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U. E. R. DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ DE BORDEAUX I
Talence, France

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Prime k -th power non-residues

by

RICHARD H. HUDSON (Durham, North Carolina)

1. Introduction and summary. Throughout this paper k will denote an integer ≥ 2 and p will be a prime such that $(k, p-1) = v_k(p) > 1$. We denote the n th prime k th power non-residue by $g_n(p, k)$, $n = 1, 2, \dots$

We attack the problem of finding an upper bound for $g_2(p, k)$ from several vantage points and we consider, in addition, the case $n > 2$.

A large number of authors have given upper bounds for $g_1(p, k)$ under varying hypotheses.

Burgess [6] has shown that for each $\delta > 0$,

$$(1.1) \quad g_1(p, 2) = O_\delta(p^{1/(4e^{1/2})+\delta}).$$

In order to avoid any misunderstanding regarding the nature of our O -estimates, we will always use the notation O_δ to indicate an implied constant depending at most on δ , while O will indicate an absolute constant.

Wang Yuan [18] generalized the method of Burgess. Namely, for each $\delta > 0$, he has shown that

$$(1.2) \quad g_1(p, k) = O_\delta(p^{1/(4e^{v-1/v})+\delta})$$

for every $v = v_k(p) \geq 2$,

$$(1.3) \quad g_1(p, k) < p^{1/12}$$

if $v_k(p) \geq 21$, and

$$(1.4) \quad g_1(p, k) < p^{(\log \log v + 2)/4 \log v}$$

if $v = v_k(p) > e^{33}$.

Wang's results (1.2), (1.3), and (1.4), essentially halve the exponent in the upper bounds for $g_1(p, k)$ given by Buchstab [5] and independently by Davenport and Erdős [8].

K. K. Norton [15] has recently generalized the above results by omitting the restriction that p be prime.

Employing analytic methods, Hua [10] and Erdős and Ko [9], have given upper bounds for $g_2(p, 2)$. In particular, Hua has shown that for each $p \geq e^{250}$,

$$(1.5) \quad g_2(p, 2) < (57600 p)^{5/16}.$$

A. Brauer [2], [4], and C. T. Whyburn [17] have given upper bounds for $g_2(p, 2)$ using purely elementary methods. The important advantage in the use of elementary methods, is that the results of Brauer and Whyburn hold for small primes as well as for large primes, an advantage which may well be crucial for a given application. Indeed, this was precisely the case in the application of Brauer's results [2] for $g_2(p, 2)$ and $g_3(p, 2)$ to the problem of determining which quadratic number fields are Euclidean.

Brauer [2] has shown that the following results hold for all p .

$$(1.6) \quad \text{If } p \equiv 3 \pmod{8}, g_2(p, 2) < 2p^{2/5} + (49/2)p^{1/5} + 7.$$

$$(1.7) \quad \text{If } p \equiv 5 \pmod{8}, g_2(p, 2) < 2^{3/5}p^{2/5} + 2^{-6/5}(25)p^{1/5} + 3.$$

C. Whyburn [17] has extended Brauer's results for $g_2(p, 2)$ in the following cases.

$$(1.8) \quad \text{If } p \equiv 7 \pmod{24}, g_2(p, 2) < (6p)^{2/5} + (86/3)(6p)^{1/5} + 59.$$

$$(1.9) \quad \text{If } p \equiv 17 \pmod{24}, g_2(p, 2) < (3p)^{2/5} + (91/6)(3p)^{1/5} + 29.$$

$$(1.10) \quad \text{If } p \equiv 23 \pmod{24}, g_2(p, 2) < (10p)^{2/5} + (27/2)(10p)^{1/5} - 1.$$

Unfortunately, neither Brauer nor Whyburn was able to give a non-trivial upper bound for $g_2(p, 2)$ if $p \equiv 1 \pmod{4}$ and $g_1(p, 2) > 3$, i.e., if $p \equiv 1 \pmod{24}$. Furthermore, to the best of our knowledge, no author has given an upper bound sharper than $O(p^{1/2})$ for $g_2(p, k)$, $v_k(p) > 2$, although it is obvious that $g_2(p, k) \leq g_2(p, 2)$ if $v_k(p)$ is even.

Let $C(p)$ denote the multiplicative group consisting of the residue classes mod p which are relatively prime to p . $C(p)$ has a proper multiplicative subgroup, $C_k(p)$, consisting of the k th power residues. The remaining $v_k(p) - 1$ cosets formed with respect to $C_k(p)$ are called classes of k th power non-residues. Let S_n denote the maximum number of consecutive integers in any of the $v_k(p) - 1$ classes of non-residues and let S denote the maximum number of consecutive integers in any of the $v_k(p)$ classes of residues or non-residues.

It follows from a paper of A. Brauer [3], that

$$(1.11) \quad S < (2p)^{1/2} + 2$$

for all p .

The author [11] has given a small improvement of (1.11), namely,

$$(1.12) \quad S_n < p^{1/2} + 2^{2/3}p^{1/3} + 2^{1/3}p^{1/6} + 1.$$

The best upper bound for S has been given by Burgess [7]. Employing non-elementary methods, he has shown that

$$(1.13) \quad S = O(p^{1/4} \log p)$$

where the implied constant is absolute.

Unfortunately, an admissible value for the implied constant in (1.13) is not known, a fact which lends significance to specific estimates for S .

In § 2 some useful lemmas will be established which give an upper bound for $g_n(p, k)$ in terms of S_n and $\prod_{r=1}^{n-1} g_r(p, k)$ if $\prod_{r=1}^{n-1} g_r(p, k)$ is a k th power non-residue, and in terms of S and $\prod_{r=1}^{n-1} g_r(p, k)$ otherwise.

In § 3 we show, by purely elementary methods, that under very general conditions; in fact, whenever $g_1(p, k) < 2p^{1/5} + 3$, that

$$(1.14) \quad g_2(p, k) < 12p^{2/5} + 42p^{1/5} + 43.$$

More generally, we show that for all p ,

$$(1.15) \quad g_2(p, k) < 4p^{7/16}(4.7 \log p)^{3/4} + 37.6p^{1/4} \log p + 1.$$

We also show that the sharper result (1.14) holds whenever $v_k(p) \geq 13$ and p is larger than a constant which can be made specific. The coefficients in (1.14) and (1.15) can be approximately cut in half if -1 is a k th power residue.

