

Primordial trispectrum from inflation

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We use the δN -formalism to describe the leading order contributions to the primordial power spectrum, bispectrum and trispectrum in multiple-field models of inflation at leading order in a perturbative expansion. In slow-roll models where the initial field fluctuations at Hubble-exit are nearly Gaussian, any detectable non-Gaussianity is expected to come from super-Hubble evolution. We show that the contribution to the primordial trispectrum can be described by two non-linearity parameters, τ_{NL} and g_{NL} , which are dependent upon the second and third derivatives of the local expansion with respect to the field values during inflation.

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I. INTRODUCTION

The simplest models of inflation predict a quasi scale invariant spectrum of nearly Gaussian, adiabatic perturbations [1]. While observations are consistent with this picture, it is important to measure any deviations from these predictions, in order to constrain the many models of inflation. Non-Gaussianity is a potentially powerful discriminant between different models. Currently most efforts to constrain the non-Gaussianity have focused on the 3-point function of perturbations, the bispectrum [2, 3, 4]. However the 4-point function, or trispectrum can also be constrained by increasingly accurate measurements [5, 6, 7].

The δN -formalism [8, 9, 10, 11] identifies the primordial curvature perturbation with a perturbation in the local value of the integrated expansion,

$$N(x) = \int_{t_i}^{t_p} H(x, t) dt, \quad (1)$$

with respect to the expansion in the background spacetime

$$\bar{N} = \int_{t_i}^{t_p} \bar{H}(t) dt. \quad (2)$$

This enables one to calculate the non-linear primordial curvature perturbation (e.g., at the epoch of primordial nucleosynthesis) in terms of initial scalar field fluctuations during inflation [11]. In particular the primordial curvature perturbation on uniform-density hypersurfaces, ζ , corresponds to the perturbation in the local expansion defined with respect to an initial spatially flat hypersurface [10]. In the long-wavelength limit where spatial gradients and anisotropic shear becomes small, the integrated expansion along a worldline can be calculated from solutions to the unperturbed Friedmann equation.

We have

$$\zeta = N(\phi^A) - \bar{N}, \quad (3)$$

where ϕ^A represents the initial values for the scalar fields on the initial spatially-flat hypersurface. If we decompose the local scalar field values into a homogeneous background value plus perturbation,

$$\phi_A = \bar{\phi}_A + \varphi_A, \quad (4)$$

we can write the curvature perturbation as a Taylor expansion

$$\zeta = \sum_n \frac{1}{n!} N_{A_1 A_2 \dots A_n} \varphi^{A_1} \varphi^{A_2} \dots \varphi^{A_n}. \quad (5)$$

There is an implicit sum over the fields for all of the repeated indices A_i . We are thus able to describe the non-Gaussianity of the primordial curvature perturbation given an initial distribution of scalar field perturbations. In practice, scalar fields during slow-roll inflation are expected to have a nearly Gaussian distribution during inflation, at least on scales slightly larger than the Hubble scale, i.e., shortly after Hubble exit [12]. In single-field inflation this is sufficient to guarantee that the non-Gaussianity of the primordial curvature perturbation is very small [13, 14, 15]. It is suppressed by slow-roll parameters since one can express derivatives of the expansion in terms of slow-roll parameters. However in multi-field models the perturbed expansion cannot necessarily be related to slow-roll parameters at Hubble-exit, and the primordial curvature perturbations may deviate strongly from a Gaussian distribution.

II. PERTURBATIVE EXPANSION

In linear perturbation theory the initial free-field fluctuations describe Gaussian random fields, which we will denote by $\varphi_1^A(t, x^i)$. In a perturbative expansion higher-order interactions and non-linear evolution lead to higher-order terms which are non-Gaussian

$$\varphi^A = \varphi_1^A + \frac{1}{2}\varphi_2^A + \frac{1}{6}\varphi_3^A + \dots \quad (6)$$

From the definition of Gaussian statistics, the n -point correlator of the Gaussian fluctuation is zero for odd n and can be reduced to (disconnected) products of two-point functions for even n .

A. Power spectrum

In Fourier space the power spectrum is given by

$$\langle \varphi_{\mathbf{k}}^A \varphi_{\mathbf{k}'}^B \rangle = C^{AB}(k) (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}'). \quad (7)$$

This is second and higher order in a perturbative expansion.

