

Primordial black holes and the deuterium abundance

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Summary. The destruction of deuterium as a result of high energy photon emission from evaporating primordial black holes is examined. The decrease in abundance is found to be comparable in magnitude with the increase predicted by Zeldovich *et al.* in their consideration of nucleon emission by black holes, indicating that the fate of deuterium in a universe containing small black holes needs further examination.

1 Introduction

In a previous paper (Lindley 1979, Paper I) it was demonstrated that a source of high energy photons in the early Universe (massive, unstable neutrinos in that case) could alter light element abundances through photodisintegration. In this paper a similar analysis is applied to the problem of primordial black hole (pbh) evaporation; there are, however, two complications in this problem. Applying a conventional elementary particle model to pbh evaporation, the temperature and lifetime of a black hole can be estimated as

$$T \approx 10^{13} \text{ MeV } (m/\text{kg})^{-1} \quad \text{and} \quad t = Am^3, \quad A \approx 10^{-18} \text{ s kg}^{-3},$$

so that if we are interested in pbh evaporation occurring before recombination (at about 10^{12} s), when the evaporation products can still interact with the matter and radiation in the Universe, then we must contend with pbh temperatures greater than roughly 10^3 MeV and extending up to 10^6 – 10^7 MeV in a rapid final explosion (Page & Hawking 1976). The first difficulty is then that the photons emitted by the pbh have energies at which the photonuclear cross-sections of interest are negligible, and we must estimate the spectrum of photons produced as thermalization proceeds. The second is that at these temperatures the pbh is capable of emitting a great variety of particles. In the following we examine only the consequences of emitted photons, but later on compare results with some obtained through consideration of emitted nucleons (see Novikov *et al.* 1979, and references therein).

Similar work has also been performed by Aly (1978), who looks at the effects on element abundances of non-thermal nuclear reactions resulting from matter–antimatter annihilation in a baryon-symmetric cosmology. However, Aly considers only the production of D and ^3He through disruptive reactions on ^4He , without taking into account possible destruction of the more fragile deuterium.

2 PBH evaporation and photodisintegration of deuterium

We wish first to estimate the average number of deuterons destroyed by a single photon emitted by a pbh. To do this we must examine the thermalization of high energy photons. To a first approximation we suppose that only electron–photon processes (pair-production, Compton and inverse Compton scattering) are important.

A photon of initial energy $E_0 \geq 10^3 \text{ MeV}$ will produce a cascade of electrons, positrons and photons, which will contain a set $\{E_i\}$ of individual photon energies. (Note that only $E_i > Q_d = 2.225 \text{ MeV}$, the threshold for $d(\gamma, n)$, need be considered, so that the set $\{E_i\}$ will contain a finite number of photons.) Each of the photons in $\{E_i\}$ will exist for a time determined by the electron–photon interaction cross-sections; there is, however, a small probability $p(E_i)$ that in this time the photon will destroy a deuteron. From Paper I this probability is given by

$$p(E_i) = \frac{n_d \sigma_d(E_i)}{n_e \sigma_T(E_i)},$$

where n_d and n_e are the densities of deuterons and electrons respectively, σ_d is the cross-section for $d(\gamma, n)$, and σ_T is the sum of the pair-production and Compton cross-sections. The value of $p(E_i)$ is always small ($\leq 10^{-6}$) so the effect of this extra process on the thermalization cascade can be ignored. The numbers of deuterons destroyed in the cascade above is then

$$dN_d = - \sum_i p(E_i),$$

the sum extending over the set $\{E_i\}$.

In fact, the set of photon energies $\{E_i\}$ will not be the same for all starting photons of the same energy E_0 . However, by considering many such cascades one can add together the individual sets $\{E_i\}$ to form a histogram, which will define an average photon distribution $N(E, E_0)dE$. This represents the number of photons in the energy range E to $E + dE$ to be found in an ‘average’ thermalization cascade resulting from one photon of energy E_0 . Finally, therefore, the average number of deuterons destroyed by one such photon is

$$dN_d = - \int_{Q_d}^{E_0} N(E, E_0) p(E) dE = - \frac{n_d}{n_e} \Sigma(E_0),$$

where

$$\Sigma(E_0) = \int_{Q_d}^{E_0} N(E, E_0) \frac{\sigma_d(E)}{\sigma_T(E)} dE.$$

There is an implicit assumption here that n_d/n_e does not change significantly over the thermalization time-scale, but since later on we constrain the total change in n_d/n_e over the whole radiation era to be small, this seems reasonable.