The results in Section 3 are more general and the proofs are simpler than the numerous special results of Brauer and Whyburn, for not only do they extend to all values of k , but they also encompass the difficult case, $p \equiv 1 \pmod{24}$.

In § 4 we turn away from elementary methods and specific estimates and use the O -estimates, (1.2), (1.3), (1.4), and (1.13). Combined with the lemmas established in Section 2 we are able to obtain upper bounds for $g_2(p, k)$ superior to (1.14).

In fact we are able to show that for each $\delta > 0$ and p "sufficiently large",

$$(1.16) \quad g_2(p, k) = O_\delta(p^{a_\nu + \delta}),$$

where, for example, $a_3 = .378354 \dots$, $a_{21} = 1/3$, and $a_\nu = 1/4$ for "large" ν .

In addition, we show that if p is a "sufficiently large" prime for which $g_1(p, k), \dots, g_{n-1}(p, k)$ are "small" (in a sense to be made precise later),

$$(1.17) \quad g_n(p, k) = O(p^{1/4} \log p).$$

For example, if $v_k(p) = 2$ and $p \not\equiv \pm 1 \pmod{24}$, or if $v_k(p) = 3$ and $p \neq x^2 + 27y^2$ so that $g_2(p, k)$ is the smallest odd cubic non-residue, then

$$(1.18) \quad g_2(p, k) = O(p^{1/4} \log p).$$

Similar results are discussed when $n > 2$.

Finally, we note that if any one of several conjectures is true, then (1.16) can be improved. For example, if the extended Riemann hypothesis is true, then for all k and "sufficiently large" p ,

$$(1.19) \quad g_2(p, k) = O(p^{1/4} \log^3 p).$$

2. Preliminaries. We often will abbreviate $g_n(p, k)$, by g_n and $v_k(p)$ by v . $[x]$ will denote the greatest integer $\leq x$. (y_1, \dots, y_n) will denote an integer interval which does not include y_1 or y_n ; $[y_1, \dots, y_n]$ will include y_1 and y_n if and only if they are integers. We assume the fact, which is trivial to verify, then $g_2(p, k) < p$ if $p \geq 5$.

LEMMA 1. Let $p \geq 5$ be a prime for which $g_1(p, k) = 2$ so that $g_2(p, k)$ is the smallest odd k -th power non-residue, and let S_n denote the maximum number of consecutive integers in any of the $v_k(p) - 1$ non-residue classes. Then

$$(2.1) \quad g_2(p, k) \leq 2S_n + 1$$

if -1 is a k -th power non-residue, and

$$(2.2) \quad g_1(p, k) \leq S_n + 1$$

if -1 is a k -th power residue.

Proof. The odd integers less than g_2 are k th power residues and, consequently,

$$(2.3) \quad (p+1)/2, (p+3)/2, \dots, (p+(g_2-2))/2$$

are $(g_2-1)/2$ consecutive integers belonging to precisely one of the $v-1$ classes of k th power non-residues, call it C .

If -1 is a k th power residue, then

$$(2.4) \quad (p-(g_2-2))/2, \dots, (p-1)/2, (p+1)/2, \dots, (p+(g_2-2))/2$$

are g_2-1 consecutive integers belonging to C .

It follows that if -1 is a k th power non-residue,

$$(2.5) \quad g_2 \leq 2S_n + 1,$$

since $(g_2-1)/2 \leq S_n$. Similarly, if -1 is a k th power residue,

$$(2.6) \quad g_2 \leq S_n + 1$$

since $g_2-1 \leq S_n$.

Combining (1.11) or (1.12) with (2.5) and (2.6) one obtains immediately an elementary upper bound for $g_2(p, k)$ which may be of some interest for small p , but which will be substantially improved in the next section.

There is clearly nothing magical about the choice $g_1 = 2$ in the above proof and, in fact, Lemma 1 generalizes in the following fashion.

LEMMA 2. Let $p \geq 5$. Then

$$(2.7) \quad g_2(p, k) \leq g_1 S_n + 1$$

if -1 is a k -th power non-residue, and

$$(2.8) \quad g_2(p, k) \leq (g_1/2) S_n + 1$$

if -1 is a k -th power residue.

Proof. Let t_1 be the largest positive integer such that $t_1 g_1 + 1 < g_2$. Then the integers

$$(2.9) \quad 1, g_1+1, \dots, t_1 g_1+1,$$

are k th power residues. Let x be the unique integer solution to the congruence $g_1 x \equiv 1 \pmod{p}$ such that $1 < x < p$. Then

$$(2.10) \quad x, x+1, \dots, x+t_1,$$

are t_1+1 consecutive integers belonging to exactly one of the $v-1$ classes of k th power non-residues, call it C' .

Let t_2 be the largest positive integer such that $1 - t_2 g_1 > -g_2$. If -1 is a k th power residue,

$$(2.11) \quad x-t_2, \dots, x-1, x, x+1, \dots, x+t_1,$$

are t_1+t_2+1 consecutive integers belonging to C' .

It follows that if -1 is a k th power non-residue,

$$(2.12) \quad g_2 \leq (t_1+1)g_1+1 \leq g_1(S_n+1),$$

since t_1 is the largest integer such that $t_1 g_1 + 1 < g_2$ and, obviously, $t_1 + 1 \leq S_n$.

Similarly, if -1 is a k th power residue,

$$(2.13) \quad g_2 \leq (g_1/2)(S_n+1)$$

since

$$1 - (t_2+1)g_1 \leq -g_2 \Rightarrow g_2 \leq (t_2+1)g_1 - 1 \Rightarrow 2g_2 \leq (t_1+t_2+2)g_1 \\ \leq (S_n+1)g_1 \Rightarrow g_2 \leq (g_1/2)(S_n+1).$$

With some minor additional complications, Lemmas 1 and 2 generalize still further.

