symmetry

MDPI

Article

Prime Cordial Labeling of Generalized Petersen Graph under Some Graph Operations

Weidong Zhao [1,†], Muhammad Naeem [2,*,†] and Irfan Ahmad [2,†]

1 School of Computer Science, Chengdu University, Chengdu 610100, China; zwdbox@cdu.edu.cn
2 Department of Mathematics and Statistics, Institute of Southern Punjab, Multan 59300, Pakistan; irfanch784@gmail.com
* Correspondence: muhammadnaeem@isp.edu.pk or naeempkn@gmail.com
† These authors contributed equally to this work.

Abstract: A graph is a connection of objects. These objects are often known as vertices or nodes and the connection or relation in these nodes are called arcs or edges. There are certain rules to allocate values to these vertices and edges. This allocation of values to vertices or edges is called graph labeling. Labeling is prime cordial if vertices have allocated values from 1 to the order of graph and edges have allocated values 0 or 1 on a certain pattern. That is, an edge has an allocated value of 0 if the incident vertices have a greatest common divisor (gcd) greater than 1. An edge has an allocated value of 1 if the incident vertices have a greatest common divisor equal to 1. The number of edges labeled with 0 or 1 are equal in numbers or, at most, have a difference of 1. In this paper, our aim is to investigate the prime cordial labeling of rotationally symmetric graphs obtained from a generalized Petersen graph $P(n,k)$ under duplication operation, and we have proved that the resulting symmetric graphs are prime cordial. Moreover, we have also proved that when we glow a Petersen graph with some path graphs, then again, the resulting graph is a prime cordial graph.

Keywords: prime cordial labeling; duplication operations; glowing of graph; generalized Petersen graph

check for updates

Citation: Zhao, W.; Naeem, M.; Ahmad, I. Prime Cordial Labeling of Generalized Petersen Graph under Some Graph Operations. *Symmetry* **2022**, *14*, 732. https://doi.org/10.3390/sym14040732

Academic Editor: Palle E.T. Jorgensen

Received: 19 February 2022
Accepted: 30 March 2022
Published: 3 April 2022

1. Introduction

Graph theory started with a solution for the Königsberg bridges problem in the 18th century by Leonhard Euler. In the beginning, it was considered a recreational subject. It took almost two centuries for its development as a mathematical subject, graph theory. Graph labelings started in the 19th century, when Arthur Cayley, the renouned British mathematician, showed that there are n^{n-2} different labeled trees of order n [1]. This subject attracted the attention of researchers in the previous century. Many conferences were organized for graph theory in 1950–1960. These conferences concluded that graph theory is an easy way for understanding mathematics, which attracts the interest of young students. These conferences were known as "Symposium on the Theory of Graphs and Its Applications", held in Smolenice (Czechoslovakia) in 1963 [2]. Graph theory conferences were held regularly in the 1970s. One member of the Rome conference was Alexander Rosa; a paper was presented by him [3]. This paper brought revolution in graph labeling. Four vertex labelings were mentioned by Rosa, described as α-valuations, β-valuations, σ-valuations and ρ-valuations. The β-valuation became graceful labeling in 1972 [4]. The main aim of graceful labeling was to minimize the values assigned to terminals. The number theory has an impact on graph labelings [5]. For many years, a lot of graph labelings have been introduced, which have an impact on many problems for their solution. The labeled graphs are working excellently in many mathematical models. For instance, radar codes and cyber security [6]. Prime cordial labeling is the assignment of values to vertices from 1 to order of graph and edge labeling induced from it where 0 and 1 valued to edges with respect to the greatest common divisor (gcd) of incident vertices. That is, incident vertices have a common divisor greater than 1 labeled with 0 and incident vertices have a common divisor equal to 1 labeled with 1. It was introduced by Sundaram et al. [7] and, in the same paper, they investigated several results on prime cordial. In this paper, we will investigate the generalized Petersen graph $P(n,k)$ under duplication operation and glowing of P_3 with $P(n,k)$ is prime cordial. All the graphs in this article are simple connected graphs and the notions and symbols are the same as those used by I. Ahmad in [8].

Definition 1. *Let $P_e(n,k)$ be a graph obtained by attaching a vertex to each edge of outer cycle of $P(n,k)$. Then, $P_e(n,k)$ has $3n$ vertices with vertex set $V(P_e(n,k)) = \{x_i, y_i, z_i : 0 \leqslant i \leqslant n-1\}$ and $5n$ edges has edge set $E(P_e(n,k)) = \{x_i x_{i+k}, x_i y_i, y_i y_{i+1}, y_i z_i, y_{i+1} z_i : 0 \leqslant i \leqslant n-1\}$. In Figure 1, we have depicted vertices and edges of this graph.*

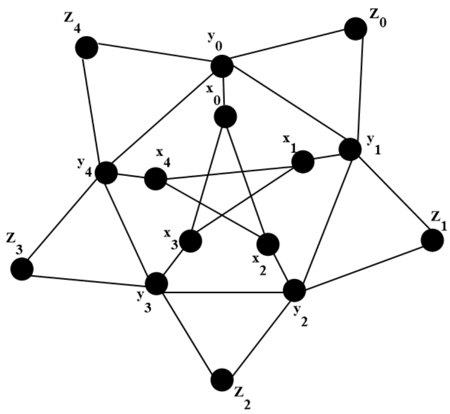

Figure 1. $P_e(5,k)$; duplication of edges by vertices.

Definition 2. *Let $P_v(n,k)_o$ be a graph obtained by duplicating outer cycle vertices $P(n,k)$ by an edge. Then, $P_v(n,k)_o$ has $4n$ vertices with vertex set $V(P_v(n,k)_o) = \{x_i, y_i : 0 \leqslant i \leqslant n-1$ and $z_i : 0 \leqslant i \leqslant 2n-1\}$ and $6n$ edges with edge set;*

$E(P_v(n,k)_o) = \{x_i x_{i+k}, x_i y_i, y_i y_{i+1}, y_i z_i, z_i z_{i+1}, y_i z_{i+1} : 0 \leqslant i \leqslant n-1\}$. As shown in Figure 2.

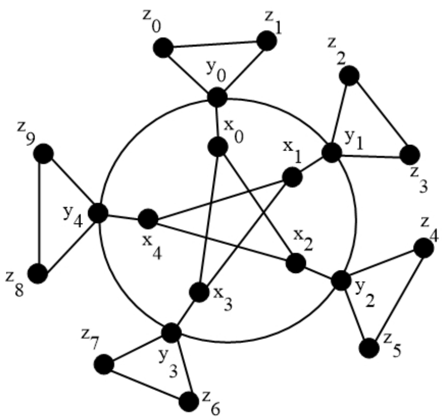

Figure 2. $P_v(5,k)_o$; outer cycle vertex duplication by an edge.

Definition 3. *Let $P_v(n,k)_i$ be a graph obtained by edge duplication of all the vertices on the inner cycle of $P(n,k)$, that is, the duplication of vertices such as x_i's of Figure 2 in $P(n,k)$. Then, the graph $P_v(n,k)_i$ has $4n$ vertices and $6n$ edges, having vertex set $V(P_v(n,k)_i) = \{x_i, z_i : 0 \leqslant i \leqslant n-1$ and $y_i : 0 \leqslant i \leqslant 2n-1\}$ and $6n$ edges with edge set;*

$E(P_v(n,k)_i) = \{x_i x_{i+k}, x_i y_i, x_i y_{i+1}, y_i y_{i+1}, x_i z_i, z_i z_{i+1}, : 0 \leqslant i \leqslant n-1$, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor\}$. As shown in Figure 3.

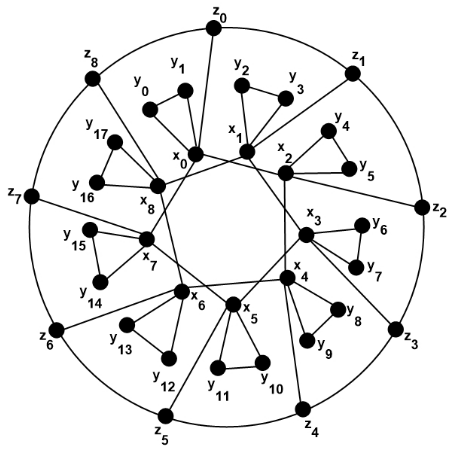

Figure 3. $P_v(9,2)_i$; inner vertex duplication by an edge.