In the Appendix a form for $\Sigma(E_0)$ is estimated and we take

$$\Sigma(E_0) = \beta \left(\frac{E_0}{E_*} \right),$$

where $E_* = 100 \text{ MeV}$ and $\beta = O(1)$.

Consider a comoving volume V , containing a mass $M(t)$ in the form of pbh’s, with $M(t)$ decreasing through evaporation. In time dt , suppose a fraction f of the evaporated mass dM emerges as high energy photons with a spectrum $\nu(E)dE$; then

$$\int E \nu(E) dE = -f dM \quad (dM < 0).$$

The number of deuterons in V destroyed is

$$dN_d = -\frac{N_d}{N_e} \int \Sigma(E) \nu(E) dE = -\frac{N_d}{N_e} \frac{f\beta}{E_*} |dM|, \quad (1)$$

the latter following from the assumed form of $\Sigma(E)$. Note also that n_d/n_e has been replaced by N_d/N_e , where the latter quantities refer to numbers of particles in V . This equation is valid when pbh evaporation is the only process changing N_d ; so as in Paper I we can integrate equation (1) over an interval $t_1 \leq t \leq t_2$, (where t_1 is the time when nucleosynthesis has just ceased, and t_2 is the time when evaporation products no longer thermalize) to obtain

$$\ln \left[\frac{N_d(t_2)}{N_d(t_1)} \right] = \ln \left[\frac{X_d(t_2)}{X_d(t_1)} \right] = -\frac{\Delta M f \beta}{N_e E_*},$$

where $\Delta M = M(t_1) - M(t_2)$ is the amount of pbh matter which has evaporated in the given time interval.

If we require that the proportion X_d of deuterium shall not be significantly altered, then we can put

$$\frac{\Delta M}{N_e} \frac{f\beta}{E_*} \lesssim 1$$

as a rough limit on ΔM . Taking $N_e \approx M_b/m_p$, where M_b is the baryonic mass in V and m_p the proton mass, then

$$\frac{\Delta M}{M_b} \lesssim \frac{1}{f\beta} \frac{E_*}{m_p} \approx \frac{0.1}{f\beta}. \quad (2)$$

In the radiation era, M_b will be much less than the energy E_γ in the form of radiation in V . Therefore if the limit (2) obtains, $\Delta M \ll E_\gamma$, so that the heating effect of pbh evaporation will be negligible (except perhaps in the case of a sudden burst of pbh evaporation near the end of the radiation era, when $M_b \approx E_\gamma$).

It is implicit in the above derivation that the destruction of deuterium should be uniform, i.e. that pbh evaporation does not scour a small volume near the pbh clear of deuterium whilst leaving everywhere else unchanged. This is easily checked; suppose for simplicity that the total pbh mass ΔM is concentrated in black holes of mass m . Then $\Delta M = Nm$, and the typical pbh separation is

$$l = \left(\frac{V}{N} \right)^{1/3}.$$

Taking $f = \beta = 1$ in equation (2), the maximum allowed value of ΔM is $\sim 0.1 M_b$, and putting $\rho_b = M_b/V$, we have

$$l = \left(\frac{m}{0.1 \rho_b} \right)^{1/3}.$$

The mean free path for a high energy photon is

$$\lambda = \frac{1}{n_e \sigma_{pp}} \approx \frac{m_p}{\rho_b \sigma_{pp}},$$

where the pair-production cross-section σ_{pp} is roughly 50 mb at 10^6 MeV.

Taking a cosmological model given by

$$\rho_b = \Omega \rho_c \left(\frac{T}{2.7 \text{ K}} \right)^3, \quad \rho_c = 5 \times 10^{-27} h_0^2 \text{ kg m}^{-3}; \quad h_0 = H_0 / 50 \text{ km s}^{-1} \text{ Mpc}^{-1},$$

$T \approx 10^{10} \text{ K } t^{-1/2}$ (during the radiation era), and with the epoch of pbh explosion given by $t = 10^{-18} \text{ m}^3$, we find

$$\frac{l}{\lambda} \sim (\Omega h_0^2)^{2/3} \left(\frac{m}{10^6 \text{ kg}} \right)^{-8/3}.$$

The cosmological evolution used is that of the standard big-bang, since the limits derived below show that the mass density in pbh form is never dynamically significant. We are concerned with $m \geq 10^7 \text{ kg}$, $\Omega \approx 0.1$, so $l/\lambda \lesssim 1$, meaning that individual pbh explosions overlap and that the injection of high energy particles into the Universe is, as required, uniform. This result also means that we can ignore possible small-scale baryon density fluctuations which may be associated with pbh formation.