At leading order in the field perturbations and in the slow-roll limit the fluctuations are independent and we have

$$C^{AB}(k) = \delta^{AB} P(k), \quad (8)$$

where δ^{AB} is the Kronecker delta-function, and the variance per logarithmic interval in k -space is given by

$$\mathcal{P}(k) = \frac{4\pi k^3}{(2\pi)^3} P(k) = \left(\frac{H_*}{2\pi} \right)^2, \quad (9)$$

where the Hubble parameter H is evaluated at Hubble-exit, $k = (aH)_*$. At zeroth order in slow-roll parameters, \mathcal{P} is independent of wavenumber, i.e. we have a scale invariant spectrum for the field fluctuations.

In general the fields are correlated at Hubble-exit, at first order in the slow-roll parameters. In the case of two field inflation, using the methods of [16], (see also [17]), we find for the power spectra and cross-correlation respectively,

$$C^{11} = P_{\varphi_1} = \frac{(2\pi)^3}{4\pi k^3} \left(\frac{H_*}{2\pi} \right)^2 \left[1 - 2 \frac{\dot{H}}{H^2} + C \left(4 \frac{\dot{H}}{H^2} + \frac{1}{m_p^2} \frac{\dot{\phi}_1^2}{H^2} - \frac{1}{m_p^2} \frac{\dot{\phi}_2^2}{H^2} - 2m_p^2 \frac{V_{\phi_1 \phi_1}}{V} \right) \right], \quad (10)$$

$$C^{22} = P_{\varphi_2} = \frac{(2\pi)^3}{4\pi k^3} \left(\frac{H_*}{2\pi} \right)^2 \left[1 - 2 \frac{\dot{H}}{H^2} + C \left(4 \frac{\dot{H}}{H^2} + \frac{1}{m_p^2} \frac{\dot{\phi}_2^2}{H^2} - \frac{1}{m_p^2} \frac{\dot{\phi}_1^2}{H^2} - 2m_p^2 \frac{V_{\phi_2 \phi_2}}{V} \right) \right], \quad (11)$$

$$C^{12} = \frac{(2\pi)^3}{4\pi k^3} \left(\frac{H_*}{2\pi} \right)^2 C \left(\frac{2}{m_p^2} \frac{\dot{\phi}_1 \dot{\phi}_2}{H^2} - 2m_p^2 \frac{V_{\phi_1 \phi_2}}{V} \right), \quad (12)$$

where $C = 2 - \ln 2 - \gamma \simeq 0.7296$ and γ is the Euler-Mascheroni constant. Overdots represent derivatives with respect to time, and $V_{\phi_1} = \partial V / \partial \phi_1$.

B. Bispectrum

The first signal of non-Gaussianity comes from the bispectrum, which at lowest order is

$$\langle \varphi^A \varphi^B \varphi^C \rangle = \langle \varphi_1^A \varphi_1^B \varphi_2^C \rangle + \text{perms}, \quad (13)$$

and therefore is fourth order in perturbations.

The bispectrum of the distribution is given by

$$\langle \varphi_{\mathbf{k}_1}^A \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \rangle \equiv B^{ABC}(k_1, k_2, k_3) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \quad (14)$$

This is fourth and higher order. This was originally calculated by Maldacena, [13] for single field inflation, and by Seery and Lidsey, [12], for multiple fields. They show that it only depends on the amplitude of the k vectors. Specifically they calculate the quantity \mathcal{A}^{ABC} at leading order in slow roll, which is related to B^{ABC} by

$$B^{ABC}(k_1, k_2, k_3) = \frac{4\pi^4}{\prod k_i^3} \mathcal{P}^2 \mathcal{A}^{ABC}(k_1, k_2, k_3). \quad (15)$$

They find $\mathcal{A}^{ABC} \sim \mathcal{O}(\epsilon^{1/2})$ so it vanishes in the slow-roll limit.

C. Trispectrum

From Wick's theorem, the first-order, Gaussian perturbations do not contribute to the connected part of the four-point function,

$$\langle \varphi_1^A \varphi_1^B \varphi_1^C \varphi_1^D \rangle_c = 0. \quad (16)$$

Note there is a disconnected part of the four-point function which is a product of two two-point functions which is only fourth order in slow roll and arises even for purely Gaussian statistics. This disconnected term is only non-zero if, e.g., the momenta satisfy $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and $\mathbf{k}_3 + \mathbf{k}_4 = 0$.

