Primes in arithmetic progressions to large moduli

by

E. BOMBIERI, J. B. FRIEDLANDER[1][2] and H. IWANIEC[1]

Institute for Advanced Study
Princeton, NJ, U.S.A.

Contents

Notations

$\Lambda(q)$—the von Mangoldt function.

$\tau(q)$—the divisor function.

$\varphi(q)$—the Euler function.

$\mu(q)$—the Möbius function.

---

(1) Supported in part by NSF grant MCS-8108814(A02).

(2) Supported in part by NSERC grant A5123.

$e(\xi)$—the additive character $e^{2\pi i\xi}$.
$\chi(n)$—a multiplicative character.
$\hat{f}$—the Fourier transform of f, i.e.,

$$\hat{f}(\eta)=\int_{-\infty}^{\infty} f(\xi)\,e(\xi\eta)\,d\xi.$$

$m\equiv a(q)$—means $m\equiv a \pmod q$.
$\bar{d}/c$—means a/c where $ad\equiv 1 \pmod c$.
$m\sim M$—means $M<m\leqslant 2M$.
$\|\alpha\|$—means L^2 norm of $\alpha=(\alpha_m)$, i.e.,

$$\|\alpha\|=\left(\sum|\alpha_m|^2\right)^{1/2}.$$

ε—any sufficiently small, positive constant, not necessarily the same in each occurrence.
B—some sufficiently large, positive constant, not necessarily the same in each occurrence.
$\mathcal{L}=\log x$.
$\pi(x;q,a)$—the number of primes $p\leqslant x$, $p\equiv a \pmod q$.
$\psi(x;q,a)=\sum_{n\leqslant x,\, n\equiv a \pmod q}\Lambda(n)$.

Some of our results depend on a variety of assumptions scattered throughout the paper. For ease of reference we list here the pages on which these are described.

Assumption	Page
(A_1)	206
(A_2)	206
(A_3)	214
(A_4)	220
(A_5)	229
(A_6)	235
(A_6^*)	237
(A_7)	239

The reader should take some caution with our use of the constant ε. Any statement including ε is meant simply as the claim that the statement is true for all sufficiently small positive ε. The meaning of "sufficiently small" may vary from one line to the next.

1. Introduction

Given an arithmetic function $f(n)$, it is natural to study its distribution in residue classes a (mod q). One focuses on the classes a with $(a, q)=1$, without restricting the generality, and expects that among these classes a reasonable function f will be uniformly distributed, such uniformity being measured by upper bounds for the magnitude of

$$\Delta_f(x; q, a) = \sum_{\substack{n\leq x\\ n\equiv a(q)}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\leq x\\ (n,q)=1}} f(n). \tag{1.1}$$

A not unreasonable goal is the estimate

$$\Delta_f(x; q, a) \ll \frac{1}{\varphi(q)} \mathscr{L}^{-A} x^{1/2} \|f\|, \tag{1.2}$$

for any $A>0$, the implied constant depending only on A, the result valid uniformly in q in a range as large as possible. In view of Cauchy's inequality it is natural to regard (1.2) as saving $\mathscr{L}^A$ from the "trivial" estimate.

The following examples illustrate the largest known ranges of q in (1.2) for some basic functions:

(i) $f(n)=\tau_k(n)$, the number of representations of n as the product of k factors, $q<x^{\theta_k-\varepsilon}$ with

$\theta_2 = \frac{2}{3}$ (C. Hooley, Ju. Linnik, A. Selberg)

$\theta_3 = \frac{1}{2} + \frac{1}{230}$ (J. Friedlander and H. Iwaniec [9])

$\theta_4 = \frac{1}{2}$ (Ju. Linnik [16])

$\theta_5 = \frac{9}{20}$, $\theta_6 = \frac{5}{12}, \ldots$ (J. Friedlander and H. Iwaniec [10]).

(ii) $f(n)=r(n)$, the number of representations of n as the sum of two squares, $q<x^{\vartheta-\varepsilon}$ with $\vartheta=2/3$ (C. Hooley, Ju. Linnik, R. A. Smith).

(iii) $f(n)=b(n)$, the characteristic function of numbers represented as the sum of two squares. Then

$$\Delta_f(x; q, a) = o\left(\frac{1}{\varphi(q)} \frac{x}{(\log x)^{1/2}}\right) \quad \text{as } x\to\infty$$

uniformly in $q<x^{\vartheta(x)}$ where $\vartheta(x)$ is any function decreasing to zero as $x\to\infty$ (H. Iwaniec [14]).

(iv) $f(n)=\Lambda(n)$, the von Mangoldt function, $q<(\log x)^A$ with any $A>0$ (Siegel-Walfisz theorem).

This last example, $\Lambda(n)$, has of course received the most attention. The Riemann hypothesis for Dirichlet's L-series implies and is implied by

$$\psi(x;q,a)=\frac{1}{\varphi(q)}x+O(x^{1/2+\varepsilon}). \tag{1.3}$$

Here the constant implied in the symbol O depends at most on ε; thus the Riemann hypothesis yields (1.2) for $q<x^{1/2-\varepsilon}$. While a proof of (1.3) seems to be out of reach by present methods, it was shown in 1965 by E. Bombieri [1] and by A. I. Vinogradov [21] that (1.2) holds for almost all $q<x^{1/2-\varepsilon}$. In the form given by Bombieri, the result yields (somewhat more than)

$$\sum_{q\leq Q}\max_{(a,q)=1}\left|\psi(x;q,a)-\frac{x}{\varphi(q)}\right| \ll x\mathcal{L}^{-A} \tag{1.4}$$

for any $A>0$ with $Q=x^{1/2}\mathcal{L}^{-B}$, where B and the implied constant depend on A alone.

It was conjectured by P. D. T. A. Elliott and H. Halberstam [3] that (1.4) may hold with $Q=x^{1-\varepsilon}$ but even the result with $Q=x^{1/2}$ has not yet been achieved. Several simplifications and generalizations of the original arguments were provided; (see, for example, [11], [20], [22], [18]). It is now known that Bombieri's mean value theorem is valid for fairly general arithmetic functions $f(n)$. This is essentially due to Y. Motohashi [18]. The crucial property required is that f can be represented as a linear combination of convolutions of two sequences $\alpha*\beta$ with the following properties.

(A_1) $\alpha=(\alpha_m)$, $m\sim M$, $M=x^{1-\vartheta}$, $\beta=(\beta_n)$, $n\sim N$, $N=x^{\vartheta}$, with $\varepsilon\leq\vartheta\leq 1-\varepsilon$.

(A_2) $\beta=(\beta_n)$, $n\sim N$ is well distributed in arithmetic progressions to small moduli, that is, for any $d\geq 1$, $k\geq 1$, $l\neq 0$, $(k,l)=1$ we have

$$\sum_{\substack{n\equiv l(k)\\(n,d)=1}}\beta_n-\frac{1}{\varphi(k)}\sum_{(n,dk)=1}\beta_n \ll \|\beta\|N^{1/2}\tau(d)^B(\log 2N)^{-A}$$

with some $B>0$ and any $A>0$, the constant implied in $\ll$ depending on A alone.

Under the conditions (A_1) and (A_2) we have

$$\sum_{q\leq Q}\max_{(a,q)=1}|\Delta_{\alpha*\beta}(x;q,a)| \ll \|\alpha\|\,\|\beta\|\,x^{1/2}\mathcal{L}^{-A} \tag{1.5}$$

with $Q=x^{1/2}\mathcal{L}^{-B}$, $B=B(A)>0$. The proof is a consequence of the large sieve inequality (see Theorem 0 below)

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)}\left|\sum_{h\leq H} C_h\chi(h)\right|^2 \ll (Q^2+H)\,\|C\|^2; \tag{1.6}$$

here Σ^* stands for summation over primitive characters.

In order to complete the proof of (1.4) it remains to represent $\Lambda(n)$ as a sum of convolutions $\alpha*\beta$ of sequences with the above properties. This is a matter of combinatorial identities which we shall discuss later.

It is the application of the large sieve inequality (1.6) that sets the limit $Q=x^{1/2}\mathcal{L}^{-B}$ and not the shape of the bilinear form $\alpha*\beta$. By this we mean that the location of ϑ in $[\varepsilon, 1-\varepsilon]$ in (A_1) is irrelevant to the proof.

In the series of papers by E. Fouvry and H. Iwaniec ([6], [4], [7], [5]) the first successful attempts were made to get mean value theorems for arithmetic progressions to moduli beyond $x^{1/2}$. The large sieve inequality (1.6) is replaced by new arguments based on the dispersion method, Fourier analysis and Kloosterman sums, the last appealing to results from the spectral theory of automorphic functions.

In these new arguments the parameter a is now forced to be (more or less) fixed so we must drop from both (1.5) and (1.4) the expression $\max_{(a,q)=1}$. Since, in most applications of (1.4), a is fixed, this causes no great concern. More serious is the fact that, for these arguments, the location of ϑ does matter.

One would like to prove, with $Q=x^{1/2+\delta}$, an estimate

$$\sum_{\substack{q\leq Q\\ (q,a)=1}} \gamma_q\,\Delta_{\alpha*\beta}(x;q,a) \ll \|\alpha\|\,\|\beta\|\,x^{1/2}\mathcal{L}^{-A} \tag{1.7}$$

for general weights γ_q and thus, in particular, for absolute values,

$$\gamma_q = \operatorname{sgn}\Delta_{\alpha*\beta}(x;q,a).$$

This cannot yet be done. The class of weights for which (1.7) can be shown depends on the range of ϑ.

In this paper we enhance the former arguments to extend substantially the range of ϑ and to work out forms that were not considered before. For technical reasons only we deal with bilinear forms which satisfy some additional constraints, see (A_4) below. From our seven theorems of this type we infer, by combinatorial arguments, the following results.

THEOREM 8. *Let $a\neq 0$, $\theta_1<1/3$, $\theta_2<1/5$, $5\theta_1+2\theta_2<2$, and $\theta_1+\theta_2<29/56$. For any numbers $\gamma_{q_1}\ll\tau(q_1)^B$, $\delta_{q_2}\ll\tau(q_2)^B$ and any $A>0$ we have*

$$\sum_{\substack{q_1\le x^{\theta_1}}}\sum_{\substack{q_2\le x^{\theta_2}\\(q_1q_2,a)=1}}\gamma_{q_1}\delta_{q_2}\left(\psi(x;q_1q_2,a)-\frac{x}{\varphi(q_1q_2)}\right)\ll x\mathcal{L}^{-A};$$

the constant implied in $\ll$ may depend at most on θ_1, θ_2, a, A and B.

THEOREM 9. *Let $a\neq 0$, $\varepsilon>0$ and $R<x^{1/10-\varepsilon}$. For any $A>0$ there exists $B=B(A)$ such that provided $QR<x\mathcal{L}^{-B}$ we have*

$$\sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le Q\\(q,a)=1}}\left(\psi(x;qr,a)-\frac{x}{\varphi(qr)}\right)\right|\ll x\mathcal{L}^{-A};$$

the constant implied in $\ll$ depends at most on ε, a and A.

COROLLARY 1 (the Titchmarsh divisor problem). *Let $a\neq 0$. For any $A>0$ we have*

$$\sum_{|a|<n\le x}\Lambda(n)\,\tau(n+a)=c_1(a)\,x\log x+c_2(a)\,x+O(x\mathcal{L}^{-A}),$$

the implied constants depending only on a and A. Here we have

$$c_1(a)=\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\prod_{p|a}\left(1-\frac{p}{p^2-p+1}\right)$$

and

$$c_2(a)=c_1(a)\left\{2\sum_{p|a}\frac{p^2\log p}{(p-1)(p^2-p+1)}-2\sum_{p}\frac{\log p}{p^2-p+1}+2\gamma-1\right\}.$$

Definition. An arithmetic function $\lambda(q)$ is called "well factorable" of "level" Q if for any $Q_1,Q_2\ge 1$, $Q_1Q_2=Q$ there exist two functions $\lambda_1(q_1)$, $\lambda_2(q_2)$ supported in $[1,Q_1]$ and $[1,Q_2]$ respectively such that

$$|\lambda_1|\le 1,\quad |\lambda_2|\le 1\quad\text{and}\quad \lambda=\lambda_1*\lambda_2.$$

The well factorable functions were introduced in connection with the modern linear sieve theory, cf. [15], [7].

THEOREM 10. *Let $a \neq 0$, $\varepsilon > 0$ and $Q = x^{4/7-\varepsilon}$. For any well factorable function $\lambda(q)$ of level Q and any $A > 0$ we have*

$$\sum_{(q,a)=1} \lambda(q) \left(\psi(x; q, a) - \frac{x}{\varphi(q)} \right) \ll x \mathscr{L}^{-A};$$

the constant implied in $\ll$ depends at most on ε, a and A.

COROLLARY 2. *Let $\pi_2(x)$ be the number of pairs of twin primes p, $p+2$ with $p \leq x$. We then have*

$$\pi_2(x) \leq (\tfrac{7}{2} + \varepsilon) Bx (\log x)^{-2}$$

where $B = 2 \prod_{p>2} (1-(p-1)^{-2})$, for any $\varepsilon > 0$ and $x \geq x_0(\varepsilon)$.

Theorems 8, 9 and Corollary 1 are new and they constitute the bulk of this paper. E. Fouvry has informed us that he has independently proved Corollary 1 and a slightly weaker version of Theorem 9. Theorem 10 and Corollary 2 improve the results of Fouvry and Iwaniec of [7] and of Fouvry [5].

In Theorem 8 the constraint $5\theta_1 + 2\theta_2 < 2$ is unnecessary if Selberg's eigenvalue conjecture [2] holds.

Our methods are capable of giving results for larger ranges of q, given good estimates for certain exponential sums. We formulate the following general conjecture.

Let $A_2(p)$, $1 \leq p < \infty$ denote the hypothesis (A_2) with $\|\beta\| N^{1/2}$ replaced by $\|\beta\|_p N^{1-1/p}$ where $\|\beta\|_p$ is the usual l_p norm.

Conjecture 1. Let (A_1), $A_2(p)$ hold, $a \neq 0$, $A > 0$, $r \geq 1$, $s \geq 1$. There exists $B_1 = B_1(A)$ such that

$$\sum_{\substack{q < x\mathscr{L}^{-B_1} \\ (q,a)=1}} |\Delta_{\alpha * \beta}(x; q, a)| \ll \|\alpha\|_r \|\beta\|_s M^{1-1/r} N^{1-1/s} \mathscr{L}^{-A},$$

the implied constant depending on ε, a, A, B, r, s.

Remark. We are led to the consideration of l_p norms because Hölder's inequality features in our arguments and because the optimal employment of this depends on the current state of the estimates for exponential sums. It is possible that Hölder's inequality could be dispensed with. This leads us to extend the conjecture to the case where r or s (or both) is ∞ and in which case we define

$$\|\alpha\|_\infty = \sup_n \tau^B(n) |\alpha(n)|.$$

The value of the above conjecture is limited due to the absence of plausible methods for attacking it. The following weaker conjecture can be reduced to the expected estimate for certain exponential sums whose arguments are rational functions in several variables. Lemma 1 is a prototype of such an estimate.

Conjecture 2. Let $\varepsilon>0$, (α), (β) satisfy (A_1), (A_2) and $|\alpha_m|\leq\tau^B(m)$, $|\beta_n|\leq\tau^B(n)$. For any $A>0$, $a\neq 0$ we have

$$\sum_{\substack{q<x^{3/4-\varepsilon}\\(q,a)=1}} |\Delta_{\alpha*\beta}(x;q,a)| \ll x\mathcal{L}^{-A},$$

the implied constant depending on ε, a, A and B.

2. Lemmas

In this section we state some results from the literature of which we shall have need. The most central to our purposes is the following estimate for sums of Kloosterman sums, cf. [2, Theorem 12].