LEMMA 3. Let n be any integer ≥ 2 and let p be any prime. If -1 is a k -th power non-residue, then

$$(2.14) \quad g_n \leq (S+1) \left(\prod_{r=1}^{n-1} g_r \right) + 1,$$

and if -1 is a k -th power residue,

$$(2.15) \quad g_n \leq S \left(\prod_{r=1}^{n-1} g_r \right) + 1.$$

Proof. Let $Z = \prod_{r=1}^{n-1} g_r$ and let t_1 be the largest non-negative integer such that $t_1 Z + 1 < g_n$. Then the integers

$$(2.16) \quad 1, Z+1, 2Z+1, \dots, t_1 Z+1,$$

are k th power residues (since their prime factorizations clearly contain only k th power residues) provided that they are not multiples of p . Of course, this additional complication arises only if $g_n > p$.

As before, let x be the unique integer solution to the congruence $Zx \equiv 1 \pmod{p}$ such that $1 < x < p$. This exists since it is clear that $(Z, p) = 1$.

Let p be a prime for which -1 is a k th power non-residue. Then

$$(2.17) \quad x, x+1, \dots, x+t_1,$$

with the possible exception of $x+t_1$, are t_1+1 consecutive integers belonging to the same coset. For if $x < p$, then $x+t_1 \leq p$ since otherwise the interval $[x, \dots, x+t_1]$ must contain $p+1$ and $p-1$ which are in different cosets. It follows that $t_1 \leq S$, and since t_1 is the largest integer such that $t_1 Z + 1 < g_n$,

$$(2.18) \quad g_n \leq (t_1+1)Z+1 \leq (S+1)Z+1.$$

Let p be a prime for which -1 is a k th power residue and let t_2 be the largest non-negative integer such that $1-t_2 Z > -g_n$. Consider the integers,

$$(2.19) \quad x-t_2, \dots, x-1, x, x+1, \dots, x+t_1.$$

If $x-t_2 \leq 0$, then clearly $x+t_1 < p$ since all the integers between 1 and p cannot belong to the same coset. It follows that $x, x+1, \dots, x+t_1$ are t_1+1 consecutive integers belonging to the same coset so that $t_1+1 \leq S$.

If $x-t_2 \geq 1$, then $x, x-1, \dots, x-t_2$ are t_2+1 consecutive elements belonging to the same coset so that $t_2+1 \leq S$.

Since $g_n \leq (t_1+1)Z+1$ and $\leq (t_2+1)Z-1$, we obtain

$$(2.20) \quad g_n \leq SZ+1.$$

If it is known that $g_n < p$, the above proof simplifies. In fact, the proof is then identical with the proof of Lemma 2 with g_2 replaced by g_n , g_1 replaced by Z , and S_n replaced by S . Consequently, one is able to obtain

$$(2.21) \quad g_n \leq \left(\prod_{r=1}^{n-1} g_r \right) (S+1)$$

if -1 is a k th power non-residue, and

$$(2.22) \quad g_n \leq \left(\prod_{r=1}^{n-1} g_r \right) ((S+1)/2)$$

if -1 is a k th power residue.

Also, it is obvious that S may be replaced by S_n in Lemma 3 if $\prod_{r=1}^{n-1} g_r$ is a k th power non-residue.

Remark 1. Some curious results, not of great interest, may be read off from the above lemmas if S, S_n , or g_n are specifically known, or if $v = p^a$, $a \geq 3/4$. For example, if $k = 2$ and $p \equiv 13 \pmod{24}$, it follows from (2.2) that $S_n \geq 4$, since $g_2 = 5$. $p = 13$ is the only known example of a prime for which $S_n > \sqrt{p}$.

If $v = p^a$, $a > 3/4$, and p is "sufficiently large", then by (1.4), $g_1 < p^\varepsilon$ (for each $\varepsilon > 0$), and it follows trivially from Lemma 2 that $g_2 = O(p^{1/4})$ since the maximum number of integers in any coset (consecutive, or otherwise) is $(p-1)/v = O(p^{1/4})$. Indeed, if $v = p^a$, $a > 8/9$, it follows from Lemmas 2 and 3 that $g_2 = O(p^{1/9})$, $g_3 = O(p^{2/9})$, and $g_4 = O(p^{1/3})$!

3. Specific estimates for $g_2(p, k)$. By a specific estimate for $g_2(p, k)$ we will mean an upper bound for $g_2(p, k)$ which is valid for all p and k , or at least for all $p > p_0$ where p_0 is explicitly stated.

THEOREM 1. *If $g_1(p, k) = 2$ so that $g_2(p, k)$ is the smallest odd k -th power non-residue, then for all $p > 2^{15}$,*

$$(3.1) \quad g_2(p, k) < 2^{3/5} + ((3125/2048)p)^{1/5} + 1$$

if -1 is a k -th power residue;

$$(3.2) \quad g_2(p, k) < 2p^{3/5} + (9/8)p^{1/5} + 1$$

if -1 is a k -th power non-residue.

Proof. Let $p > 32768$. Assume that the theorem is false and let r be an odd integer such that $1 \leq r < g_2$. Let J denote the interval

$$(3.3) \quad [p - (g_2 - 2)/2, \dots, p + (g_2 - 2)/2]$$

if -1 is a k th power residue, and

$$(3.4) \quad [(p+1)/2, \dots, p + (g_2 - 2)/2]$$

if -1 is a k th power non-residue. It follows that J contains only k th power non-residues (in fact, only integers from one of the $v-1$ classes of non-residues). Let

$$(3.5) \quad dr, (d+1)r, \dots, (d+f-1)r$$

be the integral multiples of r contained in J . Then the integers

$$(3.6) \quad d, d+1, \dots, d+f-1$$

are all k th power non-residues.

Let $\varepsilon = 1$ if -1 is a k th power residue and let $\varepsilon = 2$ if -1 is a k th power non-residue. Then $f \geq [(g_2 - 1)/\varepsilon r]$ since the interval J contains $(g_2 - 1)/\varepsilon$ consecutive integers. Let

$$(3.7) \quad C = [\delta p^{1/5}]$$

where $\delta = 2^{-11/5}$ if -1 is a k th power residue and $\delta = 1/8$ if -1 is a k th power non-residue. Note that $C \geq 1$ since $p > 8^5$. Finally, let $r = C$ if C is odd and let $r = C-1$ if C is even.