The second-order field perturbations, φ_2^A , are generated from the product (or the convolution in Fourier space) of two first order variables that have Gaussian distributions, so they do not contribute to the (connected) four-point function at lowest possible order,

$$\langle \varphi_1^A \varphi_1^B \varphi_1^C \varphi_2^D \rangle_c = 0. \quad (17)$$

Hence the leading order contribution to the connected four-point function comes from two terms, [5],

$$\langle \varphi_1 \varphi_1 \varphi_1 \varphi_3 \rangle_c \quad \text{and} \quad \langle \varphi_1 \varphi_1 \varphi_2 \varphi_2 \rangle_c. \quad (18)$$

Both of these terms are sixth-order in perturbations.

In Fourier space the connected part of the four-point function is sixth order, and given by

$$\langle \varphi_{\mathbf{k}_1}^A \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle_c \equiv T^{ABCD}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4). \quad (19)$$

This was recently calculated at leading order by Seery, Lidsey and Sloth [18]. They show this quantity depends on both the magnitude and direction of the \mathbf{k} vectors, specifically it depends on $k_i = |\mathbf{k}_i|$ and $\mathbf{k}_i \cdot \mathbf{k}_j$.

III. THE PRIMORDIAL N-POINT FUNCTIONS

So far we have calculated the two-, three- and four-point function of the field fluctuations. To link these to observations, we need to calculate the n-point functions of the primordial curvature perturbation ζ . We do this using the δN expansion for ζ , (5).

A. The primordial power spectrum

At leading order the primordial power spectra depends purely on ζ_1 , from Eq. (5),

$$\zeta_1 = N_A \varphi_1^A. \quad (20)$$

The power spectrum is thus

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = P_\zeta(k) (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}'), \quad (21)$$

where

$$P_\zeta(k) = N_A N_B C^{AB}(k). \quad (22)$$

In the slow-roll limit, (8), this reduces to

$$P_\zeta(k) = N_A N^A P(k). \quad (23)$$

B. The primordial bispectrum

To leading order in the field perturbations, the 3-point function of the curvature perturbations depends on ζ_1 , (20) and

$$\zeta_2 = N_A \varphi_2^A + N_{AB} \varphi_1^A \varphi_1^B. \quad (24)$$

The primordial bispectrum is thus

$$\begin{aligned} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = & N_A N_B N_C \langle \varphi_{\mathbf{k}_1}^A \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \rangle \\ & + \frac{1}{2} N_{A_1 A_2} N_B N_C \left[\langle (\varphi^{A_1} * \varphi^{A_2})_{\mathbf{k}_1} \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \rangle + (2 \text{ perms}) \right], \end{aligned} \quad (25)$$

where $'*'$ denotes the convolution, defined by

$$(\varphi^A * \varphi^B)_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int d^3 k' \varphi_{\mathbf{k}-\mathbf{k}'}^A \varphi_{\mathbf{k}'}^B. \quad (26)$$

Hence the bispectrum of the curvature perturbation is

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv B_\zeta(k_1, k_2, k_3) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (27)$$

where to leading order [19]

$$\begin{aligned} B_\zeta(k_1, k_2, k_3) = & N_A N_B N_C B^{ABC}(k_1, k_2, k_3) \\ & + N_A N_{BC} N_D [C^{AC}(k_1) C^{BD}(k_2) + C^{AC}(k_2) C^{BD}(k_3) + C^{AC}(k_3) C^{BD}(k_1)]. \end{aligned} \quad (28)$$

In the slow-roll limit we can write the bispectrum as [20]

$$B_\zeta(k_1, k_2, k_3) = 4\pi^4 \frac{\sum_i k_i^3}{\prod_i k_i^3} \mathcal{P}_\zeta^2 \left(\frac{-1}{4m_p^2 N_C N^C} \frac{\mathcal{F}(k_1, k_2, k_3)}{\sum_i k_i^3} + \frac{N_{AB} N^A N^B}{(N_C N^C)^2} \right), \quad (29)$$

where the form factor \mathcal{F} is defined by

$$\mathcal{F}(k_1, k_2, k_3) = \sum_i k_i^3 - \sum_{i \neq j} k_i k_j^2 - 8 \frac{\sum_{i > j} k_i^2 k_j^2}{k_1 + k_2 + k_3}. \quad (30)$$

