Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 69, n° 1 (2019), p. 81-118.
<http://aif.centre-mersenne.org/item/AIF_2019__69_1_81_0>

Ann. Inst. Fourier, Grenoble
69, 1 (2019) 81-118

PRINCIPAL BOUNDARY OF MODULI SPACES OF ABELIAN AND QUADRATIC DIFFERENTIALS

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ABSTRACT. — The seminal work of Eskin–Masur–Zorich described the principal boundary of moduli spaces of abelian differentials that parameterizes flat surfaces with a prescribed generic configuration of short parallel saddle connections. In this paper we describe the principal boundary for each configuration in terms of twisted differentials over Deligne–Mumford pointed stable curves. We also describe similarly the principal boundary of moduli spaces of quadratic differentials originally studied by Masur–Zorich. Our main technique is the flat geometric degeneration and smoothing developed by Bainbridge–Chen–Gendron–Grushevsky–Möller.

RÉSUMÉ. — Le travail fondateur d'Eskin–Masur–Zorich a décrit la limite principale des espaces de modules des différentielles abéliennes qui paramètre les surfaces plates possédant une configuration générique de petites connexions de selles parallèles prescrite. Dans cet article, nous décrivons la limite principale pour chaque configuration en terme de différentielles entrelacées sur les courbes stables pointées de Deligne–Mumford. Nous décrivons également la limite principale des espaces de modules des différentielles quadratiques étudiée à l'origine par Masur–Zorich. Nos principaux outils sont la dégénérescence géométrique plate et le lissage développés par Bainbridge–Chen–Gendron–Grushevsky–Möller.

1. Introduction

Many questions about Riemann surfaces are related to study their flat structures induced from abelian differentials, where the zeros of differentials correspond to the saddle points of flat surfaces. Loci of abelian differentials with prescribed type of zeros form a natural stratification of the moduli space of abelian differentials. These strata have fascinating geometry and can be applied to study dynamics on flat surfaces.

Keywords: Abelian differential, principal boundary, moduli space of stable curves, spin and hyperelliptic structures.
2010 *Mathematics Subject Classification:* 14H10, 14H15, 30F30, 32G15.
(*) D. Chen is partially supported by the NSF CAREER grant DMS-1350396 and Q. Chen is partially supported by the NSF grant DMS-1560830.

Given a configuration of saddle connections for a stratum of flat surfaces, Veech and Eskin–Masur ([11, 23]) showed that the number of collections of saddle connections with bounded lengths has quadratic asymptotic growth, whose leading coefficient is called the Siegel–Veech constant for this configuration. Eskin–Masur–Zorich ([12]) gave a complete description of all possible configurations of parallel saddle connections on a generic flat surface. They further provided a recursive method to calculate the corresponding Siegel–Veech constants. To perform this calculation, a key step is to describe the *principal boundary* whose tubular neighborhood parameterizes flat surfaces with short parallel saddle connections for a given configuration.

As remarked in [12], flat surfaces contained in the Eskin–Masur–Zorich principal boundary can be disconnected and have total genus smaller than that of the original stratum. Therefore, as the underlying complex curves degenerate by shrinking the short saddle connections, the Eskin–Masur–Zorich principal boundary does not directly imply the limit objects from the viewpoint of algebraic geometry. In this paper we solve this problem by describing the principal boundary in the setting of the strata compactification [4] and consequently in the Deligne–Mumford compactification.

Main Result

For each configuration we give a complete description for the principal boundary in terms of twisted differentials over pointed stable curves.

This result is a combination of Theorems 2.1 and 3.4. Along the way we deduce some interesting consequences about meromorphic differentials on $\mathbb{P}^1$ that admit the same configuration (see Propositions 2.3 and 3.8). Moreover, when a stratum contains connected components due to spin or hyperelliptic structures ([19]), Eskin–Masur–Zorich ([12]) described how to distinguish these structures nearby the principal boundary via an analytic approach. Here we provide algebraic proofs for the distinction of spin and hyperelliptic structures in the principal boundary under our setting (see Sections 4.6 and 4.7 for related results).

Masur–Zorich ([20]) described similarly the principal boundary of strata of quadratic differentials. Our method can also give a description of the principal boundary in terms of twisted quadratic differentials in the sense of [3] (see Section 5 for details).

Twisted differentials play an important role in our description of the principal boundary, so we briefly recall their definition (see [4] for more details). Given a zero type $\mu = (m_1, \ldots, m_n)$, a *twisted differential* η of type μ on an n-pointed stable curve $(C, \sigma_1, \ldots, \sigma_n)$ is a collection of (possibly meromorphic) differentials η_i on each irreducible component C_i of C, satisfying the following conditions:

(0) η has no zeros or poles away from the nodes and markings of C and η has the prescribed zero order m_i at each marking σ_i.
(1) If a node q joins two components C_1 and C_2, then $\operatorname{ord}_q \eta_1 + \operatorname{ord}_q \eta_2 = -2$.
(2) If $\operatorname{ord}_q \eta_1 = \operatorname{ord}_q \eta_2 = -1$, then $\operatorname{Res}_q \eta_1 + \operatorname{Res}_q \eta_2 = 0$.
(3) If C_1 and C_2 intersect at k nodes $q_1, \ldots, q_k$, then $\operatorname{ord}_{q_i} \eta_1 - \operatorname{ord}_{q_i} \eta_2$ are either all positive, or all negative, or all equal to zero for $i = 1, \ldots, k$.

Condition (3) provides a partial order between irreducible components that are not disjoint. If one expands it to a full order between all irreducible components of C, then there is an extra global residue condition which governs when such twisted differentials are limits of abelian differentials of type μ. A construction of the moduli space of twisted differentials can be found in [2].

