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Prime and Semiprime Bi-ideals in Ordered Semigroups

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Abstract

In this paper we shall introduce the concept of prime and semiprime bi-ideals of ordered semigroups and we give characterizations of prime bi-ideal and regular of ordered semigroups.

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1 Introduction

The concept of prime and semiprime bi-ideals of associative rings with unity was introduced by A. P. J. van der Walt [7]. In [6] H J le Roux have constructed a number of results by using prime and semiprime bi-ideals of associative rings without unity. In this paper, we define a prime and semiprime bi-ideals of ordered semigroups. Further we shall extend the results of H J le Roux [6] to an ordered semigroups. It is shown that a bi-ideal B of a po-semigroup S is prime if and only if $RL \subseteq B$, with R a right ideal of S, L a left ideal of S, implies $R \subseteq B$ or $L \subseteq B$. Moreover, let B be a prime bi-ideal of a po-semigroup S, then $H(B)$ is a prime ideal of S.

2 Preliminary Notes

In this section, we recall some basic definitions and results that are relevant for this paper.

Definition 2.1 *A po-semigroup (:ordered semigroup) is an ordered set* $(S, \leq)$ *at the same time a semigroup such that:*
$a \leq b \Rightarrow ca \leq cb$ *and* $ac \leq bc\ \forall\ c \in S$ *[1].*

The following definitions and results were due to N. Kehayopulu.

Definition 2.2 *Let* S *be a po-semigroup and* $\phi \neq A \subseteq S$. A *is called a right (resp. left) ideal of* S *[2, 3, 4, 5] if*
1) $AS \subseteq A$ *(resp.* $SA \subseteq A$*).*
2) $a \in A$, $S \ni b \leq a \Rightarrow b \in A$.
A *is called an ideal of* S *if it both a right and a left ideal of* S.

Definition 2.3 *Let* S *be a po-semigroup and* $T \subseteq S$. T *is called prime if*
$A, B \subseteq S$, $AB \subseteq T \Rightarrow A \subseteq T$ *or* $B \subseteq T$ *[5].*
Equivalently, $a, b \in S$, $ab \in T \Rightarrow a \in T$ *or* $b \in T$.

Definition 2.4 *Let* S *be a po-semigroup and* $T \subseteq S$. T *is called weakly prime if*
For all ideals A, B *of* S *such that* $AB \subseteq T$, *we have* $A \subseteq T$ *or* $B \subseteq T$ *[2, 5].*

Definition 2.5 *Let* S *be a po-semigroup and* $T \subseteq S$. T *is called semiprime if*
$A \subseteq S$, $A^2 \subseteq T \Rightarrow A \subseteq T$ *[5].*
Equivalently, $a \in S$, $a^2 \in T \Rightarrow a \in T$.

Definition 2.6 *Let* S *be a po-semigroup and* $\phi \neq Q \subseteq S$. Q *is called a quasi ideal of* S *[4] if*
1) $QS \bigcap SQ \subseteq Q$.
2) $a \in Q$, $S \ni b \leq a \Rightarrow b \in Q$.

Definition 2.7 *Let* S *be a po-semigroup and* $\phi \neq B \subseteq S$. B *is called a bi-ideal of* S *[4] if*
1) $BSB \subseteq B$.
2) $a \in B$, $S \ni b \leq a \Rightarrow b \in B$.

Notation
For $H \subseteq S$, $(H]=\{t \in S/t \leq h\ forsome\ h \in H\}$.
We denote by $I(a)$ $(resp.\ L(a),\ R(a))$ the ideal $(resp.\ left\ ideal,\ right\ ideal)$ of S generated by a. One can easily prove that:
$I(a) = (a \cup Sa \cup aS \cup SaS]$, $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$.

Definition 2.8 *Let S be a po-semigroup. S is called left (resp. right ideal) regular if*

$\forall\, a \in S\ \exists\, x \in S : a \le xa^2$ *(resp.* $a \le a^2x$*) [4, 5].*

Equivalently,

1) $a \in (Sa^2]$ *(resp.* $a \in (a^2S]$*)* $\forall\, a \in S$.

2) $A \subseteq (SA^2]$ *(resp.* $A \subseteq (A^2S]$*)* $\forall\, A \subseteq S$.

Definition 2.9 *Let S be a po-semigroup. S is called regular if*

$\forall\, a \in S\ \exists\, x \in S : a \le axa$ *[5].*

Equivalently,

1) $a \in (aSa]\ \forall\, a \in S$.

2) $A \subseteq (ASA]\ \forall\, A \subseteq S$.

