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# Prime Quasientropy and Quasichaos

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We construct a prime symmetry relation for integers that is equivalent to Goldbach's conjecture and show that numerical computations of this prime symmetry property strongly resemble a chaotic sequence. We define and examine the notions of global and local prime quasientropies. Finally, we employ the fact that the prime number sequence satisfies the property of deterministic randomness to consider its utility for the field of quantum computation.

**KEY WORDS:** prime numbers; chaotic sequence; prime sequence; global and local quasientropies; twin primes.

Goldbach's conjecture (L. Euler, personal communication, 1742) states that any even integer greater than or equal to six may be written as the sum of two primes. An equivalent statement was given by Erdös (1945) in terms of Euler functions:

$$\phi(q) + \phi(r) = 2n \tag{1}$$

Namely, for any positive integer $n$ there exist integers $q$ and $r$ such that Eq. (1) is satisfied. [The Euler function $\phi(m)$ is defined as the number of positive integers $y \leq m$ with $y$ relatively prime to $m$ (Dummit and Foote, 1991).] A generalization of Goldbach's conjecture given by Vinogradov (1937) states that any "sufficiently large" odd integer may be written as the sum of three primes. A number of theorems and statements related to this conjecture may be found in the collection compiled by Yuan (1984).

An additional equivalent of Goldbach's conjecture is a statement of *prime symmetry*. Namely, for any positive integer $x \geq 2$, there exists integer $\Delta < x$ such that $x \pm \Delta$ are prime numbers. The function $\Delta(x) \geq 0$ represents the minimum number with this property. A vector representation of the parameters of $(2a, b)$

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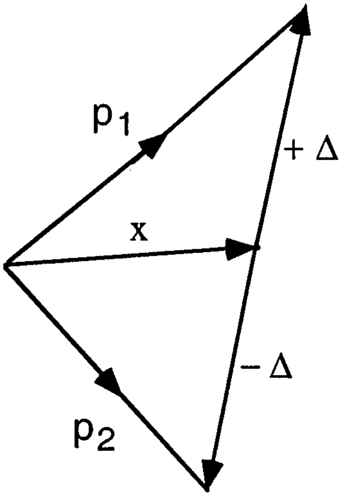


**Fig. 1.** Vector representation of the parameters in Eq. (2). This representation requires that the inside angle between $x$ and $+\Delta$ be obtuse.

is shown in Fig. 1. A numerical computation of $\Delta(x)$ is noted to have a property characteristic of a chaotic sequence. Namely, a periodic pattern of this computation at small $x$ degenerates into a mixed, aperiodic pattern at larger $x$. The concept of global and local *prime quasientropy* is defined. Local prime entropy vanishes for "twin primes." Thus, the existence of infinitely many twin primes (an open question) is equivalent to the statement that the local prime quasientropy vanishes infinitely often. Global prime entropy is noted to increase.

Note additionally that for any two primes, $p_1 \geq p_2$, there exist nonnegative integers $\Delta$, $x$ such that $\Delta \leq x$, $x > 0$, and

$$\begin{aligned} x + \Delta &= p_1, \\ x - \Delta &= p_2. \end{aligned} \tag{2}$$

Additionally,

$$\begin{aligned} 2x &= p_1 + p_2, \\ 2\Delta &= p_1 - p_2, \\ p_1^2 - p_2^2 &= 4x\Delta, \\ x^2 - \Delta^2 &= p_1 p_2. \end{aligned} \tag{3}$$

Numerical plots of $\Delta(x)$ for $x \leq 400$ reveal a symmetric structure when $x \leq 50$. However, this structure is lost at higher values (Fig. 2). With Eq. (2), we note that