Definition 4. *There is path graph P_3 and a generalized Petersen graph $P(n,k)$. In glowing operation, there is a subgraph denoted as $(P_3, P(n,k))_{ev}$. This graph has $4n$ vertices and $5n$ edges. These are $V((P_3, P(n,k))_{ev}) = \{u_i, x_i, y_i, z_i; 0 \le i \le n-1\}$ and $E((P_3, P(n,k))_{ev}) = \{u_iu_{i+k}, u_ix_i, x_ix_{i+1}, x_iy_i, y_iz_i; 0 \le i \le n-1\}$. Represented in Figure 4.*

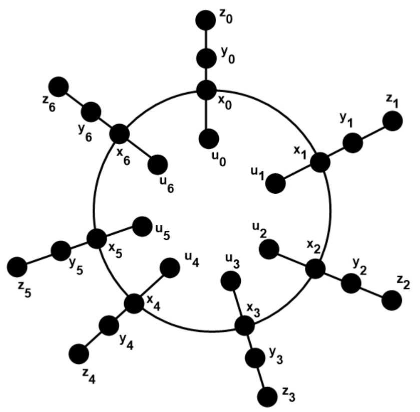

Figure 4. $(P_3, P(n,k))_{ev}$; end vertex glowing of P_3 with $P(n,k)$.

Definition 5. *Let $(P_3, P(n,k))_{mv}$ be a glowing of n copies of P_3 at its middle vertex common with $P(n,k)$ outer cycle. Glowing graph has $4n$ vertices and $5n$ edges. These are $V((P_3, P(n,k))_{mv}) = \{x_i, y_i\ ;\ 0 \le i \le n-1 \text{ and } z_i\ ;\ 0 \le i \le 2n-1\}$ and $E((P_3, P(n,k))_{mv}) = \{x_ix_{i+k}, x_iy_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}; 0 \le i \le n-1\}$ depicted in Figure 5.*

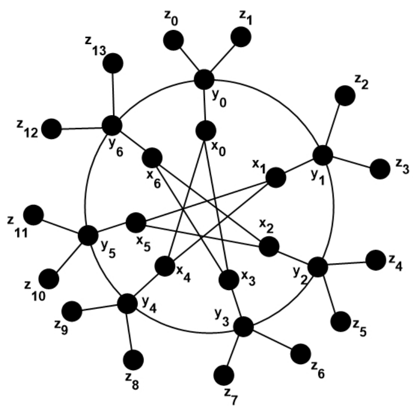

Figure 5. $(P_3, P(7,2))_{mv}$; middle vertex glowing of P_3 with $P(7,2)$.

Definition 6. *Let $V(P_v(n,k)_p$ be graph obtained by duplication of vertices by n pendant vertices in $P(n,k)$ outer cycle. There are $3n$ vertices and $4n$ edges with edge set $E(P_v(n,k)_p)$ shown in Figure 6,*

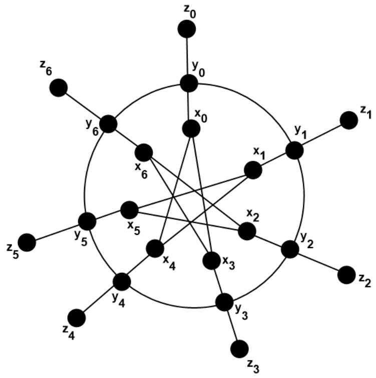

Figure 6. $P_v(7,3)_p$; Outer cycle vertices duplication by pendant vertices.

2. Main Results

In this section, we shall present the main results of the article.

Theorem 1. *The graph $P_e(n,k)$ is prime cordial for all $n \geqslant 3$ and $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$.*

Proof. Let us assume $P(n,k)$ *where* $n \geqslant 3$ *and* $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$ be generalized Petersen graph. On each outer edge, one vertex is added in such a way that duplicates edges in the outer cycle of the existing graph. There are $3n$ vertices and $5n$ edges having the vertex set $V(P_e(n,k)) = \{x_i, y_i, z_i \,:\, 0 \leqslant i \leqslant n-1\}$ and edge set $E(P_e(n,k)) = \{x_i x_{i+k}, x_i y_i, y_i y_{i+1}, y_i z_i, y_{i+1} z_i \,:\, 0 \leqslant i \leqslant n-1\}$. As we knew, in prime cordial labeling, we assign values to edges on the basis of the greatest common divisor of the incident vertices. In prime cordial labeling, assignment of values to vertices is a bijective function. Therefore, the values assigned to vertices are;

$$f(x_i) = \{3(i+1) \quad : 0 \leqslant i \leqslant n-1\}$$

$$f(y_i) = \begin{cases} 2+3i & : 0 \leqslant i \leqslant n-1 \textit{ where } i \textit{ is even;} \\ 4+3(i-1) & : 1 \leqslant i \leqslant n-1 \textit{ where } i \textit{ is odd.} \end{cases}$$

$$f(z_i) = \begin{cases} 1+3i & : 0 \leqslant i \leqslant n-1 \textit{ where } i \textit{ is even;} \\ 5+3(i-1) & : 1 \leqslant i \leqslant n-1 \textit{ where } i \textit{ is odd.} \end{cases}$$

There are five types of edges, incident to these vertices. The edge set of $P_e(n,k)$ is divided into five subsets that are mentioned below;

$$\begin{aligned} E_1 &= \{x_i x_{i+k} : 0 \leqslant i \leqslant n-1\}, \\ E_2 &= \{x_i y_i : 0 \leqslant i \leqslant n-1\}, \\ E_3 &= \{y_i y_{i+1} : 0 \leqslant i \leqslant n-1\}, \\ E_4 &= \{y_i z_i : 0 \leqslant i \leqslant n-1\}, \\ E_5 &= \{y_{i+1} z_i : 0 \leqslant i \leqslant n-1\}. \end{aligned}$$

In E_1 edge set, $gcd(f(x_i), f(x_{i+k})) > 1$. Therefore, n edges are labeled with 0.

There are two cases for labeling edge set E_2;

Case-I: For even n, there are $\frac{n}{2}$ edges incident vertices has $gcd(f(x_i), f(y_i)) > 1$. Therefore, labeled with 0 and $\frac{n}{2}$ edges incident vertices has $gcd(f(x_i), f(y_i)) = 1$. Therefore, labeled with 1.

Case-II: For odd n, there are $\frac{n-1}{2}$ edges incident vertices has $gcd(f(x_i), f(y_i)) > 1$, are labeled with 0 and $\frac{n+1}{2}$ edges incident vertices has $gcd(f(x_i), f(y_i)) > 1$ are labeled with 1.

In edge set E_3, n edges incident vertices has $gcd(f(y_i), f_{(}y_{i+1})) > 1$. Therefore, labeled with 0.

In edge set E_4, n edges incident vertices has $gcd(f(y_i), f(z_i)) = 1$. Therefore, labeled with 1.

In edge set E_5, n edges incident vertices has $gcd(f(y_{i+1}, f(z_i)) = 1$. Therefore, labeled with 1.

When n is even, there are $\frac{5n}{2}$ edges labeled with 0 and 1. In Figure 7, number of 0 and 1 are equal.

When n is odd, there are $\frac{5n-1}{2}$ edges valued 0 and $\frac{5n+1}{2}$ edges valued 1 , as shown in Figure 8.

Edges valued 0 and 1 are equal or supremum difference 1. Hence, the graph $P_e(n, k)$, and Petersen graph edges duplication is prime cordial. □

Example 1. *The graph $P_e(12, k)$ is prime cordial for $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$.*

In Figure 7, there are 36 vertices and 60 edges. Dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here, 30 edges are valued 0 and 1, respectively. Therefore, it is prime cordial.

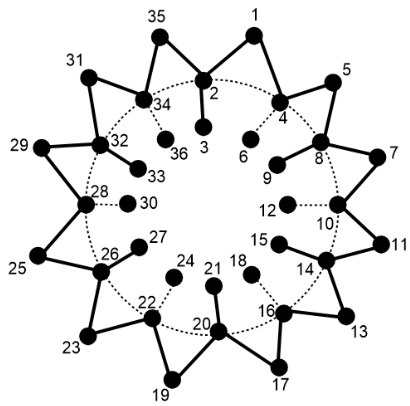

Figure 7. $P_e(12, k)$; duplication of edges by n vertices.

Example 2. *The graph $P_e(13, k)$ is prime cordial where $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$.*

In Figure 8, there are 39 vertices with 65 edges. Dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here are 33 edges valued 1 and 32 edges valued 0. This $P_e(13, k)$ has supremum difference 1 in edge labeling. Therefore, it is prime cordial.

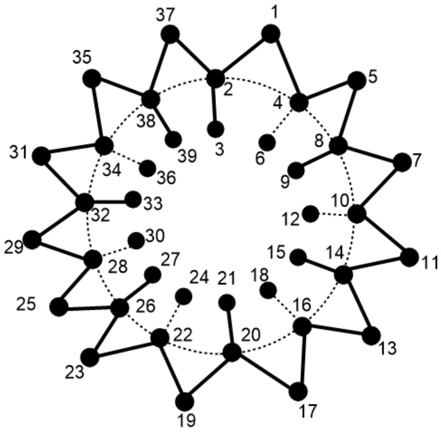

Figure 8. $P_e(13,k)$; duplication of edges by n vertices.

