PRIMITIVE COLLECTIONS AND TORIC VARIETIES

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Abstract. This paper studies Batyrev's notion of primitive collection. We use primitive collections to characterize the nef cone of a quasi-projective toric variety whose fan has convex support, a result stated without proof by Batyrev in the smooth projective case. When the fan is non-simplicial, we modify the definition of primitive collection and explain how our definition relates to primitive collections of simplicial subdivisons. The paper ends with an open problem.

Introduction. Let X be the toric variety of a fan Σ. When X is smooth and projective, Batyrev [1] defines a collection $\{\rho_1, \dots, \rho_k\}$ of 1-dimensional cones of Σ to be a *primitive collection* provided it does not span a cone of Σ but every proper subset does. Each primitive collection gives a *primitive inequality*, and one of the nice results of [1] states that the nef cone of X is defined by the primitive inequalities. For a proof, Batyrev cited the work of Oda and Park [12] and Reid [14], without giving details.

The survey article [3] by the first author notes that Batyrev's theorem applies to simplicial projective toric varieties. Casagrande [2] and Sato [15] explain how primitive collections relate to Reid's paper [14], and Kresch [9] gives a proof in the smooth case. However, a complete proof of Batyrev's result in the simplicial case has never appeared in print. In this paper, we give two proofs of Batyrev's theorem, one based on [9] and the other on [14]. We also extend the definition of primitive collection to the non-simplicial case and show that primitive collections still have the required properties. Our results apply to all quasi-projective toric varieties whose fans have convex support of maximal dimension.

NOTATION. We use standard notation and terminology for toric varieties. Let N and $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ be dual lattices of rank n with associated real vector spaces $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$.

Let $X = X_\Sigma$ be a toric variety of a fan Σ in $N_{\mathbf{R}} \cong \mathbf{R}^n$. We always assume that the support $|\Sigma|$ of Σ is convex of dimension n. Hence all maximal cones have dimension n.

Given Σ, $\Sigma(k)$ denotes the set of k-dimensional cones of Σ, and $\Sigma(k)^\circ$ is the subset of $\Sigma(k)$ consisting of k-dimensional cones not lying on the boundary of $|\Sigma|$. An *interior wall* is an element of $\Sigma(n-1)^\circ$. Also, $\sigma(k)$ denotes the set of k-dimensional faces of a cone σ.

We use the convention that ρ will denote both an element of $\Sigma(1)$ and its primitive generator in N. The torus-invariant divisor associated to ρ is denoted D_ρ.

Also recall that a piecewise linear function ϕ can be represented by giving $m_\sigma \in M_{\mathbf{R}}$ for each $\sigma \in \Sigma(n)$, i.e., $\phi(u) = \langle m_\sigma, u \rangle$ if $u \in \sigma$. We define $\mathrm{PL}(\Sigma)$ as the vector space of all piecewise linear functions on Σ. The function ϕ is well-defined in $\mathrm{PL}(\Sigma)$ if and only if the following statement holds: if τ is an interior wall and σ, σ' are the n-dimensional cones on each side of τ, then $m_\sigma - m_{\sigma'} \in \tau^\perp$. The support function ϕ of a torus-invariant Cartier divisor D satisfies $D = \sum_\rho \phi(\rho) D_\rho$. Note the absence of minus signs.

For us, ϕ is *convex* if and only if $\phi(u) + \phi(v) \geq \phi(u + v)$ for all $u, v \in |\Sigma|$. We also define $\mathrm{CPL}(\Sigma) \subset \mathrm{PL}(\Sigma)$ to be the cone consisting of all convex piecewise linear functions on Σ. A function $\phi \in \mathrm{PL}(\Sigma)$ is *strictly convex* when $\phi(u) + \phi(v) > \phi(u + v)$ for all $u, v \in |\Sigma|$ not lying in the same cone of Σ. The toric variety X is quasi-projective if and only if there exists a strictly convex $\phi \in \mathrm{PL}(\Sigma)$. When this happens, the interior of $\mathrm{CPL}(\Sigma)$ is nonempty and consists of all strictly convex piecewise linear functions in $\mathrm{PL}(\Sigma)$.

OUTLINE OF THE PAPER.   In Section 1, we give a new definition of primitive collection and state our main theorem. We also recall the nef and Mori cones and review the description of the Mori cone in terms of the wall relations coming from interior walls. In Section 2, we prove Batyrev's theorem in the simplicial case, and then we treat the non-simplicial case in Section 3. This section also studies how primitive collections for Σ relate to primitive collections for a simplicial subdivision Σ' of Σ. The final section of the paper explores an open problem dealing with the quasi-projective hypothesis.

1. Primitive collections and the main theorem.

1.1. Primitive collections. The nef cone $\mathrm{Nef}(X)$ of X is the quotient of the cone $\mathrm{CPL}(\Sigma) \subset \mathrm{PL}(\Sigma)$ by all linear functions on Σ. Thus

$$\mathrm{Nef}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}} = \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{R} \,.$$

Here is the central definition of our paper.

DEFINITION 1.1.   A subset $\{\rho_1, \ldots, \rho_k\} \subset \Sigma(1)$ is called a *primitive collection* for Σ if $\{\rho_1, \ldots, \rho_k\}$ is not contained in a single cone of Σ but every proper subset is.

REMARK 1.2.   In the smooth projective case, Batyrev defined $\{\rho_1, \ldots, \rho_k\} \subset \Sigma(1)$ to be a primitive collection when $\{\rho_1, \ldots, \rho_k\}$ does not generate a cone of Σ but every proper subset does. When Σ is smooth or more generally simplicial, this is clearly equivalent to Definition 1.1.

DEFINITION 1.3.   Let $\{\rho_1, \ldots, \rho_k\}$ be a primitive collection. We say that $\phi \in \mathrm{PL}(\Sigma)$ satisfies the *primitive inequality for* $\{\rho_1, \ldots, \rho_k\}$ if

$$\phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k) \,.$$

If $\phi \in \mathrm{PL}(\Sigma)$ is convex, i.e., if ϕ is in $\mathrm{CPL}(\Sigma)$, then ϕ clearly satisfies the primitive inequality for every primitive collection. In other words,

(1)
$$\mathrm{CPL}(\Sigma) \subset \{\phi \in \mathrm{PL}(\Sigma)\,;\, \phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma \} \,.$$

The main result of our paper is that the inclusion (1) is in fact an equality, i.e., the nef cone is defined by the primitive inequalities. Here is the precise statement.

THEOREM 1.4 (Main theorem). *Let X be a quasi-projective toric variety coming from the fan Σ in $N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$. If $|\Sigma|$ is convex of dimension n, then*

$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma)\,;\,\phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k)$$
$$\textit{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \textit{ for } \Sigma\}\,.$$

Section 2 will prove Theorem 1.4 when X is simplicial and Section 3 will treat the non-simplicial case.

1.2. The Mori cone. The proof of the main theorem will use extremal rays. Hence we need recall the Mori cone of a toric variety. Although this material is well-known to experts, we include many details since the results we need do not appear explicitly in the literature. We begin with the exact sequence

$$0 \to M_{\boldsymbol{R}} \to \mathrm{PL}(\Sigma) \to \mathrm{Pic}(X)_{\boldsymbol{R}} \to 0$$

which dualizes to

$$0 \to N_1(X) \to \mathrm{PL}(\Sigma)^* \to N_{\boldsymbol{R}} \to 0$$

where $\mathrm{PL}(\Sigma)^*$ denotes the dual of $\mathrm{PL}(\Sigma)$ and $N_1(X)$ is the dual of $\mathrm{Pic}(X)_{\boldsymbol{R}}$. The cone $\mathrm{CPL}(\Sigma) \subset \mathrm{PL}(\Sigma)$ contains the image of $M_{\boldsymbol{R}}$ and hence dualizes to a cone

$$\mathrm{NE}(X) = \mathrm{CPL}(\Sigma)^{\vee} \subset N_1(X) \subset \mathrm{PL}(\Sigma)^*\,.$$

We call $\mathrm{NE}(X)$ the *Mori cone* of X. When X is quasi-projective, $\mathrm{Nef}(X)$ has maximal dimension in $\mathrm{Pic}(X)_{\boldsymbol{R}}$, so that the Mori cone $\mathrm{NE}(X) \subset N_1(X)$ is strongly convex. The unique minimal generators of the Mori cone are called *extremal ray generators*.

We now review the combinatorial description of $\mathrm{NE}(X)$ in terms of the interior walls of Σ. The basic observation is that relations among elements of $\Sigma(1)$ give elements of $N_1(X)$. The map $\phi \in \mathrm{PL}(\Sigma) \mapsto (\phi(\rho))_\rho \in \boldsymbol{R}^{\Sigma(1)}$ gives a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\boldsymbol{R}} & \longrightarrow & \boldsymbol{R}^{\Sigma(1)} & \longrightarrow & A_{n-1}(X)_{\boldsymbol{R}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_{\boldsymbol{R}} & \longrightarrow & \mathrm{PL}(\Sigma) & \longrightarrow & \mathrm{Pic}(X)_{\boldsymbol{R}} & \longrightarrow & 0
\end{array}
$$

where $A_{n-1}(X)$ is the Chow group of $(n-1)$-cycles modulo rational equivalence. This dualizes to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n-1}(X)_{\boldsymbol{R}}^* & \longrightarrow & \boldsymbol{R}^{\Sigma(1)*} & \longrightarrow & N_{\boldsymbol{R}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \uparrow & & \\
0 & \longrightarrow & N_1(X) & \longrightarrow & \mathrm{PL}(\Sigma)^* & \longrightarrow & N_{\boldsymbol{R}} & \longrightarrow & 0\,.
\end{array}
$$

In the top row, the map $\boldsymbol{R}^{\Sigma(1)*} \to N_{\boldsymbol{R}}$ sends the standard basis element e_ρ to $\rho \in N$. Thus $A_{n-1}(X)_{\boldsymbol{R}}^*$ can be interpreted as all linear relations among the ρ's in $\Sigma(1)$, and the surjective

map $A_{n-1}(X)^*_R \to N_1(X)$ shows that every element of $N_1(X)$ comes from a linear relation among the ρ's in $\Sigma(1)$.