LEMMA 1. *Let $g_0(\xi,\eta)$ be a smooth function with compact support in $\mathbf{R}^+\times\mathbf{R}^+$. Let $C, D, N, R, S>0$ and $g(c,d)=g_0(c/C, d/D)$. For any complex numbers B_{nrs} denote*

$$\mathcal{K}(C,D,N,R,S)=\sum_{r\sim R}\sum_{s\sim S}\sum_{0<n\leq N} B_{nrs}\sum_{c}\sum_{\substack{d\\(rd,sc)=1}} g(c,d)\,e\left(n\frac{\overline{rd}}{sc}\right).$$

Then, for any $\varepsilon>0$ we have

$$\mathcal{K}(C,D,N,R,S)\ll(CDNRS)^{\varepsilon}\mathcal{I}(C,D,N,R,S)\,\|B\|$$

where

$$\mathcal{I}^2(C,D,N,R,S)=CS(RS+N)(C+DR)+C^2DS\sqrt{(RS+N)R}+D^2NRS^{-1},$$

the constant implied in $\ll$ depending at most on ε and $g(\xi,\eta)$.

The next lemma is a truncated Poisson formula.

LEMMA 2. *Let $M\geq 1$ and let $f(m)$ be a smooth function with compact support in $[-4M,4M]$ such that*

$$f^{(j)}(m)\ll M^{-j},\quad j=0,1,\ldots$$

the constant implied in $\ll$ depending on j alone. For any $H \geq q^{1+\varepsilon} M^{-1}$ we have

$$\sum_{m \equiv a(q)} f(m) = \frac{1}{q} \sum_{|h| \leq H} \hat{f}\left(\frac{h}{q}\right) e(-ah/q) + O(q^{-1})$$

the constant implied in O depending on ε alone.

LEMMA 3. *Let $a \neq 0$, $A > 0$ and $1 \leq k \leq x^{1-\varepsilon}$. We then have*

$$\sum_{\substack{|a| < m \leq x \\ m \equiv l \pmod{k}}} \tau^A(m) \tau^A(m-a) \ll \frac{x}{k} (\tau(k) \log x)^B$$

with some $B = B(A)$ and the constant implied in $\ll$ depending at most on ε, a and A.

Proof. Apply Cauchy's inequality and Theorem 2 of [19].

It is often convenient to work with numbers free of small prime factors. The following result, known in sieve theory as a "fundamental lemma", is useful for the relevant reduction.

LEMMA 4. *Let $D \geq 2$, $z = D^{1/s}$ with $s \geq 3$. There exist two sequences $\{\lambda_d^+\}\, d \leq D$ and $\{\lambda_d^-\}\, d \leq D$ such that*

$$|\lambda_d^{\pm}| \leq 1$$

$$\begin{cases} (\lambda^- * 1)(n) = (\lambda^+ * 1)(n) = 1 & \text{if } n \text{ has no prime factor} < z \\ (\lambda^- * 1)(n) \leq 0 \leq (\lambda^+ * 1)(n) & \text{otherwise} \end{cases}$$

$$\sum_{d \leq D} \lambda_d^{\pm} d^{-1} = \prod_{p < z} \left(1 - \frac{1}{p}\right) (1 + O(\exp(-s \log s))).$$

Proof. See [8].

Our next lemma is the combinatorial identity of Heath-Brown [12]. The use of similar identities to replace sums over primes by sums over divisor-like functions was, in the context of the dispersion method, originally made by Yu. V. Linnik, see [17].

LEMMA 5. *Let $J \geq 1$ and $n < 2x$. We then have*

$$\Lambda(n) = \sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{m_1, \ldots, m_j \leq x^{1/J}} \cdots \sum \mu(m_1) \ldots \mu(m_j) \sum_{n_1, \ldots, n_j m_1 \ldots m_j = n} \cdots \sum \log n_1. \tag{2.1}$$

In the next result we give rather general versions of two famous consequences of the large sieve inequality (1.6). The first of these is the Barban-Davenport-Halberstam theorem, the version of Hooley [13] being not quite sufficient for our purpose. The second is the formulation of the Bombieri-Vinogradov theorem in terms of general bilinear forms as described in (1.5).

THEOREM 0. (a) *Let* (β_n), $n\leq N$, *be any sequence of complex numbers satisfying the "Siegel-Walfisz" assumption* (A_2). *For any* $A>0$, *there exists* $B_1>0$ *such that*

$$\sum_{q\leq Q}\sum_{(a,q)=1}\left|\sum_{n\equiv a\,(\mathrm{mod}\,q)}\beta_n-\frac{1}{\varphi(q)}\sum_{(n,q)=1}\beta_n\right|^2 \ll \|\beta\|^2 N(\log N)^{-A} \tag{2.2}$$

provided that $Q\leq N(\log N)^{-B_1}$.

(b) *Let* (A_1) *and* (A_2) *hold. For any* $A>0$, *there exists* $B_1>0$ *such that*

$$\sum_{q\leq Q}\max_{(a,q)=1}|\Delta_{\alpha*\beta}(x;q,a)| \ll \|\alpha\|\,\|\beta\|\,x^{1/2}\mathscr{L}^{-A} \tag{1.5}$$

for any $Q\leq x^{1/2}\mathscr{L}^{-B_1}$.

Here in (a) *and* (b) *the implied constants depend on* A *and on the constant* B *occurring in* (A_2), *and in* (b) *also depends on the constant* ε *occurring in* (A_1).

Proof. (a) The left-hand side of (2.2) is just

$$S=\sum_{q\leq Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\left|\sum_n\beta_n\chi(n)\right|^2.$$

Let χ be induced by ψ mod f where $q=fe$, $f>1$. Since

$$\sum_n\beta_n\chi(n)=\sum_{(n,e)=1}\beta_n\psi(n),$$

and $\varphi(fe)\geq\varphi(f)\,\varphi(e)$, we have

$$\begin{aligned}S&\leq\sum_{e\leq Q}\frac{1}{\varphi(e)}\sum_{2\leq f\leq Q/e}\frac{1}{\varphi(f)}\sum_{\psi(\mathrm{mod}\,f)}{}^{*}\left|\sum_{(n,e)=1}\beta_n\psi(n)\right|^2\\&=\sum_{e\leq Q}\frac{1}{\varphi(e)}(S_e(f\leq F)+S_e(f>F)),\end{aligned}$$

say, where Σ^* is restricted to primitive characters. For $2\leq f\leq F$ we split into progressions (mod f) and apply (A_2) getting

$$\sum_{(n,e)=1} \beta_n \psi(n) \ll_{A'} \|\beta\| N^{1/2} (\log N)^{-A'} \tau^B(e) f,$$

so that

$$S_e(f \leqslant F) \ll \tau^{2B}(e) F^3 \|\beta\|^2 N (\log N)^{-2A'}. \tag{2.3}$$

For $S_e(f>F)$ we split the sum into $\ll \log Q$ intervals of the type $(V, 2V]$. The large sieve inequality gives

$$S_e(f>F) \ll \log^2 Q \sup_{F<V\leqslant Q} V^{-1}(V^2+N)\|\beta\|^2$$
$$\ll (Q+NF^{-1})\|\beta\|^2 \log^2 Q.$$

Thus, for some $B_2 = B_2(B)$ we have

$$S \ll \|\beta\|^2 (\log Q)^{B_2} \{NF^3 (\log N)^{-2A'} + NF^{-1} + Q\}.$$

Taking $B_1 = A + B_2$, $A' = 2B_1$ and $F = (\log N)^{B_1}$, we get (a).

(b) We have

$$\Delta_{\alpha*\beta}(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) \left(\sum_m \alpha_m \chi(m) \right) \left(\sum_n \beta_n \chi(n) \right).$$

We reduce to primitive characters as in (a). The left-hand side T of (1.5) is thus bounded by

$$\begin{aligned} T &\leqslant \sum_{e\leqslant Q} \frac{1}{\varphi(e)} \sum_{2\leqslant f\leqslant Q/e} \frac{1}{\varphi(f)} \sideset{}{^*}\sum_{\psi(\operatorname{mod} f)} \left| \sum_{(m,e)=1} \alpha_m \psi(m) \right| \left| \sum_{(n,e)=1} \beta_n \psi(n) \right| \\ &= \sum_{e\leqslant Q} \frac{1}{\varphi(e)} (T_e(f\leqslant F) + T_e(f>F)), \end{aligned} \tag{2.4}$$

say. By Cauchy's inequality

$$\begin{aligned} T_e(f\leqslant F) \leqslant & \left\{ \sum_{2\leqslant f\leqslant F} \frac{1}{\varphi(f)} \sideset{}{^*}\sum_{\psi(\operatorname{mod} f)} \left| \sum_{(m,e)=1} \alpha_m \psi(m) \right|^2 \right\}^{1/2} \\ & \times \left\{ \sum_{2\leqslant f\leqslant F} \frac{1}{\varphi(f)} \sideset{}{^*}\sum_{\psi(\operatorname{mod} f)} \left| \sum_{(n,e)=1} \beta_n \psi(m) \right|^2 \right\}^{1/2} \end{aligned} \tag{2.5}$$

Here, the multiple sum in the second parentheses is just $S_e(f \leqslant F)$ from (a) and to this we again apply (2.3). To the first multiple sum we apply the large sieve inequality (1.6). Together these yield, provided that $F<M^{1/2}$,

$$T_e(f\leqslant F) \ll \|\alpha\|\,\|\beta\|\, x^{1/2}\tau^B(e)\,F^{3/2}(\log N)^{-A'}. \tag{2.6}$$

We split the sum $T_e(f>F)$ into intervals $V<F\leqslant 2V$. To each of these we apply Cauchy's inequality getting an expression like (2.5). Now we apply (1.6) to both sums in parentheses. In this way we get

$$\begin{aligned} T_e(f>F) &\ll (\log^2 Q)\,\|\alpha\|\,\|\beta\| \sup_{F\leqslant V\leqslant Q} V^{-1}(V^2+M)^{1/2}(V^2+N)^{1/2} \\ &\ll (\log^2 Q)\,\|\alpha\|\,\|\beta\|\,(Q+M^{1/2}+N^{1/2}+M^{1/2}N^{1/2}F^{-1}). \end{aligned} \tag{2.7}$$

Choose $B_2(B)$ so that

$$\sum_{e\leqslant Q}\frac{\tau^B(e)}{\varphi(e)} \ll \mathscr{L}^{B_2}.$$

Take $B_1=A+2$, $F=\mathscr{L}^{B_1}(<M^{1/2})$, $A'=\frac{3}{2}A+B_2+3$. Combining (2.4), (2.6) and (2.7), we get the result.

3. A generalization of the problem

We consider the somewhat general sum

$$\mathscr{D}(M,N,Q,R)=\sum_{\substack{q\sim Q\\ (qr,a)=1}}\sum_{r\sim R}\gamma_q\delta_r\left(\sum_{\substack{m\sim M\\ mn\equiv a(qr)}}\sum_{n\sim N}\alpha_m\beta_n-\frac{1}{\varphi(qr)}\sum_{\substack{m\sim M\\ (mn,qr)=1}}\sum_{n\sim N}\alpha_m\beta_n\right) \tag{3.1}$$

with coefficients α_m, β_n satisfying (A_1), (A_2) and the coefficients γ_q, δ_r satisfying

(A_3) $|\gamma_q|\leqslant\tau(q)^B$, $|\delta_r|\leqslant\tau(r)^B$, $QR<x$.

Our aim is to prove the following upper bound

$$\mathscr{D}(M,N,Q,R)\ll_A \|\alpha\|\,\|\beta\|\,x^{1/2}\mathscr{L}^{-A} \tag{3.2}$$

with any $A>0$ under certain constraints on M, N, Q, R and on the sequences (α_m), (β_n), (γ_q), (δ_r).

By Cauchy's inequality we obtain

$$\mathscr{D}^2(M,N,Q,R) \leq \|\delta\|^2 \|\alpha\|^2 \mathscr{S}(M,N,Q,R)$$

where

$$\mathscr{S}(M,N,Q,R) = \sum_{\substack{r\sim R\\(r,am)=1}} \sum_{m\sim M} \left\{ \sum_{\substack{q\sim Q\\(q,am)=1}} \gamma_q \left(\sum_{\substack{n\sim N\\mn\equiv a(qr)}} \beta_n - \frac{1}{\varphi(qr)} \sum_{\substack{n\sim N\\(n,qr)=1}} \beta_n \right) \right\}^2$$

with the aim of showing that for any $A>0$

$$\mathscr{S}(M,N,Q,R) \ll \|\beta\|^2 x R^{-1} \mathscr{L}^{-A}. \tag{3.3}$$

Now we enlarge $\mathscr{S}(M,N,Q,R)$ a bit by introducing a smooth weight function $f(m)\geq 0$ in front of $\{\ \}^2$ such that

$$f(m)=1 \quad \text{if } m\in[M,2M]$$

$$f(m)=0 \quad \text{if } m\notin\left[\tfrac{1}{2}M,3M\right]$$

$$f^{(j)}(m) \ll M^{-j}, \quad j=0,1,\ldots.$$

In this way we obtain a smooth majorant $\mathscr{S}^*(M,N,Q,R)$. This is necessary for the application of Lemma 1 and simplifies other aspects of the treatment. Squaring out in $\mathscr{S}^*(M,N,Q,R)$ we obtain

$$\mathscr{S}^* = \mathscr{S}_1 - 2\mathscr{S}_2 + \mathscr{S}_3 \tag{3.4}$$

where $\mathscr{S}_i=\mathscr{S}_i(M,N,Q,R)$, $i=1,2,3$ are defined by

$$\mathscr{S}_1 = \sum_{(am,r)=1} \sum f(m) \left(\sum_{(q,am)=1} \gamma_q \sum_{mn\equiv a(qr)} \beta_n \right)^2.$$

$$\mathscr{S}_2 = \sum_{(am,r)=1} \sum f(m) \sum_{(am,q_1q_2)=1} \sum \frac{\gamma_{q_1}\gamma_{q_2}}{\varphi(q_2 r)} \sum_{\substack{mn_1\equiv a(q_1 r)\\(n_2,q_2 r)=1}} \sum \beta_{n_1}\beta_{n_2}$$

and

$$\mathscr{S}_3 = \sum_{(am,r)=1} \sum f(m) \left(\sum_{(am,q)=1} \frac{\gamma_q}{\varphi(qr)} \sum_{(n,qr)=1} \beta_n \right)^2.$$

Here we omitted the constraints $r\sim R$, $q\sim Q$, $n\sim N$ for notational simplicity, so they have to be remembered in the sequel.

We shall evaluate $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ separately. The above elementary arguments constitute the underlying idea of Linnik's dispersion method [17].

4. Evaluation of $\mathscr{S}_3$

We begin with the evaluation of the simplest sum. By Poisson's formula (Lemma 2) we get

$$\sum_{(m,b)=1} f(m) = \frac{\varphi(b)}{b}\hat{f}(0) + O(\tau(b)).$$

This yields

$$\mathscr{S}_3 = \hat{f}(0)X + \mathscr{R}_3 \tag{4.1}$$

where

$$X = \sum\sum_{(a,rq_1q_2)=1}\sum \Phi(q_1,q_2,r)\gamma_{q_1}\gamma_{q_2}\sum\sum_{(n_i,q_ir)=1}\beta_{n_1}\beta_{n_2}$$

and $\Phi(q_1,q_2,r)=\varphi(q_1q_2r)/q_1q_2r\varphi(q_1r)\varphi(q_2r)$. The error term $\mathscr{R}_3$ is bounded by

$$\mathscr{R}_3 \ll N\|\beta\|^2R^{-1}\mathscr{L}^B \tag{4.2}$$

which is admissible for (3.3).