Now by (1.11), (2.1), and (2.2), $g_2 < 2\sqrt{2p} + 5$, and it follows that

$$(3.8) \quad \begin{aligned} d+f-1 &\leq (p+g_2-2)/2r < (p+2\sqrt{2p}+1)/2\delta p^{1/5} \\ &< (1/2\delta)p^{4/5} + (2^{1/2}p^{3/10}/\delta) + 1 < (g_2-2)^2. \end{aligned}$$

Consequently, there exists a positive integer a such that

$$(3.9) \quad a^2 \leq d < d+f-1 < (a+1)^2.$$

For if J is not square free so that there exists an integer a such that $a^2 \in J$, then a is an even integer since $a < g_2 - 2$, by (3.8), and the square of an odd integer less than g_2 is clearly a k th power residue. But then, $a+1$ and $a-1$ are odd integers less than g_2 and so $a^2 - 1$ is a k th power residue. Since J contains only k th power non-residues, we are forced to the conclusion that if $a^2 \in J$, then $a^2 = d$.

Now let t be the largest positive integer such that $(a+1)^2 - t^2 > a^2$. Then $2a+1-t^2 > 0$ so that $t < (2a+1)^{1/2} < t+1$ and, hence, $t = [(2a+1)^{1/2}]$.

The integers

$$(3.10) \quad (a+1)^2 - v^2 \quad (v = 0, 1, 2, \dots, t),$$

divide the interval $[a^2, \dots, (a+1)^2]$ into subintervals. Furthermore, $a+1+t < g_2$. For

$$(3.11) \quad a \leq d^{1/2} < ((p+2r)/2r)^{1/2} < (p/2r)^{1/2} + 1,$$

since there must be a multiple of $2r$ between p and $p+2r$, but then

$$(3.12) \quad \begin{aligned} a+1+t &< a+1 + (2a+1)^{1/2} < a+1 + (2a)^{1/2} + 1 \\ &< p^{2/5}/(2\delta)^{1/2} + (2^{1/4}p^{1/5}/\delta^{1/4}) + 2 < g_2. \end{aligned}$$

Hence, every odd integer of the form (3.9) is a k th power residue. The number of integers lying between two consecutive odd integers of the form (3.10) is given by

$$(3.13) \quad ((a+1)^2 - v^2) - ((a+1)^2 - (v+2)^2) + 1 = 4v + 3.$$

It follows that $f \leq 4t + 3 \leq 4(2a+1)^{1/2} + 3 < 4(2a-2)^{1/2} + 4$ since it is obvious that $a > 19$ if $p > 2^{15}$.

Now, by (3.9), $(a-1)^{1/2} < (p/2r)^{1/4}$ and so

$$4(2a-2)^{1/2} + 4 < 2^{5/2}(p/2r)^{1/4} + 4.$$

Thus,

$$(3.14) \quad ((g_2-1)/er) - 1 < f < 2^{5/2}(p/2r)^{1/4} + 4.$$

If -1 is a k th power residue we obtain from (3.14),

$$(3.15) \quad \begin{aligned} g_2 &< 2^{9/4}r^{3/4}p^{1/4} + 5r + 1 < 2^{9/4}(2^{-11/5}p^{1/5})^{3/4}p^{1/4} + 5(2^{-11/5})p^{1/5} + 1 \\ &= 2^{3/5}p^{2/5} + ((3125/2048)p)^{1/5} + 1. \end{aligned}$$

If -1 is a k th power non-residue we obtain,

$$(3.16) \quad \begin{aligned} g_2 &< 2^{21/4}r^{3/4}p^{1/4} + 9r + 1 < 2^{13/4}(p^{1/5}/8)^{3/4}p^{1/4} + (9/8)p^{1/5} + 1 \\ &= 2p^{2/5} + (9/8)p^{1/5} + 1. \end{aligned}$$

The contradiction establishes the theorem.

Remark 2. We have not attempted to check (3.15) and (3.16) for $p < 32768$, although tables exist for checking such things; see, for example, Lehmer, Lehmer, and Shanks [12]. We note that (3.15) and (3.16) are, in fact, slightly sharper than (1.6) and (1.7). More significantly, we note that in the proof of Theorem 1 we only need to use the fact that J does not contain any k th power residues, rather than the stronger fact that J contains integers from only one of the $v-1$ classes of non-residues. Implications of the stronger statement will be considered in forthcoming papers.

THEOREM 2. If $2 < g_1(p, k) < 2p^{1/5} + 3$, then

$$(3.17) \quad g_2(p, k) < 6p^{2/5} + 21p^{1/5} + 37/2 \quad \text{if } -1 \text{ is a } k\text{-th power residue;}$$

$$(3.18) \quad g_2(p, k) < 12p^{2/5} + 42p^{1/5} + 43 \quad \text{if } -1 \text{ is a } k\text{-th power non-residue.}$$

Proof. Let p be > 1024 . Assume that the theorem is false and let J denote the interval

$$(3.19) \quad [p - g_2 + 1, \dots, p - 1, p + 1, \dots, p + g_2 - 1]$$

if -1 is a k th power residue, and let J denote the interval

$$(3.20) \quad [p + 1, \dots, p + g_2 - 1]$$

if -1 is a k th power non-residue.

If $g_1 < p^{1/5}$ there exists a k th power non-residue, which we will denote by n , such that $p^{1/5} < n < 2p^{1/5} + 3$ since 2 is a k th power residue. If $g_1 > p^{1/5}$ we let $n = g_1$.

Let

$$(3.21) \quad dn, (d+1)n, \dots, (d+f-1)n,$$

be the integral multiples of n contained in J .

We claim that the integers

$$(3.22) \quad d, d+1, \dots, d+f-1,$$

form a sequence of f consecutive k th power non-residues. For n is obviously a multiple of g_1 , say $n = ag_1$. Furthermore, the only integers in the

interval J which can fail to be k th power residues are integers of the form $p \pm bg_1$. But integers of the form $p \pm bg_1$ cannot be multiples of n for $p \pm bg_1 = cn = acg_1 \Rightarrow p = (ac \mp b)g_1$, contradicting the fact that p is prime. Consequently, integers of the form (3.21) are all k th power residues which, of course, implies that integers of the form (3.22) are all k th power non-residues.

Now if -1 is a k th power residue, J contains $2g_2 - 1$ consecutive integers and, thus, $f \geq [(2g_2 - 1)/n]$ since any w consecutive integers must contain at least $[w/n]$ multiples of n . If -1 is a k th power non-residue, J contains $g_2 - 1$ consecutive integers so that $f \geq [(g_2 - 1)/n]$.