Theorem 2. *The graph $P_v(n,k)_o$ is prime cordial for all $n \geqslant 3$ and $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$.*

Proof. Let us assume that $P(n,k)$ has n edges attached to each outer cycle vertex. Therefore, outer cycle vertices have duplication. There are $4n$ vertices with vertex set $V(P_v(n,k)_o) = \{x_i, y_i : 0 \leqslant i \leqslant n-1 \text{ and } z_i : 0 \leqslant i \leqslant 2n-1\}$ and these vertices are labeled as

$$\begin{aligned} f(x_i) &= 4(i+1) : 0 \leqslant i \leqslant n-1; \\ f(y_i) &= 4i+2 : 0 \leqslant i \leqslant n-1; \\ f(z_i) &= 2i+1 : 0 \leqslant i \leqslant 2n-1. \end{aligned}$$

There are $6n$ edges, $E(P_v(n,k)_o) = \{x_i x_{i+k}, x_i y_i, y_i y_{i+1}, y_i z_i, z_i z_{i+1}, y_i z_{i+1} : 0 \leqslant i \leqslant n-1\}$.

These edges are further described as;

$$\begin{aligned} E_1 &= \{x_i x_{i+k} : 0 \leqslant i \leqslant n-1\}, \\ E_2 &= \{x_i y_i : 0 \leqslant i \leqslant n-1\}, \\ E_3 &= \{y_i y_{i+1} : 0 \leqslant i \leqslant n-1\}, \\ E_4 &= \{y_i z_i : 0 \leqslant i \leqslant n-1\}, \\ E_5 &= \{z_i z_{i+1} : 0 \leqslant i \leqslant n-1\}, \\ E_6 &= \{y_i z_{i+1} : 0 \leqslant i \leqslant n-1\}. \end{aligned}$$

In the E_1 edge set, there is $gcd(f(x_i), f(x_{i+k})) > 1$; therefore, n edges are labeled with 0. In the E_2 edge set, there is $gcd(f(x_i), f(y_i)) > 1$; therefore, n edges are labeled with 0. In the E_3 edge set, there is $gcd(f(y_i), f(y_{i+1})) > 1$; therefore, n edges are labeled with 0. In the E_4 edge set, there is $gcd(f(y_i), f(z_i)) = 1$; therefore, n edges are labeled with 1. In the E_5 edge set, there is $gcd(f(z_i), f(z_{i+1})) = 1$; therefore, n edges are labeled with 1. In the E_6 edge set, there is $gcd(f(y_i), f(z_{i+1})) = 1$; therefore, n edges are labeled with 1. Hence, $3n$ edges are valued 0 and 1 as depicted in Figures 9 and 10; thus, $P_v(n,k)_o$ is prime cordial. □

Example 3. *Figure 9 represents the prime cordial labeling of $P_v(12,k)_o$. The graph $P_v(12,k)_o$ has 48 vertices and 72 edges. Dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here are 36 edges labeled with 1 and 36 edges labeled with 0. Hence, it is prime cordial.*

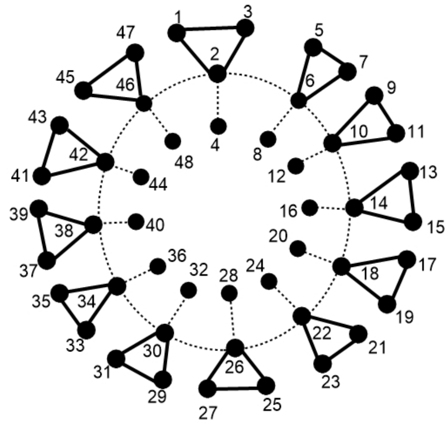

Figure 9. $P_v(12,k)_o$; duplication of outer cycle vertices by 12 edges is prime cordial.

Example 4. *Figure 10 represents the prime cordial labeling of $P_v(13,k)_o$. The graph $P_v(13,k)_o$ has 52 vertices and 78 edges. Dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here are 39 edges labeled with 1 and 39 edges labeled with 0. Hence, it is prime cordial.*

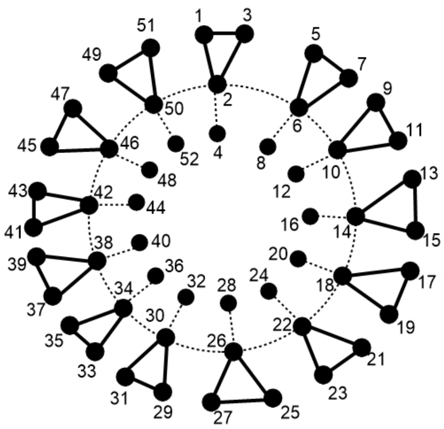

Figure 10. $P_v(13,k)_o$; duplication of outer cycle vertices is prime cordial.

Theorem 3. *The graph $P_v(n,k)_i$ is prime cordial for all $n \geqslant 3$ and $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$.*

Proof. Let us assume $n - edges$ attached to generalized Petersen graph $P(n,k)$ inner cycle vertices. This graph is denoted as $P_v(n,k)_i$. This graph $P_v(n,k)_i$ has $4n$ vertices, $V(P_v(n,k)_i) = \{x_i, z_i : 0 \leqslant i \leqslant n-1 \ and \ y_i : 0 \leqslant i \leqslant 2n-1\}$ are labeled as;

$$\begin{aligned} f(x_i) &= \{4i+2 \quad : 0 \leqslant i \leqslant n-1\}; \\ f(y_i) &= \{2i+1 \quad : 0 \leqslant i \leqslant 2n-1\}; \\ f(z_i) &= \{4(i+1) \quad : 0 \leqslant i \leqslant n-1\}. \end{aligned}$$

The edges are $6n$, $E(P_v(n,k)_i) = \{x_i x_{i+k}, x_i y_i, x_i y_{i+1}, y_i y_{i+1}, x_i z_i, z_i z_{i+1}; 0 \leqslant i \leqslant n-1$ *and* $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor\}$.

These edges are further described as;

$$
\begin{aligned}
E_1 &= \{x_i x_{i+k} \quad : 0 \leqslant i \leqslant n-1\}, \\
E_2 &= \{x_i y_i \quad : 0 \leqslant i \leqslant n-1\}, \\
E_3 &= \{x_i y_{i+1} \quad : 0 \leqslant i \leqslant n-1\}, \\
E_4 &= \{y_i y_{i+1} \quad : 0 \leqslant i \leqslant n-1\}, \\
E_5 &= \{x_i z_i \quad : 0 \leqslant i \leqslant n-1\}, \\
E_6 &= \{z_i z_{i+1} : 0 \leqslant i \leqslant n-1\}.
\end{aligned}
$$

In the E_1 edge set, we have $gcd(f(x_i), f(x_{i+k})) > 1$; therefore, n edges are labeled with 0. In the E_2 edge set, we have $gcd(f(x_i), f(y_i)) = 1$; therefore, n edges are labeled with 1. In the E_3 edge set, we have $gcd(f(x_i), f(y_i)) = 1$; therefore, n edges are labeled with 1. In the E_4 edge set, we have $gcd(f(y_i), f(y_{i+1})) = 1$; therefore, n edges are labeled with 1. In the E_5 edge set, we have $gcd(f(x_i), f(z_i)) > 1$; therefore, n edges are labeled with 0. In the E_6 edge set, we have $gcd(f(z_i), f(z_{i+1})) > 1$; therefore, n edges are labeled with 0. Hence, there are $3n$ edges labeled with 0 and 1, respectively, as shown in Figures 11 and 12; thus, $P_v(n,k)_i$ is prime cordial. □

Example 5. *The graph $P_v(8,k)_i$ is prime cordial for $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$. The graph in Figure 11 has 32 vertices and 48 edges. Dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here are 24 edges labeled with 1 and 24 edges labeled with 0. Hence, this is a prime cordial.*

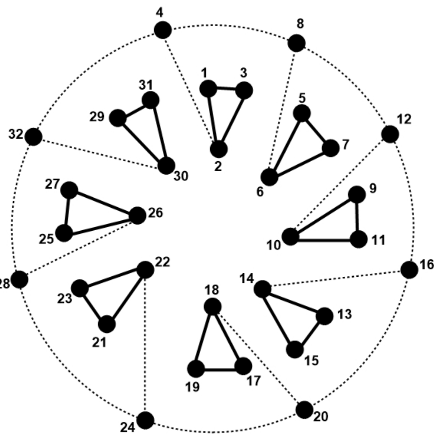

Figure 11. $P_v(8,k)_i$; inner vertex duplication by an edge.