Clearly similar limits to (2) could be derived from consideration of other photonuclear reactions (e.g. ${}^4\text{He}(\gamma, x)X$), but these would be less strict since deuterium has the lowest threshold and largest total cross-section for photodisintegration among the light elements. These other reactions are therefore not made use of here.

3 Limits on total pbh density

3.1 NARROW PBH MASS SPECTRUM

Zeldovich *et al.* (1977) (hereinafter ZSKC) consider the consequences of nucleon and anti-nucleon emission; in particular they find that such emission increases the deuterium abundance due to capture of free neutrons by protons and spallation of ${}^4\text{He}$ by high energy particles. They suppose that the pbh spectrum is concentrated around a mass m , and define limits with respect to a parameter $\alpha(m)$, given by

$$\alpha(m) = (\rho_{\text{pbh}}/\rho_u)_{t_m},$$

where ρ_u denotes the total density (matter and radiation) of the Universe, and $t_m = Am^3$ is the time at which the black holes explode. Their limits are

$$\alpha \lesssim 10^{-6} t_m^{1/6} \Omega \quad 10^7 \text{ kg} \lesssim m \lesssim 5 \times 10^7 \text{ kg},$$

$$\alpha \lesssim 10^{-11} t_m^{7/6} \Omega \quad 5 \times 10^7 \text{ kg} \lesssim m \lesssim 10^8 \text{ kg},$$

$$\alpha \lesssim 5 \times 10^{-6} t_m^{1/6} \Omega \quad 10^8 \text{ kg} \lesssim m \lesssim 10^{10} \text{ kg}.$$

Values of α below these do not increase deuterium abundance significantly.

Limit (2) can be written

$$\frac{\rho_{\text{pbh}}}{\rho_b} \lesssim \frac{0.1}{f\beta},$$

and using the relations between ρ_b , T and t given in the last section we find

$$\alpha = \left(\frac{\rho_{\text{pbh}}}{\rho_\gamma} \right) \lesssim 2 \times 10^{-7} \frac{\Omega t_m^{1/2}}{f\beta}. \quad (3)$$

Given the uncertainties in the estimates presented here and in ZSKC, it is clear that the effects of nuclear and photonuclear reactions are comparable, and the question of whether the deuterium abundance is decreased or increased will not be decided without more accurate calculations involving (perhaps unknown) details of high energy physics. In particular, the suggestion of ZSKC that a small pbh component could serve to produce an acceptable deuterium abundance in a high density universe needs further examination.

3.2 BROAD PBH MASS SPECTRUM

In this section we use directly some of the results of Carr (1975), who suggests that the mechanism of pbh formation from density fluctuations should lead to a power-law pbh spectrum extending over a large mass range. If this is the case, then the limit derived above can be compared with others at different pbh masses, and also used to put a limit on the present day value of ρ_{pbh} .

Suppose that at ' $t = 0$ ', there is, in a comoving volume V , a pbh spectrum given by

$$N_0(m_0) dm_0 = K m_0^{-\nu} dm_0.$$

Carr (1975) suggests $2 \leq \nu \leq 3$, with $\nu = 5/2$ corresponding to a universe dominated by conventional relativistic particles throughout its early stages. A pbh which has a mass m at a later time t will be the remnant of one which started with mass m_0 , where

$$m_0 = (t/A + m^3)^{1/3},$$

so at time t

$$\begin{aligned} N(m, t) dm &= N_0(m_0) dm_0 \\ &= K (t/A + m^3)^{-(2+\nu)/3} m^2 dm. \end{aligned}$$

The total pbh mass $M(t)$ in V is then

$$M(t) = \int_0^\infty m N(m, t) dm.$$

(The upper limit should not strictly be infinite, but provided ν is not close to 2, M is dominated by the low mass pbh contribution.)

Substituting for $N(m, t)$, we find

$$M(t) = K' t^{-(\nu-2)/3} \quad (\nu \neq 2), \quad (4)$$

where

$$K' = K A^{(\nu-2)/3} \int_0^\infty \frac{x^3 dx}{(1+x^3)^{(2+\nu)/3}}.$$

If M_0 is the present total pbh mass in V , then

$$\frac{M_0}{\Delta M} = \frac{t_0^{-1/6}}{(t_1^{-1/6} - t_2^{-1/6})},$$

where $\nu = 5/2$ has been used.