By (1.11), (2.7), and (2.8), $g_2 < g_1\sqrt{2p} + 2g_1 + 1$. It follows that

$$(3.23) \quad d + f - 1 \leq (p + g_2 - 1)/n < (p + g_1\sqrt{2p} + 2g_1)/p^{1/5} < (g_2 - 2)^2$$

since $g_1 < 2p^{1/5} + 3$. Consequently, there exists a positive integer a such that

$$(3.24) \quad a^2 \leq d < d + f - 1 < (a + 1)^2.$$

For if $a^2 \in [d, \dots, d + f - 1]$, then a must be a multiple of g_1 since, by (3.24), $a < g_2 - 2$, and the square of an integer less than g_2 which is not a multiple of g_1 is clearly a k th power residue, whereas $d, \dots, d + f - 1$, are all k th power non-residues. But, then, $a + 1$ and $a - 1$ are integers less than g_2 which are not multiples of g_1 and so $a^2 - 1$ is a k th power residue. As in the proof of Theorem 1, we are forced to the conclusion that if $a^2 \in [d, \dots, d + f - 1]$, then $a^2 = d$.

Now subdivide the interval $A = [a^2, \dots, (a + 1)^2]$ into the overlapping subintervals

$$(3.25) \quad A_1 = [a^2, \dots, a(a + 1)],$$

$$(3.26) \quad A_2 = [(a + 2)(a - 1), \dots, (a + 1)^2].$$

Note that if either $g_1|a$ or $g_1|a + 1$, then $g_1 \nmid a + 2$ and $g_1 \nmid a - 1$ since $g_1 > 2$. Conversely, if either $g_1|a + 2$ or $g_1|a - 1$, then $g_1 \nmid a$ and $g_1 \nmid a + 1$. Consequently, at least one of the integers $(a + 2)(a - 1) = a^2 + a - 2$ or $a(a + 1) = a^2 + a$ is a k th power residue, since $a + 2 < g_2$ by (3.23) and (3.24).

Let t_1 be the largest positive integer such that

$$(a + 1)^2 - t_1^2 > (a + 2)(a - 1).$$

Then $a + 3 - t_1^2 > 0$ so $t_1 < (a + 3)^{1/2} \leq t_1 + 1$ and, hence, $t_1 = [(a + 2)^{1/2}]$.

The integers

$$(3.27) \quad (a + 1)^2 - v^2 \quad (v = 0, 1, 2, \dots, t_1),$$

divide the interval A_2 into subintervals.

Furthermore, $a + 1 + t_1 < g_2$. For

$$(3.28) \quad a \leq d^{1/2} < ((p + n)/n)^{1/2} < (p/n)^{1/2} + 1$$

since there must be a multiple of n between p and $p + 2n$. But then

$$(3.29) \quad a + 1 + t_1 \leq a + 1 + (a + 2)^{1/2} < p^{2/5} + 2 + (p^{2/5} + 3)^{1/2} < g_2.$$

Now, if $(a + 1)^2 - v^2$ ($v = 0, 1, 2, \dots, t_1$), is a k th power non-residue, then at least one of its factors $a + 1 + v$ or $a + 1 - v$ must be a k th power non-residue and, hence, a multiple of g_1 because of (3.29). It follows that either $(a + 1 + (v + 1))(a + 1 - (v + 1))$ is a k th power residue or $(a + 1 + (v + 2))(a + 1 - (v + 2))$ is a k th power residue. For, if $a + 1 + v$ is a multiple of g_1 , then $a + 1 + (v + 1)$ and $(a + 1 + (v + 2))$ clearly are not, since $g_1 > 2$. By (3.29), then, $a + 1 + (v + 1)$ and $a + 1 + (v + 2)$ are k th power residues. But at least one of the integers $a + 1 - (v + 1)$ or $a + 1 - (v + 2)$ must be a k th power residue since both cannot be multiples of g_1 . Similarly, if $a + 1 - v$ is a multiple of g_1 , then $a + 1 - (v + 1)$, $a + 1 - (v + 2)$, and at least one of $a + 1 + (v + 1)$ and $a + 1 + (v + 2)$ are k th power residues.

Consequently, at least one of the integers

$$(a + 1)^2 - v^2, (a + 1)^2 - (v + 1)^2, \text{ or } (a + 1)^2 - (v + 2)^2$$

is a k th power residue for each $v = 0, 1, 2, \dots, t_1 - 2$. It follows that the maximum number of integers lying between k th power residues of the form (3.27) cannot exceed

$$(3.30) \quad ((a + 1)^2 - v^2) - ((a + 1)^2 - (v + 3)^2) + 1 = 6v + 8.$$

Thus, in the interval A_2 ,

$$(3.31) \quad f \leq 6t_1 + 8 \leq 6(a + 2)^{1/2} + 8 < 6(a^{1/2}) + 10$$

for $(a + 1)^2 > p/n > (1024/11) > 93$ since $p > 1024$ and $n < 2p^{1/5} + 3 \Rightarrow a \geq 9$, but $6(a + 2)^{1/2} < 6(a^{1/2}) + 2$ if $a \geq 9$.

Let t_2 be the largest positive integer such that $a(a + 1) - t_2(t_2 + 1) > a^2$. Then $t_2^2 + t_2 < a$ so that $t_2 \leq [a^{1/2}]$. The integers

$$(3.32) \quad a(a + 1) - v(v + 1) = (a + 1 + v)(a - v) \quad (v = 0, 1, 2, \dots, t_2),$$

divide the interval A_1 into subintervals and by the same argument as before, at least one of the integers

$$(a + 1 + v)(a - v), (a + 1 + (v + 1))(a - (v + 1)), \text{ or } (a + 1 + (v + 2))(a - (v + 2))$$

is a k th power residue for each $v = 0, 1, 2, \dots, t_2 - 2$. It follows that the maximum number of integers lying between k th power residues of the form (3.32) cannot exceed

$$(3.33) \quad ((a^2 + a - (v^2 + v)) - (a^2 + a - (v^2 + 7v + 12))) + 1 = 6v + 11.$$

Thus in the interval A_1 ,

$$(3.34) \quad f \leq 6t_2 + 11 \leq 6(a^{1/2}) + 11,$$

and so $f \leq 6(a^{1/2}) + 11$ in the entire interval A .