Example 6. *The graph $P_v(9,k)_i$ is prime cordial for $1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor$. The graph represented in Figure 12 has 36 vertices and 54 edges. Dark lines indicate edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Here are 27 edges labeled with 1 and 0, respectively. Hence, $P_v(9,k)_i$ is a prime cordial.*

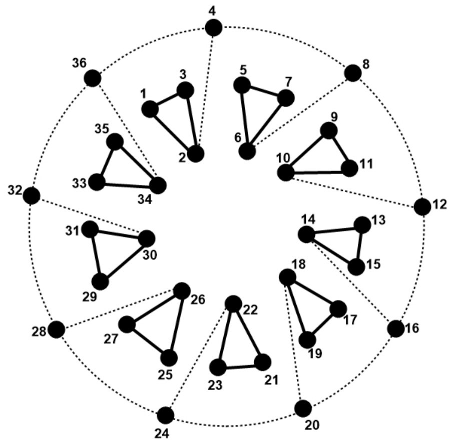

Figure 12. $P_v(9,k)_i$; inner vertex duplication by an edge.

Theorem 4. *The graph* $(P_3, P(n,k))_{ev}$ *is prime cordial for all* $n \geq 3$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Let P_3 and $P(n,k)$ be two graphs. End vertex glowing of n copies of P_3 with outer cycle of $P(n,k)$, denoted as $(P_3, P(n,k))_{ev}$. The end vertex glowing graph $(P_3, P(n,k))_{ev}$ has $4n$ vertices and $5n$ edges. The vertices are $\{u_i, x_i, y_i, z_i; 0 \leq i \leq n-1\}$. The edges are,

$$
\begin{aligned}
E_1 &= \{u_i u_{i+k}\ ;\ 0 \leq i \leq n-1 \text{ and } 1 \leq k \leqslant \lfloor \frac{n}{2} \rfloor\},\\
E_2 &= \{u_i x_i\ ;\ 0 \leq i \leq n-1\},\\
E_3 &= \{x_i x_{i+1}\ ;\ 0 \leq i \leq n-1\},\\
E_4 &= \{x_i y_i\ ;\ 0 \leq i \leq n-1\},\\
E_5 &= \{y_i z_i\ ;\ 0 \leq i \leq n-1\}.
\end{aligned}
$$

Because we knew that values assignment to vertices is a bijective function. Therefore, we have,

For all n greater or equal to 3

$$f(u_i) = \{4(i+1)\ ;\ 0 \leq i \leq n-1\}$$

Now, we will have different cases for assignment of values to the vertices x_i , y_i and z_i.

Case 1: for $n = 3$

$f(x_0) = 3, f(x_1) = 6, f(x_2) = 9$
$f(y_0) = 1, f(y_1) = 5, f(y_2) = 10$
$f(z_0) = 2, f(z_1) = 7, f(z_2) = 11$

Edge labeling is induced from vertex labeling; therefore, valuation of edges differs by 1. Depicted in Figure 13, dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Hence, it is prime cordial.

Case 2: for $n = 4$

$f(x_0) = 3, f(x_1) = 6, f(x_2) = 9, f(x_3) = 15$
$f(y_0) = 1, f(y_1) = 5, f(y_2) = 10, f(y_3) = 13$
$f(z_0) = 2, f(z_1) = 7, f(z_2) = 11, f(z_3) = 14$

Edge labeling is induced function; therefore, edges are equally labeled with 0 and 1. In Figure 13, dark lines indicate that edge is labeled with 1 and dotted line indicates that edge is labeled with 0. Hence, $(P_3, P(4,k))_{ev}$, $\forall\, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is prime cordial.

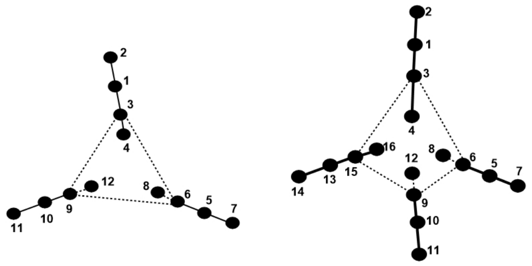

Figure 13. End vertex glowing of P_3 with $P(3,k)$ and $P(4,k)$ is prime cordial.

Case 3: for $n \equiv 0\ (mod\ 12)$

$$f(x_i) = \begin{cases} 2 & : i = 0; \\ 4i+3 & : 1 \leqslant i \leqslant \frac{n}{4}; \\ 4i+2 & : \frac{n}{4} < i \leq n-1. \end{cases}$$

$$f(y_i) = \{\ 4i+1 \quad : 0 \leq i \leq n-1.$$

$$f(z_i) = \begin{cases} 3 & : i = 0; \\ 4i+2 & : 1 \leqslant i \leqslant \frac{n}{4}; \\ 4i+3 & : \frac{n}{4} < i \leq n-1. \end{cases}$$

Edge labeling is induced function; therefore, $\frac{5n}{2}$ edges are equally valued 0 and 1. That is clear from Figure 14, where dark lines indicate that edges are labeled with 1 and dotted line indicate that edges are labeled with 0. This graph is prime cordial.

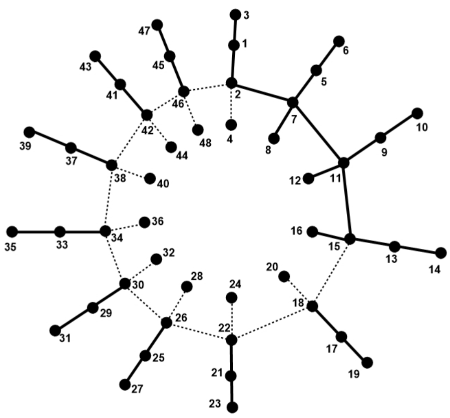

Figure 14. $(P_3, P(12,k))_{ev}$; end vertex glowing of P_3 with $P(12,k)$.

Case 4: for $n \equiv 4\ (mod\ 12)$

$$f(x_i) = \begin{cases} 1 & : i = 0; \\ 4i+3 & : 1 \leqslant i \leqslant [\frac{n}{4}] - 1; \\ 4i+2 & : [\frac{n}{4}] - 1 < i \leq n-1. \end{cases}$$

$$f(y_i) = \begin{cases} 2 & : i = 0; \\ 4i+1 & : 1 \leq i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 3 & : i = 0; \\ 4i+2 & : 1 \leqslant i \leqslant \frac{n}{4} - 1; \\ 4i+3 & : \frac{n}{4} - 1 < i \leq n-1. \end{cases}$$

Since edge labeling is induced function, then $e_f(0) = \frac{5n}{2}$ and $e_f(1) = \frac{5n}{2}$. The valued edges are equal, depicted in Figure 15.

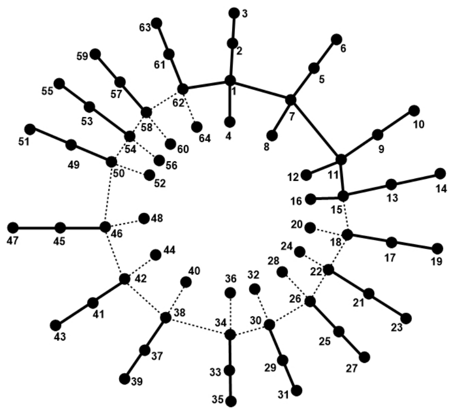

Figure 15. $(P_3, P(16,k))_{ev}$; end vertex P_3 glowing with $P(16,k)$ is prime cordial.

Case 5: for $n \equiv 6, 7\ (mod\ 12)$

$$f(x_i) = \begin{cases} 2 & : i = 0\,; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor\,; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor < i \leq n-1. \end{cases}$$

$$f(y_i) = \left\{ \ 4i+1 \quad : 0 \leq i \leq n-1. \right.$$

$$f(z_i) = \begin{cases} 3 & : i = 0\,; \\ 4i+2 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor\,; \\ 4i+3 & : \lfloor \frac{n}{4} \rfloor < i \leq n-1. \end{cases}$$

Since edge labeling is induced function, then for even n, $e_f(0) = \frac{5n}{2}$ and $e_f(1) = \frac{5n}{2}$. In addition, for odd n, $e_f(0) = \frac{5n-1}{2}$ and $e_f(1) = \frac{5n+1}{2}$. That is prime cordial, depicted in Figure 16.

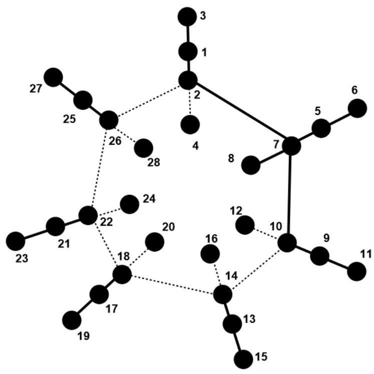

Figure 16. $(P_3, P(7,k))_{ev}$; end vertex P_3 glowing with $P(7,k)$ is prime cordial.