5. Evaluation of $\mathscr{S}_2$

We have

$$\mathscr{S}_2 = \sum\sum_{(q_1q_2,a)=1}\gamma_{q_1}\gamma_{q_2}\sum_{(n_1,q_1)=1}\beta_{n_1}\sum_{(n_2,q_2)=1}\beta_{n_2}\mathscr{S}_2(n_1,n_2,q_1,q_2)$$

where

$$\mathscr{S}_2(n_1,n_2,q_1,q_2) = \sum_{\substack{R<r\leq 2R\\(r,an_1n_2)=1}}\frac{1}{\varphi(q_2r)}\sum_{\substack{mn_1\equiv a(q_1r)\\(m,q_2)=1}} f(m).$$

The constraint $(m, q_2)=1$ is relaxed by means of the Möbius inversion formula giving

$$\sum_m f(m) = \sum_{\nu|q_2, (\nu, q_1 r)=1} \mu(\nu) \sum_{\nu m n_1 \equiv a(q_1 r)} f(\nu m). \tag{5.1}$$

The terms with $\nu > x^{2\varepsilon}$ contribute to $\mathcal{S}_2$ by Lemma 3

$$O(\|\beta\|^2 R^{-1} x^{1-\varepsilon}) \tag{5.2}$$

which is acceptable for (3.3). Let $\nu \leq x^{2\varepsilon}$. By Lemma 2 the innermost sum is equal to

$$\sum_{m \equiv a\overline{\nu n_1}(q_1 r)} f(\nu m) = \frac{1}{\nu q_1 r} \sum_{|h| \leq H_0} \hat{f}\left(\frac{h}{\nu q_1 r}\right) e\left(-ah\frac{\overline{\nu n_1}}{q_1 r}\right) + O\left(\frac{1}{QR}\right)$$

where $H_0 = x^\varepsilon QRM^{-1}$. By the 'reciprocity' relation

$$-ah\frac{\overline{\nu n_1}}{q_1 r} \equiv ah\frac{\overline{q_1 r}}{\nu n_1} - \frac{ah}{\nu n_1 q_1 r} \pmod 1$$

we get

$$\sum_m f(\nu m) = \frac{1}{\nu q_1 r} \sum_{|h| \leq H_0} \hat{f}\left(\frac{h}{\nu q_1 r}\right) e\left(ah\frac{\overline{q_1 r}}{\nu n_1}\right) + O\left(\frac{1}{QR} + \frac{1}{x^{1-\varepsilon}}\right).$$

Here the error term $O((QR)^{-1} + x^{\varepsilon-1})$ contributes to $\mathcal{S}_2$ at most $O(\|\beta\|^2 R^{-1} N x^\varepsilon)$ which is acceptable for (3.3).

We first sum up the main terms $\hat{f}(0)/\nu q_1 r$, i.e. the terms with $h=0$. The restriction $\nu \leq x^{2\varepsilon}$ can be relaxed at the cost of the error term (5.2). Having done this the resulting total sum proves to be $\hat{f}(0) X$.

The remaining terms contribute to $\mathcal{S}_2(n_1, n_2, q_1, q_2)$

$$\frac{1}{q_1 q_2} \sum_{\substack{\nu < x^{2\varepsilon} \\ \nu|q_2, (\nu, q_1)=1}} \frac{\mu(\nu)}{\nu} \sum_{0<|h| \leq H_0} \sum_{\substack{R<r\leq 2R \\ (r, a\nu n_1 n_2)=1}} \frac{1}{r^2} \frac{q_2 r}{\varphi(q_2 r)} \hat{f}\left(\frac{h}{\nu q_1 r}\right) e\left(ah\frac{\overline{q_1 r}}{\nu n_1}\right).$$

The innermost sum is essentially an incomplete Kloosterman sum. In order to estimate this we use the following result which easily follows from the A. Weil upper bound for complete Kloosterman sums

$$\sum_{\substack{1 \leq d \leq D \\ (d, ck)=1}} e\left(b\frac{\bar{d}}{c}\right) \ll \left(c^{1/2} + \frac{(b,c)}{c} D\right) (ck)^\varepsilon. \tag{5.3}$$

Using the formula

$$\frac{dq}{\varphi(dq)}=\frac{q}{\varphi(q)}\sum_{(\eta,q)=1,\,\eta^*|d}\eta^{-1}$$

where η^* denotes the product of all prime factors of η by (5.3) we first deduce that

$$\sum_{\substack{1\le d\le D\\(d,ck)=1}}\frac{dq}{\varphi(dq)}e\left(b\frac{d}{c}\right)\ll\left(c^{1/2}+\frac{(b,c)}{c}D\right)(ckqD)^{\varepsilon}$$

from which by partial summation we obtain

$$\sum_{\substack{R<r\le 2R\\(r,avn_1n_2)=1}}\frac{1}{r^2}\frac{q_2r}{\varphi(q_2r)}\hat{f}\left(\frac{h}{vq_2r}\right)e\left(ah\frac{\overline{q_1r}}{vn_1}\right)\ll\frac{M}{R^2}\left(N^{1/2}+(h,n_1)\frac{R}{N}\right)x^{\varepsilon}.$$

Gathering the above results together we conclude that

$$\mathcal{S}_2=\hat{f}(0)X+R_2 \tag{5.4}$$

with the error term R_2 bounded by

$$R_2\ll\|\beta\|^2(QR^{-1}N^{3/2}+Q)x^{\varepsilon}+\|\beta\|^2R^{-1}x^{1-\varepsilon}.$$

This bound is acceptable for (3.3) provided that

$$N<\left(\frac{x}{Q}\right)^{2/3}x^{-\varepsilon}\quad\text{and}\quad QR<x^{1-\varepsilon}.$$

These constraints will turn out to be weaker than those imposed when evaluating $\mathcal{S}_1$.

6. A truncation of $\mathcal{S}_1$

The evaluation of $\mathcal{S}_1$ is the most difficult and it involves the key arguments. Before applying them, in this section we reduce the range of the summation by elementary means. By definition we have

$$\mathcal{S}_1=\sum\sum_{(am,r)=1}f(m)\sum\sum_{(am,q'q'')=1}\gamma_{q'}\gamma_{q''}\sum\sum_{\substack{mn_1\equiv a(q'r)\\mn_2\equiv a(q''r)}}\beta_{n_1}\beta_{n_2}.$$

Letting $q_0=(q',q'')$, $q_1=q'/q_0$, $q_2=q''/q_0$ we get

$$\mathscr{S}_1 = \sum_{(a,q_0r)=1} \sum \sum_{\substack{(q_1,q_2)=1\\(a,q_1q_2)=1}} \sum \gamma_{q_0q_1}\gamma_{q_0q_2} \sum_{\substack{(n_i,q_0q_ir)=1\\n_1\equiv n_2(q_0r)}} \sum \beta_{n_1}\beta_{n_2} \sum_{m\equiv\mu(q_0q_1q_2r)} f(m)$$

where $\mu \,(\mathrm{mod}\, q_0q_1q_2r)$ is a common solution of

$$\begin{aligned} \mu n_1 &\equiv a \,(\mathrm{mod}\, q_0q_1r) \\ \mu n_2 &\equiv a \,(\mathrm{mod}\, q_0q_2r). \end{aligned} \tag{6.1}$$

Let us impose the following condition

$$x^\varepsilon R \leqslant N \tag{6.2}$$

which could already be anticipated from (3.3); it ensures us that there are enough terms in $\mathscr{S}(M,N,Q,R)$ to produce a considerable cancellation.

We first estimate trivially the contribution of terms with $q_0>Q_0$ where Q_0 will be chosen later. By Cauchy's inequality we deduce the following

$$\mathscr{S}_1(q_0>Q_0) \ll \sum_{Q_0<q_0\leqslant 2Q} \sum_{r\sim R} \sum_m f(m) \sum_{q_1} |\gamma_{q_0q_1}| \sum_{n_1\equiv a\bar{m}(q_0q_1r)} |\beta_{n_1}|^2 \sum_{q_2} |\gamma_{q_0q_2}| \sum_{n_2\equiv a\bar{m}(q_0q_2r)} 1.$$

By (A_3) and by Lemma 3 we obtain

$$\begin{aligned} \sum_{q_2}\sum_{n_2} &\ll \sum_{n\equiv a\bar{m}(q_0r)} \tau^B(mn-a)\,\tau^B(q_0) \\ &\ll N(q_0r)^{-1}\mathscr{L}^B + x^{\varepsilon/2} \ll N(Q_0R)^{-1}\mathscr{L}^B, \end{aligned}$$

provided $Q_0<x^{\varepsilon/2}$. This and (A_3) imply

$$\mathscr{S}_1(q_0>Q_0) \ll N(Q_0R)^{-1}\mathscr{L}^B \sum_m \sum_n |\beta_n|^2 \tau^B(mn-a).$$

Finally, again by Lemma 3 and by (A_1), we get

$$\mathscr{S}_1(q_0>Q_0) \ll \|\beta\|^2 x(Q_0R)^{-1}\mathscr{L}^B.$$

This bound is admissible provided

$$Q_0 = \mathscr{L}^{A+B} \tag{6.3}$$

which we henceforth assume.

In a much similar way we estimate trivially the contribution of terms with $(n_1, n_2)=n_0$ for some $n_0>N_0$, say. Namely we have

$$\mathscr{S}_1(n_0>N_0) \ll \|\beta\|^2 x(N_0R)^{-1}\mathscr{L}^B \tag{6.4}$$

provided $N_0<x^{\varepsilon/2}$. We take

$$N_0=\mathscr{L}^{A+B} \tag{6.5}$$

so (6.4) is admissible.

Now, define $\mathscr{S}_1^*(M, N, Q, R)$ to be the partial sum of $\mathscr{S}_1(M, N, Q, R)$ restricted by

$$q_0 \leqslant Q_0, \tag{6.6}$$

$$n_0 \leqslant N_0. \tag{6.7}$$

Therefore $\mathscr{S}_1^*$ differs from $\mathscr{S}_1$ by an admissible quantity (3.3).

From now on we impose a new condition on β_n, namely

(A_4) $\beta_n=0$ if n has a prime factor $\leqslant N_0$.

This assumption is not crucial (see [7]) but it greatly simplifies the congruences (6.1). Due to (A_4) and (6.7) each pair n_1, n_2 in $\mathscr{S}_1^*$ is coprime. Due to (A_4) the terms in $\mathscr{S}_1^*$ with $n_1 n_2$ not squarefree can be removed with admissible error as in (6.4). We write the resulting sum $\mathscr{S}_1^{**}$ as

$$\sum_{\substack{q_0 \leqslant Q_0\\(a,\, q_0 q_1 q_2 r)=1}} \sum_{r\sim R} \sum\sum_{(q_1, q_2)=1} \gamma_{q_0 q_1}\gamma_{q_0 q_2} \sum\sum_{\substack{(n_1 q_2,\, n_2 q_1)=1\\ n_1\equiv n_2 (q_0 r)}} \mu^2(n_1 n_2)\beta_{n_1}\beta_{n_2} \sum_{m\equiv\mu(q_0 q_1 q_2 r)} f(m). \tag{6.8}$$

To the innermost sum we apply Lemma 2 giving

$$\sum_{m\equiv\mu(q_0 q_1 q_2 r)} f(m)=\frac{1}{q_0 q_1 q_2 r}\left\{\sum_{|h|\leqslant H}\hat{f}\left(\frac{h}{q_0 q_1 q_2 r}\right)e\left(\frac{-\mu h}{q_0 q_1 q_2 r}\right)+O(1)\right\}$$

with

$$H=x^{\varepsilon}M^{-1}Q^2R. \tag{6.9}$$

Here, by (6.1), we have

$$\frac{\mu}{q_0 q_1 q_2 r}\equiv a\frac{n_1-n_2}{q_0 r}\,\frac{\overline{n_2 q_1}}{n_1 q_2}+\frac{a}{q_0 q_1 q_2 r n_1} \pmod 1$$

whence

$$e\left(\frac{-\mu h}{q_0 q_1 q_2 r}\right) = e\left(ah\frac{n_2-n_1}{q_0 r}\,\frac{\overline{n_2 q_1}}{n_1 q_2}\right) + O\left(\frac{|ah|}{q_0 q_1 q_2 r n_1}\right).$$

Since $\hat{f} \ll M$ this yields

$$\sum_{m\equiv\mu(q_0 q_1 q_2 r)} f(m) = \frac{1}{q_0 q_1 q_2 r}\sum_{|h|\leq H}\hat{f}\left(\frac{h}{q_0 q_1 q_2 r}\right) e\left(ah\frac{n_2-n_1}{q_0 r}\,\frac{\overline{n_2 q_1}}{n_1 q_2}\right) + O(x^{2\varepsilon-1}). \tag{6.10}$$

Finally, insering (6.10) into (6.8), by Lemma 3 we obtain

$$\mathcal{S}_1 = \hat{f}(0)\,\mathscr{X} + \mathscr{R}_1 + O(\|\beta\|^2 x R^{-1}\mathscr{L}^{-A}) \tag{6.11}$$

provided

$$NQ^2R < x^{2-\varepsilon}, \tag{6.12}$$

where

$$\mathscr{X} = \sum_{q_0\leq Q_0}\ \sum_{\substack{r\sim R\\ (a,q_0q_1q_2r)=1}}\ \sum\sum_{(q_1,q_2)=1} \frac{\gamma_{q_0q_1}\gamma_{q_0q_2}}{q_0 q_1 q_2 r} \sum\sum_{\substack{(n_1q_2,n_2q_1)=1\\ n_1\equiv n_2(q_0r)}} \mu^2(n_1n_2)\beta_{n_1}\beta_{n_2} \tag{6.13}$$

and

$$\mathscr{R}_1 = \sum_{q_0\leq Q_0}\ \sum_{\substack{r\sim R\\ (a,q_0q_1q_2r)=1}}\ \sum\sum_{(q_1,q_2)=1} \frac{\gamma_{q_0q_1}\gamma_{q_0q_2}}{q_0 q_1 q_2 r} \sum\sum_{\substack{(n_1q_2,n_2q_1)=1\\ n_1\equiv n_2(q_0r)}} \mu^2(n_1n_2)\beta_{n_1}\beta_{n_2}$$
$$\times \sum_{1\leq|h|\leq H}\hat{f}\left(\frac{h}{q_0 q_1 q_2 r}\right) e\left(ah\frac{n_2-n_1}{q_0 r}\,\frac{\overline{n_2 q_1}}{n_1 q_2}\right). \tag{6.14}$$

Now it remains to evaluate $\mathscr{X}$ and to estimate $\mathscr{R}_1$.

7. Evaluation of $\mathscr{X}$

In this section we prove that $\mathscr{X}$ is asymptotically equal to X, apart from the admissible error term

$$O(\|\beta\|^2 NR^{-1}\mathscr{L}^{-A}). \tag{7.1}$$

This result is essentially of the type of the Barban-Davenport-Halberstam theorem and rests on Theorem 0.

We remove from $\mathscr{X}$ the factor $\mu^2(n_1 n_2)$ and the condition $(n_1, n_2)=n_0=1$ at the cost of the admissible error term (7.1). The arguments are the same as those for (6.4). Thus

$$\mathscr{X}=\sum_{\substack{q_0\le Q_0\\ (a,q_0q_1q_2r)=1}}\sum_{r\sim R}\sum\sum_{(q_1,q_2)=1}\frac{\gamma_{q_0q_1}\gamma_{q_0q_2}}{q_0q_1q_2r}\sum_{l(q_0r)}{}^{*}\Bigg(\sum_{\substack{n\equiv l(q_0r)\\ (n,q_1)=1}}\beta_n\Bigg)\Bigg(\sum_{\substack{n\equiv l(q_0r)\\ (n,q_2)=1}}\beta_n\Bigg)$$

$$+O(\|\beta\|^2NR^{-1}\mathscr{L}^{-A}),$$

where Σ^* stands for the summation over the primitive residue classes.