Now, if -1 is a k th power residue,

$$(3.35) \quad f \geq [(2g_2 - 1)/n] > ((2g_2 - 1)/n) - 1 > ((2g_2 - 1)/(2p^{1/5} + 3)) - 1$$

since $n < 2p^{1/5} + 3$. Clearly $a^2 < p/n < p^{4/5}$ since $n > p^{1/5}$ so that $a^{1/2} < p^{1/5}$.

It follows that

$$(3.36) \quad \begin{aligned} (2g_2 - 1)/(2p^{1/5} + 3) &< 6(a^{1/2}) + 12 < 6p^{1/5} + 12 \\ &\Rightarrow 2g_2 - 1 < 6(2p^{1/5} + 3)p^{1/5} + 12(2p^{1/5} + 3) = 12p^{2/5} + 42p^{1/5} + 36 \\ &\Rightarrow g_2 < 6p^{2/5} + 21p^{1/5} + 37/2. \end{aligned}$$

The contradiction establishes (3.17) for $p > 1024$. If $p < 1024$, $g_1 \leq 7$ since $g_1 < 2p^{1/5} + 3 < 11$ so by (1.11) and (2.8),

$$g_2 < (7/2)\sqrt{2p} + 8 < 6p^{2/5} + 21p^{1/5} + 37/2.$$

If -1 is a k th power non-residue,

$$(3.37) \quad f \geq [(g_2 - 1)/n] > ((g_2 - 1)/n) - 1 > ((g_2 - 1)/(2p^{1/5} + 3)) - 1.$$

Clearly $(a-1)^2 < p/n < p^{4/5}$ so that $(a-1)^{1/2} < p^{1/5}$. Furthermore,

$$6(a^{1/2}) + 11 < 6(a-1)^{1/2} + 13 < 6p^{1/5} + 13 \quad \text{since } a > 9.$$

It follows that

$$(3.38) \quad (g_2 - 1)/(2p^{1/5} + 3) < 6p^{1/5} + 14 \Rightarrow g_2 < 12p^{2/5} + 42p^{1/5} + 43.$$

The contradiction establishes (3.18) for $p > 1024$. If $p < 1024$, $g_1 \leq 7$, so by (1.11) and (2.7),

$$g_2 < 7\sqrt{2p} + 15 < 12p^{2/5} + 42p^{1/5} + 43.$$

Theorem 2 generalizes Whyburn's results (1.8), (1.9), and (1.10), even when $v = 2$. For if $g_1(p, 2) > 3$, Whyburn was only able to obtain a non-trivial upper bound for $g_2(p, 2)$ if -1 is a quadratic non-residue. Consequently, Theorem 2 is more general than (1.8), (1.9), and (1.10), even when $v = 2$. Theorem 3, combined with a remarkable specific estimate given recently by K. K. Norton [15] for $g_1(p, k)$, namely,

$$(3.39) \quad g_1(p, k) < 4.7p^{1/4} \log p,$$

will yield a non-trivial upper bound for $g_2(p, k)$ for all p and k . We omit details in the proof of Theorem 3 which are identical with arguments already established in the proof of Theorem 2.

THEOREM 3. If $g_1(p, k) > 2p^{1/5} + 3$, then

$$(3.40) \quad g_2(p, k) < 2g_1^{3/4} p^{1/4} + 3g_1 + 1/2$$

if -1 is a k -th power residue:

$$(3.41) \quad g_2(p, k) < 4g_1^{3/4} p^{1/4} + 8g_1 + 1$$

if -1 is a k -th power non-residue.

Proof. Let p be ≥ 257 . Assume that the theorem is false and let J be defined as in Theorem 2. Let

$$(3.42) \quad dg_1, (d+1)g_1, \dots, (d+f-1)g_1$$

be the integral multiples of g_1 contained in J .

By the same reasoning that was used in the proof of Theorem 2 the integers

$$(3.43) \quad d, d+1, \dots, d+f-1$$

form a sequence of f consecutive k th power non-residues.

Also, of course, $f \geq [(2g_2 - 1)/g_1]$ if -1 is a k th power residue, and $f \geq [(g_2 - 1)/g_1]$ if -1 is a k th power non-residue.

As before, $g_2 < g_1\sqrt{2p} + 2g_1 + 1$ implies that

$$(3.44) \quad d+f-1 \leq (p+g_2-1)/g_1 < (p+g_1\sqrt{2p}+2g_1)/2p^{1/5} < (g_2-2)^2$$

since it is well known that $g_1 < p^{1/2}$, and so there exists a positive integer a such that

$$(3.45) \quad a^2 \leq d < d+f-1 < (a+1)^2.$$

Define t_1, t_2 , the interval A , and the subintervals A_1 and A_2 as in Theorem 2. Recall that $t_1 = [(a+2)^{1/2}]$ and that $t_2 \leq [a^{1/2}]$. It is easy to verify that $a+1+t_1 < g_2$ and so, of course, $a+1+t_2 < g_2$.

However, a new and useful fact is available to us if $g_1 > 2p^{1/5} + 3$. Namely, we claim that

$$(3.46) \quad 2[(a+2)^{1/2}] + 1 < g_1 \quad \text{and} \quad 2[a^{1/2}] + 2 < g_1.$$

For if $p > 512$,

$$(3.47) \quad \begin{aligned} (a-1)^2 < p/g_1 &\Rightarrow a-1 < (p/2p^{1/5})^{1/2} \Rightarrow a+2 < (p^{2/5}/2^{1/2}) + 3 \\ &\Rightarrow 2(a+2)^{1/2} + 2 < 2((p^{2/5}/2^{1/2}) + 3)^{1/2} + 2 < 2^{3/4} p^{1/5} + 7/2 < g_1 \end{aligned}$$

since

$$((p^{2/5}/2^{1/2}) + 3)^{1/2} < (p^{1/5}/2^{1/4}) + 3/4 \quad \text{if } p \geq 257.$$

Clearly (3.46) follows from (3.47).