We note the following Lemma.

Lemma 2.10 *For an ordered semigroup S, we have*

1)$A \subseteq (A]\ \forall\, A \subseteq S$.

2) If $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

3)$(A]\,(B] \subseteq (AB]\ \forall\, A, B \subseteq S$.

4)$((A]] = (A]\ \forall\, A \subseteq S$.

5) For every left (resp. right) ideal or bi-ideal T of S, we have $(T] = T$.

6)$((A]\,(B]] = (AB]\ \forall\, A, B \subseteq S$ *(cf.[2, the Lemma]).*

The following results were due to N. Kehayopulu.

Result 2.11 *Let S be a po-semigroup and T be an ideal of S. The following are equivalent:*

1) T is weakly prime.

2) If $a, b \in S$ such that $(aSb] \subseteq T$, then $a \in T$ or $b \in T$.

3) If $a, b \in S$ such that $I\,(a)\,I\,(b) \subseteq T$, then $a \in T$ or $b \in T$.

4) If A, B are right ideals of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.

5) If A, B are left ideals of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.

6) If A a right ideal, B a left ideal of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$ [2].

Result 2.12 *An ideal T of a po-semigroup S is weakly semiprime if and only if one of the following four equivalent conditions holds in S:*

1) For every $a \in S$ such that $(aSa] \subseteq T$, we have $a \in T$.

2) For every $a \in S$ such that $(I\,(a))^2 \subseteq T$, we have $a \in T$.

3) For every right ideal A of S such that $A^2 \subseteq T$, we have $A \subseteq T$.

4) For every left ideal B of S such that $B^2 \subseteq T$, we have $B \subseteq T$ [2].

Result 2.13 *An ideal of a po-semigroup is prime if and only if it is both semiprime and weakly prime. In commutative po-semigroups the prime and weakly prime ideals coincide.*

3 Main Results

In this section, we introduce prime and semiprime bi-ideals of ordered semigroups and obtain some properties of it.

Definition 3.1 *Let S be a po-semigroup. A bi-ideal B of S is called prime if*

$xSy \subseteq B \Rightarrow x \in B$ *or* $y \in B$.
Equivalently,
the subsets $C, D \subseteq S, CSD \subseteq B \Rightarrow C \subseteq B$ *or* $D \subseteq B$.

Definition 3.2 *Let S be a po-semigroup. A bi-ideal B of S is called semiprime if*

$xSx \subseteq B \Rightarrow x \in B$.
Equivalently,
a subset $C \subseteq S, CSC \subseteq B \Rightarrow C \subseteq B$.

Now we shall generalize the results on associative ring without unity found in [6] for an ordered semigroups.

Proposition 3.3 *A bi-ideal B of a po-semigroup S is prime if and only if $RL \subseteq B$, with R a right ideal of S, L a left ideal of S, implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Let B be a prime bi-ideal of a po-semigroup S and $RL \subseteq B$. Suppose $R \nsubseteq B$. For all $x \in L$ and $r \in R \backslash B$, we have $rSx \subseteq RL \subseteq B$.

Since B is prime and $r \notin B$, we have $x \in B$ for all $x \in L$, so $L \subseteq B$.

Conversely, suppose $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$ for any right ideal R of S and any left ideal L of S. Let $x, y \in S$ such that $xSy \subseteq B$. Then $(xS]\,(Sy] \subseteq (xS^2y] \subseteq (xSy] \subseteq (B] = B$.

Since $(xS]$ is a right ideal of S and $(Sy]$ a left ideal of S, we have $(xS] \subseteq B$ or $(Sy] \subseteq B$. Suppose $(xS] \subseteq B$. Then $x^2 \in B$.

Consider $R\,(x)$ and $L\,(x)$, the right ideal of S and left ideal of S generated by x in S, respectively. Now $R\,(x)\,L\,(x) = (x \cup xS]\,(x \cup Sx] \subseteq ((x \cup xS)\,(x \cup Sx)] = (x^2 \cup xSx \cup xSx \cup xSSx] \subseteq (x^2 \cup xSx]$. Let z be any element of the product $R\,(x)\,L\,(x)$. Then $z \in (x^2 \cup xSx]$. So $z \leq t$ for some $t \in x^2 \cup xSx$. If $t = x^2$, then $z \leq x^2 \in B$ i.e. $z \in B$, since B is a bi-ideal of S. If $t = xyx$ forsome $y \in S$, then $z \leq xyx \in xSx \subseteq xS \subseteq (xS] \subseteq B$ i.e. $z \in B$, since B is a bi-ideal of S. Hence $R\,(x)\,L\,(x) \subseteq B$.