By (A_2), Theorem 0 yields

$$\sum_{k\le K}\sum_{l(k)}{}^{*}\Bigg|\sum_{\substack{n\equiv l(k)\\ (n,q)=1}}\beta_n-\frac{1}{\varphi(k)}\sum_{(n,qk)=1}\beta_n\Bigg|^2\ll\|\beta\|^2N(\log N)^{-A} \tag{7.2}$$

for any $A>0$ provided that

$$K\le N(\log N)^{-B(A)}, \tag{7.3}$$

the constant implied in $\ll$ depending on A alone. Applying (7.2) with $K=2Q_0R$ (as we may by (6.2) and (6.3)) we deduce by (A_3) that

$$\mathscr{X}=\mathscr{X}_0+O(\|\beta\|^2NR^{-1}\mathscr{L}^{-A}) \tag{7.4}$$

where

$$\mathscr{X}_0=\sum_{\substack{q_0\le Q_0\\ (a,q_0q_1q_2r)=1}}\sum_{r\sim R}\sum\sum_{(q_1,q_2)=1}\frac{\gamma_{q_0q_1}\gamma_{q_0q_2}}{q_0q_1q_2r\varphi(q_0r)}\sum_{\substack{(n_1,q_0q_1r)=1\\ (n_2,q_0q_2r)=1}}\sum\beta_{n_1}\beta_{n_2}.$$

Finally, extending the summation over all q_0 we get

$$\mathscr{X}_0=X+O(\|\beta\|^2NR^{-1}\mathscr{L}^{-A}), \tag{7.5}$$

the error term being estimated by the same arguments as those for (6.2). Gathering together (7.4) and (7.5) we get what we claimed.

Now, if we insert the results (4.1), (4.2), (5.1), (5.2) and (6.11) into (3.4) we see that the main terms $\hat{f}(0)X$ disappear throughout and we are left with $\mathscr{R}_1$ and with a couple of admissible error terms (satisfying (3.3)). In the three next sections we give three treatments of $\mathscr{R}_1$ getting the admissible upper bound

$$\mathscr{R}_1 \ll \|\beta\|^2 R^{-1} x^{1-\varepsilon} \tag{7.6}$$

for different ranges of the parameters M, N, Q, R.

For notational simplicity we write β_n in place of $\mu^2(n)\beta_n$ remembering that from now on the support of β_n occurring in $\mathscr{R}_1$ is restricted to squarefree integers.

8. Estimation of $\mathscr{R}_1$. First method

The method begins with the arrangement of $\mathscr{R}_1$ in the following way

$$\mathscr{R}_1 = \sum \cdots \sum \left| \sum_h \sum_{q_1} \right|. \tag{8.1}$$

In order to separate the variables h, q_1 from the remaining ones we first compute

$$\hat{f}(h/q_0 q_1 q_2 r) = q_0 q_2 r \int_{-\infty}^{\infty} f(\xi q_0 q_2 r)\, e(\xi h/q_1)\, d\xi.$$

Next we put $k=(n_2-n_1)/q_0 r$, thus $1 \leqslant |k| \leqslant K$ where $K=N/R$ and $(q_0 k, n_1 n_2)=1$, $n_1 \equiv n_2 (q_0 |k|)$. Then, by (6.14) it follows that

$$|\mathscr{R}_1| \leqslant 4 \sum_{q_0 \leqslant Q_0} \sum_{1 \leqslant k \leqslant K} \sum_{q_2} |\gamma_{q_0 q_2}| \sum_{\substack{(n_1 q_2, n_2)=1 \\ n_1 \equiv n_2 (q_0 k)}} \sum |\beta_{n_1} \beta_{n_2}|$$

$$\times \int_{-\infty}^{\infty} f(\xi q_0 q_2 r) \left| \sum_{1 \leqslant h \leqslant H} \sum_{(q_1, a n_1 q_2)=1} \frac{\gamma_{q_0 q_1}}{q_1} e\left(\frac{\xi h}{q_1}\right) e\left(ahk \frac{\overline{n_2 q_2}}{n_1 q_2}\right) \right| d\xi.$$

Hence, by Cauchy's inequality and by (A_3) we get

$$\mathscr{R}_1 \ll x^{\varepsilon} M Q^{-3/2} R^{-1} \|\beta\|^2 \sup \mathscr{A}^{1/2}(4NQ, 2N, K, H, 2Q) \tag{8.2}$$

where by defintion

$$\mathscr{A}(C, D, K, H, Q) = \sum_{1 \leqslant c \leqslant C} \sum_{1 \leqslant d \leqslant D} \sum_{1 \leqslant k \leqslant K} \left| \sum_{1 \leqslant h \leqslant H} \sum_{1 \leqslant q \leqslant Q} \alpha(h, q)\, e\left(ahk \frac{\overline{dq}}{c}\right) \right|^2,$$

the supremum being taken over all coefficients $\alpha(h, q)$ with

$$|\alpha(h, q)| \leq 1. \tag{8.3}$$

Now, we appeal to Lemma 1 to infer the following

LEMMA 6. *Let $C, D, H, K, Q \geq 1$, $a \neq 0$ and $\varepsilon > 0$. We then have*

$$\mathscr{A}(C, D, K, H, Q) \ll (CDHKQ)^{\varepsilon}\{CDHKQ + H(KQ)^{1/2}(H+Q)^{1/2} \times [C(Q^2+HKQ)(C+DQ^2)+C^2DQ\sqrt{Q^2+HKQ}+D^2HKQ^3]^{1/2}\} \tag{8.4}$$

the constant implied in $\ll$ depending on ε and a only.

Proof. Clearly, it suffices to prove (8.4) for a modified sum having the variables c, d reduced by a smooth weight function $g(c, d)$ as described in Lemma 1. Squaring and changing the order of summation we represent $\mathscr{A}(C, D, K, H, Q)$ as $\mathscr{K}(C, D, |a|HKQ, Q^2, 1)$ (cf. Lemma 1) with the coefficients

$$B_{nrs} = B_{nr} = \sum_{\substack{1 \leq q_1, q_2 \leq Q \\ q_1 q_2 = r}}\sum \; \sum_{\substack{1 \leq h_1, h_2 \leq H \\ ak(h_1 q_2 - h_2 q_1) = n}}\sum \; \sum_{1 \leq k \leq K} \alpha(h_1, q_1)\bar{\alpha}(h_2, q_2).$$

The terms on the diagonal ($n=0$) are not covered by Lemma 1, they contribute trivially

$$\ll CDK \sum_{1 \leq q_1, q_2 \leq Q}\sum \; \sum_{\substack{1 \leq h_1, h_2 \leq H \\ h_1 q_2 = h_2 q_1}}\sum 1 \ll CDHKQ (\log 2HQ)^4.$$

The terms off the diagonal ($n \neq 0$), by Lemma 1, contribute

$$\ll (CDHKQ)^{\varepsilon} \mathscr{I}(C, D, |a|HKQ, Q^2, 1)\,\|B\|$$

where

$$\|B\|^2 = \sum_{n \geq 1}\sum_{r \geq 1} |B_{nr}|^2$$

$$\ll (HKQ)^{\varepsilon} K \sum_{1 \leq q_1, q_2 \leq Q}\sum \; \sum_{l} \left(\sum_{\substack{1 \leq h_1, h_2 \leq H \\ h_1 q_2 - h_2 q_1 = l}} 1 \right)^2$$

$$= (HKQ)^\varepsilon K \# \{q_1, q_2, h_1, h_2, h_3, h_4; (h_1-h_3)\, q_2 = (h_2-h_4)\, q_1\}$$
$$\ll (HKQ)^\varepsilon K(H^2Q^2+H^3Q).$$

Gathering the above results together we obtain (8.4).

Assume that

$$N^2Q < x^{1-\varepsilon}. \tag{8.5}$$

Then $HK \ll Q$, so Lemma 6 yields

$$\mathscr{A}(4NQ, 2N, K, H, 2Q) \ll x^\varepsilon\{N^3M^{-1}Q^4+N^{3/2}M^{-1}Q^{11/2}R^{1/2}+N^2M^{-1}Q^5R^{1/2}\}.$$

Combining this with (8.2) we end up with

$$\mathscr{R}_1 \ll \|\beta\|^2R^{-1}x^{1/2+\varepsilon}\{NQ^{1/2}+N^{1/4}Q^{5/4}R^{1/4}+N^{1/2}QR^{1/4}\}.$$

This bound satisfies (7.6) provided (8.5) and

$$NQ^5R < x^{2-\varepsilon}, \quad N^2Q^4R < x^{2-\varepsilon}. \tag{8.6}$$

Concluding the investigations of this section we formulate our results as

THEOREM 1. *Suppose* (A_1)–(A_4) *hold. We then have* (3.3) *provided*

$$x^\varepsilon R < N < x^{-\varepsilon}\min\{x^{1/2}Q^{-1/2}, x^2Q^{-5}R^{-1}, xQ^{-2}R^{-1/2}\}.$$

9. Estimation of $\mathscr{R}_1$. Second method

The method begins with the arrangement of $\mathscr{R}_1$ in the following way

$$\mathscr{R}_1 = \sum \cdots \sum \left|\sum_h \sum_{n_2}\right|. \tag{9.1}$$

To separate the variables h, n_2 from the remaining ones requires more effort than in the first method (8.1).

We first wish to get rid of the condition $(r,a)=1$; to this end we appeal to the Möbius formula

$$\sum_{\delta|(a,r)} \mu(\delta) = \begin{cases} 1 & \text{if } (r,a)=1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence we get

$$\mathscr{R}_1 = \sum_{\delta | a} \mu(\delta) \sum_{\substack{q_0 \leq Q_0 \\ (q_0, a)=1}} \sum_{\delta r \sim R} \sum_{\substack{(q_1, q_2)=1 \\ (a, q_1 q_2)=1}} \sum \frac{\gamma_{q_0 q_1} \gamma_{q_0 q_2}}{\delta q_0 q_1 q_2 r}$$
$$\times \sum_{\substack{(n_1 q_2, n_2 q_1)=1 \\ n_1 \equiv n_2 (\delta q_0 r)}} \sum \beta_{n_1} \beta_{n_2} \sum_{1 \leq |h| \leq H} \hat{f}\left(\frac{h}{\delta q_0 q_1 q_2 r}\right) e\left(ah \frac{n_2 - n_1}{\delta q_0 r} \frac{\overline{n_2 q_1}}{n_1 q_2}\right). \tag{9.2}$$

Now we change the variable r into k defined by

$$n_2 - n_1 = \delta q_0 r k, \tag{9.3}$$

$$(\delta q_0 k, n_1 n_2) = 1 \quad \text{and} \quad n_2 \equiv n_1 (\delta q_0 k). \tag{9.4}$$

The remaining condition $\delta r \sim R$ is interpreted as

$$q_0 |k| R < |n_2 - n_1| \leq 2 q_0 |k| R. \tag{9.5}$$

In particular we have

$$1 \leq |k| \leq K \tag{9.6}$$

where now $K = N / q_0 R$.

We wish to separate the variables h, n_2 from the remaining ones. We detect the conditions (9.4) by means of multiplicative characters $\chi \pmod k$ and $\psi \pmod{\delta q_0}$, i.e., we appeal to the following orthogonality relations

$$\frac{1}{\varphi(k)} \sum_{\chi \,(\mathrm{mod}\, k)} \bar{\chi}(n_1) \chi(n_2) = \begin{cases} 1 & \text{if } n_1 \equiv n_2 \pmod k,\ (n_1 n_2, k) = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$\frac{1}{\varphi(\delta q_0)} \sum_{\psi \,(\mathrm{mod}\, \delta q_0)} \bar{\psi}(n_1) \psi(n_2) = \begin{cases} 1 & \text{if } n_1 \equiv n_2 \pmod{\delta q_0},\ (n_1 n_2, \delta q_0) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{9.7}$$

Next we detect the condition (9.5) by means of additive characters, i.e., we appeal to the following integral relation

$$\int_0^1 e((n_2 - n_1)\alpha) F(\alpha)\, d\alpha = \begin{cases} 1 & \text{if } \delta r \sim R \\ 0 & \text{otherwise} \end{cases} \tag{9.8}$$

where

$$F(\alpha)=\sum_{q_0|k|R<|l|\leq 2q_0|k|R} e(\alpha l) \ll \min\{N, \|\alpha\|^{-1}\}$$

therefore

$$\int_0^1 |F(\alpha)|\, d\alpha \ll \log 2N. \tag{9.9}$$

Finally, in order to separate the variables in $\hat{f}$, we write

$$\begin{aligned}\hat{f}(h/\delta q_0 q_1 q_2 r) &= \delta q_0 q_1 q_2 r \int_{-\infty}^{\infty} f(\xi \delta q_0 q_1 q_2 r)\, e(\xi h)\, d\xi \\ &= \delta q_0 q_1 q_2 r \int_0^Y f(\xi \delta q_0 q_1 q_2 r)\, e(\xi h)\, d\xi\end{aligned}$$

with $Y=3q_0 M/Q^2 R$, because f has compact support in $\left[\frac{1}{2}M, 3M\right]$. Hence by the inversion formula

$$\begin{aligned}\hat{f}(h/\delta q_0 q_1 q_2 r) &= \delta q_0 q_1 q_2 r \int_0^Y \int_{-\infty}^{\infty} \hat{f}(\eta)\, e(\xi \eta \delta q_0 q_1 q_2 r)\, e(\xi h)\, d\eta\, d\xi \\ &= \delta q_0 k r \int_0^Y \int_{-\infty}^{\infty} \hat{f}\left(\frac{\eta k}{q_1 q_2}\right) e(\xi \eta (n_2 - n_1))\, e(\xi h)\, d\eta\, d\xi.\end{aligned} \tag{9.10}$$

Here we notice that $|\hat{f}(\eta)| \ll \min\{M, \eta^{-2}M^{-1}\}$, so

$$\int_{-\infty}^{\infty} \left|\hat{f}\left(\frac{\eta k}{q_1 q_2}\right)\right| d\eta = \frac{q_1 q_2}{|k|} \int_{-\infty}^{\infty} |\hat{f}(\eta)|\, d\eta \ll \frac{q_1 q_2}{|k|}. \tag{9.11}$$

Now collecting the formulas (9.2)–(9.11) we conclude that

$$\begin{aligned}\mathscr{R}_1 \ll Y(\log 2N) \sum_{\delta | a} \sum_{q_0 \leq Q_0} \sum_{1\leq k \leq K} \frac{1}{\varphi(\delta q_0)\varphi(k)} \sum_{\psi(\delta q_0)} \sum_{\chi(k)} \sum\sum_{(q_1, q_2)=1} |\gamma_{q_0 q_1} \gamma_{q_0 q_2}| \\ \times \sum_{(n_1, q_1)=1} |\beta_{n_1}| \sum_{1\leq |h| \leq H} \left| \sum_{(n_2, n_1 q_2)=1} \beta(h, n_2)\, \psi\chi(n_2)\, e\left(ahk \frac{\overline{n_2 q_1}}{n_1 q_2}\right)\right|\end{aligned} \tag{9.12}$$

with some $|\beta(h, n_2)| = |\beta_{n_2}|$. Hence by Cauchy's inequality and by Lemma 3

$$\mathscr{R}_1 \ll x^{\varepsilon} M N^{1/2} Q^{-1/2} R^{-3/2} \|\beta\|\, \mathscr{B}_{\beta}^{1/2}(4NQ, 2Q, NR^{-1}, H, 2N) \tag{9.13}$$

where by definition

$$\mathcal{B}_\beta(C,D,K,H,N)=\sum_{1\le c\le C}\sum_{1\le d\le D}\sum_{1\le k\le K}\frac{1}{\varphi(k)}\sum_{\chi(k)}\left|\sum_{1\le h\le H}\sum_{1\le n\le N}\beta(h,n)\chi(n)e\left(ahk\frac{\overline{nd}}{c}\right)\right|^2 \tag{9.14}$$

with some coefficients $\beta(h,n)$ such that $|\beta(h,n)|\le|\beta_n|$.