Combining with limit (2), we find

$$\frac{M_0}{M_b} = \left(\frac{\rho_{\text{pbh}}}{\rho_b} \right)_0 \lesssim \frac{0.1}{f\beta} \frac{t_0^{-1/6}}{(t_1^{-1/6} - t_2^{-1/6})}.$$

Putting $t_0 \approx 10^{18}$ s, $t_1 \approx 1000$ s (just after nucleosynthesis), and $t_2 \approx 10^{12}$ s (recombination), this limit becomes

$$\left(\frac{\rho_{\text{pbh}}}{\rho_b}\right)_0 \lesssim \frac{3 \times 10^{-4}}{f\beta}. \quad (5)$$

If $\rho_b \approx 0.1 \rho_c$, and taking $f\beta \sim 1$, then $(\rho_{\text{pbh}})_0 \lesssim 3 \times 10^{-5} \rho_c$.

Page & Hawking (1976) look at the high energy photon background to get a limit from pbh emission since recombination. They suggest

$$m_* \frac{dn}{dm} \lesssim N \left(\frac{m}{m_*}\right)^{-\nu} \quad \text{with} \quad N = 10^4 \text{ pc}^{-3},$$

where $m_* \approx 10^{12}$ kg is the mass of a pbh which just survives until today and dn/dm is the pbh spectrum. Taking $\nu = 5/2$ again we get

$$(\rho_{\text{pbh}})_0 = 2Nm_* \lesssim 8 \times 10^{-34} \text{ kg m}^{-3} \approx 2 \times 10^{-7} \rho_c.$$

In the standard case of $\nu = 5/2$ (corresponding to an equation of state with $p = 1/3 \rho$ in the early Universe) the limit from recent pbh emission is stronger. However, if we take the extreme case $\nu = 3$ (which requires $p = \rho$, a maximally stiff equation of state during the era of pbh formation), it is easily seen that the two limits coincide, both demanding $(\rho_{\text{pbh}})_0 \lesssim 10^{-7} \rho_c$. Also, if any deviation from a pure power law pbh spectrum is supposed the relation between the two limits can be changed almost arbitrarily.

4 PBH evaporation during nucleosynthesis

The limit (2) obtained on ΔM applies to pbh evaporation occurring after nucleosynthesis, when non-thermal photodisintegration is the only factor affecting X_d . In this section we ask whether it is possible to obtain a limit on pbh density from evaporation during nucleosynthesis.

In the case of a narrow pbh spectrum, limit (3) on $\alpha(m)$ is the requirement that the sudden change in X_d due to the more or less instantaneous injection of high energy photons at time t_m be small. At a first guess one might suppose that such a change in X_d would have an equally adverse effect during nucleosynthesis as after, and therefore the limit (3) can be tentatively applied through the nucleosynthesis era, down to $t_m \approx 100$ s, $m \approx 5 \times 10^6$ kg.

If there is a broad pbh mass range, then it produces a continuous change in X_d . Limit (2) follows from requiring the integrated change in X_d , during $t_1 \leq t \leq t_2$, to be small. We can ask whether nucleosynthesis will be affected by comparing the value of \dot{X}_d due to pbh evaporation with the rate of change of X_d during nucleosynthesis.

Taking equation (1) and putting X_d instead of N_d , we obtain

$$\frac{dX_d}{dt} = -\frac{X_d}{N_e} \frac{f\beta}{E_*} \left| \frac{dM}{dt} \right| = -\frac{X_d}{t} \frac{(\nu-2)}{3} \frac{f\beta}{E_*} \frac{M}{N_e},$$

the latter following from $M \propto t^{-(\nu-2)/3}$.

With $E_* \approx 0.1 m_p$, and $N_e \approx M_b/m_p$ as before, then

$$\frac{d \log X_d}{d \log t} = -\frac{(\nu-2)}{3} 10 f\beta \frac{M}{M_b} \approx -\frac{\rho_{\text{pbh}}}{\rho_b},$$

taking all the constants together to be roughly unity. Using the limit (5) and the relation between M and t , then

$$\frac{\rho_{\text{pbh}}}{\rho_{\text{b}}} \simeq 3 \times 10^{-4} \left(\frac{t}{t_0} \right)^{-1/6} \simeq 0.3 t^{-1/6}$$

gives the maximum pbh density which does not disturb X_{d} after nucleosynthesis.