C. The primordial trispectrum

From the discussion in sec. II C the four-point function of the curvature perturbation at leading order will depend on ζ_1 , (20), ζ_2 , (24), and

$$\zeta_3 = N_A \varphi_3^A + N_{AB} (\varphi_1^A \varphi_2^B + \varphi_2^A \varphi_1^B) + N_{ABC} \varphi_1^A \varphi_1^B \varphi_1^C \quad (31)$$

The four-point function at leading order is

$$\begin{aligned} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c &= N_A N_B N_C N_D \langle \varphi_{\mathbf{k}_1}^A \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle_c \\ &+ \frac{1}{2} N_{A_1 A_2} N_B N_C N_D \left[\langle (\varphi^{A_1} * \varphi^{A_2})_{\mathbf{k}_1} \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle + (3 \text{ perms}) \right] \\ &+ \frac{1}{4} N_{A_1 A_2} N_{B_1 B_2} N_C N_D \left[\langle (\varphi^{A_1} * \varphi^{A_2})_{\mathbf{k}_1} (\varphi^{B_1} * \varphi^{B_2})_{\mathbf{k}_2} \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle + (5 \text{ perms}) \right] \\ &+ \frac{1}{3!} N_{A_1 A_2 A_3} N_B N_C N_D \left[\langle (\varphi^{A_1} * \varphi^{A_2} * \varphi^{A_3})_{\mathbf{k}_1} \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle + (3 \text{ perms}) \right]. \end{aligned} \quad (32)$$

All of the terms shown are sixth order in the field perturbations.

The first term of the expansion above is the intrinsic 4-point function of the fields, as calculated, [18]. The disconnected part of this term would only give a contribution if the sum of any two k vectors is zero, e.g. if $\mathbf{k}_1 + \mathbf{k}_2 = 0$. We will exclude this case, which is equivalent to neglecting parallelograms of the wavevectors.

The second term of (32) consists of permutations of terms of the form

$$\frac{1}{2} N_{A_1 A_2} N_B N_C N_D \frac{1}{(2\pi)^3} \int d^3 q \langle \varphi_{\mathbf{q}}^{A_1} \varphi_{\mathbf{k}_1 - \mathbf{q}}^{A_2} \varphi_{\mathbf{k}_2}^B \varphi_{\mathbf{k}_3}^C \varphi_{\mathbf{k}_4}^D \rangle \quad (33)$$

This five-point function is zero for the first-order, Gaussian, perturbations, hence the leading order contribution is sixth-order, due to the second order contribution of one of the fields. Hence we use Wick's theorem to split the 5-point function in to lower point functions. There is no contribution to (33) from the split into a four-point and a one-point function. Only the split into a two-point and three-point function gives a contribution. However the first possible contraction in (33), $\langle \varphi_{\mathbf{k}_1 - \mathbf{q}}^{A_1} \varphi_{\mathbf{q}}^{A_2} \rangle$, does not contribute since it is only non-zero when $\mathbf{k}_1 = 0$. Therefore we can reduce the above term into a power spectra and a trispectrum in 6 different ways, which gives three distinct pairs of terms. In total the second term of (32) is

$$N_{A_1 A_2} N_B N_C N_D [C^{A_1 B}(k_1) B^{A_2 BC}(k_{12}, k_3, k_4) + (11 \text{ perms})] (2\pi)^3 \delta^3(\mathbf{k}_t), \quad (34)$$

where we use the shortened notation $k_{ij} = |\mathbf{k}_i + \mathbf{k}_j|$ and $\mathbf{k}_t = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4$. The 12 permutations come from having 3 distinct choices for the indices of the wavenumber k_{ij} (only three distinct choices because $k_{ij} = k_{ji}$ and $k_{12} = k_{34}$ etc). We then choose which two wavenumbers form the remaining arguments of B^{ABC} , either k_i, k_j or the other pair of wavenumbers, and finally we choose which of the two indices i or j is attached to the wavenumber k_i that is the argument of C .