$$\Delta(x) = 0 \ \text{ at } \ x = p_1 = p_2 \tag{4}$$

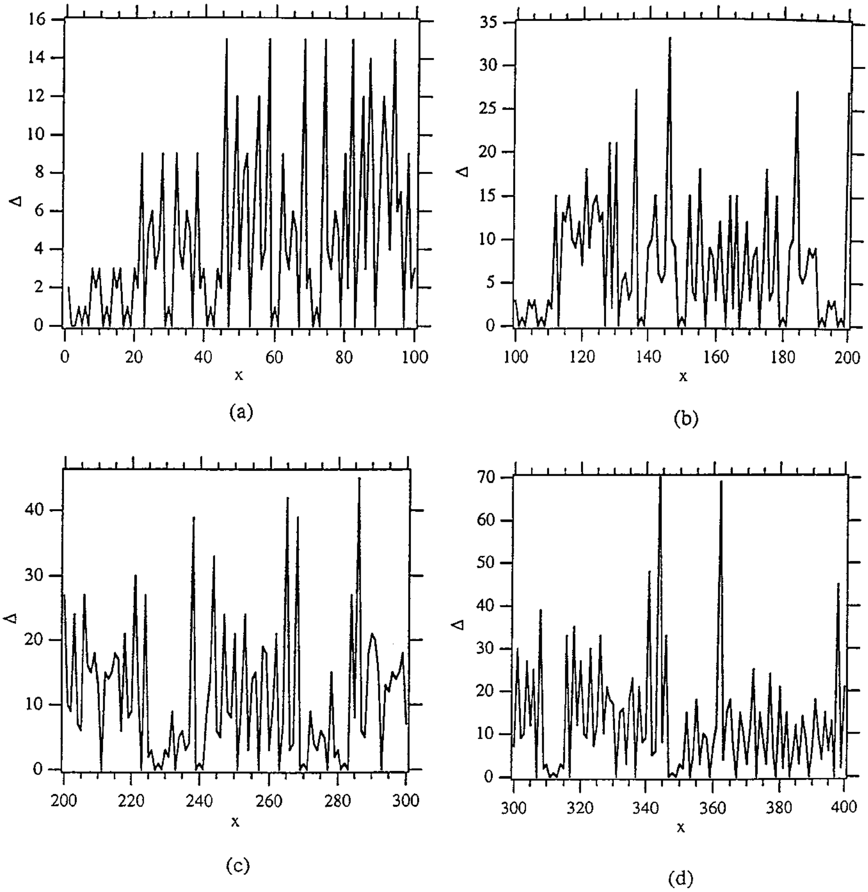


**Fig. 2.** Read-outs for (a) $0 \leq x \leq 100$, (b) $100 \leq x \leq 200$, (c) $200 \leq x \leq 300$, (d) $300 \leq x \leq 400$.

so that the zeros of $\Delta(x)$ occur at prime values of $x$. The property that there are arbitrarily large gaps in the series of primes (Niven and Zuckerman, 1990) applies equally to the intervals between the zeros of $\Delta(x)$. Furthermore, the segmented periodic form of these curves at small $x$ and aperiodic structure at higher values of $x$ is characteristic of chaotic development of a dynamical system in time. Here, one imagines that the $x$ axis is the measure of a discrete time sequence. From Fig. 2. one notes that a chaotic structure (Haake, 2001; Reichl, 1992) of $\Delta(x)$ enters for $x \gtrapprox 90$. As $\Delta(x)$ is related to prime numbers, the chaotic character of this function is consistent with the quasichaotic property of the prime-number sequence (Liboff and Wong, 1998). This quasichaotic property describes the fact that a histogram of the interval between nearest-neighbor primes very roughly follows the Wigner distribution. Data is said to be "quasichaotic" if its plot is well-approximated by either a Wigner or Poisson distribution (Haake, 2001; Liboff and Wong, 1998; Reichl, 1992).

With this link to the Wigner distribution, there is an accompanying connection to quantum chaos and random matrix theory. Quantum chaos of a system may be described in terms of the theory of Gaussian distributions of random matrices and the spectral statistics of the spacing of eigenvalues of these matrices (Berry and Tabor, 1977; Haake, 2001; Liboff and Seidman, 1993; Mehta, 1967; Porter, 1965; Reichl, 1992). In the formalism stemming from symmetries of the given system, three classes of matrices emerge: real symmetric, Hermitian, and real quaternion, which in turn may be diagonalized by orthogonal, unitary, and symplectic similarity transformations, respectively. Corresponding distributions of nearest-neighbor eigenvalues carry the acronyms: GOE, GUE, and GSE. The first of these (Gaussian orthogonal ensemble) is also called the Wigner–Dyson (Dyson, 1962; Wigner, 1967) or, simply, the Wigner distribution. A histogram of nearest-neighbor increments of eigenvalues which roughly resembles any of these distributions reflects a nonintegrable system whereas a Poisson-like distribution is indicative of an integrable system. Thus for example, nearest-neighbor increments of the energy spectrum of the circular quantum billiard gives a Poisson-like histogram whereas the energy spectrum of chaotic billiards such as the stadium or Sinai billiard give Wigner-like histogram (Bohigas *et al.*, 1984; Liboff and Wong, 1998; McDonald and Kaufman, 1979).

The fact that spectral densities of chaotic systems reflect level repulsion leads to physical intepretations of the prime number sequence. For example, studies of nuclear resonances in $U^{238}$ under neutron bombardment exhibit a level repulsion related to the Wigner distribution (Gutzwiller, 1990). Studies addressing the applicability of prime numbers to physics have also suggested a relation between prime-number sequences and the spectra of excited nuclei (Cipra, 1996; Michell *et al.*, 1991). This example was discussed by Liboff and Wong (1998).