Case 6: for $n \equiv 1, 2, 3, 5, 9, 10, 11\ (mod\ 12)$

$$f(x_i) = \begin{cases} 1 & : i = 0\,; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor - 1\,; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor - 1 < i \leq n-1. \end{cases}$$

$$f(y_i) = \begin{cases} 2 & : i = 0\,; \\ 4i+1 & : 1 \leq i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 3 & : i = 0; \\ 4i+2 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor - 1; \\ 4i+3 & : \lfloor \frac{n}{4} \rfloor - 1 < i \leq n-1. \end{cases}$$

Since edge labeling is induced function, then for even n, labeled edges have equal 0 and 1. In addition, for odd, labeled edges with 0 and 1 only have a difference of 1. Hence, the graph is prime cordial. There are Figures 17 and 18,

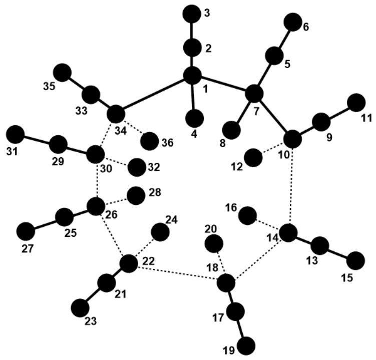

Figure 17. $(P_3, P(9,k))_{ev}$; end vertex P_3 glowing with $P(9,k)$ is prime cordial.

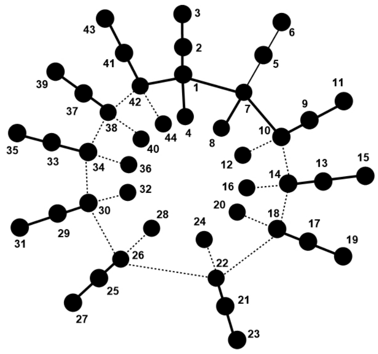

Figure 18. $(P_3, P(11,k))_{ev}$; end vertex P_3 glowing with $P(11,k)$ is prime cordial.

Case 7: for $n \equiv 8\ (mod\ 12)$

$$f(x_i) = \begin{cases} 4i+3 & : 0 \leqslant i \leqslant \frac{n}{4}\ ; \\ 4i+2 & : \frac{n}{4} < i \leq n-1. \end{cases}$$

$$f(y_i) = \begin{cases} 4i+1 & : 0 \leq i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 4i+2 & : 0 \leqslant i \leqslant \frac{n}{4}; \\ 4i+3 & : \frac{n}{4} < i \leq n-1. \end{cases}$$

Since edge labeling is induced function, $\frac{5n}{2}$ edges are valued 0 and 1, respectively. That is clear from Figure 19, where dark lines indicate that edges are labeled with 1 and dotted line indicates that edges are labeled with 0. This graph is prime cordial.

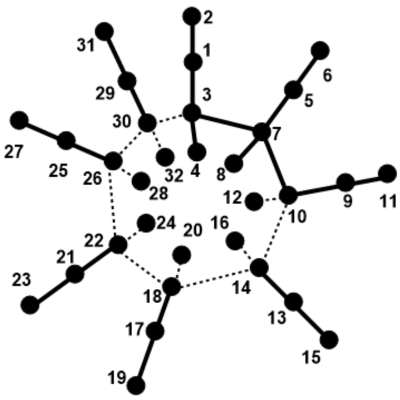

Figure 19. $(P_3, P(8,k))_{ev}$; end vertex glowing of P_3 with $P(8,k)$.

Hence, $(P_3, P(n,k))_{ev}$ *for* $n \geq 3$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is prime cordial. □

Theorem 5. *The graph* $(P_3, P(n,k))_{mv}$ *is prime cordial for all* $n \geq 3$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Let us assume P_3 and $P(n,k)$ are path graph and generalized Petersen graph, respectively. P_3 has middle vertex glowing with $P(n,k)$ outer cycle n vertices, denoted as $(P_3, P(n,k))_{mv}$. The middle vertex glowing graph $(P_3, P(n,k))_{mv}$ has $4n$ vertices and $5n$ edges. The vertices are $\{x_i, y_i\ ;\ 0 \leq i \leq n-1\ and\ z_i\ ;\ 0 \leq i \leq\ 2n-1\}$.

The edges are as following,

$$
\begin{aligned}
E_1 &= \{x_i x_{i+k};\ 0 \leq i \leq n-1\ and\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor\},\\
E_2 &= \{x_i y_i;\ 0 \leq i \leq n-1\},\\
E_3 &= \{y_i y_{i+1};\ 0 \leq i \leq n-1\},\\
E_4 &= \{y_i z_i;\ 0 \leq i \leq n-1\},\\
E_5 &= \{y_i z_{i+1};\ 0 \leq i \leq n-1\}.
\end{aligned}
$$

For prime cordial labeling, we assign values to vertices from 1 to order of graph and edge labeling induced from it. Therefore, we assign values to vertices as given below;

$$f(x_i) = \{4(i+1);\ 0 \leq i \leq n-1\}$$

Now, we will have different cases for assignment of values to the vertices y_i and z_i.

Case 1: for $n = 3$

$f(y_0) = 3, f(y_1) = 6, f(y_2) = 9$
$f(z_0) = 1, f(z_1) = 2, f(z_2) = 5, f(z_3) = 7, f(z_4) = 10, f(z_5) = 11.$

Since edge labeling is induced function, valued edges have supremum difference 1, as shown in Figure 20. Hence, $(P_3, P(3,k))_{mv}\ \forall\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is prime cordial.

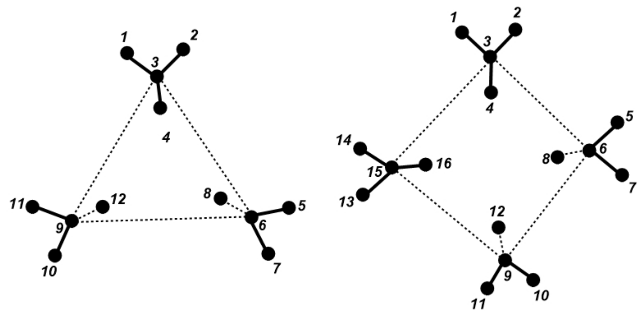

Figure 20. Middle vertex glowing of three copies of P_3 with $P(3,k)$ and $P(4,k)$ is prime cordial.

Case 2: for $n = 4$

$f(y_0) = 3, f(y_1) = 6, f(y_2) = 9, f(y_3) = 15$
$f(z_0) = 1, f(z_1) = 2, f(z_2) = 5, f(z_3) = 7, f(z_4) = 10, f(z_5) = 11, f(z_6) = 13, f(z_7) = 14.$

Since edge labeling is induced function, valued edges are equally labeled with 0 and 1. As shown in Figure 20, dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. Hence, $(P_3, P(4,k))_{mv}\ \forall\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is prime cordial.

Case 3: for $n \equiv 0\ (mod\ 12)$

$$
f(y_i) = \begin{cases} 2 & : i = 0\,; \\ 4i+3 & : 1 \leqslant i \leqslant \frac{n}{4}\,; \\ 4i+2 & : \frac{n}{4} < i \leq n-1. \end{cases}
$$

$$f(z_i) = \begin{cases} 2i+1 & : 0 \leq i \leq 2n-1 \text{ and } i \text{ is even } ; \\ 3 & : i = 1 ; \\ 2i & : 3 \leqslant i \leqslant [\frac{n}{2}] + 1 \text{ and } i \text{ is odd } ; \\ 2i+1 & : [\frac{n}{2}] + 1 < i \leq 2n-1 \text{ and } i \text{ is odd } . \end{cases}$$

Since edge labeling is induced function, $\frac{5n}{2}$ edges are valued 0 and 1 equally as shown in Figure 21. The graph is prime cordial.

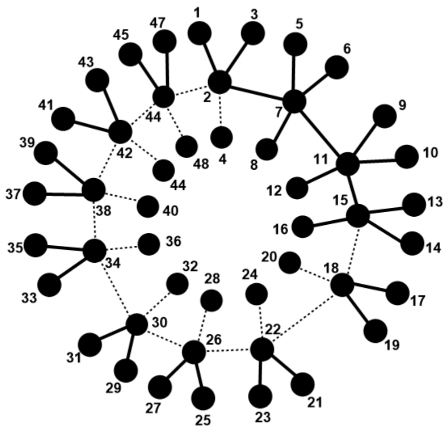

Figure 21. $(P_3, P(12,k))_{mv}$; middle vertex P_3 glowing with $P(12,k)$ is prime cordial.

Case 4: for $n \equiv 1 \ (mod\ 12)$

$$f(y_i) = \begin{cases} 1 & : i = 0 ; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor - 1 ; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0 ; \\ 3 & : i = 1 ; \\ 2i+1 & : 2 \leq i \leq 2n-1 \text{ and } i \text{ is even } ; \\ 2i & : 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor \text{ and } i \text{ is odd } ; \\ 2i+1 & : \lfloor \frac{n}{2} \rfloor + 1 < i \leq 2n-1 \text{ and } i \text{ is odd } . \end{cases}$$

Since edge labeling is induced function, $\frac{5n-1}{2}$ edges are valued 0 and $\frac{5n+1}{2}$ edges are valued 1. The edges labeled with 0 and 1 have maximum difference of 1. The graph is prime cordial.

