be the ideal generated by H. If $g \in J_g = \bigcup_{h \in G} (h)$, then $g \in (h)$ for some $h \in H$. But then $(g) = (h)$ or $(g) \subset M(h)$, both of which contradict the choice of G. Hence J_g is properly contained in M.

Let P be a prime ideal minimal over J_g. Since $M = J_g \cup (g)$, we have $P = J_g \cup (P \cap (g)) = J_g \cup (P : (g))(g)$. Hence either $P = J_g$ or $P = M$. If $P = M$, choose n least such that $M^n \subset J_g$. Since $M^n = M^n \cap (\bigcup_{h \in H} (h)) = \bigcup_{h \in H} (M^n \cap (h)) = \bigcup_{h \in H} (M^n : (h))(h)$, and since M^n is not contained in MJ_g, it follows that $M^n : (h) = S$, for some $h \in H$. But then $h \in M \setminus M^2$ implies $n = 1$, a contradiction. Hence $P = J_g$.

From $M = J_g \cup (g) = P \cup (g)$, we get $M^3 = (P \cup (g))^3 = M(P^2 \cup (g^2)) \neq 0$, whence $P(g) \subset ((g^2) \cup P^2) \cap P = (P \cap (g^2)) \cup P^2 = P(g^2) \cup P^2 = P((g^2) \cup P)$. Since $g \notin (g^2) \cup P$, it follows that $P((g^2) \cup P) = 0$. But then $P^2 = J_g^2 = 0$.

Fix $h \in G \setminus \{g\}$. Since g is an arbitrary element of G, it follows that J_h is also prime and that $J_h^2 = 0$. But then $g \in J_h \subset J_g$, a contradiction. It is now clear that G has only one element g, and $M = (g)$.

Hence M is principal in either case.

Let A be any nonzero ideal of S. Choose n least such that A is not contained in M^{n+1}. Then $A = A \cap M^n = (A : M^n) M^n$, so $A : M^n = S$ and $A = M^n$. Hence every ideal of S is principal and every nonzero ideal is a power of M.

Note 4. It is clear that a principally reduced semigroup S in which every nonzero ideal is a power of M has the property that every nonzero ideal is a unique product of primaries. It is also clear that any semigroup S in which $M^2 = 0$ has this property.

References

[1] *D. D. Anderson:* Noetherian rings in which every ideal is a product of primaries, Canad. Math. Bull. *23* (1980), 457—459.

[2] *M. Satyanarayana:* Commutative primary semigroups, Czech. Math. J. *22* (1972), 509—516.

[3] *J. Wells:* Weak Noether lattices and cancellative properties in Noether lattices, Ph. D. Dissertation, Univ. of Iowa, 1970.

Author's address: The University of Iowa, Iowa City, Iowa 522 42, U.S.A.