Now, we appeal to Lemma 1 to infer the following

LEMMA 7. *Let $C,D,H,K,N\ge 1$, $a\neq 0$ and $\varepsilon>0$. Then, for any complex numbers β_n we have*

$$\begin{aligned}\mathcal{B}(C,D,K,H,N)\ll (CDKHN)^\varepsilon\Big\{&CDHK\sum|\beta_n|^2\\&+\big[C(N^2+HKN)(C+DN^2)+C^2DN\sqrt{N^2+HKN}+D^2HKN^3\big]^{1/2}\\&\times\Big[H^2(HK+N)\sum_n\varrho(n)|\beta_n|^4\Big]^{1/2}\Big\},\end{aligned} \tag{9.15}$$

where the implied constant may depend on a and ε and where $\varrho(n)=\Sigma_{\alpha\beta=n}(\alpha,\beta)$.

Proof. As before it suffices to prove the result for a sum modified by a smooth weight function $g(c,d)$. Squaring and changing the order of summation we transform $\mathcal{B}(C,D,K,H,N)$ into $\mathcal{K}(C,D,|a|KHN,N^2,1)$ (cf. Lemma 1) with the coefficients

$$B_{lr}=\sum_{\substack{n_1n_2=r\\(k,n_1n_2)=1,\,n_1\equiv n_2(k)}}\sum_{1\le k\le K}\ \sum_{\substack{1\le h_1,h_2\le H\\a(h_1n_2-h_2n_1)k=l}}\sum\beta(h_1,n_1)\bar\beta(h_2,n_2).$$

The terms on the diagonal ($l=0$) are trivially found to contribute

$$\ll (HKN)^\varepsilon CDHK\sum|\beta_n|^2. \tag{9.16}$$

The terms off the diagonal ($l\neq 0$), by Lemma 1, contribute

$$\ll (CDHKN)^\varepsilon \mathcal{I}(C,D,|a|HKN,N^2,1)\,\|B\| \tag{9.17}$$

where $\|B\|^2\,\Sigma\,\Sigma\,|B_{lr}|^2$.

Let $B'(l,n^2)$ be the contribution to B_{lr} of the terms with $n_1=n_2=n$ in case $r=n^2$ and $n|l$, and let $B''(l,r)$ be the contribution to B_{lr} of all remaining terms.

For the first sum we have

$$B'(l, n^2) \ll |\beta_n|^2 \# \{k, h_1, h_2;\ a(h_1-h_2)kn = l\}$$
$$\ll |\beta_n|^2 H(HKN)^\varepsilon$$

and $n|l$ or else the sum is void. Hence

$$\sum_l \sum_{n|l} |B'(l, n^2)|^2 \ll (HKN)^\varepsilon H^3 K \sum_n |\beta_n|^4.$$

For the second sum we have

$$B''(l, r) \ll (HKN)^\varepsilon (H+N) N^{-1} \sum_{n_1 n_2 = r} (n_1, n_2) |\beta_{n_1} \beta_{n_2}|.$$

Hence

$$\sum_l \sum_r |B''(l, r)|^2 \ll (HKN)^\varepsilon (H+N) H^2 N^{-1} \sum_{n_1 n_2 = n_3 n_4} (n_1, n_2) |\beta_{n_1} \beta_{n_2} \beta_{n_3} \beta_{n_4}|$$
$$\ll (HKN)^\varepsilon (H+N) H^2 \sum_n \varrho(n) |\beta_n|^4$$

by Hölder's inequality.

Gathering together the above estimates we conclude that

$$\|B\|^2 \ll (HKN)^\varepsilon H^2 (HK+N) \sum_n \varrho(n) |\beta_n|^4. \tag{9.18}$$

Finally, we complete the proof of Lemma 7 by (9.16), (9.17), Lemma 1 and (9.18).

Remark. In the circumstances of $\mathcal{R}_1$ the n's are squarefree, so $\varrho(n) = \tau(n) \ll n^\varepsilon$. This fact that n's are squarefree will be useful (not crucial) in other situations as well.

Let us assume that

$$(\mathrm{A}_5)\quad N^{1-\varepsilon} \sum_n |\beta_n|^4 \ll \left(\sum_n |\beta_n|^2 \right)^2.$$

Assume also that

$$Q^2 N < x^{1-\varepsilon}. \tag{9.19}$$

Then $HK < N$, so Lemma 7 yields

$$\mathcal{B}_\beta(4NQ, 2Q, NR^{-1}, H, 2N) \ll \|\beta\|^2 x^\varepsilon Q^3 N^2 M^{-1} \{Q + RN^{1/2} + RQ^{1/2}\}. \tag{9.20}$$

By (9.12) and (9.20) we get

$$\mathcal{R}_1 \ll \|\beta\|^2 x^\varepsilon M^{1/2} N^{3/2} Q^{1/2} R^{-3/2}\{Q^{1/2}+RN^{1/4}+RQ^{1/4}\}.$$

This bound satisfies (7.6) provided $Q^2N^2<x^{1-\varepsilon}R$, $QN^{5/2}<x^{1-\varepsilon}$, and $Q^{3/2}N^2<x^{1-\varepsilon}$.

Concluding the investigations of this section we formulate our results as

THEOREM 2. *Suppose* (A_1)–(A_5) *hold. We then have* (3.3) *provided*

$$x^\varepsilon R < N < x^{-\varepsilon} \min\left\{\left(\frac{xR}{Q^2}\right)^{1/2}, \left(\frac{x}{Q}\right)^{2/5}, \left(\frac{x^2}{Q^3}\right)^{1/4}\right\}.$$

10. Estimation of $\mathcal{R}_1$. Third method

This method does not depend on the factorization of the moduli qr; in other words we assume that

$$R=1. \tag{10.1}$$

This method was first applied by E. Fouvry in [5]; it begins with the arrangement of $\mathcal{R}_1$ in the following form

$$\mathcal{R}_1 = \sum \cdots \sum \left|\sum_h \sum_{n_1} \sum_{n_2}\right|. \tag{10.2}$$

In order to separate the variables in $\hat{f}$ we write

$$\hat{f}\left(\frac{h}{q_0 q_1 q_2}\right) = q_0 q_1 q_2 \int_{-\infty}^{\infty} f(\xi q_0 q_1 q_2)\, e(\xi h)\, d\xi;$$

by (6.14) we get

$$\mathcal{R}_1 \ll \sum_{q_0 \le Q_0} \sum\sum_{(q_1,q_2)=1} |\gamma_{q_0q_1}\gamma_{q_0q_2}|$$
$$\times \int_0^{3q_0MQ^{-2}} \left| \sum_{1\le h\le H} e(\xi h) \sum\sum_{\substack{(n_1q_2,\, n_2q_1)=1\\ n_1\equiv n_2(q_0)}} \beta_{n_1}\beta_{n_2}\, e\left(ah\frac{n_2-n_1}{q_0}\,\frac{\overline{n_2q_1}}{n_1q_2}\right)\right| d\xi.$$

Hence, by Cauchy's inequality we get

$$\mathcal{R}_1 \ll x^{\varepsilon} M Q^{-1} \mathcal{C}_{\beta}^{1/2}(Q, H, N) \tag{10.3}$$

where $\mathcal{C}_\beta(Q,H,N)$ is given by

$$\sum_{(q_1,q_2)=1}\sum g(q_1,q_2)\left|\sum_{1\le h\le H} e(\xi h) \sum_{\substack{(n_1q_2,n_2q_1)=1\\ n_1\equiv n_2 (\mathrm{mod}\, q_0),\, n_1\neq n_2}}\sum \beta_{n_1}\beta_{n_2} e\left(ah\frac{n_2-n_1}{q_0}\frac{\overline{n_2q_1}}{n_1q_2}\right)\right|^2$$

with some $q_0\geq 1$ and some real ξ. Here $g(q_1,q_2)$ is a smooth function supported in $\left[\frac{1}{2}q_0^{-1}Q, 3q_0^{-1}Q\right]\times\left[\frac{1}{2}q_0^{-1}Q, 3q_0^{-1}Q\right]$.

Now we appeal to Lemma 1 to infer the following

LEMMA 8. *Let $H, N, Q\geq 1$, $a\neq 0$, $q_0\geq 1$. Assume* (A_5) *and that $\beta_n=0$ if n is not squarefree. We then have*

$$\mathcal{C}_\beta(Q,H,N) \ll (HNQ)^{\varepsilon}\|\beta\|^4\{HQ^2+(\sqrt{H}+N)HNQ[N^4+HN^3+QN+Q\sqrt{HN}]^{1/2}\}; \tag{10.4}$$

the constant implied in $\ll$ may depend on ε and a at most.

Proof. Squaring and changing the order of summation we get

$$\mathcal{C}_\beta(Q,H,N) = \sum_{1\le h_1,h_2\le H} e((h_1-h_2)\xi) \sum_{\substack{(n_1,n_2)=1\\ n_1\equiv n_2(q_0)}} \beta_{n_1}\beta_{n_2} \sum_{\substack{(n_3,n_4)=1\\ n_3\equiv n_4(q_0)}} \overline{\beta_{n_3}\beta_{n_4}}$$
$$\times \sum_{\substack{(n_2q_1,n_1q_2)=1\\ (n_4q_1,n_3q_2)=1}}\sum g(q_1,q_2)\, e\left(ah_1\frac{n_2-n_1}{q_0}\frac{\overline{n_2q_1}}{n_1q_2} - ah_2\frac{n_4-n_3}{q_0}\frac{\overline{n_4q_1}}{n_3q_2}\right).$$

Denote $\delta_1=(n_1,n_4)$, $\delta_2=(n_2,n_3)$; thus the exponent is equal to (mod 1)

$$\left(ah_1\frac{n_2-n_1}{q_0}\frac{n_3n_4}{\delta_1\delta_2} - ah_2\frac{n_4-n_3}{q_0}\frac{n_1n_2}{\delta_1\delta_2}\right)\frac{\overline{q_1n_2n_4/\delta_1\delta_2}}{q_2n_1n_3}$$

(we recall that n's are squarefree, so the above expression is well defined). This transforms $\mathcal{C}_B(Q,H,N)$ into four sums of the type $\mathcal{K}(3Q, 3Q, 8|a|HN^3, N_1^2, N_2^2)$ where $N_j=N$ or $2N$, see Lemma 1, with the coefficients

$$B_{lrs}=\sum_{N<n_1,n_2,n_3,n_4\leq 2N}\beta_{n_1}\beta_{n_2}\overline{\beta_{n_3}\beta_{n_4}}\sum_{1\leq h_1,h_2\leq H}\sum e((h_1-h_2)\,\xi),$$

the variables of summation being restricted by the conditions $n_1 n_3=s$, $n_2 n_4=(n_1, n_4)(n_2, n_3)r$, $(n_1, n_2)=(n_3, n_4)=1$, $n_1\equiv n_2(q_0)$, $n_3\equiv n_4(q_0)$ and

$$ah_1(n_2-n_1)\,n_3\,n_4-ah_2(n_4-n_3)\,n_1\,n_2=(n_1\,n_3, n_2\,n_4)\,q_0\,l. \tag{10.5}$$

The terms on the diagonal ($l=0$) are not covered by Lemma 1; they contribute trivially

$$\ll Q^2\sum_{(n_1,n_2)=(n_3,n_4)=1}|\beta_{n_1}\beta_{n_2}\beta_{n_3}\beta_{n_4}|\sum_{\substack{1\leq h_1,h_2\leq H\\ h_1(n_2-n_1)\,n_3\,n_4=h_2(n_4-n_3)\,n_1\,n_2}}1.$$

By Cauchy's inequality the above does not exceed

$$Q^2\sum_{(n_1,n_2)=1}|\beta_{n_1}\beta_{n_2}|^2\sum_{\substack{n_3,n_4\sim N\\ h_1(n_2-n_1)\,n_3\,n_4=h_2(n_4-n_3)\,n_1\,n_2}}\sum_{1\leq h_1,h_2\leq H}1.$$

Given h_2, n_1, n_2 we find that $n_3\,n_4|h_2\,n_1\,n_2$, so there are $\ll(HN)^\varepsilon$ values of h_1, n_3, n_4. Hence, the terms on the diagonal ($l=0$) contribute

$$\ll (HN)^\varepsilon HQ^2\|\beta\|^4. \tag{10.6}$$

The terms off the diagonal ($l\neq 0$), by Lemma 1, contribute

$$\ll (HNQ)^\varepsilon\mathcal{I}(3Q, 3Q, 8|a|HN^3, 4N^2, 4N^2)\,\|B\| \tag{10.7}$$

where

$$\|B\|^2=\sum_{l\geq 1}\sum_r\sum_s|B_{lrs}|^2.$$

We first give an upper bound for B_{lrs}. By (10.5) we infer that h_1 is determined modulo $n_1\,n_2/(n_1\,n_2, n_3\,n_4)$, whence

$$|B_{lrs}|\ll(HN^{-2}+1)\sum_{n_1 n_3=s}\sum_{\substack{n_2 n_4=(n_1 n_3, n_2 n_4)r\\ (n_1,n_2)=(n_3,n_4)=1}}(n_1\,n_2, n_3\,n_4)\,|\beta_{n_1}\beta_{n_2}\beta_{n_3}\beta_{n_4}|$$

$$=B^+_{rs},$$

say. Thus,

$$\|B\|^2 \ll \sum_{r,s} B^+_{rs} \sum_{l} |B_{lrs}|$$

$$\ll (HN^{-2}+1)\,H^2 {\sum}^* |\beta_{n_1}\beta_{n_3}\beta_{m_1}\beta_{m_3}| \sum |\beta_{n_2}\beta_{n_4}\beta_{m_2}\beta_{m_4}|\,(n_1, n_2, n_3, n_4)$$

where Σ^* means that the range of summation is restricted by

$$n_1 n_3 = m_1 m_3,$$

$$n_2 n_4 (m_1 m_3, m_2 m_4) = m_2 m_4 (n_1 n_3, n_2 n_4),$$

$$(n_1, n_2) = (n_3, n_4) = (m_1, m_2) = (m_3, m_4) = 1.$$

Using (A_5) and two applications of Cauchy's inequality we get

$$\|B\|^2 \ll N^{\varepsilon}(HN^{-2}+1)\,H^2\|\beta\|^8. \tag{10.8}$$

Combining (10.6), (10.7), (10.8) and Lemma 1 we obtain

$$\mathscr{C}_\beta(Q, H, N) \ll (HNQ)^{\varepsilon}\|\beta\|^4$$
$$\times \left\{ HQ^2 + \left(\frac{\sqrt{H}}{N}+1\right) H[Q^2N^4(N^4+HN^3)+Q^3N^3\sqrt{N^4+HN^3}+Q^2HN^3]^{1/2} \right\}$$

completing the proof of Lemma 8.

From Lemma 8 and (10.3) it follows that

$$\mathscr{R}_1 \ll \|\beta\|^2 x^{\varepsilon} M Q^{-1} \left\{ \frac{Q^4}{M} + \left(\frac{Q}{\sqrt{M}}+N\right) \frac{Q^3N}{M} \left[N^4 + \frac{Q^2N^3}{M} + QN + Q^2\sqrt{\frac{N}{M}} \right]^{1/2} \right\}^{1/2}.$$

This bound satisfies (7.6) provided $Q^2<x^{1-\varepsilon}N$, $QN^3<x^{1-\varepsilon}$, $Q^2N^3<x^{3/2-\varepsilon}$, $QN<x^{2/3-\varepsilon}$, and $Q^4N^3<x^{5/2-\varepsilon}$. Assuming that $N<x^{1/6}$ these inequalities all follow from only two, the first and the last ones. Concluding the investigations of this section we formulate our results as

THEOREM 3. *Suppose* (A_1)–(A_5) *hold. Let* $R=1$. *We then have* (3.3) *provided*

$$x^{\varepsilon-1}Q^2 < N < x^{5/6-\varepsilon}Q^{-4/3}.$$

Remark. We may omit the restriction $N<X^{1/6}$ because otherwise $Q<x^{1/2-\varepsilon}$ and the result follows from the large sieve inequality (see (1.5)).