Wagoner (1973) gives a graph showing the time evolution of element abundances for a typical case followed by his reaction network. It can be seen that the crucial period is around $t_{\text{c}} \simeq 300\text{s}$, when X_{d} reaches a peak and other elements are being rapidly built up. The non-thermal contribution to X_{d} can be estimated from the foregoing, and gives

$$\left| \left(\frac{d \log X_{\text{d}}}{d \log t} \right)_{t_{\text{c}}} \right| \lesssim 0.1.$$

The slope of Wagoner's graph gives directly the logarithmic rate of change, and the value around the peak of X_{d} can be estimated as

$$\left| \frac{d \log X_{\text{d}}}{d \log t} \right| \simeq 5-10.$$

This means that a power law pbh spectrum obeying limit (2) is unlikely to have a significant effect on nucleosynthesis, since the thermal rate of change of X_{d} exceeds the non-thermal by one or two orders of magnitude.

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References

- Aly, J. J., 1978. *Astr. Astrophys.*, **64**, 273.
 Blumenthal, G. R. & Tucker, W. H., 1974. *X-Ray Astronomy*, ch. 3, eds Giacconi, R. & Gursky, H., D. Reidel, Dordrecht, Holland.
 Carr, B. J., 1975. *Astrophys. J.*, **201**, 1.
 Heitler, W., 1954. *The Quantum Theory of Radiation*, 3rd edn, Clarendon Press, Oxford.
 Lindley, D., 1979. (Paper I) *Mon. Not. R. astr. Soc.*, **188**, 15P.
 Novikov, I. D., Polnarev, A. G., Starobinskii, A. A. & Zeldovich, Ya. B., 1979. *Astr. Astrophys.*, **80**, 104.
 Page, D. N. & Hawking, S. W., 1976. *Astrophys. J.*, **206**, 1.
 Tucker, W. H., 1975. *Radiation Processes in Astrophysics*, MIT Press, Cambridge, Massachusetts.
 Wagoner, R. V., 1973. *Astrophys. J.*, **179**, 343.
 Zeldovich, Ya. B., Starobinskii, A. A., Khlopov, M. Yu. & Chechetkin, V. M., 1977. (ZSKC), *Soviet Astr. Lett.*, **3**, 110.

Appendix: Thermalization of high energy photons

We wish to find the number $N(E, E_0)dE$ of photons produced during the thermalization of a photon with initial energy $E_0 \gtrsim 10^3\text{MeV}$. The scattering processes considered are:

(i) for photons, electron–positron pair-production (PP) and Compton scattering (CS). Energetically one can expect production of other pairs (mesons, nucleons etc.) but we shall simply suppose that the lightest component dominates. For $E > E_1 \approx 120$ MeV PP dominates over CS.

(ii) for electrons, inverse Compton scattering dominates (see e.g. the appropriate formula in Tucker 1975), but there are two regimes to be considered. If $\mathcal{E} = \gamma m_e c^2$ is the electron's energy, and $\epsilon \approx kT$ is the energy of thermal background photons, then the scattering in the electron's rest frame is relativistic or non-relativistic according as $\gamma\epsilon > m_e c^2$ or $< m_e c^2$. In the latter case, we have Thomson scattering (Th), and the energy of the scattered photon is $\sim \gamma^2 \epsilon < \mathcal{E}$, whereas in the former case, the Klein–Nishina (KN) cross-section must be used, and the scattered photon takes up most of the electron's energy (see Blumenthal & Tucker 1974). The critical value of \mathcal{E} is $\mathcal{E}_c = m_e^2 c^4 / \epsilon$.

The cascade of electrons and photons can be analysed in two energy ranges:

(a) For $E, \mathcal{E} > E_{\min} = \max(E_1, \mathcal{E}_c)$ only KN and PP occur, and an approximate analytic form for $N(E, E_0)$ can be obtained. For PP, we assume that the distribution in energy \mathcal{E} of electrons is rectangular in $0 \leq \mathcal{E} \leq E$, where E is the photon energy. The number distribution of electrons from a single photon is then

$$\nu(\mathcal{E}, E) d\mathcal{E} = \frac{2d\mathcal{E}}{E},$$

so that

$$\int_0^E \nu(\mathcal{E}, E) d\mathcal{E} = 2.$$

If there is a distribution $N(E, E_0) dE$ ($E \leq E_0$) of photons, it will be related to the electron distribution by

$$N_e(\mathcal{E}) d\mathcal{E} = d\mathcal{E} \int_{\mathcal{E}}^{E_0} \nu(\mathcal{E}, E) N(E, E_0) dE = 2d\mathcal{E} \int_{\mathcal{E}}^E \frac{N(E, E_0) dE}{E}.$$