Interior walls of Σ give the following linear relations. Given an interior wall τ, let σ and σ' be the n-dimensional cones on each side of τ, i.e., $\tau = \sigma \cap \sigma'$. Pick $n - 1$ linearly independent vectors $\rho_1, \ldots, \rho_{n-1}$ in $\tau(1)$ and pick vectors $\rho_n \in \sigma(1) \setminus \tau(1)$, $\rho_{n+1} \in \sigma'(1) \setminus \tau(1)$. Then there is a nontrivial relation

$$(2) \qquad \sum_{i=1}^{n+1} a_i \rho_i = 0, \quad a_i \in \boldsymbol{Q}, \; a_n, a_{n+1} > 0,$$

where the final condition holds since ρ_n and ρ_{n+1} lie on opposite sides of the wall. Hence the coefficients $a_1, \ldots, a_{n+1}$ are unique up to multiplication by a positive constant. Let $a_\tau \in \boldsymbol{R}^{\Sigma(1)*}$ have components $(a_\tau)_{\rho_i} = a_i$ for $i = 1, \ldots, n + 1$ and $(a_\tau)_\rho = 0$ otherwise. Using the above diagram, we see that $a_\tau \in A_1(X)_R$.

DEFINITION 1.5.   Depending on the context, we use the term *wall relation* to refer to the equation (2), the vector $a_\tau \in A_{n-1}(X)^*_R$, or its image $l_\tau \in N_1(X)$.

Notice that in the non-simplicial case, a given wall can have many choices for the $\rho_1, \ldots, \rho_{n+1}$ in the wall relation (2), while in the simplicial case, $\rho_1, \ldots, \rho_{n+1}$ are uniquely determined by the wall.

THEOREM 1.6.   *Let Σ be a fan in $N_R \cong \boldsymbol{R}^n$ with convex support of dimension n.*

1.   *For $\tau \in \Sigma(n-1)^\circ$, the different choices of the wall relation* (2) *all give the same $l_\tau \in N_1(X)$ up to a positive constant.*

2.   *The Mori cone in $N_1(X)$ is given by*

$$\mathrm{NE}(X) = \sum_{\tau \in \Sigma(n-1)^\circ} \boldsymbol{R}_{\geq 0} l_\tau.$$

PROOF.   This follows from Oda and Park [12, Thm. 2.3]. We give the details since their definition of l_τ differs from ours.

Let $\tau = \sigma \cap \sigma'$ and pick a wall relation $\sum_{i=1}^{n+1} a_i \rho_i = 0$, $a_n, a_{n+1} > 0$ as in (2). Rescaling by a positive constant, we may assume $a_{n+1} = 1$, so that

$$(3) \qquad \rho_{n+1} = -a_1 \rho_1 - \cdots - a_n \rho_n.$$

Then $l_\tau \in \mathrm{PL}(\Sigma)^*$ is the linear functional on $\mathrm{PL}(\Sigma)$ given by

$$l_\tau(\phi) = a_1 \phi(\rho_1) + \cdots + a_{n-1} \phi(\rho_{n-1}) + a_n \phi(\rho_n) + \phi(\rho_{n+1})$$
$$= a_1 \langle m_{\sigma'}, \rho_1 \rangle + \cdots + a_{n-1} \langle m_{\sigma'}, \rho_{n-1} \rangle + a_n \langle m_\sigma, \rho_n \rangle + \langle m_{\sigma'}, \rho_{n+1} \rangle$$

since $\phi(u) = \langle m_{\sigma'}, u \rangle$ for $u \in \sigma'$ and $\phi(\rho_n) = \langle m_\sigma, \rho_n \rangle$ by $\rho_n \in \sigma$. However, (3) implies that

$$\langle m_{\sigma'}, \rho_{n+1} \rangle = \langle m_{\sigma'}, -a_1 \rho_1 - \cdots - a_{n-1} \rho_{n-1} - a_n \rho_n \rangle$$
$$= -a_1 \langle m_{\sigma'}, \rho_1 \rangle - \cdots - a_{n-1} \langle m_{\sigma'}, \rho_{n-1} \rangle - a_n \langle m_{\sigma'}, \rho_n \rangle.$$

Hence the above formula for $l_\tau(\phi)$ simplifies to

$$l_\tau(\phi) = a_n\langle m_\sigma, \rho_n\rangle - a_n\langle m_{\sigma'}, \rho_n\rangle = \langle m_\sigma - m_{\sigma'}, a_n\rho_n\rangle.$$

Note that $a_n\rho_n \in \sigma \setminus \tau$. Since $m_\sigma - m_{\sigma'} \in \tau^\perp$ and $(\sigma + \mathrm{span}(\tau))/\mathrm{span}(\tau)$ is 1-dimensional, it follows that up to a positive constant,

$$l_\tau(\phi) = \langle m_\sigma - m_{\sigma'}, v\rangle$$

for *any* $v \in \sigma \setminus \tau$. This proves the first part of the theorem and also shows that our l_τ agrees with the l_τ appearing in the statement of [12, Thm. 2.3]. Then the second part follows immediately from [12, Thm. 2.3]. □

1.3. **Primitive relations.**   Let $P = \{\rho_1, \ldots, \rho_k\}$ be a primitive collection for Σ. Then $\rho_1 + \cdots + \rho_k$ lies in some unique minimal cone σ of Σ. Pick a subset $S \subset \sigma(1)$ satisfying

(4) $$\rho_1 + \cdots + \rho_k = \sum_{\rho \in S} b_\rho \rho, \quad b_\rho > 0, \; S \text{ linearly independent}.$$

The equation (4) gives the vector $a_P \in \mathbf{R}^{\Sigma(1)*}$ defined by

$$(a_P)_\rho = \begin{cases} 1 & \rho \in P \setminus S \\ 1 - b_\rho & \rho \in P \cap S \\ -b_\rho & \rho \in S \setminus P \\ 0 & \text{otherwise}. \end{cases}$$

From (4), it follows that $a_P \in A_{n-1}(X)_{\mathbf{R}}^*$.

DEFINITION 1.7.   Depending on the context, we use the term *primitive relation* to refer to the equation (4), the vector $a_P \in A_{n-1}(X)_{\mathbf{R}}^*$, or its image $l_P \in N_1(X)$ under the map $A_{n-1}(X)_{\mathbf{R}}^* \to N_1(X)$.

When X is smooth, the case $\rho \in P \cap S$ cannot occur [1, Prop. 3.1], but happens frequently in the simplicial case. In this case, we bound b_ρ as follows.

LEMMA 1.8.   *The coefficients in the primitive relation* (4) *satisfy* $b_\rho < 1$ *when* $\rho \in P \cap S$. *Hence* P *is determined by the positive entries of* a_P.

PROOF.   Suppose $\rho_1 \in S$ and $b_{\rho_1} \geq 1$. Subtracting ρ_1 from each side of (4) gives

$$\rho_2 + \cdots + \rho_k = (b_{\rho_1} - 1)\rho_1 + \sum_{\rho \in S \setminus \{\rho_1\}} b_\rho \rho.$$

Since $\rho_2, \ldots, \rho_k$ lie in a cone of Σ, this equation implies that $\rho_2, \ldots, \rho_k$ and $S \setminus \{\rho_1\}$ lie in the same cone of Σ. Adding in ρ_1 shows that P lies in a cone of Σ, which is impossible. □

The minimal cone σ containing $\rho_1 + \cdots + \rho_k$ need not be simplicial, so there may be many subsets S satisfying (4). But when there are many choices for a_P, they all give the same element $l_P \in N_1(X)$, as shown by the following proposition.

PROPOSITION 1.9.   *Let* $P = \{\rho_1, \ldots, \rho_k\}$ *be a primitive collection for* Σ *and let* $l_P \in N_1(X)$ *be defined as above. Then*:

1. *When regarded as an element of* $\mathrm{PL}(\Sigma)^*$, l_P *is the linear functional on* $\mathrm{PL}(\Sigma)$ *defined by*

$$\phi \mapsto \phi(\rho_1) + \cdots + \phi(\rho_k) - \phi(\rho_1 + \cdots + \rho_k).$$

2. $l_P \in \mathrm{NE}(X)$.

PROOF. Let $\sigma \in \Sigma$ be the smallest cone containing $\rho_1 + \cdots + \rho_k$. Since ϕ is linear on σ and $S \subset \sigma(1)$, we obtain

$$l_P(\phi) = \sum_\rho (a_P)_\rho \phi(\rho) = \sum_{i=1}^k \phi(\rho_i) - \sum_{\rho \in S} b_\rho \phi(\rho)$$
$$= \sum_{i=1}^k \phi(\rho_i) - \phi(\sum_{\rho \in S} b_\rho \rho) = \sum_{i=1}^k \phi(\rho_i) - \phi(\sum_{i=1}^k \rho_i).$$

This proves the first part of the proposition, and then the second part follows immediately from the first part and (1) since $\mathrm{NE}(X) = \mathrm{CPL}(\Sigma)^\vee$. □

We can formulate Theorem 1.4 in terms of primitive relations as follows.

PROPOSITION 1.10. *Let* X *be the toric variety of the fan* Σ *in* $N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$. *If* $|\Sigma|$ *is convex of dimension* n, *then the following are equivalent*:

1. $\mathrm{CPL}(\Sigma) = \left\{ \phi \in \mathrm{PL}(\Sigma) \, ; \, \phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k) \text{ for all primitive} \right.$
collections $\{\rho_1, \ldots, \rho_k\}$ *for* $\Sigma\}$.