Subdivide the interval A_2 by the integers

$$(3.48) \quad (a+1)^2 - v^2 \quad (v = 0, 1, \dots, t_1).$$

As before, $(a+1)^2 - v^2$ can only be a k th power non-residue if at least one of its factors $a+1+v$ or $a+1-v$ is a multiple of g_1 . Unlike before, if either $a+1+v$ or $a+1-v$ is a multiple of g_1 , then, by using (3.46), we can show that neither $a+1+(v+1)$ nor $a+1-(v+1)$ can be multiples of g_1 .

For if $a+1+v$ is a multiple of g_1 , then $(a+1+v) - (a+1-(v+1)) = 2v+1 \leq 2t_1+1 = [(a+2)^{1/2}] + 1 < g_1$, by (3.46), and so $a+1-(v+1)$ cannot be a multiple of g_1 . Obviously $a+1+(v+1)$ is not a multiple of g_1 either. Similarly if $a+1-v$ is a multiple of g_1 , then $a+1+(v+1) - (a+1-v) = 2v+1 < g_1$ so that $a+1+(v+1)$ is not a multiple of g_1 and, of course, $a+1-(v+1)$ cannot be either. It follows that at least every other (rather than every third) integer of the form $(a+1)^2 - v^2$ ($v = 0, 1, 2, \dots, t_1$) is a k th power residue. Consequently, the maximum number of integers lying between k th power residues of the form (3.48) cannot exceed

$$(3.49) \quad ((a+1)^2 - v^2) - ((a+1)^2 - (v+2)^2) + 1 = 4v + 3.$$

Thus, in the interval A_2 ,

$$(3.50) \quad f \leq 4t_1 + 3 \leq 4(a+2)^{1/2} + 3 < 4(a^{1/2}) + 5,$$

for $a+1 > (p/g_1)^{1/2} \Rightarrow a+1 > p^{1/4}$ (since it is well known that $g_1 < p^{1/2}$) $\Rightarrow a \geq 4$ since $p \geq 257$.

Subdivide the interval A_1 by the integers

$$(3.51) \quad a(a+1) - v(v+1) = (a+1+v)(a-v) \quad (v = 0, 1, 2, \dots, t_2).$$

Analogous to the previous argument we can show, using (3.46), that if either of the factors $a+1+v$ or $a-v$ is a multiple of g_1 , then neither of the factors $(a+1+(v+1))$ nor $(a-(v+1))$ can be multiples of g_1 .

For if $a+1+v$ is a multiple of g_1 , then $(a+1+v) - (a-(v+1)) = 2v+2 \leq 2t_2+2 \leq 2[(a^{1/2})] + 2 < g_1$, by (3.46). Similarly, if $a-v$ is a multiple of g_1 , then $(a+1+(v+1)) - (a-v) = 2v+2 < g_1$.

It follows that the maximum number of integers lying between k th power residues of the form (3.51) cannot exceed

$$(3.52) \quad (a^2 + a - (v^2 + v)) - (a^2 + a - (v^2 + 5v + 6)) + 1 = 4v + 5.$$

Thus, in the interval A_1 ,

$$(3.53) \quad f \leq 4t_2 + 5 \leq 4(a)^{1/2} + 5,$$

and so $f \leq 4(a^{1/2}) + 5$ in the entire interval A .

Now, if -1 is a k th power residue,

$$(3.54) \quad f \geq [(2g_2 - 1)/g_1] > (2g_2 - 1)/g_1 - 1.$$

Also

$$a^2 < p/g_1 \Rightarrow a^{1/2} < (p/g_1)^{1/4}.$$

It follows that

$$(3.55) \quad (2g_2 - 1)/g_1 < 4(a^{1/2}) + 6 \Rightarrow 2g_2 - 1 < 4g_1^{3/4} p^{1/4} + 6g_1 \\ \Rightarrow g_2 < 2g_1^{3/4} p^{1/4} + 3g_1 + 1/2.$$

The contradiction establishes (3.40) for $p > 257$. If $p < 257$, Theorem 3 is vacuously true since g_1 is never greater than $2p^{1/5} + 3$. For if $p < 31$, then $g_1 < p^{1/2}$ so that $g_1 \leq 5$, but $2p^{1/5} + 3 > 5$. If $37 \leq p \leq 257$, $2p^{1/5} + 3 > 7$, but it is easily checked that $g_1 \leq 7$ using, e.g., Nagell [14] and well known bounds for $g_1(p, 2)$.

If -1 is a k th power non-residue,

$$(3.56) \quad f \geq [(g_2 - 1)/g_1] > (g_2 - 1)/g_1 - 1.$$

Furthermore,

$$(a-1)^2 < p/g_1 \Rightarrow (a-1)^{1/2} < (p/g_1)^{1/4}.$$

Now

$$4(a^{1/2}) + 5 < 4(a-1)^{1/2} + 7 \quad \text{since } a \geq 4.$$

It follows that

$$(3.57) \quad (g_2 - 1)/g_1 < 4(a-1)^{1/2} + 8 \Rightarrow g_2 < 4g_1^{3/4} p^{1/4} + 8g_1 + 1.$$

The contradiction establishes (3.41) for $p > 257$ and, as noted, the theorem holds vacuously if $p < 257$.

COROLLARY 1. For every $k \geq 2$ and every p such that $(k, p-1) > 1$,

$$(3.58) \quad g_2(p, k) < 2p^{7/16} (3.9 \log p)^{3/4} + 11.7p^{1/4} \log p + 1/2$$

if -1 is a k -th power residue, and

$$(3.59) \quad g_2(p, k) < 4p^{7/16} (4.7 \log p)^{3/4} + 37.6p^{1/4} \log p + 1$$

if -1 is a k -th power non-residue.

Proof. Noting that Norton [15] has shown that the coefficient 4.7 in (3.39) can be replaced by 3.9 if -1 is a k th power residue, the proof follows immediately from (3.39), (3.40), and (3.41), together with Theorems 1 and 2.

Remark 3. Norton [15] has also given the specific estimate

$$(3.60) \quad g_1(p, k) < (p^{1/2} \log p)^B$$

where

$$B = \exp\{-1 + v^{-1} + 6/\log p + 20/\log^2 p\}.$$

(3.60) is sharper than (3.39) if $v > 3$ and p is larger than a calculable constant c (depending on v). Correspondingly, Corollary 1 can be improved if $v > 3$. In fact, it follows from (3.60) that if $v = v_k(p) \geq 13$ and $p > e^{4440}$,



then $g_1(p, k) < p^{1/5}$ and, consequently, $g_2(p, k)$ is bounded by the quadratic polynomials in $p^{1/5}$, (3.17) and (3.18).