11. Special case. I

In this section we consider the following sum

$$\Delta(M,N,L,R)=\sum_{\substack{r\sim R\\(r,a)=1}}\left|\sum_{\substack{m\sim M\\mnl\equiv a(r)}}\sum_{n\sim N}\sum_{l\sim L}\alpha_m\beta_n\lambda_l-\frac{1}{\varphi(r)}\sum_{\substack{m\sim M\\(mnl,r)=1}}\sum_{n\sim N}\sum_{l\sim L}\alpha_m\beta_n\lambda_l\right| \qquad (11.1)$$

where $M,N,L,R\geqslant 1$, $MNL=x$ with the aim of showing that

$$\Delta(M,N,L,R)\ll\|\alpha\|\,\|\beta\|\,\|\lambda\|\,x^{1/2}\mathscr{L}^{-A} \qquad (11.2)$$

with any $A>0$ under certain constraints on M,N,L,R and the sequences (α_m), (β_n), (λ_l).

The expression (11.1) is a particular case of (3.1), namely

$$\Delta(M,N,L,R)=\mathscr{D}(M,NL,1,R)$$

with an obvious interpretation of the coefficients. We adopt the hypotheses (A_2), (A_4) for (β_n) and the hypothesis (A_4) for λ_l, so the results of Sections 3–7 can be applied. Therefore, our problem reduces to the estimation of $\mathscr{R}_1$ which is given by (see (6.14))

$$\mathscr{R}_1=\sum_{\substack{r\sim R\\(r,a)=1}}\frac{1}{r}\sum_{1\leqslant|h|\leqslant H}\hat f(h/r)\sum_{\substack{(n_1l_1,n_2l_2)=1\\n_1l_1\equiv n_2l_2(r)}}\beta_{n_1}\beta_{n_2}\lambda_{l_1}\lambda_{l_2}e\left(ahk\frac{\overline{n_2l_2}}{n_1l_1}\right)$$

where $H=x^{\varepsilon}M^{-1}R$ (see (6.9)) and $k=(n_1l_1-n_2l_2)/r$. To estimate $\mathscr{R}_1$ we apply the method of Section 9 obtaining (compare with (9.12))

$$\mathscr{R}_1\ll x^{\varepsilon}MR^{-1}\sum_{\delta|a}\sum_{1\leqslant k\leqslant K}\frac{1}{\varphi(\delta)\varphi(k)}\sum_{\psi(\mathrm{mod}\,\delta)}\sum_{\chi(\mathrm{mod}\,k)}\sum_{n_1}\sum_{l_1}\sum_{l_2}|\beta_{n_1}\lambda_{l_1}\lambda_{l_2}|$$
$$\times\left|\sum_{1\leqslant|h|\leqslant H}\sum_{n_2}\beta(h,n_2)\,\psi\chi(n_2)\,e\left(ahk\frac{\overline{n_2l_2}}{n_1l_1}\right)\right|$$

with some $|\beta(h,n_2)|=|\beta_{n_2}|$ and $K=4NLR^{-1}$. Hence by Cauchy's inequality we get (compare with (9.13), (9.14))

$$\mathscr{R}_1\ll x^{\varepsilon}MR^{-1}K^{1/2}\|\beta\|\,\|\lambda\|^2\mathscr{B}_\beta^{1/2}(4NL,2L,2NLR^{-1},H,2N). \qquad (11.3)$$

For an estimate of $\mathscr{B}_\beta$ we appeal to Lemma 7. Let us assume that

$$L < x^{-\varepsilon}M, \tag{11.4}$$

then $HK<N$, so Lemma 7 yields

$$\mathcal{B}_\beta \ll x^\varepsilon M^{-1}N^2L(L^2+RN^{1/2}+RL^{1/2})\,\|\beta\|^2. \tag{11.5}$$

Finally (11.3) and (11.5) yield

$$\mathcal{R}_1 \ll x^\varepsilon M^{1/2}N^{3/2}LR^{-3/2}(L+R^{1/2}N^{1/4}+R^{1/2}L^{1/4})\,\|\beta\|^2\,\|\lambda\|^2.$$

We require this bound to be $\ll \|\beta\|^2\|\lambda\|^2R^{-1}x^{1-\varepsilon}$ (compare with (7.6)) which is satisfied under (11.4) and the following conditions:

$$N^2L^3 < x^{1-\varepsilon}R \tag{11.6}$$

$$N^4L^3+N^5L^2 < x^{2-\varepsilon}. \tag{11.7}$$

Then we have (11.2) provided

$$x^\varepsilon R < NL \tag{11.8}$$

(compare with (6.2)). Notice that (11.8) and (11.6) imply (11.4).

Concluding the investigations of this section we formulate our results as

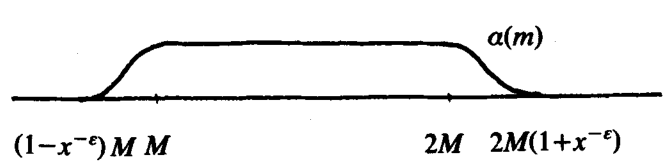
THEOREM 4. *Suppose* (A_2)–(A_5) hold for (β_n), *that* (A_4) *holds for* (λ_l), *and that* (11.6), (11.7) *and* (11.8) *also hold. Then we have* (11.2).

12. Special case. II

In this section we consider $\mathcal{D}(M,N,Q,R)$ with the special coefficients

(A_6) $\alpha_m \equiv 1$.

Now there is no point to using the dispersion method because we can execute the summation over m immediately by applying Poisson's formula (Lemma 2). We assume the hypotheses (A_1) and (A_3). Before we proceed to estimate $\mathcal{D}(M,N,Q,R)$ we replace α_m by a smooth function $\alpha(m)$, say, whose graph is



The relevant correction in $\mathcal{D}(M, N, Q, R)$ is bounded by (apply Lemma 3)

$$O(\|\beta\| N^{1/2} M x^{-\varepsilon/2}). \tag{12.1}$$

Now, applying Lemma 2 we get

$$\sum_{m\equiv a\bar{n}(qr)} \alpha(m) = \frac{1}{qr} \sum_{|h|\leq H} \hat{\alpha}\left(\frac{h}{qr}\right) e\left(-ah\frac{\bar{n}}{qr}\right) + O(Q^{-1}R^{-1})$$

with $H = x^{\varepsilon} QRM^{-1}$ and

$$\frac{1}{\varphi(qr)} \sum_{(m, qr)=1} \alpha(m) = \frac{1}{qr} \hat{\alpha}(0) + O(Q^{-1}R^{-1}\tau(qr)).$$

Hence

$$\mathcal{D}(M, N, Q, R) = \sum_{\substack{q\sim Q \\ (qr, a)=1}} \sum_{r\sim R} \frac{\gamma_q \delta_r}{qr} \sum_{\substack{n\sim N \\ (n, qr)=1}} \beta_n \sum_{1\leq|h|\leq H} \hat{\alpha}\left(\frac{h}{qr}\right) e\left(-ah\frac{\bar{n}}{qr}\right) + O(\|\beta\| M^{1/2} x^{1/2-\varepsilon/2}).$$

In order to separate the variables in $\hat{\alpha}$ we write

$$\hat{\alpha}\left(\frac{h}{qr}\right) = q \int_{-\infty}^{\infty} \alpha(\xi q)\, e\left(\frac{\xi h}{r}\right) d\xi$$

giving

$$\mathcal{D}(M, N, Q, R) \ll \|\beta\| M^{1/2} x^{1/2-\varepsilon/2} + x^{2\varepsilon} RH^{-1} \sum_{q\sim Q} \sum_{n\sim N} |\beta_n| \left| \sum_{1\leq h\leq H} \sum_{r\sim R,\, (r, a)=1} \frac{\delta_r}{r} e\left(\frac{\xi h}{r}\right) e\left(-ah\frac{\bar{n}}{qr}\right) \right|$$

with some real ξ. Hence by Cauchy's inequality

$$\mathcal{D}(M, N, Q, R) \ll \|\beta\| M^{1/2} x^{1/2-\varepsilon/2} + \|\beta\| x^{\varepsilon} M Q^{-1/2} R^{-1} \mathcal{E}^{1/2}(2Q, 2N, H, 2R) \tag{12.2}$$

where

$$\mathcal{E}(C, D, H, R) = \sum_{1\leq c\leq C} \sum_{1\leq d\leq D} \left| \sum_{1\leq h\leq H} \sum_{1\leq r\leq R} \delta(h, r)\, e\left(ah\frac{\bar{d}}{cr}\right) \right|^2$$

with $|\delta(h, r)| \leq 1$. We have the following

LEMMA 9. *Let $a \neq 0$, $C, D, H, R \geq 1$. We then have*

$$\mathscr{E}(C,D,H,R) \ll (CDHR)^{\varepsilon}\{H^2+CDHR+HR^{1/2}(H+R)^{1/2} \times [D(R^2+HR)(D+CR^2)+D^2CR\sqrt{R^2+HR}+C^2HR^3]^{1/2}\}$$

the constant implied in $\ll$ *depending on* ε *and* a *only.*

Proof. This easily follows from Lemma 6 because

$$\mathscr{E}(C,D,H,R) \leq \mathscr{A}(D,C,1,H,R)+O((|a|H\log 2RCD)^2).$$

To see this apply the "reciprocity" relation $\bar{d}/cr \equiv -\overline{cr}/d+1/cdr \pmod 1$ giving

$$e\left(ah\frac{\bar{d}}{cr}\right)=e\left(-ah\frac{\overline{cr}}{d}\right)+O\left(\frac{|a|h}{cdr}\right).$$

This completes the proof.

Let us assume that

$$x^{\varepsilon}Q<M. \tag{12.3}$$

We then have $H<R$ and by Lemma 9

$$\mathscr{E}(2Q,2N,H,2R) \ll x^{\varepsilon}\{Q^2R^2NM^{-1}+QR^2M^{-1}[QNR^4+QN^2R^2+Q^3R^4M^{-1}]^{1/2}\}.$$

Hence, by (12.2) we get

$$\mathscr{D}(M,N,Q,R) \ll \|\beta\| x^{1/2-\varepsilon}M^{1/2} \tag{12.4}$$

which is admissible, provided (12.3) and $x^{\varepsilon}QR^4<xM$, $x^{\varepsilon}QR^2<M^2$, and $x^{\varepsilon}Q^3R^4<x^2M$.

Concluding the investigations of this section we put or results into

THEOREM 5. *Let* (A_1), (A_3) *and* (A_6) *hold. Let* $a \neq 0$ *and* $\varepsilon>0$. *We then have* (12.4) *provided*

$$M>x^{\varepsilon}\max\{Q,x^{-1}QR^4,Q^{1/2}R,x^{-2}Q^3R^4\}. \tag{12.5}$$

Now, by means of Lemma 4 we extend Theorem 5 to the following sequences

(A_6^*) $\alpha_m=\begin{cases}1 & \text{if } (m,P(z))=1\\ 0 & \text{otherwise}\end{cases}$

where $P(z)=\prod_{p<z}p$ and z is a sufficiently small number.

Without loss of generality we may assume that

$$\beta_n \geq 0,$$

otherwise, consider the two sequences $\beta_n^+=\max\{0,\beta_n\}$ and $\beta_n^-=\min\{0,\beta_n\}$ separately. Then

$$\mathscr{E}(M,N;q,a):=\sum\sum_{mn\equiv a(q)}\alpha_m\beta_n-\frac{1}{\varphi(q)}\sum\sum_{(mn,q)=1}\alpha_m\beta_n$$

$$\leq\sum_{\substack{d|P(z)\\(d,q)=1}}\left\{\sum_{\substack{dm\sim M\\dmn\equiv a(q)}}\lambda_d^+\sum_n\beta_n-\frac{1}{\varphi(q)}\sum_{\substack{dm\sim M\\(dmn,q)=1}}\lambda_d^-\sum_n\beta_n\right\}$$

$$=E^+(M,N;q,a)+\Delta(M,N;q),$$

say where

$$E^+(M,N;q,a):=\sum_{\substack{d|P(z)\\(d,q)=1}}\lambda_d^+\left\{\sum_{\substack{dm\sim M\\dmn\equiv a(q)}}\sum_n\beta_n-\frac{1}{\varphi(q)}\sum_{\substack{dm\sim M\\(dmn,q)=1}}\sum_n\beta_n\right\}$$

and

$$\Delta(M,N;q):=\sum_{\substack{d|P(z)\\(d,q)=1}}(\lambda_d^+-\lambda_d^-)\sum_{\substack{dm\sim M\\(m,q)=1}}\frac{1}{\varphi(q)}\sum_{(n,q)=1}\beta_n.$$

Analogously, we have

$$E(M,N;q,a)\geq E^-(M,N;q,a)-\Delta(M,N;q).$$

From the above estimates it follows that

$$|E(M,N;q,a)|\leq\sum_{\substack{(d,q)=1\\d\leq D}}\left|\sum_{\substack{dm\sim M\\dmn\equiv a(q)}}\sum_n\beta_n-\frac{1}{\varphi(q)}\sum_{\substack{dm\sim M\\(dmn,q)=1}}\sum_n\beta_n\right|+|\Delta(M,N;q)|.$$

The inner sum can be interpreted as $E(d^{-1}M,dN;q,a)$ with the coefficients $\alpha_m'=1$ and $\beta_n'=\beta_{n/d}$ for $n\equiv 0(d)$, $\beta_n'=0$ for $n\not\equiv 0(d)$.

Let us choose $D=x^\varepsilon$. Then, given d with $1\leq d<D$, Theorem 5 is applicable for $\mathscr{D}(d^{-1}M,dN,Q,R)$ giving (see (12.4))

$$\mathscr{D}(d^{-1}M,dN,Q,R)\ll\|\beta\|x^{1/2-\varepsilon}d^{-1/2}M^{1/2}$$

whence

$$\sum_{d\leq D}\mathscr{D}(d^{-1}M,dN,Q,R)\ll\|\beta\|x^{1/2-\varepsilon}(DM)^{1/2}\ll\|\beta\|x^{(1-\varepsilon)/2}M^{1/2}$$

provided (12.5) holds (with ε replaced by 2ε).

Now it remains to estimate $\Delta(M, N; q)$. We have

$$\frac{1}{\varphi(q)} \sum_{\substack{m \sim d^{-1}M \\ (m,q)=1}} 1 = \frac{M}{dq} + O\left(\frac{\tau(q)}{\varphi(q)}\right)$$

whence

$$\Delta(M, N; q) = \left\{ \frac{M}{q} \left(\sum_{\substack{d \mid P(z) \\ (d,q)=1}} \frac{\lambda_d^+}{d} - \sum_{\substack{d \mid P(z) \\ (d,q)=1}} \frac{\lambda_d^-}{d} \right) + O\left(\frac{\tau(q)}{\varphi(q)}\right) \right\} \sum_{(n,q)=1} \beta_n$$

$$\ll \left(\frac{M}{q} \exp(-s \log s) + \frac{\tau(q)}{\varphi(q)} \right) \sum_{(n,q)=1} \beta_n$$

where $s = \log D / \log z$. Taking $z \leqslant \exp(\log x / \log\log x)$ we get $s \geqslant \varepsilon \log\log x$, so

$$\Delta(M, N; q) \ll \mathscr{L}^{-A} Q^{-1} M \sum_{(n,q)=1} \beta_n$$

which is admissible.