Now we also assume that in KN an electron of energy \mathcal{E} puts all its energy into a photon of energy $E = \mathcal{E}$. Then for each electron in $N_e(\mathcal{E}) d\mathcal{E}$, there corresponds a photon of the same energy in $N(E, E_0) dE$ and the two distributions must be in fact identical. This gives

$$N(E, E_0) = 2 \int_E^{E_0} \frac{N(E', E_0) dE'}{E'} + \delta(E - E_0),$$

where the δ -function is the source term; a single photon at $E = E_0$. The solution to this is simply

$$N(E, E_0) = 2E_0 E^{-2} + \delta(E - E_0); \quad E_{\min} \leq E \leq E_0.$$

(b) For $E < E_{\min}$, there is a mixture of KN and Th for electrons and of PP and CS for photons. The electron and photon distributions are linked through a pair of coupled integral equations, but their form becomes complex and no simple analytic solution is possible. We resort instead to a crude numerical simulation in which each scattering process is typified by a fixed energy change as follows:

(i) for photons, if $E > E_1$, then PP occurs, and two electrons of energy $\mathcal{E} = E/2$ are produced. For $E < E_1$, CS dominates, and the scattered photon energy E is given by the

average

$$E' = \frac{1}{\sigma_{\text{CS}}} \int E'' \frac{d\sigma(E)}{dE''} dE'',$$

where the total and differential cross-sections σ_{CS} and $d\sigma/dE$ are to be found in Heitler (1954) (the integration can be performed analytically). In addition, in CS, an electron of energy $\mathcal{E} = E - E'$ is taken from the thermal background.

(ii) for electrons, an inverse Compton scattered photon E is produced, where

$$E = \gamma^2 \mathcal{E} \quad \mathcal{E} < \mathcal{E}_c,$$

$$E = \mathcal{E} \quad \mathcal{E} > \mathcal{E}_c.$$

The electron is left with $\mathcal{E}' = \mathcal{E} - E$.

These processes are easily incorporated into a computer routine, and given an initial specification of photon energies, thermalization can be followed down to 2.225 MeV, with photon energies being recorded in a histogram (with logarithmic intervals) to give a numerical distribution of photons. The starting distribution of photons was obtained from the analytic expression in Section (a) above. All photons with $E_{\text{min}} < E < 2E_{\text{min}}$ will produce electrons with $\mathcal{E} \leq E_{\text{min}}$, which in turn give photons with $E' \leq E_{\text{min}}$. The procedure used was to divide the range E_{min} to $2E_{\text{min}}$ into 20 discrete, equally spaced, energy values, populate them in accordance with the analytic formula for $N(E, E_0)dE$, and then to follow the thermalization and create the histogram as described above.

Having thus obtained an estimate for $N(E, E_0)dE$, we can simply put

$$\Sigma(E_0) = \int_{Q_d}^{E_0} N(E, E_0) \frac{\sigma_d(E)}{\sigma_T(E)} dE \approx \sum_i N(E_i, E_0) \frac{\sigma_d(E_i)}{\sigma_T(E_i)},$$

where the sum goes over the histogram energy bins.

The data for $\sigma_d(E)$ is essentially unchanged from Paper I, though with some small modifications (a few per cent, at high and low energies) from recent experimental data (E. W. Lees, private communication). It can be seen from Section (a) that there is a basic proportionality of $N(E, E_0)$ with E_0 , so that $\Sigma(E_0) \propto E_0$ too. The only other variation is in the value of ϵ (which affects E_{min}). Table A1 shows values of ϵ , the corresponding background temperature and $\beta(\epsilon)$, where

$$\Sigma(E_0) = \beta(\epsilon) \frac{E_0}{E_*}, \quad E_* = 100 \text{ MeV}.$$

It can be seen that β varies by one order of magnitude while ϵ changes by five; we therefore take β to be a constant of order unity to simplify calculations in Section 2.

Table A1.

$\log(\epsilon/\text{MeV})$	0	-1	-2	-3	-4	-5
$\log(T/\text{K})$	10	9	8	7	6	5
β	1.6	1.6	0.83	0.35	0.23	0.18