Continuing this argument for the second and third terms of (32), we find the connected part of the trispectrum of the curvature perturbation is

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c \equiv T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4), \quad (35)$$

where

$$\begin{aligned} T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= N_A N_B N_C N_D T^{ABCD}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\ &+ N_{A_1 A_2} N_B N_C N_D [C^{A_1 B}(k_1) B^{A_2 BC}(k_{12}, k_3, k_4) + (11 \text{ perms})] \\ &+ N_{A_1 A_2} N_{B_1 B_2} N_C N_D [C^{A_2 B_2}(k_{13}) C^{A_1 C}(k_3) C^{B_1 D}(k_4) + (11 \text{ perms})] \\ &+ N_{A_1 A_2 A_3} N_B N_C N_D [C^{A_1 B}(k_2) C^{A_2 C}(k_3) C^{A_3 D}(k_4) + (3 \text{ perms})]. \end{aligned} \quad (36)$$

IV. GAUSSIAN SCALAR FIELDS

If the scalar field perturbations are independent, Gaussian random fields, as we expect shortly after Hubble-exit during inflation in the slow-roll limit [12, 13] then the bispectrum for the fields, B^{ABC} , and connected part of the trispectrum, T^{ABCD} , both vanish.

In this case the bispectrum of the primordial curvature perturbation (28) at leading (fourth) order, can be written as

$$B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{6}{5} f_{NL} [P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)] . \quad (37)$$

where the dimensionless non-linearity parameter¹ is given by [11]

$$f_{NL} = \frac{5}{6} \frac{N_A N_B N^{AB}}{(N_C N^C)^2} . \quad (38)$$

The trispectrum (36) in this case reduces to

$$\begin{aligned} T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = & N_{AB} N^{AC} N^B N_C [P(k_{13})P(k_3)P(k_4) + (11 \text{ perms})] \\ & + N_{ABC} N^A N^B N^C [P(k_2)P(k_3)P(k_4) + (3 \text{ perms})] , \end{aligned} \quad (39)$$

Hence we can write the trispectrum as

$$\begin{aligned} T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = & \tau_{NL} [P_\zeta(k_{13})P_\zeta(k_3)P_\zeta(k_4) + (11 \text{ perms})] \\ & + \frac{54}{25} g_{NL} [P_\zeta(k_2)P_\zeta(k_3)P_\zeta(k_4) + (3 \text{ perms})] . \end{aligned} \quad (40)$$

where comparing the above two expressions, and using (23) we see

$$\tau_{NL} = \frac{N_{AB} N^{AC} N^B N_C}{(N_D N^D)^3} , \quad (41)$$

$$g_{NL} = \frac{25}{54} \frac{N_{ABC} N^A N^B N^C}{(N_D N^D)^3} . \quad (42)$$

The expression for τ_{NL} from multiple fields was given in the arXiv version of [21]. Note that we have factored out products in the trispectrum with different k dependence in order to define the two k independent non-linearity parameters τ_{NL} and g_{NL} . This gives the possibility that observations may be able to distinguish between the two parameters [5].

A. Single field dependence

In many cases there is single direction in field-space, φ , which is responsible for perturbing the local expansion, $N(\varphi)$, and hence generating the primordial curvature perturbation, ζ . For example this would be the inflaton field in single field models of inflation, or it could be the late-decaying scalar field in the curvaton scenario [22].

In the case where a single field dominates, the curvature perturbation (5) is given by

$$\zeta = N' \varphi + \frac{1}{2} N'' \varphi^2 + \frac{1}{6} N''' \varphi^3 + \dots , \quad (43)$$

where we use the shorthand $N' = dN/d\varphi$. If in addition we assume that the field perturbation is purely Gaussian, $\varphi = \varphi_1$, then the non-Gaussianity of the primordial perturbation has a simple “local form” where the full non-linear perturbation at any point in real space, $\zeta(\mathbf{x})$, is a local function of a single Gaussian random field, φ_1 . Thus we can write [4, 23]

$$\zeta = \zeta_1 + \frac{3}{5} f_{NL} \zeta_1^2 + \frac{9}{25} g_{NL} \zeta_1^3 + \dots , \quad (44)$$

where ζ_1 is Gaussian because it is directly proportional to the initial Gaussian field perturbation, φ_1 , and the dimensionless non-linearity parameters, f_{NL} and g_{NL} , are given by

$$f_{NL} = \frac{5}{6} \frac{N''}{(N')^2} , \quad (45)$$

$$g_{NL} = \frac{25}{54} \frac{N'''}{(N')^3} , \quad (46)$$

¹ Some papers use a different sign convention for f_{NL} . For example, Refs. [11, 20] use the opposite sign convention.