2. $\mathrm{NE}(X) = \sum_P \boldsymbol{R}_{\geq 0} l_P$, *where the sum is over all primitive collections for* Σ.

PROOF. This follows easily from Proposition 1.9 and $\mathrm{NE}(X) = \mathrm{CPL}(\Sigma)^\vee$. □

The strategy for proving Theorem 1.4 in the simplicial case is the observation, implicit in [14], that every minimal generator of $\mathrm{NE}(X)$ is a primitive relation l_P for some primitive collection P. Then Theorem 1.4 for simplicial fans follows immediately from Proposition 1.10. We give the details of this argument in Section 2.

1.4. Curves and the Mori cone. An interior wall τ gives a complete torus-invariant curve $V(\tau) \cong \boldsymbol{P}^1$ in X. Let

$$c_\tau : \mathrm{Pic}(X)_{\boldsymbol{R}} \to \boldsymbol{R}$$

denote the linear functional that sends an $\boldsymbol{R}$-Cartier divisor D to the intersection product $V(\tau) \cdot D$. Thus

$$c_\tau \in \mathrm{Pic}(X)_{\boldsymbol{R}}^* = N_1(X).$$

Up to a positive multiple, this gives the same class as the wall relation $l_\tau \in N_1(X)$ from Definition 1.5. Although this result is well-known to experts, we include a proof for completeness.

PROPOSITION 1.11. *Let* Σ *be a fan in* $N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$ *with convex support of dimension* n. *For each* $\tau \in \Sigma(n-1)^\circ$, $c_\tau \in N_1(X)$ *is a positive multiple of the wall relation* l_τ *appearing in Theorem* 1.6.

PROOF. When Σ is simplicial, we have $\tau(1) = \{\rho_1, \ldots, \rho_{n-1}\}$ and as in the proof of Theorem 1.6, we have the wall relation

$$a_1 \rho_1 + \cdots + a_n \rho_n + \rho_{n+1} = 0.$$

Since Σ is simplicial, the divisors D_ρ corresponding to $\rho \in \Sigma(1)$ are $\boldsymbol{Q}$-Cartier, so that $V(\tau) \cdot D_\rho$ is defined. By [14, (2.7)], we have

$$(5) \qquad V(\tau) \cdot D_\rho = \begin{cases} 0 & \rho \notin \{\rho_1, \ldots, \rho_{n+1}\} \\ a_i\, V(\tau) \cdot D_{\rho_{n+1}} & \rho = \rho_i,\ i \in \{1, \ldots, n\} \\ V(\tau) \cdot D_{\rho_{n+1}} > 0 & \rho = \rho_{n+1}\,. \end{cases}$$

The proof in [14] assumes Σ is simplicial and complete and τ is any wall; the argument applies without change when Σ is simplicial and τ is an interior wall.

For the general case, we use the well-known fact that Σ has a simplicial refinement Σ' such that $\Sigma'(1) = \Sigma(1)$ (see Corollary 3.2 and Remark 3.3). If X' is the toric variety of Σ', then we have a proper map

$$X' \xrightarrow{\pi} X\,.$$

Let τ' be an interior wall of Σ' contained in τ, and let $V(\tau')$ and $V(\tau)$ be the corresponding curves in X' and X. The induced map $\pi|_{V(\tau')} : V(\tau') \to V(\tau)$ has degree $d = [\boldsymbol{Z}(\tau \cap N) : \boldsymbol{Z}(\tau' \cap N)]$, which implies that $\pi_* V(\tau') = d\, V(\tau)$. Let D be a Cartier divisor on X. By the projection formula,

$$V(\tau) \cdot D = (1/d)\, \pi_* V(\tau') \cdot D = (1/d)\, V(\tau') \cdot \pi^* D\,.$$

If we write $D = \sum_\rho \alpha_\rho D_\rho$ on X, then $\pi^* D = \sum_\rho \alpha_\rho D_\rho$ on X' since $\Sigma'(1) = \Sigma(1)$. If $a_{\tau'} = (a_\rho)_\rho$ is the wall relation of τ' coming from (2), then up to a positive constant,

$$V(\tau') \cdot \pi^* D = \sum_\rho a_\rho \alpha_\rho$$

since Σ' is simplicial. However, the wall relation for τ' is one of the (possibly many) wall relations for τ, i.e., $a_{\tau'}$ is one of the possible choices for a_τ. Then the formula

$$V(\tau) \cdot D = \frac{1}{d} V(\tau') \cdot \pi^* D = \frac{1}{d} \sum_\rho a_\rho \alpha_\rho$$

shows (again up to a positive constant) that the class of $V(\tau)$ in $N_1(X)$ is the image of a_τ in $N_1(X)$. In other words, c_τ equals l_τ up to a positive constant, as claimed. $\square$

We conclude our discussion of the Mori cone explaining how our definition of $\mathrm{NE}(X)$ relates to the standard geometric approach. Since X need not be complete, we work in the relative context. Let U be the affine toric variety of the strongly convex cone $|\Sigma|/(|\Sigma| \cap (-|\Sigma|))$. This gives a proper toric morphism $X \to U$.

Following [10] or [14], the Mori cone of $X \to U$ is defined as follows. Let $Z_1(X/U)$ be the free group generated by irreducible curves in X that map to a point in U. Then we have a natural pairing

$$Z_1(X/U) \times \mathrm{Pic}(X) \to \boldsymbol{Z}\,.$$

By restricting to torus-invariant curves coming from interior walls, one sees easily that this pairing is nondegenerate with respect to $\mathrm{Pic}(X)$, i.e., if a Cartier divisor satisfies $C \cdot D = 0$ for

all torus-invariant curves C coming from interior walls, then $[D] = 0$ in $\mathrm{Pic}(X)$. It follows that the above pairing induces a perfect pairing

$$N_1(X/U) \times \mathrm{Pic}(X)_{\boldsymbol{R}} \to \boldsymbol{R}.$$

Thus $N_1(X/U)$ is what we call $N_1(X)$. Dropping the U from the notation is reasonable since in our situation U is determined functorially by X.

Finally, $\mathrm{NE}(X/U) \subset N_1(X)$ is the cone generated by irreducible curves in X that map to a point in U, and the *Mori cone* is its closure $\overline{\mathrm{NE}}(X/U)$ in $N_1(X)$. Then Theorem 1.6 and Proposition 1.11 easily imply that

$$\mathrm{NE}(X) = \mathrm{NE}(X/U) = \overline{\mathrm{NE}}(X/U) = \sum_{\tau \in \Sigma(n-1)^{\circ}} \boldsymbol{R}_{\geq 0} c_{\tau},$$

where $c_{\tau} \in N_1(X)$ is the class of the torus-invariant curve $V(\tau)$ associated to τ. This is the *Relative Toric Cone Theorem*.

REMARK 1.12. The Relative Toric Cone Theorem is stated for toric morphisms $X \to S$ by Matsuki [10, Thm. 14-1-4] or Reid [14, (1.7)]. As pointed out by Fujino and Sato in [6, Ex. 4.3], this fails when the torus action on S has no fixed points. They give the easy example of the projection map $X = \boldsymbol{C}^* \times \boldsymbol{P}^1 \to S = \boldsymbol{C}^*$. The fibers of this map are never torus-invariant, so that torus-invariant curves cannot generate $N_1(X/S)$. Fortunately, the Relative Toric Cone Theorem holds for our map $X \to U$ because $|\Sigma| \subset N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$ is convex of dimension n.

2. The simplicial case. A nice feature of the simplicial case is that $N_1(X) = A_{n-1}(X)^*_{\boldsymbol{R}}$. Hence an interior wall τ gives $a_{\tau} = l_{\tau}$, and a primitive collection P gives $a_P = l_P$.

2.1. Primitive collections and extremal walls.   Let τ be an interior wall of a simplicial fan Σ with $\tau(1) = \{\rho_1, \ldots, \rho_{n-1}\}$, and let ρ_n and ρ_{n+1} be the generators that are needed to span the cones on each side of the wall. The uniquely determined wall relation is

$$(6) \qquad \sum_{i=1}^{n+1} a_i \rho_i = 0, \quad a_n > 0, \ a_{n+1} = 1, \ a_i \in \boldsymbol{Q},$$

by the discussion following (2).

PROPOSITION 2.1. *Let Σ be a quasi-projective simplicial fan with convex support of dimension n. Let τ be an extremal wall, meaning that the wall relation* (6) *generates an extremal ray $l_{\tau} \in NE(X)$. Then:*

   1. $P = \{\rho_i \,; a_i > 0\}$ *is a primitive collection for Σ.*
   2. *In $\boldsymbol{R}^{\Sigma(1)*}$, the primitive relation a_P of P and wall relation a_{τ} of τ are equal up to a positive constant.*

PROOF. We give two proofs. The first is based on [9, Thm. 2.4], which assumes that X is smooth and complete. We adapt the argument to the simplicial case as follows.

We first make a useful observation about convex support functions. If ϕ is convex and $\sigma \in \Sigma$, we can change ϕ by a linear function so that

(7) $$\phi(\rho) = 0 , \quad \rho \in \sigma(1) \quad \text{and} \quad \phi(\rho) \geq 0 , \quad \rho \notin \sigma(1) .$$

Now take an extremal wall τ with wall relation (6). Consider the set

$$P = \{\rho_i ; a_i > 0\} = \{\rho ; (a_\tau)_\rho > 0\} .$$

We will prove that P is a primitive collection whose primitive relation l_P equals l_τ up to a positive constant. Recall that $a_P = l_P$ and $a_\tau = l_\tau$ since Σ is simplicial.

