Primitive Central Idempotents in Rational Group Algebras

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$\mathbb{Q}[G]$: the rational group algebra of G.

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Aim of the talk

In this talk, I give a survey of the known results so far on computation of the primitive central idempotents of the rational group algebra $\mathbb{Q}[G]$ of a finite group G and also some of my recent results (with Passi) in this area.

Classical method of determining primitive central idempotents of $\mathbb{Q}[G]$.

The classical method is in two steps

(i) First compute the primitive central idempotents of the complex group algebra $\mathbb{C}[G]$ using the character table of G.

(ii) Then compute the primitive central idempotents of $\mathbb{Q}[G]$ using a group action.

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Primitive central idempotents of $\mathbb{C}[G]$

The set $Irr(G)$ of complex irreducible characters of G $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{C}[G]$

Given $\chi \in Irr(G)$,

$$e(\chi) = \frac{1}{o(G)} \sum_{g \in G} \chi(g^{-1})g$$

is a primitive central idempotent of $\mathbb{C}[G]$, called the primitive central idempotent associated with χ.

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Primitive central idempotents of $\mathbb{Q}[G]$, from those of $\mathbb{C}[G]$.

- $Aut(\mathbb{C})$ acts on $\mathbb{C}[G]$ by acting on the coefficients. i.e.

$$\sigma.\sum a_g g = \sum \sigma(a_g)g, \quad (\sigma \in Aut(\mathbb{C}), \sum a_g g \in \mathbb{C}[G]).$$

- $Aut(\mathbb{C})$ acts on $Irr(G)$ by

$$\sigma.\chi = \sigma o \chi, \quad (\sigma \in \mathbb{A}, \chi \in Irr(G)).$$

Orbit of both χ and $e(\chi)$ can be obtained by applying the elements of $Gal(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively.

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The set of distinct orbits of $Irr(G)$ under the action of $Aut(\mathbb{C})$. $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{Q}[G]$

Corresponding to an orbit of χ

$$e_{\mathbb{Q}}(\chi) = \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi)$$

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Idempotents from subgroups.

Let H be a subgroup of a finite group G. Then

$$\hat{H} = \frac{1}{o(H)} \sum_{h \in H} h$$

is an idempotent of $\mathbb{Q}[G]$, called the idempotent determined by H.

$\hat{H}$ is central if and only if H is normal in G.

Note that $\hat{G}$ is always a primitive central idempotent of $\mathbb{Q}[G]$, which we call the trivial primitive central idempotent of $\mathbb{Q}[G]$

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A Question.

Is it possible to obtain primitive central idempotents of $\mathbb{Q}[G]$ from the subgroup structure of G?

We see progress in this direction

Sehgal(1978), Topics in Group Rings, Proposition 1.16

It was shown by Sehgal that if G is a finite abelian group, then the primitive of $\mathbb{Q}[G]$ can be written as a linear combination of the idempotents of the form $\hat{H}$, $H \leq G$.

Later Jespers Leal and Milies gave explicit description of the primitive (central) idempotents of $\mathbb{Q}[G]$, G a finite abelian group.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is a cyclic group of prime power order (Jespers, Leal and Milies)

Let $G =< x >$ be a cyclic group of order p^m, $m \geq 1$, p a prime. Let

$$G = G_0 \supseteq G_1 \supseteq ... \supseteq G_m = 1$$

be the descending chain of all the subgroups of G; $G_i =< x^{p^i} >$, then the primitive (central) idempotents of the rational group algebra $\mathbb{Q}[G]$ are

$$e_0 = \hat{G} \quad and \quad e_i = \hat{G}_i - \hat{G_{i-1}},\ 1 \leq i \leq m.$$

If G is any finite abelian group, then by decomposing G as a product of cyclic $p-$ groups, the primitive central idempotents of $\mathbb{Q}[G]$ can be obtained.