Case 5: for $n \equiv 2 \ (mod\ 12)$

$$f(y_i) = \begin{cases} 1 & : i = 0 ; \\ 4i+3 & : 1 \leqslant i \leqslant [\frac{n}{4}] - 1 ; \\ 4i+2 & : [\frac{n}{4}] - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0 ; \\ 3 & : i = 1 ; \\ 2i+1 & : 2 \leq i \leq 2n-1 \text{ and } i \text{ is even } ; \\ 2i & : 3 \leqslant i \leqslant [\frac{n}{2}] - 2 \text{ and } i \text{ is odd } ; \\ 2i+1 & : [\frac{n}{2}] - 2 < i \leq 2n-1 \text{ and } i \text{ is odd } . \end{cases}$$

From it, there are $\frac{5n}{2}$ edges labeled with 0 and 1 equally. The graph is prime cordial.

Case 6: for $n \equiv 3 \ (mod\ 12)$

$$f(y_i) = \begin{cases} 1 & : i = 0 ; \\ 4i+3 & : 1 \leqslant i \leqslant [\frac{n}{4}] - 1 ; \\ 4i+2 & : [\frac{n}{4}] - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3 & : i = 1\,; \\ 2i+1 & : 2 \leq i \leq 2n-1 \; and \;\; i \;\; is \;\; even \;\; ; \\ 2i & : 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2 \; and \;\; i \;\; is \;\; odd \;\; ; \\ 2i+1 & : \lfloor \frac{n}{2} \rfloor - 2 < i \leq 2n-1 \; and \;\; i \;\; is \;\; odd\,. \end{cases}$$

Since edge labeling is induced function, $\frac{5n+1}{2}$ edges are valued 0 and $\frac{5n-1}{2}$ are valued 1 . Edges labeled with maximum 1 difference.

Case 7: for $n \equiv 4 \; (mod \; 12)$

$$f(y_i) = \begin{cases} 1 : i = 0\,; \\ 4i+3 : 1 \leqslant i \leqslant [\frac{n}{4}] - 1\,; \\ 4i+2 : [\frac{n}{4}] - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3 & : i = 1\,; \\ 2i+1 & : 2 \leq i \leq 2n-1 \; and \;\; i \;\; is \;\; even \;\; ; \\ 2i & : 3 \leqslant i \leqslant [\frac{n}{2}] - 1 \; and \;\; i \;\; is \;\; odd \;\; ; \\ 2i+1 & : [\frac{n}{2}] - 1 < i \leq 2n-1 \; and \;\; i \;\; is \;\; odd\,. \end{cases}$$

Since edge labeling is induced function, $\frac{5n}{2}$ edges are valued 0 and 1, respectively. The graph is prime cordial.

Case 8: for $n \equiv 5, 9 \; (mod \; 12)$

$$f(y_i) = \begin{cases} 1 : i = 0\,; \\ 4i+3 : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor - 1\,; \\ 4i+2 : \lfloor \frac{n}{4} \rfloor - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3 & : i = 1\,; \\ 2i+1 & : 2 \leq i \leq 2n-1 \; and \;\; i \;\; is \;\; even \;\; ; \\ 2i & : 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 1 \; and \;\; i \;\; is \;\; odd \;\; ; \\ 2i+1 & : \lfloor \frac{n}{2} \rfloor - 1 < i \leq 2n-1 \; and \;\; i \;\; is \;\; odd\,. \end{cases}$$

Since edge labeling is induced function, $\frac{5n-1}{2}$ labeled with 0 and $\frac{5n+1}{2}$ labeled with 1 . That is, $|e_f(0) - e_f(1)| = 1$. As shown in Figure 22, dark lines indicate that edges are labeled with 1 and dotted lines indicate that edges are labeled with 0. The graph is prime cordial.

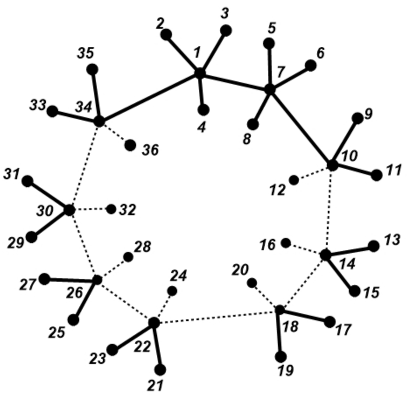

Figure 22. $(P_3, P(9,k))_{mv}$; middle vertex P_3 glowing with $P(9,k)$ is prime cordial.

Case 9: for $n \equiv 6, 10 \; (mod \; 12)$

$$f(y_i) = \begin{cases} 1 & : i = 0\,; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor - 1\,; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor - 1 < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3 & : i = 1\,; \\ 2i+1 & : 2 \le i \le 2n-1 \textit{ and } i \textit{ is even }; \\ 2i & : 3 \leqslant i \leqslant \frac{n}{2} \textit{ and } i \textit{ is odd }; \\ 2i+1 & : \frac{n}{2} < i \le 2n-1 \textit{ and } i \textit{ is odd }. \end{cases}$$

Since edge labeling is induced function. Hence, $\frac{5n}{2}$ edges are labeled by 0 and 1 equally, as shown in Figure 23. The graph is prime cordial.

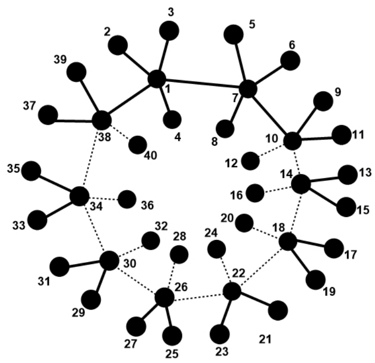

Figure 23. $(P_3, P(10,k))_{mv}$; middle vertex P_3 glowing with $P(10,k)$ is prime cordial.

Case 10: for $n \equiv 7\ (mod\ 12)$

$$f(y_i) = \begin{cases} 2 & : i = 0\,; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor\,; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor < i \le n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2i+1 & : 0 \le i \le 2n-1 \textit{ where } i \textit{ is even }; \\ 3 & : i = 1\,; \\ 2i & : 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor \textit{ and } i \textit{ is odd }; \\ 2i+1 & : \lfloor \frac{n}{2} \rfloor < i \le 2n-1 \textit{ where } i \textit{ is odd }. \end{cases}$$

Since edge labeling is induced function, then $e_f(0) = \frac{5n+1}{2}$ and $e_f(1) = \frac{5n-1}{2}$, represented in Figure 24. This is prime cordial.

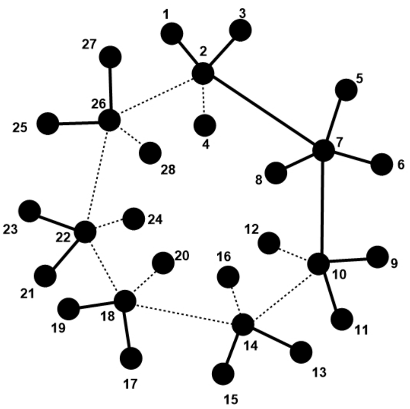

Figure 24. $(P_3, P(7,k))_{mv}$; middle vertex P_3 glowing with $P(7,k)$ is prime cordial.

Case 11: for $n \equiv 8\ (mod\ 12)$

$$f(y_i) = \begin{cases} 4i+3 & : 0 \leqslant i \leqslant \frac{n}{4}\,; \\ 4i+2 & : \frac{n}{4} < i \le n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 2i+1 & : 0 \leq i \leq 2n-1 \text{ } and \text{ } i \text{ } is \text{ } even \text{ } ; \\ 2 & : i = 1 \text{ } ; \\ 2i & : 3 \leqslant i \leqslant [\frac{n}{2}] - 1 \text{ } and \text{ } i \text{ } is \text{ } odd \text{ } ; \\ 2i+1 & : [\frac{n}{2}] - 1 < i \leq 2n-1 \text{ } and \text{ } i \text{ } is \text{ } odd \text{ } . \end{cases}$$

Since edge labeling is induced function, then $\frac{5n}{2}$ is equally valued with 0 and 1, as depicted in Figure 25. It is prime cordial.

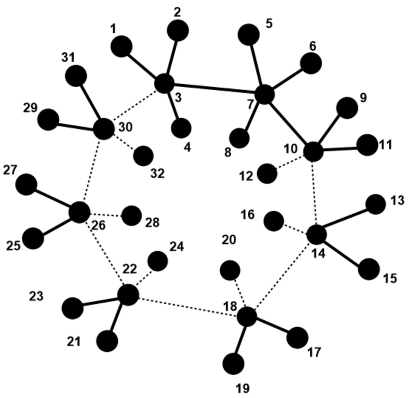

Figure 25. $(P_3, P(8,k))_{mv}$; middle vertex P_3 glowing with $P(8,k)$ is prime cordial.