We first prove by contradiction that $P \not\subseteq \sigma(1)$ for all $\sigma \in \Sigma$. Suppose $P \subseteq \sigma(1)$ and take a strictly convex support function ϕ. We may assume that ϕ is of the form (7). Since $\phi(\rho) = 0$ for $\rho \in \Sigma(1)$, we have

$$l_\tau(\phi) = \sum_{\rho \notin \sigma(1)} (a_\tau)_\rho \phi(\rho) .$$

However, $\phi(\rho) \geq 0$ by (7), and $P \subseteq \sigma(1)$ implies $(a_\tau)_\rho \leq 0$ for $\rho \notin \sigma(1)$. It follows that $l_\tau(\phi) \leq 0$, which is impossible since ϕ is strictly convex and $a_\tau = l_\tau$ is in $\mathrm{NE}(X) \setminus \{0\}$. Thus no cone of Σ contains all rays in P.

It follows that some subset $Q \subseteq P$ is a primitive collection. This gives the primitive relation $a_Q = l_Q \in N_1(X)$, and we also have $a_\tau \in N_1(X)$. Let

$$\beta = a_\tau - \lambda a_Q \in N_1(X) ,$$

where $\lambda > 0$. We claim that if λ is sufficiently small, then

(8) $$\{\rho ; \beta_\rho < 0\} \subseteq \{\rho ; (a_\tau)_\rho < 0\} .$$

To prove this, suppose that $\beta_\rho < 0$ and $(a_\tau)_\rho \geq 0$. Then the definition of β forces $(a_Q)_\rho > 0$, so that ρ is in Q by Lemma 1.8. Combined with $Q \subseteq P$, we see that $(a_\tau)_\rho > 0$ by the definition of P. But we can clearly choose λ sufficiently small so that

$$(a_\tau)_\rho > \lambda(a_Q)_\rho \quad \text{whenever} \quad (a_\tau)_\rho > 0 .$$

This inequality and the definition of β imply $\beta_\rho > 0$, a contradiction.

We next claim that β is in $\mathrm{NE}(X)$. By (8), we have

$$\{\rho ; \beta_\rho < 0\} \subseteq \{\rho ; (a_\tau)_\rho < 0\} \subseteq \tau(1) ,$$

where the second inclusion follows from (6) and the definition of a_τ. Now let ϕ be convex. By (7) with $\sigma = \tau$, we may assume

$$\phi(\rho) = 0 \text{ for } \rho \in \tau(1) \quad \text{and} \quad \phi(\rho) \geq 0 \text{ for } \rho \notin \tau(1) .$$

Then

$$\beta(\phi) = \sum_{\rho \notin \tau(1)} \phi(\rho)\beta_\rho \geq 0 ,$$

where the final inequality follows since $\phi(\rho) \geq 0$ and $\beta_\rho < 0$ can happen only when $\rho \in \tau(1)$. This proves that β is in $\mathrm{NE}(X)$.

Since $a_Q = l_Q$ is in NE(X) by Proposition 1.9, the equation

$$a_\tau = \lambda a_Q + \beta$$

expresses $a_\tau = l_\tau$ as a sum of elements of NE(X). But l_τ is extremal. This forces a_Q and β to lie in the ray generated by a_τ. Since a_Q is nonzero, a_τ is a positive multiple of a_Q. In particular, they have the same positive entries. Then $P = Q$ follows from the definition of P and Lemma 1.8. This completes the first proof.

The second proof begins with the extremal wall relation (6). For $i = 1, \ldots, n+1$, set

$$\Delta_i = \text{Cone}(\rho_1, ..., \hat{\rho}_i, ..., \rho_{n+1}) .$$

In [14], Reid proved that

$$(9) \qquad \bigcup_{a_i > 0} \Delta_i = \text{Cone}(\rho_i \, ; \, i = 1, \ldots, n+1)$$

(see the lemma on [14, p. 403]) and

$$(10) \qquad \Delta_i \in \Sigma(n) \quad \text{whenever } a_i > 0$$

(see (5) and [14, Cor. 2.10]). Reid assumed that Σ is simplicial and complete. His proofs generalize to our situation without change—see [18].

Let $I = \{i \in \{1, \ldots, n+1\} \, ; \, a_i > 0\}$, so that $P = \{\rho_i \, ; \, i \in I\}$. In order to prove that P is a primitive collection, we first show that $\text{Cone}(\rho_i \, ; \, i \in I)$ is not a cone in Σ. So assume $\text{Cone}(\rho_i \, ; \, i \in I) \in \Sigma$ and consider the relation

$$\sum_{i \in I} a_i \rho_i = \sum_{i \in I^c} -a_i \rho_i \, ,$$

where the coefficients on the left are positive. Then $\sum_{i \in I} a_i \rho_i$ lies in the relative interior of the cone $\text{Cone}(\rho_i \, ; \, i \in I) \in \Sigma$, but $\sum_{i \in I^c} -a_i \rho_i$ lies in the wall $\tau \in \Sigma$ since $n, n+1 \in I$ and $a_i \leq 0$ for $i \in I^c$. It follows that $\text{Cone}(\rho_i \, ; \, i \in I) \subset \tau$, which is a contradiction since ρ_n and ρ_{n+1} do not lie in the wall.

Now we show that every proper subset of P generates a cone of Σ. Let K be any proper subset of I. Then $\text{Cone}(\rho_i \, ; \, i \in K)$ is a face of Δ_j for any $j \in I \setminus K$. But $\Delta_j \in \Sigma$ by (10). Hence $P = \{\rho_i \, ; \, i \in I\}$ is a primitive collection.

Finally, we consider the primitive relation of P, which can be written

$$(11) \qquad \sum_{i \in I} \rho_i = \sum_{\rho \in \sigma(1)} b_\rho \rho \, ,$$

where σ is the minimal cone of Σ containing $\sum_{i \in I} \rho_i$. Since $\text{Cone}(\rho_1, \ldots, \rho_{n+1}) = \bigcup_{i \in I} \Delta_i$ and $\Delta_i \in \Sigma$ by (9) and (10), it follows that (11) is a relation among $\rho_1, \ldots, \rho_{n+1}$. This relation is unique up to a nonzero constant since Σ is simplicial. Thus a_P is a nonzero multiple of a_τ, necessarily positive by Lemma 1.8 and the definition of P. $\qquad\qquad \square$

REMARK 2.2. Batyrev clearly knew this proposition, though it is not stated explicitly in [1]. Proposition 2.1 is closely related to Theorem 1.5 in Casagrande's paper [2] and appears implicitly in the remarks preceeding Proposition 2.2 in Sato's paper [15].

2.2. The main theorem. We can now prove the simplicial case of our main theorem.

THEOREM 2.3. *Let Σ be a simplicial quasi-projective fan in $N_R \cong \mathbf{R}^n$ with convex support of dimension n. Then the cone $\mathrm{CPL}(\Sigma)$ is defined by the primitive inequalities*, *i.e.*,

$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma)\,;\, \phi(\rho_1 + \cdots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$
$$\textit{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \textit{ for } \Sigma\}\,.$$

PROOF. By Proposition 1.10 it suffices to show that the primitive relations l_P generate the Mori cone. We already know that l_P is in $\mathrm{NE}(X)$ (Proposition 1.9) and that $\mathrm{NE}(X)$ is generated by the extremal wall relations l_τ (Theorem 1.6). Furthermore, $\mathrm{NE}(X)$ is generated by extremal wall relations since X is quasi-projective. Hence it suffices to show that every extremal wall relation is a primitive relation. This is what we proved in Proposition 2.1, and the theorem follows. □

Here is an example of Theorem 2.3.

EXAMPLE 2.4. Figure 1 shows the complete simplicial fan Σ in $\mathbf{R}^3$ with five minimal cone generators:

$$\rho_0 = (0, 0, -1)\,, \quad \rho_1 = (1, 1, 1)\,, \quad \rho_2 = (1, -1, 1)\,, \quad \rho_3 = (-1, -1, 1)\,, \quad \rho_4 = (-1, 1, 1)$$

and six maximal cones:

$$\sigma_1 = \mathrm{Cone}(\rho_0, \rho_1, \rho_2)\,, \quad \sigma_2 = \mathrm{Cone}(\rho_0, \rho_2, \rho_3)\,, \quad \sigma_3 = \mathrm{Cone}(\rho_0, \rho_3, \rho_4)\,,$$
$$\sigma_4 = \mathrm{Cone}(\rho_0, \rho_4, \rho_1)\,, \quad \sigma_5 = \mathrm{Cone}(\rho_1, \rho_2, \rho_4)\,, \quad \sigma_6 = \mathrm{Cone}(\rho_2, \rho_3, \rho_4)\,.$$
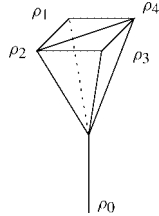

FIGURE 1. Simplicial fan in $\mathbf{R}^3$.