Concluding the above investigations we formulate

THEOREM 5*. *Theorem 5 holds if* (A_6) *is replaced by* (A_6^*) *subject to*

$$z \leqslant z_0 = \exp(\log x / \log\log x).$$

13. Special case. III

In this section we consider $\mathscr{D}(M, N, Q, R)$ with the special coefficients

(A_7) $\gamma_q = 1$ for $Q < q \leqslant Q_1$ with $Q < Q_1 \leqslant 2Q$, $QR < x\mathscr{L}^{-B}$.

Without loss of generality we may assume that

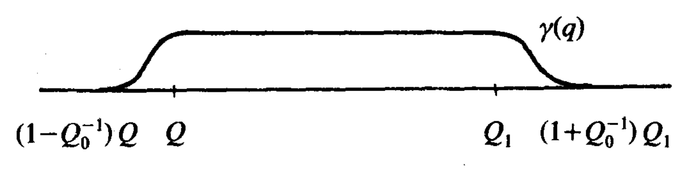
$$Q^2 R \leqslant x. \tag{13.1}$$

To see this note that we are dealing with the equation

$$mn = a + qrs$$

with $m \sim M$, $n \sim N$, $Q < q \leqslant Q_1$, $r \sim R$ where $MN = x \sim QRs$. Since both q and s are counted with weight 1 (see (A_7)) they appear symmetrically in $\mathscr{D}$ except that s runs through an interval dependent on m, n, r. By a subdivision argument we remove this dependence with an admissible error term, since $s \sim S = x/QR > \mathcal{L}^B$, this lower bound being crucial to the argument. Now, we have either $Q^2R \leqslant x$ or $S^2R \leqslant x$ and without loss of generality we assume (13.1).

We replace the coefficients γ_q by a smooth function $\gamma(q)$, say, whose graph is



The relevant correction in $\mathcal{D}(M, N, Q, R)$ is bounded by (apply Lemma 3)

$$O(\|\alpha\|\,\|\beta\|\, x^{1/2}\mathcal{L}^B Q_0^{-1/2})$$

which is admissible (see (3.2) and (6.3)). Now we adopt the hypotheses (A_1)–(A_4) and (A_7) so the results of Sections 3–7 can be granted. Therefore our problem (to prove (3.2)) reduces to the estimation of $\mathcal{R}_1$ (see (6.14)). To this end we appeal directly to Lemma 1. Writing

$$\hat{f}\left(\frac{h}{q_0 q_1 q_2 r}\right) = q_0 q_1 q_2 \int_{-\infty}^{\infty} f(\xi q_0 q_1 q_2)\, e(\xi h/r)\, d\xi$$

we get

$$\mathcal{R}_1 = \sum_{\substack{q_0 \leqslant Q_0 \\ (q_0, a)=1}} \int_{-\infty}^{\infty} \sum_{\substack{(q_1, q_2)=1 \\ (q_1 q_2, a)=1}} \gamma(q_0 q_1)\gamma(q_0 q_2) f(\xi q_0 q_1 q_2)$$
$$\times \sum_{\substack{r \sim R \\ (r, a)=1}} \frac{1}{r} \sum\sum_{\substack{(n_1 q_2, n_2 q_1)=1 \\ n_1 \equiv n_2\, (q_0 r)}} \beta_{n_1}\beta_{n_2} \sum_{1 \leqslant |h| \leqslant H} e\left(\frac{\xi h}{r}\right) e\left(ah \frac{n_2 - n_1}{q_0 r} \frac{\overline{n_2 q_1}}{n_1 q_2}\right) d\xi.$$

The condition $(q_1 q_2, a) = 1$ can be removed by means of the Möbius formula (as in Section 9)

$$\sum_{\delta \mid (a, q_1 q_2)} \mu(\delta) = \begin{cases} 1 & \text{if } (a, q_1 q_2) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\mathcal{R}_1 = \sum_{\delta_1\delta_2|a} \mu(\delta_1\delta_2) \sum_{\substack{q_0 \le Q_0 \\ (q_0,a)=1}} \int_{-\infty}^{\infty} \sum_{(q_1,q_2)=1} \gamma(\delta_1 q_0 q_1)\gamma(\delta_2 q_0 q_2) f(\xi\delta_1\delta_2 q_0 q_1 q_2)$$

$$\times \sum_{\substack{r\sim R \\ (r,a)=1}} \frac{1}{r} \sum\sum_{\substack{(n_1q_2,n_2q_1)=1 \\ n_1\equiv n_2\,(q_0 r)}} \beta_{n_1}\beta_{n_2} \sum_{1\le|h|\le H} e\left(\frac{\xi h}{r}\right) e\left(\frac{ah}{\delta_1\delta_2}\frac{n_2-n_1}{q_0 r}\frac{\overline{n_2 q_1}}{n_1 q_2}\right) d\xi. \tag{13.2}$$

For fixed δ_1, δ_2, q_0 and ξ, the sum over the remaining variables is of the type $\mathcal{K}(C,D,N,R,S)$ of Lemma 1 with $C\to Q/\delta_2 q_0$, $D\to Q/\delta_1 q_0$, $N\to|a|HN/\delta_1\delta_2 q_0 R$, $R\to N$, and $S\to N$, and with the coefficients B_{nrs} interpreted by

$$B_{ln_2n_1} = \beta_{n_1}\beta_{n_2} \sum_{\substack{r\sim R,\,(r,a)=1 \\ q_0 r|(n_1-n_2)}} \frac{1}{r} \sum_{\substack{1\le|h|\le H \\ ah(n_2-n_1)=\delta_1\delta_2 q_0 rl}} e(\xi h/r).$$

We have $|B_{ln_1n_2}| \ll x^{\varepsilon}R^{-1}|\beta_{n_1}\beta_{n_2}|$ whence

$$\begin{aligned} \|B\|^2 &:= \sum_l \sum_{n_1} \sum_{n_2} |B_{ln_1n_2}|^2 \\ &\ll x^{\varepsilon}R^{-1} \sum_{n_1}\sum_{n_2}\sum_l |\beta_{n_1}\beta_{n_2}B_{ln_1n_2}| \\ &\ll x^{\varepsilon}R^{-2}H\|\beta\|^4. \end{aligned} \tag{13.3}$$

Moreover, by (13.1) we have $Q\le\sqrt{x}$ and so

$$\mathcal{J}^2\left(\frac{Q}{\delta_2 q_0}, \frac{Q}{\delta_1 q_0}, \frac{|a|HN}{\delta_1\delta_2 q_0 R}, N, N\right) \ll x^{\varepsilon}Q^2N^4 + x^{\varepsilon}Q^3N^{5/2}. \tag{13.4}$$

Combining (13.1)–(13.4), by Lemma 1, we infer that

$$\begin{aligned} \mathcal{R}_1 &\ll x^{\varepsilon}MQ^{-2}\mathcal{J}\|B\| \\ &\ll x^{\varepsilon}MQ^{-2}(QN^2+Q^{3/2}N^{5/4})R^{-1}(Q^2RM^{-1})^{1/2}\|\beta\|^2. \end{aligned}$$

This bound satisfies (7.6) provided $N^3R<x^{1-\varepsilon}$ and $N^{3/2}QR<x^{1-\varepsilon}$. The latter condition is a consequence of the former and (13.1).

Summarizing this section we have

THEOREM 6. *Let* (A_1)–(A_4) *and* (A_7) *hold. Let* $a\neq 0$ *and* $\varepsilon>0$. *We then have* (3.2) *provided*

$$x^\varepsilon R<N<x^{-\varepsilon}(x/R)^{1/3}.$$

14. Special case. IV

In this section we consider the following sums

$$\Delta(M,N,L,Q,R)=\sum_{\substack{r\leq R\\(r,a)=1}}\sum_{\substack{l\leq L\\(l,r)=1}}\left|\sum_{\substack{q\leq Q\\(q,al)=1}}\left(\sum_{\substack{m\leq M\\lmn\equiv a(qr)}}\sum_{n\leq N}1-\frac{1}{\varphi(qr)}\sum_{\substack{m\leq M\\(mn,qr)=1}}\sum_{n\leq N}1\right)\right|$$

where $M,N,L,Q,R\geq 1$, $LMN=x$, showing that

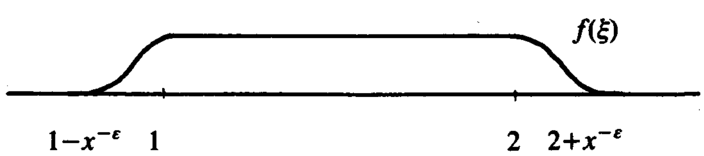
$$\Delta(M,N,L,Q,R)\ll x^{1-\varepsilon} \tag{14.1}$$

subject to certain constraints on M,N,L,Q,R.

By elementary arguments (familiar from the previous sections) the problem reduces to estimating sums of the type

$$\Delta_0(M,N,L,Q,R)=\sum_{\substack{r\sim R\\(r,a)=1}}\sum_{\substack{l\sim L\\(l,r)=1}}\delta_{lr}$$
$$\times\left\{\sum_{(q,al)=1}\gamma(q)\left(\sum_{\substack{m\\lmn\equiv a(qr)}}\sum_{n}\alpha(m)\beta(n)-\frac{1}{\varphi(qr)}\sum_{\substack{m\\(mn,qr)=1}}\sum_{n}\alpha(m)\beta(n)\right)\right\}$$

with $|\delta_{lr}|\leq 1$ and $\alpha(m)=f(m/M)$, $\beta(n)=f(n/N)$, $\gamma(q)=f(q/Q)$ where $f(\xi)$ is a smooth function whose graph is



By Lemma 2 we have

$$\sum_{m\equiv a\overline{ln}(qr)} \alpha(m) = \frac{\hat{\alpha}(0)}{qr} + \frac{1}{qr}\sum_{1\leq|h|\leq H} \hat{\alpha}\left(\frac{h}{qr}\right) e\left(-ah\frac{\overline{ln}}{qr}\right) + O\left(\frac{x^{\varepsilon}}{QR}\right)$$

with $H = x^{\varepsilon}QRM^{-1}$ and

$$\sum_{(m,qr)=1} \alpha(m) = \frac{\varphi(qr)}{qr}\hat{\alpha}(0) + O(x^{\varepsilon}).$$

Hence

$$\Delta_0(M,N,L,Q,R) \tag{14.2}$$
$$= \sum_{r\sim R}\sum_{\substack{l\sim L\\(qr,aln)=1}} \delta_{lr}\sum_q \gamma(q)\sum_n \beta(n)\frac{1}{qr}\sum_{1\leq|h|\leq H}\hat{\alpha}\left(\frac{h}{qr}\right)e\left(-ah\frac{\overline{ln}}{qr}\right) + O(x^{1+\varepsilon}M^{-1}).$$

Here the error term is admissible provided

$$M > x^{2\varepsilon}. \tag{14.3}$$

We have

$$e\left(-ah\frac{\overline{ln}}{qr}\right) = e\left(a^2h\frac{\overline{qr}}{aln}\right) + O(x^{\varepsilon-1})$$

so after separation of variables in $\hat{\alpha}$ by means of

$$\hat{\alpha}\left(\frac{h}{qr}\right) = q\int_{-\infty}^{\infty}\alpha(\xi q)\,e\left(\frac{\xi h}{r}\right)d\xi$$

we transform (14.2) into

$$\Delta_0(M,N,L,Q,R)$$
$$\ll \int_0^{3M/Q}\left|\sum_r\sum_l\sum_h \frac{\delta_{lr}}{r}e\left(\frac{\xi h}{r}\right)\sum_q\sum_n \gamma(q)\,\alpha(\xi q)\,\beta(n)\,e\left(a^2h\frac{\overline{qr}}{aln}\right)\right| d\xi + O(x^{1-\varepsilon})$$
$$\ll M(QR)^{-1}\mathcal{K}(N,Q,a^2H,R,|a|L) + x^{1-\varepsilon}$$

where $\mathcal{K}(C,D,N,R,S)$ is the expression from Lemma 1 with bounded coefficients. By Lemma 1

$$\mathcal{K} \ll x^{\varepsilon}\mathcal{I}\|B\|$$

where

$$\|B\|^2 \ll HLR$$

and

$$\mathcal{J}^2 \ll NL(LR+H)(N+QR)+N^2QL\sqrt{(LR+H)R}+Q^2HRL^{-1}.$$

Assume

$$N<x^{\varepsilon}QR \quad \text{and} \quad Q^2R<x. \tag{14.4}$$

Then $H<x^{\varepsilon}LR$, so

$$\mathcal{J} \ll x^{\varepsilon}(QRLN)^{1/2}(LR+NL^{1/2})^{1/2}$$

and

$$\Delta_0 \ll x^{\varepsilon}M(Qr)^{-1}(QRLN)^{1/2}(LR+NL^{1/2})^{1/2}\left(\frac{QR^2L}{M}\right)^{1/2}+x^{1-\varepsilon}$$

$$\ll x^{1-\varepsilon}$$

provided

$$LR<X^{1/2-\varepsilon}, \tag{14.5}$$

and

$$L^{1/2}R<Mx^{-\varepsilon}. \tag{14.6}$$

The conditions (14.4) can be removed without loss of generality. To this end we assume that $N<M$, so $N<x^{1/2}<x^{\varepsilon}QR$ because otherwise the result follows from (1.5). The assumption $Q^2R<x$ can be removed as in Theorem 6. Thus we have

THEOREM 7. *If* (14.5) *and* (14.6) *hold, then we have* (14.1).

As in Section 12, using Lemma 4 we can extend the result to sums $\Delta^*(M,N,L,Q,R)$ say, where the variables l,m,n are free of prime factors $p<z$ ($\leqslant z_0$). This gives

THEOREM 7*. *If* (14.5) *and* (14.6) *hold then*

$$\Delta^*(M,N,L,Q,R) \ll x(\log x)^{-A}.$$

15. Proof of Theorem 8

We first make the trivial observation that it is enough to prove the following

$$\sum_{\substack{q_1\sim Q_1 \\ (q_1q_2,a)=1}}\sum_{q_2\sim Q_2}\gamma_{q_1}\delta_{q_2}\left(\sum_{\substack{x\leqslant n<2x \\ n\equiv a(q_1q_2) \\ (n,P(z))=1}}\Lambda(n)-\frac{1}{\varphi(q_1q_2)}\sum_{\substack{x\leqslant n<2x \\ (n,q_1q_2P(z))=1}}\Lambda(n)\right) \ll x\mathcal{L}^{-A} \tag{15.1}$$

with

$$Q_1 = x^{\theta_1},\ Q_2 = x^{\theta_2}$$
$$\theta_1 < 1/3 - \varepsilon,\ \theta_2 < 1/5 - \varepsilon \tag{15.2}$$
$$\theta_1 + \theta_2 < 29/56 - \varepsilon,\ 5\theta_1 + 2\theta_2 < 2 - \varepsilon$$

and

$$\theta_1 + \theta_2 > 1/2 - \varepsilon, \tag{15.3}$$

the last constraint (15.3) being imposed because in the opposite case the estimate (15.1) follows from the Bombieri mean-value theorem (1.4). Here z is any number $<x$. We take

$$z = z_0 = \exp(\log x/\log\log x) \tag{15.4}$$

but, in fact, any number z with $N_0 < z < z_0$ (see (6.5), (A_4) and Theorem 5*) would be equally good.