The present description of the primes is reminiscent of exponential divergence in chaotic dynamical systems. In this context, one may define a global "quasientropy" of primes $\sigma$ on the interval $[2, a]$ by

$$\sigma = \ln\left[\max_{x\in[2,a]} \Delta(x)\right] \tag{5a}$$

or, equivalently,

$$\sigma = \ln\left[\max_{x\in[2,a]} \frac{[p_1(x) - p_2(x)]}{2}\right] \tag{5b}$$

This quasientropy increases monotonically as the system evolves in "time" (i.e., as $a$ increases). This increase corresponds to the increase in the maximum increment

$p_1 - p_2$ in the interval $[2, a]$ with growth of $a$ and is consistent with the notion of global disorder in the primes.

As has been demonstrated numerically for the interval $(0, 10^{35})$ (Liboff and Wong, 1998), the most prevalent interval between nearest-neighboring primes is six. Over a subsequent interval, this value jumps to a larger interval, etc. (M. V. Berry, personal communication, 1999). We use the notation $\Delta p_m$ to label prevalent nearest-neighbor increments and $L_m$ to label the relevant covering interval. A corresponding form may be defined for the probability of finding $\Delta p_m$ in a random sampling of nearest-neighbor increments in the covering increment $L_m$. This form is given by

$$P(\Delta p) = \exp(-|\Delta p - \Delta p_m|), \quad p \in L_m. \tag{6}$$

The function $P$ satisfies $P \leq 1$, with equality holding when $\Delta p = \Delta p_m$.

Recall that the entropy $S$ of a system composed of $\Omega$ states is given by Callen (1963)

$$S = k \ln \Omega, \tag{7}$$

where $k$ is a constant. It follows that the entropy of two systems with respective state variables $\Omega_1$ and $\Omega_2$ is given by

$$S = k \ln(\Omega_1 \Omega_2) \tag{8a}$$

$$= k \ln \Omega_1 + k \ln \Omega_2 \tag{8b}$$

$$= S_1 + S_2 \geq 0. \tag{8c}$$

In the case of global prime quasientropy, the number of states $\Omega$ is represented by

$$\Omega = \max_{x \in [2,a]} \frac{[p_1(x) - p_2(x)]}{2} \tag{9}$$

Thus, prime quasientropy is an additive, positive function. Casting the present situation in the language of state variables as in (9) also allows one to consider the combined quasientropy of sets of two primes $x, y$, where $x \in \{2, \ldots, a\}$ and $y \in \{2, \ldots, b\}$. The prime quasientropy of a system vanishes if and only if each subsystem contains exactly one state. That is, $\Omega_1 = \Omega_2 = 1$ and $S = 0$.

$$\bar{\sigma} = \ln\left[\frac{p_1(x) - p_2(x)}{2}\right] \tag{10}$$

If $p_1 > p_2 + 2$, then $\bar{\sigma} > 0$ and $\bar{\sigma} = 0$ if and only if $p_1(x) - p_2(x) = 2$, which is the case of "twin primes." The open conjecture that there are infinitely many twin primes is hence equivalent to the statement that the local prime quasientropy vanishes infinitely often. Note also that for the degenerate case $p_1(x) = p_2(x) = x$, (i.e., $\Delta = 0$), $\bar{\sigma}$ is not defined, as quasientropy is a positive entity.

Returning to Fig. 2, we remark that one of its most striking features is its resemblance to time-series of chaotic trajectories in classically chaotic systems (see, for example, Murray and Dermott, 1999, pp. 305, 412). Moreover, as one requires an exponentially long algorithm to factor numbers (and hence to determine if a number is prime) with a classical computer, the prime number sequence is also consistent with the notion of chaos as “deterministic randomness” (Ford and Ilg, 1992; Ford and Mantica, 1992). As a quantum computer can be used to factor numbers (Shor, 1994), it can likewise be used to resolve the prime number sequence. In this spirit, it seems that a quantum computer could also be used to resolve the exponential sensitivity of chaotic systems and hence to make long-time predictions for systems that currently defy predictability after some “Liapunov time” (Strogatz, 1994).

In sum, we presented a prime symmetry relation that is equivalent to Goldbach’s conjecture. Numerical computations of this prime symmetry property of integers were found to strongly resemble a chaotic sequence. The concept of global and local “prime quasientropies” were defined. A local form of this entity was found to be zero for the case that the prime increment has the value 2. With this property, an equivalent statement to the twin prime conjecture was given. Finally, we used the fact that the prime number sequence satisfies the property of deterministic randomness to consider its utility for the field of quantum computation.

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