This fan is easily seen to be projective, and primitive collections are:

$$P_1 = \{\rho_1, \rho_3\}\,, \quad P_2 = \{\rho_0, \rho_2, \rho_4\}\,,$$

so that $\phi \in \mathrm{PL}(\Sigma)$ is convex if and only if

$$\phi(\rho_1) + \phi(\rho_3) \geq \phi(\rho_1 + \rho_3)$$
$$\phi(\rho_0) + \phi(\rho_2) + \phi(\rho_4) \geq \phi(\rho_0 + \rho_2 + \rho_4)\,.$$

To get a more concrete characterization, we use the associated primitive relations:

$$P_1 : \rho_1 + \rho_3 = \rho_2 + \rho_4$$

$$P_2 : \rho_0 + \rho_2 + \rho_4 = \frac{1}{2}\rho_2 + \frac{1}{2}\rho_4 \,.$$

Let D_i be the torus-invariant divisor associated to ρ_i. Then the divisor $D = \sum_{i=0}^{4} a_i D_i$ corresponds to the support function ϕ satisfying $\phi(\rho_i) = a_i$. It follows that D is nef if and only if

$$a_1 + a_3 \geq a_2 + a_4$$

$$a_0 + a_2 + a_4 \geq \frac{1}{2}a_2 + \frac{1}{2}a_4 \,, \quad \text{i.e.,} \quad 2a_0 + a_2 + a_4 \geq 0 \,.$$

In contrast, Σ has 9 walls, so Theorem 1.6 describes $\mathrm{NE}(X)$ using 9 generators, corresponding to 9 wall inequalities defining $\mathrm{CPL}(\Sigma) \subset \mathrm{PL}(\Sigma)$. Fortunately, these can be simplified considerably. We denote by $\tau_{i,j}$ the wall that is spanned by ρ_i and ρ_j. By abuse of notation we will also call $\tau_{i,j}$ the corresponding class in $\mathrm{NE}(X)$. One can compute that the 9 walls fall into three groups:

$$\tau_{2,4}$$
$$\tau_{1,2} \equiv \tau_{3,4} \equiv \tau_{2,3} \equiv \tau_{1,4} \equiv 4\tau_{0,1} \equiv 4\tau_{0,3}$$
$$\tau_{0,2} \equiv \tau_{0,4} \equiv 2\tau_{1,2} + 2\tau_{2,4} \,.$$

Hence $\tau_{2,4}$ and $\tau_{1,2} \equiv \cdots \equiv 4\tau_{0,3}$ give the extremal rays of the Mori cone, while $\tau_{0,2} \equiv \tau_{0,4}$ do not give an extremal ray. One can check that the primitive collection P_1 generates the same ray as $\tau_{2,4}$ and P_2 generates the same ray as $\tau_{1,2}$. $\qquad\square$

Example 2.4 is nice because there were few primitive collections. However, the following example shows that primitive collections vastly outnumber interior walls in general.

EXAMPLE 2.5. Let Σ be a complete fan in $\mathbf{R}^2$ with $r \geq 4$ minimal generators, say $\rho_1, \ldots, \rho_r$, arranged counterclockwise around the origin. Then there are r walls, all interior. One easily checks that the primitive collections are given by $P = \{\rho_i, \rho_j\}$ for $i < j$ and ρ_i, ρ_j not adjacent. Hence the fan Σ has

$$\binom{r}{2} - r = \frac{r(r-3)}{2} \,,$$

primitive collections. This is greater than the number of walls provided $r \geq 6$. $\qquad\square$

3. The non-simplicial case.

3.1. Simplicial refinements. In order to prove our main theorem in the non-simplicial case, we will need the following result on the existence of simplicial refinements with special properties.

THEOREM 3.1. *Let Σ be a quasi-projective fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ with convex support of dimension n and fix $P \subset \Sigma(1)$ such that $P \cap \sigma(1)$ is linearly independent for all $\sigma \in \Sigma$.*

Then there exists a quasi-projective simplicial refinement Σ' of Σ satisfying $\Sigma'(1) = \Sigma(1)$ such that $P \cap \sigma(1)$ generates a cone in Σ' for all $\sigma \in \Sigma$.

PROOF. Since Σ is quasi-projective, we can find $\phi \in \mathrm{CPL}(\Sigma)$ which is strictly convex. We first modify ϕ so that it takes positive values on $\Sigma(1)$. To see why this is possible, consider the cone

$$\hat{\sigma} = \mathrm{Cone}((\rho, \phi(\rho)) \,;\, \rho \in \Sigma(1)) \subset N_R \times R \,.$$

Since ϕ is strictly convex for Σ, it follows that $\hat{\sigma}$ is a strongly convex cone with minimal generators given by $(\rho, \phi(\rho))$ for $\rho \in \Sigma(1)$. Hence we can find $(m, \mu) \in M_R \times R$ such that

$$\langle m, \rho \rangle + \mu \phi(\rho) > 0 \quad \text{for all} \ \rho \in \Sigma(1) \,.$$

Replacing ϕ with $\langle m, - \rangle + \mu \phi$, we may assume $\phi(\rho) > 0$ for all $\rho \in \Sigma(1)$, as claimed.

It follows that for each $\rho \in \Sigma(1)$, there is a unique $\mathbf{v}_\rho \in \rho$ such that $\phi(\mathbf{v}_\rho) = 1$. Given $\sigma \in \Sigma$, one sees easily that

$$Q_\sigma = \{ \mathbf{v} \in \sigma \,;\, \phi(\mathbf{v}) = 1 \}$$

is a polytope with vertices $\mathbf{v}_\rho$ for $\rho \in \sigma(1)$.

To create Σ', assign a weight w_ρ to each $\rho \in \Sigma(1)$ as follows:

- For $\rho \in P$, set $w_\rho = 1$.
- For $\rho \in \Sigma(1) \setminus P$, pick $0 < w_\rho < 1$ generic. The exact meaning of generic will be explained in the course of the proof.

We will use the weights w_ρ to triangulate Q_σ following a variant of the method used in [7, p. 215, Example 1.1]. Consider

$$G_{\sigma,w} = \mathrm{Conv}(0, w_\rho \mathbf{v}_\rho \,;\, \rho \in \sigma(1)) \,.$$

Since the vectors $w_\rho \mathbf{v}_\rho$ lie on the 1-dimensional rays of σ, it is easy to see that the vertices of $G_{\sigma,w}$ consist of the origin and the points $w_\rho \mathbf{v}_\rho$ for $\rho \in \sigma(1)$. Furthermore, the faces of $G_{\sigma,w}$ not containing the origin project to a polyhedral subdivision of Q_σ. Projecting from the origin in N_R, we get a refinement Σ_σ of σ that satisfies $\Sigma_\sigma(1) = \sigma(1)$. Figure 2 on the next page shows two 3-dimensional cones σ, each with a set $P \cap \sigma(1)$ and a choice of weights giving the polytope $G_{\sigma,w}$ inside σ.

We claim that the fans Σ_σ have the following three properties:

A. $P \cap \sigma(1)$ generates a cone of Σ_σ for all $\sigma \in \Sigma$.
B. If τ is a face of $\sigma \in \Sigma$, then Σ_τ is the refinement of τ induced by Σ_σ.
C. If the w_ρ are sufficiently generic for $\rho \in \Sigma(1) \setminus P$, then Σ_σ is simplicial for all $\sigma \in \Sigma$.

Proof of A. Since ϕ is linear on σ, there is an affine hyperplane H_σ such that $Q_\sigma = H_\sigma \cap \sigma$. Since $w_\rho \leq 1$ for all ρ, the polytope $G_{\sigma,w}$ lies on the side of H_σ containing the origin, and the intersection $H_\sigma \cap G_{\sigma,w}$ is clearly the convex hull of the points $\mathbf{v}_\rho$ for $\rho \in P \cap \sigma(1)$ by the choice of the weights w_ρ. It follows easily that $P \cap \sigma(1)$ generates a cone of Σ_σ.
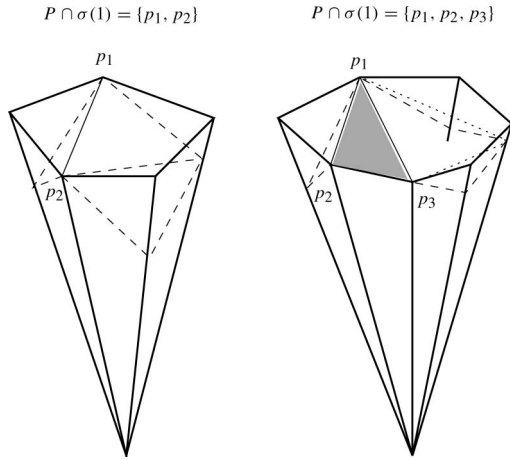
$$P \cap \sigma(1) = \{p_1, p_2\} \qquad\qquad P \cap \sigma(1) = \{p_1, p_2, p_3\}$$

FIGURE 2.   Two examples of σ, $P \cap \sigma(1)$, and a choice of weights.

Proof of B.   If τ is a face of σ, then one easily sees that

$$G_{\tau,w} = G_{\sigma,w} \cap \tau .$$

From here, B follows immediately.

Proof of C.   We may assume $\sigma \in \Sigma(n)$. For $\rho \in \sigma(1)$, write

$$\mathbf{v}_\rho = (a_1^\rho, \dots, a_n^\rho) \in \mathbf{R}^n.$$

When we have $\mathbf{v}_{\rho_1}, \mathbf{v}_{\rho_2}, \dots$, we write instead

$$\mathbf{v}_{\rho_i} = (a_1^{(i)}, \dots, a_n^{(i)})$$

and we set $w_i = w_{\rho_i}$.

Now suppose that Σ_σ is non-simplicial for some choice of weights w_ρ. This implies that $G_{\sigma,w}$ has a face F of dimension $n-1$ not containing the origin that is not an $(n-1)$-simplex. It follows that F has at least $n+1$ vertices. Pick $n+1$ vertices of F as follows. We first pick those vertices of F of the form $\mathbf{v}_\rho = w_\rho \mathbf{v}_\rho$ for $\rho \in P \cap \sigma(1)$. There are at most n such vertices since $P \cap \sigma(1)$ is linearly independent. Since $\dim(F) = n-1$, we can extend them to affinely independent vertices $w_1 \mathbf{v}_{\rho_1}, \dots, w_n \mathbf{v}_{\rho_n}$. Since F is non-simplicial, we can pick one more, $w_{n+1} \mathbf{v}_{\rho_{n+1}}$. Note that $w_{n+1} = w_{\rho_{n+1}}$ where $\rho_{n+1} \notin P$.

Now consider the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & w_1 a_1^{(1)} & \cdots & w_1 a_n^{(1)} \\ \vdots & \vdots & & \vdots \\ 1 & w_{n+1} a_1^{(n+1)} & \cdots & w_{n+1} a_n^{(n+1)} \end{pmatrix} .$$

By construction, the vectors $w_1\mathbf{v}_{\rho_1}, \ldots, w_n\mathbf{v}_{\rho_n}$ are affinely independent, but once we add the last vector, $w_1\mathbf{v}_{\rho_1}, \ldots, w_{n+1}\mathbf{v}_{\rho_{n+1}}$ are affinely dependent since they lie in a $(n-1)$-dimensional face.