Remark 4. Although, in theory, elementary methods similar to those used in Theorems 1, 2, and 3 are applicable to $g_n, n > 2$, the additional complications, even for g_3 , are overwhelming.

We remark that it follows immediately from an elementary result of Rédei ([16], p. 151) that for each integer $n \geq 1$ there exists an integer m such that for every "sufficiently large" prime p with $g_1(p, 2) > m, g_n(p, 2) < 2\sqrt{p}/\sqrt{3}$. In fact, $g_n(p, 2) < 2\sqrt{p}/\sqrt{3}$ for every "sufficiently large" prime p in the arithmetic progression $ax + b$ where $a = 4 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots m$ and $b = 1$ since for every prime in this progression, $g_1(p, 2) > m$. There are, of course, infinitely many such primes since $(a, b) = 1$. Unfortunately, however, this still leaves us a long way from showing that $g_n(p, 2) < 2\sqrt{p}/\sqrt{3}$ for every "sufficiently large" prime. In the next section we turn away from elementary methods, and we see that $g_n(p, k) = O(p^{1/4} \log p)$ for all "sufficiently large" p for which g_1, \dots, g_{n-1} assume preassigned values.

4. *O-estimates.* It follows from Wang's result (1.2), and Theorems 1 and 2, that for each $v \geq 2$ and all "sufficiently large" p ,

$$(4.1) \quad g_2(p, k) < 6p^{2/5} + 21p^{1/5} + 37/2$$

if -1 is a k th power residue;

$$(4.2) \quad g_2(p, k) < 12p^{2/5} + 42p^{1/5} + 43$$

if -1 is a k th power non-residue.

Our purpose in this section is to improve (4.1), (4.2), for "large" p through the use of the lemmas in Section 2, Burgess's result (1.13), and Wang's results (1.2), (1.3), and (1.4).

THEOREM 4. Let $a_v = 1/4 + 1/(4e^{v-1/v}), v \geq 2$. Then, for each $\delta > 0$,

$$(4.3) \quad g_2(p, k) = O_\delta(p^{a_v+\delta}) \quad \text{if } v \geq 2;$$

$$(4.4) \quad g_2(p, k) = O_\delta(p^{1/3+\delta}) \quad \text{if } v \geq 21;$$

$$(4.5) \quad g_2(p, k) = O_\delta(p^{1/4 + ((\log \log v + 2)/4 \log v) + \delta}) \quad \text{if } v > e^{33}.$$

Proof. Immediate from (1.2), (1.3), (1.4), (1.13), and Lemma 2; $\log p$ in (1.13) is, of course, swallowed up by p^δ since $\log p = o(p^\delta)$.

Computing several values of a_n , we obtain,

- $a_2 = .378354 \dots$,
- $a_3 = .362332 \dots$,
- $a_7 = .356093 \dots$,
- $\dots \dots \dots$
- $a_{19} = .346940 \dots$

If $v > e^{33}, (\log \log v + 2)/4 \log v < .04164$ so that

$$(4.6) \quad g_2(p, k) = O_\delta(p^{b_v+\delta})$$

where $b_v = 1/4 + (\log \log v + 2)/4 \log v < .29164$.

In fact, replacing δ by $\delta/2$ in (4.5), and noting that there exists a $v_0 > e^{33}$ such that $(\log \log v_0 + 2)/4 \log v_0 < \delta/2$, we immediately obtain the following theorem for "large" v .

THEOREM 5. Let v_1 be any integer $\geq v_0$. Then for all "sufficiently large" p with $(k, p-1) = v_1$,

$$(4.7) \quad g_2(p, k) = O_\delta(p^{1/4+\delta}) \quad \text{for each } \delta > 0.$$

The following theorem is very useful when $g_{n-1}(p, k), n \geq 2$, is small and an upper bound for $g_n(p, k)$ is sought.

THEOREM 6. Let b be a positive integer ≥ 2 . Then there exists p_0 such that if p is any prime $\geq p_0$ for which $g_{n-1}(p, k) = b$,

$$(4.8) \quad g_n(p, k) = O(p^{1/4} \log p).$$

Proof. Immediate from (1.13) and Lemma 3.

Some illustrative applications of Theorem 6 are given below.

If $v_k(p) = 2$, then

$$(4.9) \quad g_2(p, k) = O(p^{1/4} \log p) \quad \text{if } p \not\equiv \pm 1 \pmod{24} \quad \text{since } g_1 \leq 5,$$

$$(4.10) \quad g_3(p, k) = O(p^{1/4} \log p) \quad \text{if } p \equiv \pm 5 \pmod{24} \quad \text{since } g_2 = 3;$$

if $v_k(p) = 3$ and $p \neq x^2 + 27y^2$, then

$$(4.11) \quad g_2(p, k) = O(p^{1/4} \log p),$$

since $g_1 = 2$, etc.

Remark 5. K. K. Norton ([15], p. 26) suggested that the method used in obtaining the specific estimate (3.39) may generalize to yield an admissible value for the implied absolute constant in (1.13). Obviously, this would give fresh significance to Theorem 6 since examples of the type (4.9), (4.10), and (4.11) would become specific estimates.

Finally, we note that a number of conditional results for $g_1(p, k)$ sharper than (1.2), (1.3), and (1.4) have been given, and because of the nature of Lemma 2, these obviously lead to conditional improvements of Theorem 4. For example, Linnik [13] has given a specific function $f(\epsilon)$ such that for each $\epsilon > 0$ and all sufficiently large N , there are at most $f(\epsilon)$ primes p in the interval $[N^\epsilon, N]$ for which $g_1(p, 2) > p^\epsilon$. Ankeny [1] has shown that, conditional on the truth of the extended Riemann hypothesis, $g_1(p, 2) = O(\log^2 p)$.

Lemma 2 combined with Ankeny's result yields the following conditional result.

THEOREM 7. *If the extended Riemann hypothesis is true, then for all p and k ,*

$$(4.12) \quad g_2(p, k) = O(p^{1/4} \log^3 p).$$

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