Case 12: for $n \equiv 11$ $(mod\ 12)$

$$f(y_i) = \begin{cases} 2 & : i = 1 \text{ } ; \\ 4i+3 & : 1 \leqslant i \leqslant \lfloor \frac{n}{4} \rfloor \text{ } ; \\ 4i+2 & : \lfloor \frac{n}{4} \rfloor < i \leq n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 1 & : i = 0 \text{ } ; \\ 3 & : i = 1 \text{ } ; \\ 2i+1 & : 2 \leq i \leq 2n-1 \text{ } and \text{ } i \text{ } is \text{ } even \text{ } ; \\ 2i & : 3 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor \text{ } and \text{ } i \text{ } is \text{ } odd \text{ } ; \\ 2i+1 & : \lfloor \frac{n}{2} \rfloor < i \leq 2n-1 \text{ } and \text{ } i \text{ } is \text{ } odd \text{ } . \end{cases}$$

Since edge labeling is induced function, the edges $\frac{5n+1}{2}$ are valued with 0 and $\frac{5n-1}{2}$ are valued with 1. There is maximum 1 difference in valued edges. As shown in Figure 26. The graph is prime cordial.

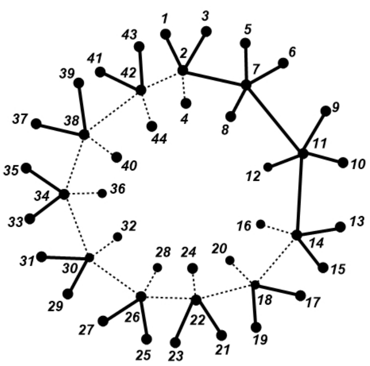

Figure 26. $(P_3, P(11,k))_{mv}$; middle vertex P_3 glowing with $P(11,k)$ is prime cordial.

Hence, $(P_3, P(n,k))_{mv}$ for all $n \geq 3$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ is prime cordial. □

Theorem 6. *The graph* $P_v(n,k)_p$ *is prime cordial for all* $n \geq 3$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Let $P(n,k)$ be a generalized Petersen graph. There are n copies of pendant vertices attached to outer cycle of $P(n,k)$.

There are $3n$ vertices and size $4n$. Described as,

$$V(P_v(n,k)_p) = \{x_i, y_i, z_i;\ 0 \le i \le n-1\},$$

$$E(P_v(n,k)_p) = \{x_i x_{i+k}, x_i y_i, y_i y_{i+1}, y_i z_i;\ 0 \le i \le n-1 \ and\ 1 \le k \le \lfloor \frac{n}{2} \rfloor\}.$$

In prime cordial labeling, values assigned to vertices is a bijective function from 1 to order of graph and edge labeling induce from it. Therefore, values are assigned to vertices as;

$$f(x_i) = \{3i+3;\ 0 \le i \le n-1\}$$

For $f(y_i)$ and $f(z_i)$, we have following cases:

Case 1: for $n = 3$

$f(y_0) = 1, f(y_1) = 4, f(y_2) = 8$
$f(z_0) = 5, f(z_1) = 2, f(z_2) = 7$

Since edge labeling is induced function, edges are equally labeled with 0 and 1. As shown in Figure 27, $P_v(3,k)_p$ is prime cordial.

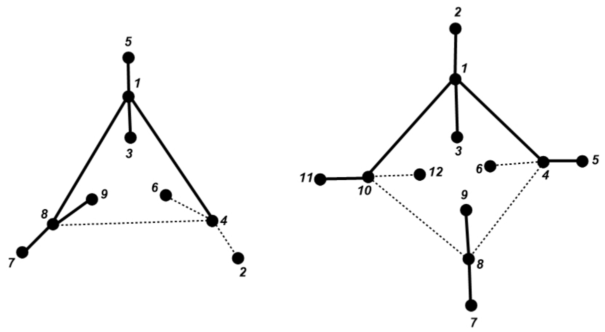

Figure 27. Outer cycle vertices duplication by pendant vertices in $P(3,k)$ and $P(4,k)$ is prime cordial.

Case 2: for $n = 4$

$f(y_0) = 1, f(y_1) = 4, f(y_2) = 8, f(y_3) = 10$
$f(z_0) = 2, f(z_1) = 5, f(z_2) = 7, f(z_3) = 11.$

Since edge labeling is induced function, 0 and 1 are equally assigned to edges. As shown in Figure 27, $P_v(4,k)_p$ is prime cordial.

Case 3: for $n \equiv 0\ (mod\ 6)$

$$f(y_i) = \begin{cases} 2 & : i = 0\,; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-3}{3}\ when\ \ i\ \ is\ \ odd\ \ ; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-3}{3}\ when\ \ i\ \ is\ \ even\ \ ; \\ 3i+2 & : \frac{n-3}{3} < i \leqslant n-1\ when\ \ i\ \ is\ \ even\ \ ; \\ 3i+1 & : \frac{n-3}{3} < i \leqslant n-1\ when\ \ i\ \ is\ \ odd\ . \end{cases}$$

$$f(z_i) = \begin{cases} 1 & : i = 0\,; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-3}{3}\ when\ \ i\ \ is\ \ odd\ \ ; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-3}{3}\ when\ \ i\ \ is\ \ even\ \ ; \\ 3i+1 & : \frac{n-3}{3} < i \leqslant n-1\ when\ \ i\ \ is\ \ even\ \ ; \\ 3i+2 & : \frac{n-3}{3} < i \leqslant n-1\ when\ \ i\ \ is\ \ odd\ . \end{cases}$$

This labeling is explained in Figure 28.

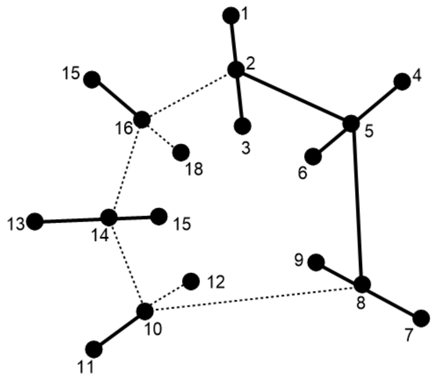

Figure 28. $P_v(6,k)_p$; outer vertex $P(6,k)$ duplication by pendant vertices is prime cordial.

Case 4: for $n \equiv 1\ (mod\ 6)$

$$f(y_i) = \begin{cases} 2 & : i = 0; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-4}{3}\ when\ i\ is\ odd; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-4}{3}\ when\ i\ is\ even; \\ 3i+2 & : \frac{n-4}{3} < i \leqslant n-1\ and\ when\ i\ is\ even; \\ 3i+1 & : \frac{n-4}{3} < i \leqslant n-1\ and\ when\ i\ is\ odd. \end{cases}$$

$$f(z_i) = \begin{cases} 1 & : i = 0; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-4}{3}\ when\ i\ is\ odd; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-4}{3}\ when\ i\ is\ even; \\ 3i+1 & : \frac{n-4}{3} < i \leqslant n-1\ when\ i\ is\ even; \\ 3i+2 & : \frac{n-4}{3} < i \leqslant n-1\ when\ i\ is\ odd. \end{cases}$$

This labeling is explained in Figure 29.

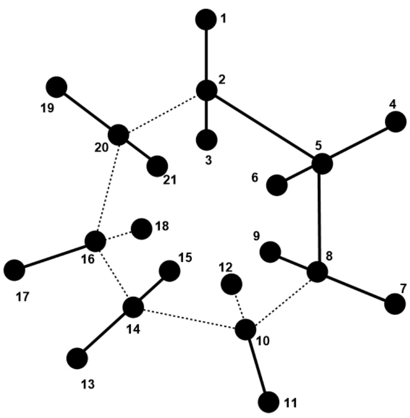

Figure 29. $P_v(7,k)_p$; outer vertex $P(7,k)$ duplication by pendant vertices is prime cordial.

Case 5: for $n \equiv 2\ (mod\ 6)$

$$f(y_i) = \begin{cases} 1 & : i = 0\,; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-5}{3}\ where\ i\ is\ odd\ ; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-5}{3}\ where\ i\ is\ even\ ; \\ 3i+2 & : \frac{n-5}{3} < i \leqslant n-1\ where\ i\ is\ even\ ; \\ 3i+1 & : \frac{n-5}{3} < i \leqslant n-1\ where\ i\ is\ odd\ . \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-5}{3} \; where \;\; i \;\; is \;\; odd \;\; ; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-5}{3} \; where \;\; i \;\; is \;\; even \;\; ; \\ 3i+1 & : \frac{n-5}{3} < i \leqslant n-1 \; where \;\; i \;\; is \;\; even \;\; ; \\ 3i+2 & : \frac{n-5}{3} < i \leqslant n-1 \; where \;\; i \;\; is \;\; odd \; . \end{cases}$$

This labeling is explained in Figure 30.