From our assumption it follows that $R\,(x) \subseteq B$ or $L\,(x) \subseteq B$ and hence $x \in B$. Similarly, if $(Sy] \subseteq B$, then $y \in B$. Hence B is a prime ideal of S.

Proposition 3.4 *A prime bi-ideal of a po-semigroup S is prime one-sided ideal of S.*

Proof. Let B be a prime bi-ideal of a po-semigroup S. It is only necessary to show that B is a one-sided ideal of S.

Clearly, $(BS](SB] \subseteq (BS^2B] \subseteq (BSB] \subseteq (B] = B$.

Since $(BS]$ is a right and $(SB]$ a left ideal of S, we have, from proposition 3.3, that $(BS] \subseteq B$ or $(SB] \subseteq B$ i.e. $BS \subseteq B$ or $SB \subseteq B$, since $BS \subseteq (BS]$, $SB \subseteq (SB]$. Assume $x \in B$, $S \ni y \leq x$. Then $y \in B$, since B is a bi-ideal of S. Hence B is a one-sided ideal of S.

Remark

Let B be any bi-ideal of a po-semigroup S and let $L(B) = \{x \in B / Sx \subseteq B\}$ and $H(B) = \{y \in L(B) / yS \subseteq L(B)\}$.

Lemma 3.5 *For any bi-ideal B of a po-semigroup S the set $L(B) = \{x \in B / Sx \subseteq B\}$ is a left ideal of S.*

Proof. If $x \in L(B)$ and $z \in S$, then $zx \in Sx \subseteq B$ and $Szx \subseteq SSx \subseteq Sx \subseteq B$.

Choose $x \in L(B)$ such that $S \ni y \leq x$. Then $y \in B$, since $L(B) \subseteq B$ and B is a bi-ideal of S. Since $y \leq x$ and S is a po-semigroup, we have $zy \leq zx \ \forall\ z \in S$. So $zy \leq zx \in Sx \subseteq B$ i.e. $zy \in B \ \forall\ z \in S$, since B is a bi-ideal of S. Thus $Sy \subseteq B$ implies $y \in L(B)$. Hence $L(B)$ is a left ideal of S.

Proposition 3.6 *If B is any bi-ideal of a po-semigroup S, then $H(B)$ is the (unique) largest two-sided ideal of S contained in B.*

Proof. As in the lines of [6], we prove $xy, yx \in H(B)$.

Since $L(B) \subseteq B$ and $H(B) \subseteq L(B)$, we have that $H(B) \subseteq B$. We now show that $H(B)$ is a two-sided ideal of S.

Let $x \in H(B)$ and $y \in S$. Then $x \in B$ and x is also an element of $L(B)$, we have that $Sx \subseteq B$ and $xS \subseteq L(B)$.

Then $yx \in Sx \subseteq B$. So $yx \in B$. Furthermore $Syx \subseteq Sx \subseteq B$. So $yx \in L(B)$. Also $xy \in xS \subseteq L(B)$. Hence $xy \in L(B)$.

Now we shall show that xy and $yx \in H(B)$. $xyS \subseteq xS \subseteq L(B)$. Hence $xy \in H(B)$. $yxS \subseteq SxS \subseteq SL(B) \subseteq L(B)$, since $L(B)$ is a left ideal of S. Hence $yx \in H(B)$.

Let $x \in H(B)$, $S \ni y \leq x$. Then $y \in L(B)$, since $H(B) \subseteq L(B)$ and $L(B)$ is a left ideal of S. Since $y \leq x$ and S is a po-semigroup, we have $yz \leq xz \ \forall\ z \in S$. So $yz \leq xz \in xS \subseteq L(B)$ i.e. $yz \in L(B) \ \forall\ z \in S$, since $L(B)$ is a left ideal of S. Thus $yS \subseteq L(B)$ implies $y \in H(B)$. Hence $H(B)$ is a two-sided ideal of S.

Let I be any ideal of S and $I \subseteq B$, and let u be an arbitrary element of I. Then $u \in B$ and $Su \subseteq I \subseteq B$. Hence $I \subseteq L(B)$.

Furthermore $u \in L(B)$ and $uS \subseteq I \subseteq L(B)$. This implies that $u \in H(B)$ and hence $I \subseteq H(B)$.