The numerical factors in Eq. (44) arise because the original definition is given in terms of the Bardeen potential on large scales (in the matter dominated era, md), $\Phi_{H\text{md}} = (3/5)\zeta_1$, so we have [4, 5, 6],

$$\frac{3}{5}\zeta = \Phi_{H\text{md}} + f_{NL}\Phi_{H\text{md}}^2 + g_{NL}\Phi_{H\text{md}}^3 + \dots \quad (47)$$

The primordial bispectrum and trispectrum are then given by Eqs. (37) and (40), where the non-linearity parameters f_{NL} and g_{NL} , given in Eqs. (38) and (42), reduce to Eqs. (45) and (46) respectively, and τ_{NL} given in Eq. (41) reduces to

$$\tau_{NL} = \frac{(N'')^2}{(N')^4} = \frac{36}{25}f_{NL}^2. \quad (48)$$

Notice that the bispectrum depends linearly on ζ_2 while the trispectrum has both a quadratic dependence upon ζ_2 and a linear dependence on ζ_3 . Thus τ_{NL} is proportional to f_{NL}^2 (shown in [5] using the Bardeen potential, and in [24] using this notation). However the trispectrum could be large even when the bispectrum is small because of the g_{NL} term [5, 23].

1. Inflaton scenario

In the case of standard single field inflation, where the primordial curvature perturbation is generated solely by the inflaton field, we can calculate the non-linearity parameters f_{NL} and g_{NL} in terms of the slow-roll parameters at Hubble-exit. Because the large-scale perturbations are adiabatic, ζ is non-linearly conserved on large scales [10, 25, 26] and the derivatives of the expansion, N' , N'' and N''' can be calculated at Hubble-exit. Using the definition (1), we find

$$N' = \frac{\bar{H}}{\dot{\bar{\phi}}} \simeq \frac{1}{\sqrt{2}} \frac{1}{m_p} \frac{1}{\sqrt{\epsilon}} \sim \mathcal{O}(\epsilon^{-\frac{1}{2}}), \quad (49)$$

$$N'' \simeq -\frac{1}{2} \frac{1}{m_p^2} \frac{1}{\epsilon} (\eta - 2\epsilon) \sim \mathcal{O}(1), \quad (50)$$

$$N''' \simeq \frac{1}{\sqrt{2}} \frac{1}{m_p^3} \frac{1}{\epsilon\sqrt{\epsilon}} \left(\epsilon\eta - \eta^2 + \frac{1}{2}\xi^2 \right) \sim \mathcal{O}(\epsilon^{\frac{1}{2}}), \quad (51)$$

where we have used the potential slow roll parameters

$$\epsilon \equiv \frac{m_p^2}{2} \left(\frac{V'}{V} \right)^2, \quad (52)$$

$$\eta \equiv m_p^2 \frac{V''}{V}, \quad (53)$$

$$\xi^2 \equiv m_p^4 \frac{V'V'''}{V^2}. \quad (54)$$

Hence the non-linearity parameters for single field inflation (45–46) are given by

$$f_{NL} = \frac{5}{6}(\eta - 2\epsilon), \quad (55)$$

$$\tau_{NL} = (\eta - 2\epsilon)^2, \quad (56)$$

$$g_{NL} = \frac{25}{54}(2\epsilon\eta - 2\eta^2 + \xi^2). \quad (57)$$

Note however that we have not calculated the full bispectrum and trispectrum at leading order in slow roll, because we assumed that the initial field fluctuations were Gaussian. If we included the contribution from the non-Gaussianity of the fields at Hubble exit, the bispectrum would have one extra term (19) and the trispectrum would have two extra terms, (36). The extra term for the trispectrum is at the same order in slow roll, because [12] $B^{ABC}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \sim \mathcal{O}(\epsilon^{\frac{1}{2}})$ and the second term of (36) is also of the same order. However Seery, Lidsey and Sloth find [18], $T^{ABCD}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \sim \mathcal{O}(1)$ which means the first term of (36) is suppressed by one less order in slow roll than the other three terms. However they find the contribution of this term is still too small to be observable,

even in the multiple field case [18]. All of these extra terms from the non-Gaussian field fluctuations are momentum dependent, while all of the non-linearity parameters are independent of momentum.