It follows that this matrix has rank exactly n. Since the weights w_i are nonzero, the same is true for the matrix

$$
M = \begin{pmatrix} w_1^{-1} & a_1^{(1)} & \cdots & a_n^{(1)} \\ \vdots & \vdots & & \vdots \\ w_{n+1}^{-1} & a_1^{(n+1)} & \cdots & a_n^{(n+1)} \end{pmatrix}.
$$

The determinant of M must vanish. The resulting linear equation in the w_i^{-1} gives a necessary condition for Σ_σ to be non-simplicial. Furthermore, the affine independence of $w_1\mathbf{v}_{\rho_1}, \ldots,$ $w_n\mathbf{v}_{\rho_n}$ guarantees that w_{n+1}^{-1} actually appears in the determinant. As noted above, we also have $w_{n+1}^{-1} = w_\rho^{-1}$ for some $\rho \in \sigma(1) \setminus P$.

Since $\Sigma(n)$ is finite, we get a finite system of non-trivial linear equations in the w_ρ^{-1} for $\rho \in \sigma(1) \setminus P$ that give necessary conditions for some Σ_σ to be non-simplicial. If we pick the weights w_ρ to avoid these finitely many subspaces, the resulting subdivisions Σ_σ will be all simplicial. This completes the proof of C.

Now pick weights w_ρ that satisfy conditions A–C and consider the set of cones

$$
\Sigma' = \bigcup_{\sigma \in \Sigma} \Sigma_\sigma.
$$

Then Σ' is a fan that refines Σ by B and satisfies $\Sigma'(1) = \Sigma(1)$. Furthermore, Σ' is simplicial by C. Finally, given $\sigma \in \Sigma$, $P \cap \sigma(1)$ generates a cone of Σ' by A. Hence the proof of the theorem will be complete once we prove that Σ' is quasi-projective.

Since Σ is quasi-projective, it suffices to prove that the induced map $X_{\Sigma'} \to X_\Sigma = X$ is projective. The latter happens when Σ' has a piecewise linear function $\varphi \in \mathrm{PL}(\Sigma')$ which is strictly convex relative to Σ, meaning that for all $\sigma \in \Sigma$, $\varphi|_\sigma$ is strictly convex with respect to the subfan $\{\sigma' \in \Sigma'; \sigma' \subset \sigma\}$ (see [8, p. 27, (*)] and Theorem 10]).

We construct the desired $\varphi : |\Sigma'| = |\Sigma| \to \mathbf{R}$ by setting

$$
\varphi(\rho) = w_\rho^{-1}\phi(\rho) \quad \text{for } \rho \in \Sigma'(1) = \Sigma(1),
$$

and extending linearly on each cone $\sigma' \in \Sigma'$. This gives a well-defined function in $\mathrm{PL}(\Sigma')$ since Σ' is simplicial. Assuming ϕ is rational, we can also assume that the w_ρ are rational. Hence we can assume that φ is rational as well.

We claim that φ is strictly convex with respect to $\Sigma_\sigma = \{\sigma' \in \Sigma'; \sigma' \subset \sigma\}$ for each $\sigma \in \Sigma$. To see this, first observe that

$$
\varphi(w_\rho\mathbf{v}_\rho) = 1 \quad \text{for all } \rho \in \sigma(1),
$$

since $\phi(\mathbf{v}_\rho) = 1$. It follows that inside σ, the inequality $\varphi \leq 1$ defines

$$
G_{\sigma, w} = \mathrm{Conv}(0, w_\rho\mathbf{v}_\rho ; \rho \in \sigma(1)).
$$

The convexity of $G_{\sigma,w}$ implies that, if $u, v \in \sigma$, then

$$\varphi(u) + \varphi(v) \geq \varphi(u + v),$$

with equality if and only if u, v lie in the same cone of Σ_σ. To prove this, we may assume $u, v \neq 0$, so that

$$u = \lambda u_0, \quad v = \mu v_0, \quad \text{where } \lambda, \mu > 0 \text{ and } \varphi(u_0) = \varphi(v_0) = 1.$$

Then $(\lambda/(\lambda + \mu))u_0 + (\mu/(\lambda + \mu))v_0$ is in $G_{\sigma,w}$, so that

$$\varphi(u + v) = (\lambda + \mu)\varphi\left(\frac{\lambda}{\lambda + \mu}u_0 + \frac{\mu}{\lambda + \mu}v_0\right) \leq \lambda + \mu = \varphi(u) + \varphi(v).$$

It is equally easy to show that equality occurs exactly when u, v lie in the same cone of Σ_σ. Hence φ has the required properties, which completes the proof of the theorem. $\qquad\square$

COROLLARY 3.2. *If Σ is a quasi-projective fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ with convex support of dimension n, then there exists a quasi-projective simplicial refinement Σ' with the same 1-dimensional generators.*

PROOF. Apply Theorem 3.1 with $P = \emptyset$. $\qquad\square$

REMARK 3.3. This corollary has other proofs, including Fujino [5] (via the toric Mori program) and Thompson [17] (via stellar subdivision).

3.2. **The main theorem.** We can now prove the non-simplicial case of our main theorem.

THEOREM 3.4. *Let Σ be a non-simplicial quasi-projective fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ with convex support of dimension n. Then the cone $\mathrm{CPL}(\Sigma)$ is defined by the primitive inequalities, i.e.,*

$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma) \, ; \, \phi(\rho_1 + \cdots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma\}.$$

PROOF. By Corollary 3.2, Σ has a quasi-projective simplicial refinement Σ' satisfying $\Sigma'(1) = \Sigma(1)$. Then observe that

$$\mathrm{CPL}(\Sigma) = \mathrm{PL}(\Sigma) \cap \mathrm{CPL}(\Sigma')$$

and that

$$\mathrm{CPL}(\Sigma') = \{\phi \in \mathrm{PL}(\Sigma') \, ; \, \phi(\rho_1 + \ldots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma'\}$$

since Σ' is simplicial and quasi-projective. Hence by Theorem 2.3,

(12)
$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma) \, ; \, \phi(\rho_1 + \cdots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma'\}.$$

We divide primitive collections $P = \{\rho_1, \ldots, \rho_k\}$ for Σ' into two types:

$$\text{Type A: } P \subset \sigma(1) \text{ for some } \sigma \in \Sigma$$

$$\text{Type B: } P \not\subset \sigma(1) \text{ for all } \sigma \in \Sigma \,.$$

Note that if $\phi \in \text{PL}(\Sigma)$, then $\phi(\rho_1 + \cdots + \rho_k) = \phi(\rho_1) + \cdots + \phi(\rho_k)$ when $P = \{\rho_1, \ldots, \rho_k\}$ is a Type A primitive collection for Σ'. Hence these can be omitted in (12), so that

(13)
$$\text{CPL}(\Sigma) = \{\phi \in \text{PL}(\Sigma) \,;\, \phi(\rho_1 + \cdots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$
$$\text{for all Type B primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma'\} \,.$$

However, a Type B primitive collection P for Σ' is a primitive collection for Σ. This is easy to prove. First, P is not contained in any cone of Σ by the definition of Type B, and second, every proper subset of P is contained in a cone of Σ' and hence lies in a cone of Σ since Σ' refines Σ. It follows that

$$\phi(\rho_1 + \cdots + \rho_k) \leq \phi(\rho_1) + \cdots + \phi(\rho_k)$$

is a primitive inequality for Σ whenver $P = \{\rho_1, \ldots, \rho_k\}$ is a Type B primitive collection for Σ'.

Hence (13) shows that a subset of the primitive inequalities for Σ define $\text{CPL}(\Sigma)$ inside $\text{PL}(\Sigma)$. Using the inclusion (1), the theorem now follows immediately. $\square$

Here is an example to illustrate Theorem 3.4 and its proof.

EXAMPLE 3.5.   Figure 3 shows the complete non-simplicial fan Σ in $\boldsymbol{R}^3$ with five minimal generators:

$$\rho_0 = (0, 0, -1)\,,\quad \rho_1 = (1, 1, 1)\,,\quad \rho_2 = (1, -1, 1)\,,\quad \rho_3 = (-1, -1, 1)\,,\quad \rho_4 = (-1, 1, 1)$$

and five maximal cones:

$$\sigma_1 = \text{Cone}(\rho_0, \rho_1, \rho_2)\,,\quad \sigma_2 = \text{Cone}(\rho_0, \rho_2, \rho_3)\,,\quad \sigma_3 = \text{Cone}(\rho_0, \rho_3, \rho_4)\,,$$
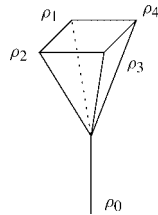$$\sigma_4 = \text{Cone}(\rho_0, \rho_4, \rho_1)\,,\quad \sigma_5 = \text{Cone}(\rho_1, \rho_2, \rho_3, \rho_4)\,.$$

FIGURE 3.   Non-simplicial fan in $\boldsymbol{R}^3$.

The fan Σ is quasi-projective, and its primitive collections are:

$$P_1 = \{\rho_0, \rho_1, \rho_3\}\,,\quad P_2 = \{\rho_0, \rho_2, \rho_4\}\,.$$

A first observation is that if we used Batyrev's definition of primitive collection in this case, we would want every proper subset of P_1 and P_2 to generate a cone of Σ. This clearly isn't true, and in fact this example has *no* primitive collections if we use Batyrev's definition. This explains why Definition 1.1 is the correct definition in the non-simplicial case.