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Let G is an Abelian p-group. If H is a subgroup of G such that G/H is cyclic and L/H is the minimal subgroup of G/H, then

$$e_H = \widehat{H} - \widehat{L}$$

is a primitive central idempotent of $\mathbb{Q}[G]$ with $\mathbb{Q}[G]e_H \cong \mathbb{Q}(\zeta_d)$, where $d = o(G/H)$ and ζ_d is primitive dth root of unity.

Moreover any non trivial primitive central idempotent $\mathbb{Q}[G]$ is of this type.

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Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G)$:= the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

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- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
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Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$ associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of $\varepsilon(K,H)$)

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My Recent Joint Work with I.B.S.Passi

Definition

A complex irreducible character χ of a finite group G, with an affording representation ρ, is defined to have the property $\mathcal{P}$ if for all $g \in G$, either $\chi(g) = 0$ or all the eigen-values of $\rho(g)$ have the same order.

We have derived explicit expression for the primitive central idempotent of $\mathbb{Q}[G]$ associated with a complex irreducible character having the property $\mathcal{P}$. Several consequences of this result are then obtained.

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Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(*i*) *All the eigen-values of* $\rho(g)$ *are of the same order.*

(*ii*) $\overline{\rho}$ *maps all primitive central idempotents of the rational group algebra* $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ *to zero, except the idempotent* $\epsilon(\langle \ker(\chi)g \rangle, 1)$, *which gets mapped to the identity matrix .*

(*ii*) $\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ *is a sum of faithful irreducible characters of* $\langle \ker(\chi)g \rangle$.

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Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(*i*) *All the eigen-values of* $\rho(g)$ *are of the same order.*

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Theorem 1 (Bakshi-Passi, 2010)

Let G be a finite group and χ a complex irreducible character of a group G satisfying the property $\mathcal{P}$. Then the primitive central idempotent $e_{\mathbb{Q}}(\chi)$ of $\mathbb{Q}[G]$ associated with χ is given by

$$e_{\mathbb{Q}}(\chi) = \frac{1}{\sum_{g\in G,\chi(g)\neq 0}\left(\frac{\mu(d(g))}{\phi(d(g))}\right)^2}\sum_{g\in G,\,\chi(g)\neq 0}\frac{\mu(d(g))}{\phi(d(g))}g,$$

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key Lemmas

Lemma 1

Let ζ be a primitive n-th root of unity, $n \geq 1$. Then

$$\sum_{1 \leq i \leq n,\, (i,\, n)=1} \zeta^i = \mu(n).$$

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Let G be a group of order n and ζ be a primitive n-th root of unity. If $\chi \in \mathrm{Irr}(G)$ and $g \in G$ are such that all the eigen-values of $\rho(g)$ have the same order, then

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Proof of the Theorem

Let ζ be a primitive o(G)-th root of unity. From the classical method used to compute the primitive central idempotents of $\mathbb{Q}[G]$ associated with χ, we have

$$\begin{aligned} e_{\mathbb{Q}}(\chi) &= \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \\ &= \frac{\chi(1)}{o(G)} \sum_{g \in G} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \\ &= \frac{\chi(1)}{o(G)} \sum_{g \in G, \chi(g) \neq 0} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \end{aligned}$$

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Since $e_{\mathbb{Q}}(\chi)^2 = e_{\mathbb{Q}}(\chi)$, we obtain, by comparing the coefficient of 1 on both sides of this equation we get that

$$\frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} = \frac{1}{\sum_{g\in G,\chi(g)\neq 0}\left(\frac{\mu(d(g))}{\phi(d(g))}\right)^2},$$

which completes the proof of our theorem. □

Some Consequences

Notation.

Let $H \trianglelefteq G$, and $K/H = \langle Ha \rangle$ a cyclic subgroup of G/H.

- If $K \neq H$ and $o(K/H) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, p_i's distinct primes, α_i's ≥ 1, we define

$$E_{H,\,K} := \frac{(p_1 - 1)(p_2 - 1) \cdots (p_n - 1)}{o(H) p_1 p_2 \dots p_n} \sum_{g \in L} \frac{\mu(d(g))}{\phi(d(g))} g,$$

  where, for any $g \in L$, $d(g)$ is the order of g modulo H and L/H is the subgroup of K/H of order $p_1 p_2 \dots p_n$.