Proposition 3.7 *Let B be a prime bi-ideal of a po-semigroup S. Then $H(B)$ is a weakly prime ideal of S.*

Proof. Let B be a prime bi-ideal of a po-semigroup S. Since B is a bi-ideal of S, we have, from proposition 3.6, that $H(B)$ is a two-sided ideal of S. We now show that an ideal $H(B)$ of S is weakly prime.

Let $a, b \in S$ such that $I(a)I(b) \subseteq H(B)$ for any two-sided ideals $I(a)$ and $I(b)$ of S generated by a and b in S, respectively. From proposition 3.3 it follows that $I(a) \subseteq B$ or $I(b) \subseteq B$, since $I(a)I(b) \subseteq B$. From proposition 3.6, we have, that $H(B)$ is the largest ideal in B. Hence $I(a) \subseteq H(B)$ or $I(b) \subseteq H(B)$. This implies that $a \in H(B)$ or $b \in H(B)$ and hence by theorem 2.11 that $H(B)$ is weakly prime.

Proposition 3.8 *Let B be a semiprime bi-ideal of a po-semigroup S. Then $L^2 \subseteq B$ (or $R^2 \subseteq B$) implies $L \subseteq B$ (or $R \subseteq B$) for any left ideal L (or right ideal R) of S.*

Proof. The proof is same as in the proposition 10 of [6].

Proposition 3.9 *Let B be a semiprime bi-ideal of a po-semigroup S. Then $H(B)$ is a weakly semiprime ideal of S.*

Proof. Let B be a semiprime bi-ideal of a po-semigroup S. Since B is a bi-ideal of S, we have, from proposition 3.6, that $H(B)$ is a two-sided ideal of S. We now show that an ideal $H(B)$ of S is weakly semiprime.

Let $a \in S$ such that $(I(a))^2 \subseteq H(B)$ for any two-sided ideal $I(a)$ of S generated by a in S. From proposition 3.8 it follows that $I(a) \subseteq B$, since $(I(a))^2 \subseteq B$. From proposition 3.6, we have, that $H(B)$ is the largest ideal in B. Hence $I(a) \subseteq H(B)$. This implies that $a \in H(B)$ and hence by theorem 2.12 that $H(B)$ is weakly semiprime.

Proposition 3.10 *Let B be a semiprime bi-ideal of a po-semigroup S. Then B is a quasi ideal of S.*

Proof. Assume $y \in BS \cap SB$. Then $y \in BS$ and $y \in SB$. $ySy \subseteq (BS)S(SB) \subseteq BSB \subseteq B$. Since B is a semiprime bi-ideal of S, we have $y \in B$. Hence $BS \cap SB \subseteq B$.

Next, let $x \in B$, $S \ni y \leq x$. Then $y \in B$, since B is a bi-ideal of S. Hence B is a quasi-ideal of S.

Proposition 3.11 *A po-semigroup S regular if and only if every bi-ideal in S is semiprime.*

Proof. Let S be a regular po-semigroup and B be any bi-ideal of S. Suppose $aSa \subseteq B$ for $a \in S$. Then there exists $x \in S$ such that $a \leq axa$, since S is regular. But $axa \in aSa \subseteq B$ i.e. $axa \in B$. Since $axa \in B$, $S \ni a \leq axa$ and B is a bi-ideal of S, we have $a \in B$ and so B is semiprime.

Conversely, suppose that every bi-ideal of S is semiprime. Let $a \in S$. It is clear that $(aSa]$ is a bi-ideal of S. Hence $(aSa]$ is semiprime for any $a \in S$. Since $aSa \subseteq (aSa]$ and $(aSa]$ is semiprime, we have $a \in (aSa]$. This implies that $a \leq axa$ for some $x \in S$ and hence S is regular.

Proposition 3.12 *A commutative po-semigroup S regular if and only if every ideal of S is semiprime.*

Proof. Let S be a regular commutative po-semigroup and I be an ideal of S. Suppose $a^2 \in I$ for $a \in S$. Then there exists $x \in S$ such that $a \leq axa$, since S is regular. But $a \leq axa = a(xa) = a(ax) = a^2x \in IS \subseteq I$.This implies that $a \in I$, since I is an ideal of S. Hence I is semiprime.

Conversely, suppose that every ideal of S is semiprime. Let $a \in S$. It is clear that $(a^2S]$ is an ideal of S. Hence $(a^2S]$ is semiprime for any $a \in S$. Since $a^4 \in (a^2S]$, $(a^2S]$ is semiprime, we have $a^2 \in (a^2S]$ implies $a \in (a^2S]$ i.e. $a \leq a^2x$ for some $x \in S$. This implies that $a \leq aax = axa$ for some $x \in S$. Hence S is regular.

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