Theorem 3.4 states that $\mathrm{CPL}(\Sigma) \subset \mathrm{PL}(\Sigma)$ is defined by the primitive inequalities coming from the primitive collections P_1 and P_2. However, the proof of the theorem shows that we need only one. To see why, consider the simplicial refinement Σ' of Σ given by subdividing non-simplicial cone σ_5 along $\mathrm{Cone}(\rho_2, \rho_4)$. This gives the fan pictured in Example 2.4. The fan Σ' has the same generators $\rho_0, \ldots, \rho_4$ as Σ, and the primitive collections for Σ' are

$$P_1' = \{\rho_1, \rho_3\}, \quad P_2 = \{\rho_0, \rho_2, \rho_4\}.$$

One easily checks that P_1' is of Type A and P_2 is of Type B in the sense defined in the proof of Theorem 3.4, and hence is a primitive collection for Σ. By (13), $\mathrm{CPL}(\Sigma)$ is defined by P_2, so that $\phi \in \mathrm{PL}(\Sigma)$ is convex if and only if

$$\phi(\rho_0) + \phi(\rho_2) + \phi(\rho_4) \geq \phi(\rho_0 + \rho_2 + \rho_4).$$

It is interesting to note that the Type A primitive collection $P_1' = \{\rho_1, \rho_3\}$ also plays an important role. The primitive relation of P_1' is

$$\rho_1 + \rho_3 = \rho_2 + \rho_4.$$

Now take $\phi \in \mathrm{PL}(\Sigma)$. As noted in the proof of Theorem 3.4, this Type A primitive collection gives the *equality*

$$(14) \qquad\qquad \phi(\rho_1) + \phi(\rho_3) = \phi(\rho_1 + \rho_3),$$

which by the above primitive relation implies

$$\phi(\rho_1) + \phi(\rho_3) = \phi(\rho_2) + \phi(\rho_4).$$

It is easy to see that this equality defines $\mathrm{PL}(\Sigma)$ inside of $\mathrm{PL}(\Sigma')$. In other words, $\phi \in \mathrm{PL}(\Sigma')$ lies in $\mathrm{PL}(\Sigma)$ if and only if it satisfies (14) coming from the Type A primitive collection for Σ'.

If we turn our attention to the other primitive collection $P_1 = \{\rho_0, \rho_1, \rho_3\}$ for Σ, then one easily sees that $\phi \in \mathrm{PL}(\Sigma)$ is convex if and only if

$$\phi(\rho_0) + \phi(\rho_1) + \phi(\rho_3) \geq \phi(\rho_0 + \rho_1 + \rho_3).$$

This follows by considering the other simplicial refinement of Σ obtained by subdividing σ_5 along $\mathrm{Cone}(\rho_1, \rho_3)$.   □

Example 3.5 has some interesting features:

- Every primitive collection for Σ comes from a Type B primitive collection for a simplicial refinement Σ' of Σ satisfying $\Sigma'(1) = \Sigma(1)$.
- For each such refinement Σ' of Σ, the Type A primitive collections for Σ' define $\mathrm{PL}(\Sigma) \subset \mathrm{PL}(\Sigma')$.

We will see below that these properties hold in general.

3.3.   Properties of primitive collections.   We begin with the following useful property of primitive collections.

PROPOSITION 3.6.   *Let Σ be a fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ such that Σ has convex support of dimension n. If P is a primitive collection for Σ, then every proper subset Q of P is linearly independent.*

PROOF.   We use induction on $|Q|$. If $|Q| = 1$ there is nothing to show. Now assume that $|Q| = k + 1$, $k \geq 1$, and that every k-element subset of Q is linearly independent.

We show that Q is linearly independent by contradiction. Hence suppose Q is linearly dependent. Then our induction hypothesis implies that the subspace span(Q) has dimension k. Define $\widetilde{\Sigma} = \{\sigma \cap \text{span}(Q) \, ; \, \sigma \in \Sigma\}$. We omit the straightforward proof that $\widetilde{\Sigma}$ is a fan in span(Q).

Now fix $\rho \in Q$ and let σ_ρ be the minimal cone of Σ containing $P \setminus \{\rho\}$. Notice that σ_ρ does not contain ρ since P is a primitive collection. Also let σ_Q be the minimal cone of Σ containing Q. The cones $\widetilde{\sigma}_Q = \sigma_Q \cap \text{span}(Q)$ and $\widetilde{\sigma}_\rho = \sigma_\rho \cap \text{span}(Q)$ are in the fan $\widetilde{\Sigma}$ and $\widetilde{\sigma}_Q \neq \widetilde{\sigma}_\rho$ since ρ is contained in $\widetilde{\sigma}_Q$ but not in $\widetilde{\sigma}_\rho$. Therefore, their intersection is at most $(k-1)$-dimensional. On the other hand, the intersection contains k linearly independent vectors

$$Q \setminus \{\rho\} \subset \widetilde{\sigma}_Q \cap \widetilde{\sigma}_\rho \, ,$$

which is a contradiction.   □

COROLLARY 3.7.   *Let Σ be a fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ such that $|\Sigma|$ is convex support of dimension n. Then every primitive collection for Σ has at most $n + 1$ elements.*

PROOF.   This follows immediately from Proposition 3.6 since any maximal proper subset $Q = P \setminus \{\rho\}$ is linearly independent and hence has at most n elements. Therefore $P = Q \cup \{\rho\}$ has at most $n + 1$ elements.   □

REMARK 3.8.   Proposition 3.6 and Corollary 3.7 are trivial in the simplicial case.

3.4.   Type A description of PL(Σ).   Let Σ be a fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ with convex support of dimension n, and let Σ' be a simplicial refinement with $\Sigma(1) = \Sigma'(1)$. Given $\sigma \in \Sigma$, let $\Sigma_\sigma = \{\sigma' \in \Sigma' \, ; \, \sigma' \subset \sigma\}$. The following convexity result will be useful.

LEMMA 3.9.   *Let σ be a non-simplicial cone in Σ and take an interior wall τ' of Σ_σ with $\tau = \sigma_1' \cap \sigma_2'$, $\sigma_1', \sigma_2' \in \Sigma_\sigma(n)$. Then $\sigma_1' \cup \sigma_2'$ is convex.*

PROOF.   Since $\tau'(1) \subset \sigma(1)$, τ' divides σ into two convex subcones σ^+, σ^- with $\tau = \sigma^+ \cap \sigma^-$. We may assume $\sigma_1' \subset \sigma^+$, $\sigma_2' \subset \sigma^-$. Given $u \in \sigma_1'$, $v \in \sigma_2'$, it follows easily that the line segment $\overline{uv}$ lies in $\sigma_1' \cup \sigma_2'$.   □

COROLLARY 3.10.   *In the situation of Lemma 3.9, let P be the two element set*

$$P = (\sigma_1'(1) \cup \sigma_2'(1)) \setminus \tau'(1) \, .$$

Thus P consists of the generators of σ_1', σ_2' not lying in the wall $\tau' = \sigma_1' \cap \sigma_2'$. Then P is a primitive collection for Σ'.

PROOF. First note that P is contained in neither σ_1' nor σ_2'. Since P is contained in the convex set $\sigma_1' \cup \sigma_2'$, it follows that P is contained in no cone of Σ'. Thus P is a primitive collection for Σ' since it has only two elements. □

As in the proof of Theorem 3.4, a primitive collection for Σ' has Type A when it is contained in a cone of Σ. Hence the primitive collection for Σ' constructed in Corollary 3.10 has Type A. The idea is that these two element primitive collections define $\mathrm{PL}(\Sigma)$ inside $\mathrm{PL}(\Sigma')$.

PROPOSITION 3.11. *Let Σ be a fan in $N_{\mathbf{R}} \cong \mathbf{R}^n$ with convex support of dimension n and let Σ' be a simplicial refinement with $\Sigma(1) = \Sigma'(1)$. Then*

$$\mathrm{PL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma') \, ; \, \phi(\rho_1 + \rho_2) = \phi(\rho_1) + \phi(\rho_2) \text{ for all}$$
$$\text{Type A primitive collections } \{\rho_1, \rho_2\} \text{ for } \Sigma'\} \, .$$

PROOF. The inclusion $\subset$ is obvious since elements of $\mathrm{PL}(\Sigma)$ are linear on cones of Σ and a Type A primitive collection is contained in such a cone.

For the opposite inclusion, take $\phi \in \mathrm{PL}(\Sigma')$ such that $\phi(\rho_1 + \rho_2) = \phi(\rho_1) + \phi(\rho_2)$ for all two element Type A primitive collections for Σ'. For each $\sigma' \in \Sigma'(n)$, there is $m_{\sigma'} \in M_{\mathbf{R}}$ such that $\phi(u) = \langle m_{\sigma'}, u \rangle$ for $u \in \sigma'$. It suffices to show that $m_{\sigma_1'} = m_{\sigma_2'}$ for cones σ_1', σ_2' that lie in the same cone σ of Σ and intersect in a wall $\sigma_1' \cap \sigma_2' = \tau'$. This is the situation of Corollary 3.10, where

$$\sigma_1'(1) \cup \sigma_2'(1) = \tau'(1) \cup \{\rho_1, \rho_2\}$$

and $P = \{\rho_1, \rho_2\}$ is a two element Type A primitive collection for Σ'. We label the elements of P so that $\rho_1 \in \sigma_1'$ and $\rho_2 \in \sigma_2'$.

Since $\sigma_1' \cup \sigma_2'$ is convex by Lemma 3.9, it contains $\rho_1 + \rho_2$. We may assume $\rho_1 + \rho_2 \in \sigma_2'$ without loss of generality. Then

$$\langle m_{\sigma_1'}, \rho_1 \rangle = \phi(\rho_1) = -\phi(\rho_2) + \phi(\rho_1 + \rho_2) = -\langle m_{\sigma_2'}, \rho_1 \rangle + \langle m_{\sigma_2'}, \rho_1 + p_2 \rangle = \langle m_{\sigma_2'}, \rho_1 \rangle \, .$$

Since $m_{\sigma_1'} - m_{\sigma_2'} \in \tau'^{\perp}$, it follows that $m_{\sigma_1'} = m_{\sigma_2'}$. This completes the proof. □

3.5. *Primitive collections supported on simplicial refinements.* In the fan Σ pictured in Figure 3 in Example 3.5, we saw that every primitive collection for Σ comes from a primitive collection for a simplicial subdivision of Σ.