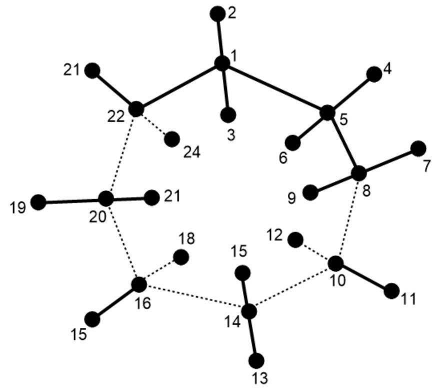

Figure 30. $P_v(8,k)_p$; outer vertex $P(8,k)$ duplication by pendant vertices is prime cordial.

Case 6: for $n \equiv 3\ (mod\ 6)$

$$f(y_i) = \begin{cases} 1 & : i = 0\,; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-6}{3} \; when \;\; i \;\; is \;\; odd \;\; ; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-6}{3} \; when \;\; i \;\; is \;\; even \;\; ; \\ 3i+2 & : \frac{n-6}{3} < i \leqslant n-1 \; and \;\; when \;\; is \;\; even \;\; ; \\ 3i+1 & : \frac{n-6}{3} < i \leqslant n-1 \; when \;\; i \;\; is \;\; odd \; . \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-6}{3} \; when \;\; i \;\; is \;\; odd \;\; ; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-6}{3} \; when \;\; i \;\; is \;\; even \;\; ; \\ 3i+1 & : \frac{n-6}{3} < i \leqslant n-1 \; when \;\; i \;\; is \;\; even \;\; ; \\ 3i+2 & : \frac{n-6}{3} < i \leqslant n-1 \; when \;\; i \;\; is \;\; odd \; . \end{cases}$$

This labeling is explained in Figure 31.

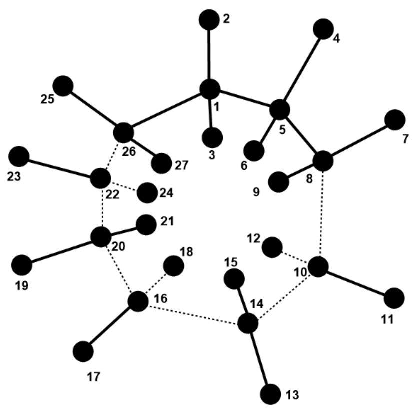

Figure 31. $P_v(9,k)_p$; outer vertex $P(9,k)$ duplication by pendant vertices is prime cordial.

Case 7: for $n \equiv 4\ (mod\ 6)$

$$f(y_i) = \begin{cases} 1 & : i = 0\,; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-4}{3} \text{ when } i \text{ is odd }; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-4}{3} \text{ when } i \text{ is even }; \\ 3i+2 & : \frac{n-4}{3} < i \leqslant n-1 \text{ when } i \text{ is even }; \\ 3i+1 & : \frac{n-4}{3} < i \leqslant n-1 \text{ when } i \text{ is odd }. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-4}{3} \text{ when } i \text{ is odd }; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-4}{3} \text{ when } i \text{ is even }; \\ 3i+1 & : \frac{n-4}{3} < i \leqslant n-1 \text{ when } i \text{ is even }; \\ 3i+2 & : \frac{n-4}{3} < i \leqslant n-1 \text{ when } i \text{ is odd }. \end{cases}$$

This labeling is explained in Figure 32.

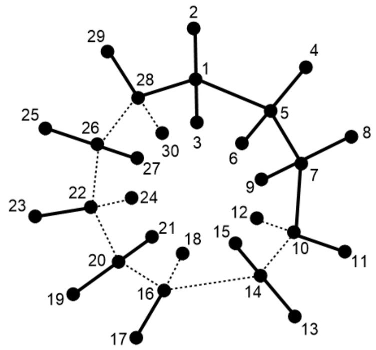

Figure 32. $P_v(10,k)_p$; outer vertex $P(10,k)$ duplication by pendant vertices is prime cordial.

Case 8: for $n \equiv 5 \ (mod\ 6)$

$$f(y_i) = \begin{cases} 1 & : i = 0\,; \\ 3i+2 & : 1 \leqslant i \leqslant \frac{n-5}{3} \text{ when } i \text{ is odd }; \\ 3i+1 & : 2 \leqslant i \leqslant \frac{n-5}{3} \text{ when } i \text{ is even }; \\ 3i+2 & : \frac{n-5}{3} < i \leqslant n-1 \text{ when } i \text{ is even }; \\ 3i+1 & : \frac{n-5}{3} < i \leqslant n-1 \text{ when } i \text{ is odd }. \end{cases}$$

$$f(z_i) = \begin{cases} 2 & : i = 0\,; \\ 3i+1 & : 1 \leqslant i \leqslant \frac{n-5}{3} \text{ when } i \text{ is odd }; \\ 3i+2 & : 2 \leqslant i \leqslant \frac{n-5}{3} \text{ when } i \text{ is even }; \\ 3i+1 & : \frac{n-5}{3} < i \leqslant n-1 \text{ when } i \text{ is even }; \\ 3i+2 & : \frac{n-5}{3} < i \leqslant n-1 \text{ when } i \text{ is odd }. \end{cases}$$

This labeling is explained in Figure 33.

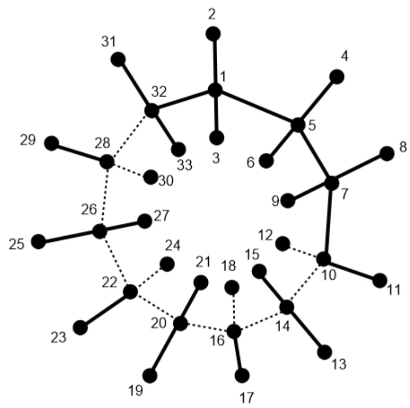

Figure 33. $P_v(11,k)_p$; outer vertex $P(11,k)$ duplication by pendant vertices is prime cordial.

Hence, the graph $P_v(n,k)_p$ is prime cordial for all $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. $\square$

3. Conclusions

In this article, we have studied the prime cordial labeling of a generalized Petersen graph under duplication operation, and we have proved that it is a prime cordial graph after duplication. We have also proved that glowing of path graph P_3 to $P(n,k)$ at a common vertex is prime cordial. These graphs invite the interest of researchers for their application in communication. Binary digits belong to Boolean Algebra, and in our results, edges are labeled with 0 and 1. These binary digits assignment to edges attracts the attention of computer programmers for data communication. Therefore, these results are fruitful for switching and data communication.

Author Contributions: Writing—original draft preparation, W.Z., M.N. and I.A.; writing—review and editing, W.Z., M.N. and I.A. All authors have read and agreed to the published version of the manuscript.

Funding: We thank the reviewers for their constructive comments in improving the quality of this paper. This work was supported by the National Key Research and Development Program under Grant 2018YFB0904205.

Institutional Review Board Statement: This research was conducted by Mr. Irfan Ahmad (Reg. No.: MPMT-023R19-24) in his M.Phil thesis under the supervision of Muhammad Naeem and funded by Weidong Zhao in the Department of Mathematics and Statistics, Institute of Southern Punjab, Multan.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest

References

1. Wilson, R.J. *History Graph Theory From: Handbook Graph Theory*; CRC Press: Boca Raton, FL, USA, 2013.
2. Chartrand, G.; Egan, C.; Zhang, P. *How to Label a Graph*; Springer Briefs in Mathematics: New York, NY, USA, 2019; pp. 19–23.
3. Rosa, A. *On Certain Valuations of the Vertices of a Graph, Theory of Graphs*; Gordon and Breach: New York, NY, USA, 1967; pp. 349–355.
4. Golomb, S.W. *How to Number a Graph, Graph Theory and Computing*; Academic Press: New York, NY, USA, 1972; pp. 23–37.
5. Beineke, L.W.; Hegde, S.M. Strongly Multiplicative Graphs. *Discuss. Math. Graph Theory* **2001**, *21*, 63–75. [CrossRef]
6. Liang, Z.H.; Bai, Z.L. On the odd harmonious graphs with applications. *J. App. Math. Comput.* **2008**, *29*, 105–116. [CrossRef]
7. Sundaram, M.; Ponraj, R.; Somasundram, S. Prime Cordial Labeling of Graphs. *J. Ind. Acad. Math.* **2005**, *27*, 373–390.
8. Ahmad, I. Muhammad Naeem, Prime Cordial Labeling of Generalized Petersen Graph under Some Graph Operations. MPhil Thesis, Institute of Southern Punjab, Multan, Pakistan, 2021.