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$\chi \in \text{Irr}(G)$,

$Z(\chi)/\ker(\chi) := Z(G/\ker(\chi))$.

It is known that $Z(G/\ker(\chi))$ *is cyclic and* $\chi(1)^2 \leq [G : Z(\chi)]$.

We determine the primitive central idempotent of $\mathbb{Q}[G]$ associated with χ provided $\chi(1)^2 = [G : Z(\chi)]$.

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Corollary 1

Let χ be an irreducible character of a group G of degree $\sqrt{[G : Z(\chi)]}$.

Then

$$e_{\mathbb{Q}}(\chi) = E_{\ker(\chi),\, Z(\chi)}.$$

If, in addition, $o(Z(\chi)/\ker(\chi)) = p^k$ for some prime p and $k \geq 1$, then

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Corollary 2

If $\chi \in \mathrm{Irr}(\chi)$ with $\chi(1)^2 = [G : Z(G)]$, then $e_{\mathbb{Q}}(\chi) = E_{\mathrm{ker}(\chi), Z(G)}$.

Corollary 3

If $\chi \in \mathrm{Irr}(G)$ such that $G/Z(\chi)$ is abelian, then $E_{\ker(\chi),Z(\chi)}$ is the primitive central idempotent of $\mathbb{Q}[G]$ associated with χ.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
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Extra special p-groups

Recall that a finite p-group G is said to be extra special if

(i) $G' = Z(G)$,

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It is known that every extra special p-group has order p^{2n+1} for some positive integer n and the irreducible complex representations of an extra special p-group G of order p^{2n+1} are given as follows:

(i) There are exactly p^{2n} irreducible representations of dimension 1; these representations are just the ones corresponding to the representations of the abelian group G/G'.

(ii) There are exactly $p-1$ faithful irreducible characters χ_i of degree p^n, which vanish outside $Z(G)$ and satisfy $\chi_i|_{Z(G)} = p^n\lambda_i$, λ_i a faithful linear character of $Z(G)$.

Note that all of these characters have the property $\mathcal{P}$.

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If χ is one of the linear characters as listed in (i) above and $H = \ker(\chi)$, then, G/H is cyclic and by Theorem 1 and arguing as in Corollary 1, it follows that

$$e_{\mathbb{Q}}(\chi) = \begin{cases} \hat{G} & \textit{if } \ker(\chi) = G \\ \hat{H} - \hat{L} & \textit{if } \ker(\chi) \neq G, \end{cases}$$

where L/H is the subgroup of G/H of order p. Also we note that for any normal subgroup H of G with G/H cyclic, there is a linear character in the list (i) above with $\ker(\chi) = H$.

If χ is any of the $p-1$ faithful irreducible characters as listed in (ii) above, then by Theorem 1, it turns out that $e_{\mathbb{Q}}(\chi) = 1 - Z(\hat{G})$. Note that the primitive central idempotents $\hat{G}$, $1 - Z(\hat{G})$, $\hat{H} - \hat{L}$, where H runs through all proper subgroups of G such that G/H is cyclic and L/H is the subgroup of G/H of order p, are all distinct. Therefore, we have the following:

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Theorem 2

If G is an extra special p-group, then all the primitive central idempotents of $\mathbb{Q}[G]$ are given by

$$\hat{G},\ 1 - Z(\hat{G})\ \textit{and}\ \hat{H} - \hat{L},$$

where H runs through all the proper normal subgroups H of G with G/H cyclic and L/H is the unique subgroup of G/H of order p.

CM_{p-1} groups

Recall that a p-group G is called a CM_{p-1}-group if every proper normal subgroup H of G with $Z(G/H)$ cyclic is the kernel of exactly $p-1$ irreducible characters of G.