In general, if Σ' is a simplicial subdivision of Σ with $\Sigma'(1) = \Sigma(1)$, we say that a primitive collection P for Σ is *supported* on Σ' if P is also a primitive collection for Σ'. We now prove that all primitive collections for Σ are supported on such simplicial subdivisions. Here is the precise result.

PROPOSITION 3.12. *Let Σ be a quasi-projective fan in $N_R \cong R^n$ with convex support of dimension n and let P be a primitive collection for Σ. Then there exists a quasi-projective simplicial refinement Σ' with $\Sigma'(1) = \Sigma(1)$ such that P is a primitive collection for Σ'.*

PROOF. By Proposition 3.6, every proper subset P is linearly independent. In particular, if $\sigma \in \Sigma$, then $P \cap \sigma(1)$ is a proper subset of P (since P is a primitive collection) and hence is linearly independent. Thus we can apply Theorem 3.1 to obtain a quasi-projective simplicial refinement Σ' such that $P \cap \sigma(1)$ generates a cone of Σ' for all $\sigma \in \Sigma$.

We claim that P is a primitive collection for Σ'. First note that if P were contained in a cone of Σ', then it would be contained in a cone of Σ, which we know to be false. Now let Q be a proper subset of P. Then Q is contained in a cone $\sigma \in \Sigma$, so that $Q \subset P \cap \sigma(1)$. Since $P \cap \sigma(1)$ generates a cone of Σ', it follows that Q is contained in a cone of Σ'. Hence P is a primitive collection for Σ'. □

REMARK 3.13. When Σ is non-simiplicial, it may be impossible to find a *single* simplicial refinement Σ' such that *every* primitive collection for Σ is also primitive for Σ'. In Figure 3 from Example 3.5, we see two primitive collections $P_1 = \{\rho_0, \rho_1, \rho_3\}$ and $P_2 = \{\rho_0, \rho_2, \rho_4\}$, but there is no simplicial refinement Σ' of Σ with $\Sigma'(1) = \Sigma(1)$ that supports both P_1 and P_2.

4. Is quasi-projective necessary?

In this section we explore an open question about primitive collections. In [2], Casagrande raises the question of whether CPL(Σ) is defined by primitive inequalities when Σ is not quasi-projective. Here is a classic example.

EXAMPLE 4.1. The following example of a non-projective smooth complete fan is taken from Oda [13, p. 84, Example] (see also [11]). Consider the fan Σ in R^3 with seven minimal generators:

$$\rho_1 = (-1, 0, 0), \quad \rho_2 = (0, -1, 0), \quad \rho_3 = (0, 0, -1), \quad \rho_4 = (1, 1, 1),$$
$$\rho_5 = (1, 1, 0), \quad p_6 = (0, 1, 1), \quad \rho_7 = (1, 0, 1).$$

The cones of Σ are obtained by projecting from the origin through the triangulated polytope shown in Figure 4 on the next page. The fan Σ has 15 walls and 10 maximal cones.

The seven primitive collections for Σ and their associated primitive relations are:

$$\{\rho_2, \rho_4\}: \rho_2 + \rho_4 = \rho_7$$
$$\{\rho_1, \rho_4\}: \rho_1 + \rho_4 = \rho_6$$
$$\{\rho_2, \rho_5\}: \rho_2 + \rho_5 = \rho_3 + \rho_7$$
$$\{\rho_3, \rho_6\}: \rho_3 + \rho_6 = \rho_1 + \rho_5$$
$$\{\rho_3, \rho_4\}: \rho_3 + \rho_4 = \rho_5$$
$$\{\rho_1, \rho_7\}: \rho_1 + \rho_7 = \rho_2 + \rho_6$$
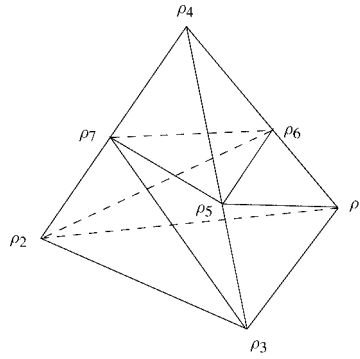$$\{\rho_5, \rho_6, \rho_7\}: \rho_5 + \rho_6 + \rho_7 = 2\rho_4.$$

FIGURE 4.   Non quasi-projective example.

By (1), a convex function $\phi \in \mathrm{CPL}(\Sigma)$ satisfies the primitive inequalities:

$$\phi(\rho_2) + \phi(\rho_4) \geq \phi(\rho_7)$$
$$\phi(\rho_1) + \phi(\rho_4) \geq \phi(\rho_6)$$
$$\phi(\rho_5) + \phi(\rho_2) \geq \phi(\rho_3) + \phi(\rho_7)$$
$$(15) \qquad \phi(\rho_3) + \phi(\rho_6) \geq \phi(\rho_1) + \phi(\rho_5)$$
$$\phi(\rho_3) + \phi(\rho_4) \geq \phi(\rho_5)$$
$$\phi(\rho_1) + \phi(\rho_7) \geq \phi(\rho_2) + \phi(\rho_6)$$
$$\phi(\rho_5) + \phi(\rho_6) + \phi(\rho_7) \geq 2\phi(\rho_4) \,.$$

Notice that adding up the third, fourth and sixth inequalities yields an equality, hence we have 3 equalities:

$$\phi(\rho_2) + \phi(\rho_5) = \phi(\rho_3) + \phi(\rho_7)$$
$$\phi(\rho_3) + \phi(\rho_6) = \phi(\rho_1) + \phi(\rho_5)$$
$$\phi(\rho_1) + \phi(\rho_7) = \phi(\rho_2) + \phi(\rho_6) \,.$$

To see what this says about the nef cone $\mathrm{Nef}(X)$, note that

$$\mathrm{Nef}(X) \cong \{\phi \in \mathrm{CPL}(\Sigma) \,;\, \phi(\rho_1) = \phi(\rho_2) = \phi(\rho_3) = 0\} \,.$$

Assume $\phi(\rho_1) = \phi(\rho_2) = \phi(\rho_3) = 0$. Then the three equalities give $\phi(\rho_5) = \phi(\rho_6) = \phi(\rho_7)$. Define $a = \phi(\rho_4)$ and $b = \phi(\rho_5) = \phi(\rho_6) = \phi(\rho_7)$. Then inequalities (15) imply $a \geq b$ and $3b \geq 2a$. It follows that $\mathrm{Nef}(X)$ is contained in the 2-dimensional cone pictured in Figure 5 on the next page.

Since $\mathrm{Pic}(X)_{\boldsymbol{R}}$ has dimension 4 and $\mathrm{Nef}(X)$ has dimension at most two, we see that X is non-projective since the nef cone does not have maximal dimension.

It is also easy to see that the cone in Figure 5 actually equals the nef cone $\mathrm{Nef}(X)$—just show that the generators of this cone are nef. For example, when $a = b > 0$, note that
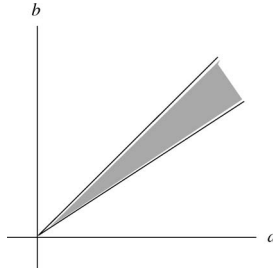
FIGURE 5. Cone defined by primitive inequalities.

Σ is a refinement of the complete fan Σ_0 with 1-dimensional generators ρ_1, ρ_2, ρ_3, ρ_4. The toric variety of Σ_0 is $\boldsymbol{P}^3$, and the class corresponding to $a = b > 0$ is the pullback of an ample divisor on $\boldsymbol{P}^3$, hence nef on X. For $3b = 2a > 0$, one proceeds similarly by noting that Σ is a refinement of the projective non-simplical fan Σ_1 with 1-dimensional generators ρ_1, ρ_2, ρ_3, ρ_5, ρ_6, ρ_7. $\qquad\square$

Other more substantial examples can be found in Chapter 7 of Scaramuzza's thesis [16]. Based on this, we make the following conjecture, which we credit to Casagrande.

CONJECTURE 4.2 (Casagrande). *Let X be a simplicial toric variety coming from the fan Σ in $N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$. If $|\Sigma|$ is convex of dimension n, then*

$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma) \,; \phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma\} \,.$$

Besides the evidence provided by numerous examples, we also have the theoretical result of Casagrande [2, Thm. 5.6], which states that if a smooth complete non-projective toric variety X has a toric blow-up $Y \to X$ with Y projective, then Conjecture 4.2 holds for X. However, the proofs of the simplicial case given in Theorem 2.3 make essential use of extremal rays, which exist only in the quasi-projective case. It is likely that some significantly new ideas will be needed to prove Conjecture 4.2 in general.

We extend Conjecture 4.2 to the non-simplicial case as follows.

CONJECTURE 4.3. *Let X be a non-simplicial toric variety of a fan Σ in $N_{\boldsymbol{R}} \cong \boldsymbol{R}^n$ such that $|\Sigma|$ is convex of dimension n. Then*:

$$\mathrm{CPL}(\Sigma) = \{\phi \in \mathrm{PL}(\Sigma) \,; \phi(\rho_1) + \cdots + \phi(\rho_k) \geq \phi(\rho_1 + \cdots + \rho_k)$$
$$\text{for all primitive collections } \{\rho_1, \ldots, \rho_k\} \text{ for } \Sigma\} \,.$$

Furthemore, every primitive collection for Σ is supported on a simplicial refinement Σ' of Σ satisfying $\Sigma'(1) = \Sigma(1)$.

Given a non-simplicial fan Σ, the first part Conjecture 4.3 follows from Conjecture 4.2 and the existence of a simplicial refinement Σ' satisfying $\Sigma'(1) = \Sigma(1)$. The latter is proved

in [17]. However, the final assertion of Conjecture 4.3 requires a version of Theorem 3.1 that doesn't assume that Σ is quasi-projective.

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