In single field inflation, ζ is conserved at all orders on superhorizon scales. Therefore no evolution of the bispectrum and trispectrum is possible after Hubble exit. Hence neither will be detectable in cosmic microwave background or large scale structure experiments, but from the 21 cm background there is the possibility that $f_{NL} \sim 0.01$ might be observable [27].

2. Curvaton scenario

In the curvaton scenario a weakly-coupled field (the curvaton field, χ) which is light, but subdominant during inflation comes to contribute a significant fraction of the energy density of the universe sometime after inflation. After it eventually decays, it is the fluctuations in this field that produce the primordial curvature perturbation, ζ [22].

In general, the energy density of the curvaton is some function of the field value at Hubble-exit, $\rho_\chi \propto g^2(\chi_*)$, and hence the primordial curvature perturbation when the curvaton decays is of local form (44). In the sudden-decay approximation the non-linearity parameters are [11, 28]

$$f_{NL} = \frac{5}{4r} \left(1 + \frac{gg''}{g'^2} \right) - \frac{5}{3} - \frac{5r}{6}, \quad (58)$$

and [23]

$$g_{NL} = \frac{25}{54} \left[\frac{9}{4r^2} \left(\frac{g^2 g'''}{g'^3} + 3 \frac{gg''}{g'^2} \right) - \frac{9}{r} \left(1 + \frac{gg''}{g'^2} \right) + \frac{1}{2} \left(1 - 9 \frac{gg''}{g'^2} \right) + 10r + 3r^2 \right], \quad (59)$$

and τ_{NL} satisfies (48), where the parameter r , is given by

$$r = \left[\frac{3\rho_\chi}{3\rho_\chi + 4\rho_r} \right]_{\text{decay}}, \quad (60)$$

where ρ_χ is the density of the curvaton field and ρ_r is the density of radiation and hence r satisfies $0 < r \leq 1$.

One can obtain significant non-Gaussianity if the curvaton does not dominate the total energy density of the Universe when it decays, $r \ll 1$, in which case we have

$$f_{NL} \simeq \frac{5}{4r} \left(1 + \frac{gg''}{g'^2} \right), \quad (61)$$

$$g_{NL} \simeq \frac{25}{24r^2} \left(\frac{g^2 g'''}{g'^3} + 3 \frac{gg''}{g'^2} \right). \quad (62)$$

One obtains a significant bispectrum, $f_{NL} \gg 1$ for $r \ll 1$ if $gg''/g'^2 \neq -1$. On the other hand if $gg''/g'^2 \simeq -1$ the bispectrum can be small even for $r \ll 1$ [29] and the first signal of non-Gaussianity could come from the trispectrum through $g_{NL} \gg 1$ [23].

V. CONCLUSIONS

In this paper we have given a general expression characterising the distribution of the primordial curvature perturbation due to an initial distribution of scalar field fluctuations during multiple-field inflation, using the δN -formalism.

We have given expressions for the power spectrum, bispectrum and trispectrum including all terms at leading order in a perturbative expansion, allowing for scale dependence and cross-correlations of the fields. In particular the connected part of the primordial trispectrum consists of four terms each with a different momentum dependence. One term depends upon the intrinsic connected part of the trispectrum of the field fluctuations and one term depends upon the bispectrum of the fields.

At lowest order in a slow-roll expansion the field fluctuations shortly after Hubble-exit are expected to be Gaussian and scale-invariant. In this case Lyth and Rodriguez [11] showed that the bispectrum from multiple-field inflation can be parameterised by a single non-linearity parameter f_{NL} dependent upon the second-derivatives of the local expansion, N , with respect to the field values, given in Eq. (38). The connected part of the trispectrum can be parameterised by two further non-linearity parameters, τ_{NL} and g_{NL} , given in Eqs. (41) and (42). τ_{NL} is another

function of the second derivatives of N [21], whereas g_{NL} is dependent on the third derivative. In the particular case where only one field generates the primordial perturbation we have $\tau_{NL} \propto f_{NL}^2$ [5, 24]. However g_{NL} can give rise to a connected part of the trispectrum at the same order [23] and hence constraints on the primordial bispectrum do not necessarily constrain the primordial trispectrum in such models.

In the case of standard single field inflation the trispectrum will be too small to ever be observed, but in alternative models such as DBI inflation [30] or the curvaton scenario [23] the trispectrum may be observable and could be an important test of such models.

Note added: While writing up this work, similar results appeared on the arXiv [31].

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