Next, we apply Lemma 5 with $J=7$ for any n from (15.1). By an obvious partition of the range of summation in (2.1) into $O((\log x)^{14(A_1+1)})$ intervals we reduce the problem to estimating the following sums

$$\mathscr{E}(M_1, \ldots, M_j | N_1, \ldots, N_j) = \sum_{\substack{q_1 \sim Q_1 \\ (q_1 q_2, a) = 1}} \sum_{q_2 \sim Q_2} \gamma_{q_1} \delta_{q_2} \Delta(M_1, \ldots, M_j | N_1, \ldots, N_j; q_1 q_2, a)$$

where

$$\Delta(M_1, \ldots, M_j | N_1, \ldots, N_j; q, a)$$
$$= \sum_{\substack{m_1 \ldots m_j n_1 \ldots n_j \equiv a(q) \\ m_i \in \mathscr{M}_i,\ n_i \in \mathscr{N}_i}}^{*} \mu(m_1) \ldots \mu(m_j) - \frac{1}{\varphi(q)} \sum_{\substack{(m_1 \ldots m_j n_1 \ldots n_j, q) = 1 \\ m_i \in \mathscr{M}_i,\ n_i \in \mathscr{N}_i}}^{*} \mu(m_1) \ldots \mu(m_j).$$

Here Σ^* means that the summation is restricted to numbers free of prime factors $<z$, and $\mathscr{M}_i$, $\mathscr{N}_i$ are intervals of the type

$$\mathscr{M}_i = [(1-\Delta) M_i, M_i), \quad \mathscr{N}_i = [(1-\Delta) N_i, N_i)$$

with

$$M_1 \ldots M_j N_1 \ldots N_j = x, \quad M_1, \ldots, M_j < x^{1/7}$$

and

$$\Delta=\mathscr{L}^{-A_1}$$

where A_1 is a sufficiently large constant depending on A. Our aim is to show that

$$\mathscr{E}(M_1,\dots,M_j|N_1,\dots,N_j)\ll x\mathscr{L}^{-A_2} \tag{15.5}$$

for any $A_2>0$ (actually $A_2=A+14(A_1+1)$ suffices). To this end we shall appeal to Theorems 1, 2, 3, 4 and 5*. Accordingly we need to represent $\Delta(M_i|N_i;q,a)$ as a bilinear or trilinear form; this can be arranged once one groups the intervals $\mathscr{M}_1,\dots,\mathscr{M}_j$, $\mathscr{N}_1,\dots,\mathscr{N}_j$ into two or three disjoint sets. The resulting coefficients a_m, b_m are convolutions of the truncated Möbius function and of the constant function 1; in particular they satisfy the hypothesis of Theorems 1, 2, 3, 4 and 5*. In this case the hypothesis (A_2) (the most crucial one) is a consequence of the Siegel-Walfisz theorem.

What is left to do is of a combinatorial nature; we must show that there exists a decomposition of each product

$$M_1\dots M_jN_1\dots N_j=x,$$

into blocks in the range of the variables in our previous theorems. We begin the construction by introducing the following notation:

$$M_i=x^{\mu_i},\quad N_i=x^{\nu_i}$$

with

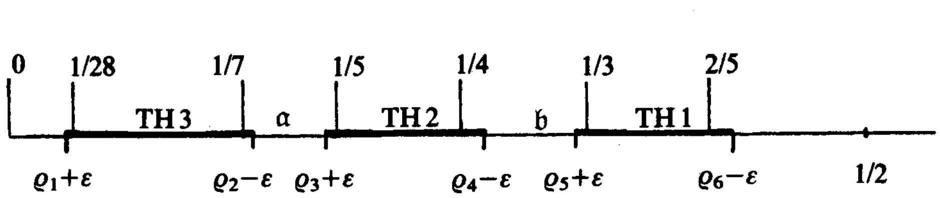
$$0\leqslant\mu_j\leqslant\dots\leqslant\mu_1\leqslant\tfrac{1}{7},\quad 0\leqslant\nu_j\leqslant\dots\leqslant\nu_1,\quad \mu_1+\dots+\mu_j+\nu_1+\dots+\nu_j=1. \tag{15.6}$$

Next, put

$$\varrho_1=2(\theta_1+\theta_2)-1,\quad \varrho_2=\tfrac{5}{6}-\tfrac{4}{3}(\theta_1+\theta_2),\quad \varrho_3=\theta_2,$$

$$\varrho_4=\min\{\tfrac{1}{2}+\tfrac{1}{2}\theta_2-\theta_1,\ \tfrac{2}{5}(1-\theta_1),\ \tfrac{1}{2}-\tfrac{3}{4}\theta_1\},\quad \varrho_5=\theta_1,\quad \varrho_6=\tfrac{1}{2}(1-\theta_2)$$

and introduce the intervals



If (15.6) has a partial sum located in one of the intervals TH1, TH2, TH3 then Theorems 1, 2, 3 are applicable respectively. Therefore, let us assume that there is no partial sum of (15.6) located in the prescribed intervals.

Notice that $\varrho_6-\varepsilon\geq\varepsilon+\max\{\theta_1,\theta_1+4\theta_2-1,\frac{1}{2}\theta_1+\theta_2,3\theta_1+4\theta_2-2\}$, so if $\nu_1>\varrho_6-\varepsilon$ then Theorem 5* is applicable. Consequently we may assume now that

$$\nu_1<\varrho_5+\varepsilon. \tag{15.7}$$

Next, notice that $2(\varrho_1+\varepsilon)<\varrho_2-\varepsilon$, so the terms of (15.6) which are $<\varrho_2-\varepsilon$ give in total τ with

$$\tau<\varrho_1+\varepsilon, \tag{15.8}$$

of course τ contains all $\mu_1,\dots,\mu_j$ and possibly some of the ν_i's. The remaining ν's must be located either in

$$\mathfrak{a}=[\varrho_2-\varepsilon,\varrho_3+\varepsilon]$$

or in

$$\mathfrak{b}=[\varrho_4-\varepsilon,\varrho_5+\varepsilon].$$

Any two numbers ν',ν'' in $\mathfrak{a}$ give $\varrho_3+\varepsilon<\nu'+\nu''<\varrho_6-\varepsilon$ so $\nu'+\nu''$ must be in $\mathfrak{b}$. Moreover, τ together with any ν''' from $\mathfrak{b}$ give $\tau+\nu'''<\varrho_5+\varepsilon+\varrho_1+\varepsilon<\varrho_6-\varepsilon$, so $\tau+\nu'''$ must be in $\mathfrak{b}$. From the above discussion it follows that we can arrange (15.6) as a sum of partial sums

$$\lambda_1+\dots+\lambda_k=1,\quad \lambda_1\geq\dots\geq\lambda_k$$

each but at most one located in $\mathfrak{b}$, the exceptional one being in $\mathfrak{a}$. In fact the exceptional one must exist because otherwise we would have $3<k<4$ which is impossible. Hence we conclude that the situation is the following:

$$k=4;\quad \lambda_1,\lambda_2,\lambda_3\in\mathfrak{b};\quad \lambda_4\in\mathfrak{a}.$$

In this situation it turns out that Theorem 4 is applicable with $N=x^{\lambda_3}$, $L=x^{\lambda_2}$. We verify the hypothesis (11.6), (11.7) and (11.8) as follows:

$$\begin{aligned}2\lambda_3+3\lambda_2&\leq\tfrac{5}{3}(\lambda_1+\lambda_2+\lambda_3)=\tfrac{5}{3}(1-\lambda_4)\\&\leq\tfrac{5}{3}(1-\varrho_2+\varepsilon)<1+\theta_1+\theta_2-\varepsilon\quad\text{because }\theta_1+\theta_2>\tfrac{1}{2}-\varepsilon.\\4\lambda_3+3\lambda_2&\leq\tfrac{7}{3}(\lambda_1+\lambda_2+\lambda_3)=\tfrac{7}{3}(1-\lambda_4)\\&\leq\tfrac{7}{3}(1-\varrho_2+\varepsilon)<2-\varepsilon\quad\text{because }\theta_1+\theta_2<29/56.\end{aligned}$$

$$\lambda_2+\lambda_3 \geqslant 2(\varrho_4-\varepsilon) > \theta_1+\theta_2+\varepsilon \quad \text{because } 5\theta_1+2\theta_2<2 \text{ and } \theta_1<1/3,\ \theta_2<1/5.$$

This completes the proof of Theorem 8.

Remark. The inequality $\theta_1+\theta_2<29/56$ cannot be improved by a refinement of the combinatorial arguments used in the proof because of the case $\nu_1=\ldots=\nu_7=1/7$.

16. Proof of Theorem 9

The proof is much the same as that of Theorem 8; the difference is that we appeal to Theorems 6 and 7* instead of 1, 2, 3, 4 and 5*.

If (15.6) has a partial sum, say λ, with

$$\theta_2+\varepsilon<\lambda<\tfrac{1}{3}(1-\theta_2)-\varepsilon \quad (\text{recall that } \theta_2<\tfrac{1}{10}) \tag{16.1}$$

then Theorem 6 is applicable. Therefore, suppose (15.6) has no partial sum in (16.1). Notice that $2(\theta_2+\varepsilon)<\frac{1}{3}(1-\theta_2)-\varepsilon$, so the terms of (15.6) which are $<\frac{1}{3}(1-\theta_2)-\varepsilon$ give in total, say τ, with

$$\tau\leqslant\theta_2+\varepsilon, \tag{16.2}$$

of course τ contains all $\mu_1,\ldots,\mu_j$ and possibly some of ν_i's. Hence we conclude that (15.6) can be partitioned as follows

$$\nu_1+\ldots+\nu_k+\tau=1,$$

with $\nu_1\geqslant\ldots\geqslant\nu_k>\frac{1}{3}(1-\theta_2)-\varepsilon$, $\theta_2+\varepsilon>\tau$. This implies that $1\leqslant k\leqslant 3$. We now apply Theorem 7* with $M=x^{\nu_1}$ and

$$N=\begin{cases} x^{\nu_2} & \text{if } k\geqslant 2 \\ 1 & \text{otherwise}\end{cases}$$

so

$$L=\frac{x}{MN}=\left\{\begin{matrix} x^{1-\nu_1-\nu_2} & \text{if } k\geqslant 2 \\ x^{\tau} & \text{otherwise}\end{matrix}\right\}\leqslant x^{1-2(1-\theta_2)/3+2\varepsilon}$$

We verify the hypotheses (14.5) and (14.6) as follows:

$$LR<x^{1-2(1-\theta_2)/3+\theta_2+3\varepsilon}<x^{1/2-\varepsilon}$$

and

$$L^{1/2}RM^{-1} < x^{1/2-(1-\theta_2)/3+\theta_2-(1-\theta_2)/3+2\varepsilon} < x^{-\varepsilon};$$

since $\theta_2 < 1/10$, the previous inequalities hold for sufficiently small ε. This completes the proof of Theorem 9.

Corollary 1 is an immediate consequence of Theorem 9 with $R=1$.

17. Proof of Theorem 10

The proof is again similar to that of Theorem 8. The difference is only in combinatorial arguments, which, due to the well factorable weights $\lambda(q)$ are more flexible. We apply Theorems 1 and 2 with

$$R = x^{-\varepsilon}N \quad \text{and} \quad Q \leqslant x^{4/7-4\varepsilon}N^{-1}.$$

Theorem 1 is applicable if

$$x^{2/7-\varepsilon} < N < x^{3/7+\varepsilon} \tag{17.1}$$

and Theorem 2 is applicable if

$$x^{1/7-\varepsilon} < N < x^{2/7+\varepsilon} \tag{17.2}$$

We now adopt the arguments of Section 15 up to the formula (15.6). If there exists a partial sum of (15.6), say λ, with

$$\tfrac{1}{7} \leqslant \lambda \leqslant \tfrac{3}{7} \tag{17.3}$$

then the results (17.1) and (17.2) complete the proof. Suppose there is no partial sum of (15.6) in (17.3). All the terms μ_i and those $\lambda_i \leqslant 1/7$ give in total, say τ, with

$$\tau \leqslant \tfrac{1}{7}. \tag{17.4}$$

Hence $\nu_1 \geqslant 3/7$ and Theorem 5* is applicable with

$$M = N_1 = x^{\nu_1} \geqslant x^{3/7}$$

and

$$Q, R \leqslant x^{2/7-2\varepsilon}.$$

This completes the proof of Theorem 10.

Corollary 2 is an immediate consequence of Theorem 10 and of the linear sieve result of [15].

Added in proof (September 26, 1985). Recently the authors have extended the methods of this paper to prove the following

THEOREM. *Let $a \neq 0$, $A>0$ and $x \geq 3$. We then have*

$$\sum_{\substack{(q,a)=1 \\ q<\sqrt{x}(\log x)^A}} \left| \pi(x;q,a) - \frac{\operatorname{li} x}{\varphi(q)} \right| \ll x \frac{(\log\log x)^B}{(\log x)^3}$$

where B is an absolute constant and the constant implied in $\ll$ depends at most on a and A.

References

[1] BOMBIERI, E., On the large sieve. *Mathematika*, 12 (1965), 201–225.

[2] DESHOUILLERS, J.-M. & IVANIEC, H., Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.*, 70 (1982), 219–288.

[3] ELLIOTT, P. D. T. A. & HALBERSTAM, H., A conjecture in prime number theory. *Symp. Math.*, 4 (*INDAM Rome*, 1968–69), 59–72.

[4] FOUVRY, E., Répartition des suites dans les progressions arithmétiques. *Acta Arith.*, 41 (1982), 359–382.

[5] — Autour du théorème de Bombieri-Vinogradov. *Acta Math.*, 152 (1984), 219–244.

[6] FOUVRY, E. & IWANIEC, H., On a theorem of Bombieri-Vinogradov type. *Mathematika*, 27 (1980), 135–172.

[7] — Primes in arithmetic progressions. *Acta Arith.*, 42 (1983), 197–218.

[8] FRIEDLANDER, J. & IWANIEC, H., On Bombieri's asymptotic sieve. *Ann. Scuola Norm. Sup. Pisa Cl. Sci (4)*, 5 (1978), 719–756.

[9] — Incomplete Kloosterman sums and a divisor problem. *Ann. Math.*, 121 (1985), 319–350.

[10] — The divisor problem for arithmetic progressions. To appear in *Acta Arith.*

[11] GALLAGHER, P. X., Bombieri's mean value theorem. *Mathematika*, 14 (1967), 14–20.

[12] HEATH-BROWN, D. R., Prime numbers in short intervals and a generalized Vaughan identity. *Canad. J. Math.*, 34 (1982), 1365–1377.

[13] HOOLEY, C., On the Barban-Davenport-Halberstam Theorem III. *J. London Math. Soc. (2)*, 10 (1975), 249–256.

[14] IWANIEC, H., The half dimensional sieve. *Acta Arith.*, 29 (1976), 67–95.

[15] — A new form of the error term in the linear sieve. *Acta Arith.*, 37 (1980), 307–320.

[16] LINNIK, JU. V., All large numbers are sums of a prime and two squares (a problem of Hardy and Littlewood), II. *Mat. Sb.* 53 (1961), 3–38; *Amer. Math. Soc. Transl.*, 37 (1964), 197–240.

[17] — *The dispersion method in binary additive problems*. Amer. Math. Soc., Providence, 1963.

[18] MOTOHASHI, Y., An induction principle for the generalization of Bombieri's Prime Number Theorem. *Proc. Japan Acad.*, 52 (1976), 273–275.

[19] SHIU, P., A Brun-Titchmarsh theorem for multiplicative functions. *J. Reine Angew. Math.*, 313 (1980), 161–170.

[20] VAUGHAN, R. C., An elementary method in prime number theory. *Acta Arith.*, 37 (1980), 111–115.

[21] VINOGRADOV, A. I., On the density hypothesis for Dirichlet *L*-functions. *Izv. Akad. Nauk SSSR Ser. Math.*, 29 (1965), 903–934; correction *ibid.* 30 (1966), 719–720.

[22] WOLKE, D., Über mittlere Verteilung der Werte zahlentheoretischer Funktionen auf Restklassen I. *Math. Ann.*, 202 (1973), 1–25; II *ibid.*, 204 (1973), 145–153.

Received April 24, 1984
Received in revised form August 14, 1984