It is known that every non-principal irreducible character of a CM_{p-1}-group G satisfies $\chi(1)^2 = [G : Z(\chi)]$.

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Thus as an immediate consequence of Corollary 1, we have the following :

Theorem 3

If G is a CM_{p-1}-group, then all the primitive central idempotents of $\mathbb{Q}[G]$ are given by

$$\hat{G} \text{ and } \hat{H} - \hat{L},$$

where H runs over all proper normal subgroups of G with $Z(G/H)$ cyclic and L/H is the unique subgroup of $Z(G/H)$ of order p.

Nilpotent groups of class ≤ 2

For an abelian group G, it is known that there is 1-1 correspondence between the primitive (central) idempotents of $\mathbb{Q}[G]$ and the (normal) subgroups H of G such that G/H is cyclic; the following theorem extends this result to nilpotent groups of class ≤ 2.

Theorem 4 Let G be a nilpotent group of class ≤ 2. Then there is 1-1 correspondence between the primitive central idempotents of $\mathbb{Q}[G]$ and the normal subgroups H of G such that $Z(G/H)$ is cyclic. Moreover, for a normal subgroup H of G with $Z(G/H)$ cyclic, $E_{H,Z}$ is the primitive central idempotent of $\mathbb{Q}[G]$ which is associated with it, where $Z/H = Z(G/H)$.

Nilpotent groups of class ≤ 2

For an abelian group G, it is known that there is 1-1 correspondence between the primitive (central) idempotents of $\mathbb{Q}[G]$ and the (normal) subgroups H of G such that G/H is cyclic; the following theorem extends this result to nilpotent groups of class ≤ 2.

Theorem 4 Let G be a nilpotent group of class ≤ 2. Then there is 1-1 correspondence between the primitive central idempotents of $\mathbb{Q}[G]$ and the normal subgroups H of G such that $Z(G/H)$ is cyclic. Moreover, for a normal subgroup H of G with $Z(G/H)$ cyclic, $E_{H,Z}$ is the primitive central idempotent of $\mathbb{Q}[G]$ which is associated with it, where $Z/H = Z(G/H)$.

Monomial irreducible characters

K be a subgroup of a group G

χ_K : a linear character on K

χ_K^G : the induced character on G (such a character of G is called a monomial character)

If $\chi_K^G \in \mathrm{Irr}(G)$, Oliveri, Rio and Simon(2004) obtained an expression of the primitive central idempotent $e_{\mathbb{Q}}(\chi_K^G)$ of $\mathbb{Q}[G]$ associated with χ_K^G.

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Here is a more simplified expression of $e_{\mathbb{Q}}(\chi_K^G)$, when K and $\ker(\chi_K)$ are both normal subgroups of G.

Theorem 5 Let χ_K^G be a monomial irreducible character of a group G. If K and $H = \ker(\chi_K)$ are both normal in G, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,K}$$

. If, in addition, $o(K/H) = p^k$ for some prime p and $k \geq 1$, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,K} = \hat{H} - \hat{L},$$

where L/H is the unique subgroup of K/H of order p.

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Shoda Pair, (Olivieri, Rio and Simon, 2004)

A pair (H, K) of subgroups of G is called a Shoda pair if it satisfies the following conditions:

(i) $H \trianglelefteq K$,

(ii) K/H *is cyclic, and*

(iii) *if* $g \in G$ *and* $[K, g] \cap K \subseteq H$, *then* $g \in K$.

By Shoda's Theorem (Shoda, 1933), if χ is a linear character of a subgroup K of G with kernel H, then the induced character χ^G is irreducible if and only if (H, K) is a Shoda pair (Olivieri, Rio and Simon, 2004).

As an immediate consequence of Theorem 5, we have the following:

Corollary

If (H, K) is a Shoda pair in G with $H, K \trianglelefteq G$, then $E_{H, K}$ is a primitive central idempotent of $\mathbb{Q}[G]$.

It may be noted that our expressions for primitive central idempotents in these results are quite simple and, as such, may possibly be of help in further studies.

THANK YOU