By using η on all maximum components and forgetting its scales on components of smaller order, [4] describes a strata compactification in the Hodge bundle over the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$. As remarked in [4], if one forgets η and only keeps track of the underlying pointed stable curve $(C, \sigma_1, \ldots, \sigma_n)$, it thus gives the (projectivized) strata compactification in $\overline{\mathcal{M}}_{g,n}$. Hence our description of the principal boundary in terms of twisted differentials determines the corresponding boundary in the Deligne–Mumford compactification. To illustrate our results, we will often draw such stable curves in the Deligne–Mumford boundary.

For an introduction to flat surfaces and related topics, we refer to the surveys [8, 24, 25]. Besides [4], there are several other strata compactifications, see [13] for an algebraic viewpoint, [9, 16] for a log geometric viewpoint and [21] for a flat geometric viewpoint. Algebraic distinctions of spin and hyperelliptic structures in the boundary of strata compactifications are also discussed in [7, 9, 14].

This paper is organized as follows. In Sections 2 and 3 we describe the principal boundary of type I and of type II, respectively, following the roadmap of [12]. In Section 4 we provide algebraic arguments for distinguishing spin and hyperelliptic structures in the principal boundary. Finally in Section 5 we explain how one can describe the principal boundary of strata of quadratic differentials by using twisted quadratic differentials.

Throughout the paper we also provide a number of examples and figures to help the reader quickly grasp the main ideas.

Notation

We denote by μ the singularity type of differentials, by $\mathcal{H}(\mu)$ the stratum of abelian differentials of type μ and by $\mathcal{Q}(\mu)$ the stratum of quadratic differentials of type μ. An n-pointed stable curve is generally denoted by $(C, \sigma_1, \dots, \sigma_n)$. We use (C, η) to denote a twisted differential on C. The underlying divisor of a differential η is denoted by (η). Configurations of saddle connections are denoted by $\mathcal{C}$ and all configurations considered in this paper are admissible in the sense of [12].

Acknowledgements. We thank Matt Bainbridge, Alex Eskin, Quentin Gendron, Sam Grushevsky, Martin Möller and Anton Zorich for inspiring discussions on related topics. We also thank the referee for carefully reading the paper and many useful comments.

2. Principal boundary of type I

2.1. Configurations of type I: saddle connections joining distinct zeros

Let C be a flat surface in $\mathcal{H}(\mu)$ with two chosen zeros σ_1 and σ_2 of order m_1 and m_2, respectively. Suppose C has precisely p homologous saddle connections $\gamma_1, \dots, \gamma_p$ joining σ_1 and σ_2 such that the following conditions hold:

- All saddle connections γ_i are oriented from σ_1 to σ_2 with identical holonomy vectors.
- The cyclic order of $\gamma_1, \dots, \gamma_p$ at σ_1 is clockwise.
- The angle between γ_i and γ_{i+1} is $2\pi(a'_i + 1)$ at σ_1 and $2\pi(a''_i + 1)$ at σ_2, where $a'_i, a''_i \geqslant 0$.

Then we say that C has a *configuration of type* $\mathcal{C} = (m_1, m_2, \{a'_i, a''_i\}_{i=1}^p)$. We emphasis here that this configuration $\mathcal{C}$ is defined with the two chosen zeros σ_1 and σ_2. If $p = 1$, we also denote the configuration by $\mathcal{C} = (m_1, m_2)$ for simplicity. Since the cone angle at σ_i is $2\pi(m_i + 1)$ for $i = 1, 2$, we necessarily have

$$\sum_{i=1}^{p}(a'_i + 1) = m_1 + 1 \quad \text{and} \quad \sum_{i=1}^{p}(a''_i + 1) = m_2 + 1. \tag{2.1}$$

2.2. Graphs of configurations

Given two fixed zeros σ_1 and σ_2 and a configuration $\mathcal{C} = (m_1, m_2, \{a_i', a_i''\}_{i=1}^p)$ as in the previous section, to describe the dual graphs of the underlying nodal curves in the principal boundary of twisted differentials, we introduce the *configuration graph* $G(\mathcal{C})$ as follows:

(1) The set of vertices is $\{v_R, v_1, \cdots, v_p\}$.
(2) The set of edges is $\{l_1, \cdots, l_p\}$, where each l_i joins v_i and v_R.
(3) We associate to v_R the subset of markings $L_R = \{\sigma_1, \sigma_2\}$ and to each v_i a subset of markings $L_i \subset \{\sigma_j\}$ such that $L_R \sqcup L_1 \sqcup \cdots \sqcup L_p$ is a partition of $\{\sigma_1, \ldots, \sigma_n\}$.
(4) We associate to each v_i a positive integer $g(v_i)$ such that

$$\sum_{i=1}^{p} g(v_i) = g \quad \text{and} \quad \sum_{\sigma_j \in L_i} m_j + (a_i' + a_i'') = 2g(v_i) - 2.$$

Figure 2.1 shows a pointed nodal curve whose dual graph is of type $G(\mathcal{C})$:

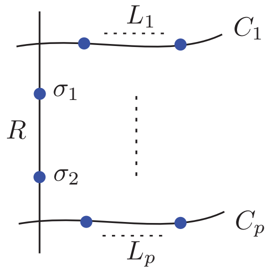

Figure 2.1. A curve with dual graph of type $\mathcal{C}$.

2.3. The principal boundary of type I

Denote by $\Delta(\mu, \mathcal{C})$ the space of twisted differentials η satisfying the following conditions:

- The underlying dual graph of η is given by $G(\mathcal{C})$, with nodes q_i and components C_i corresponding to l_i and v_i, respectively.
- The component R corresponding to the vertex v_R is isomorphic to $\mathbb{P}^1$ and contains only σ_1 and σ_2 among all the markings.
- Each C_i has markings labeled by L_i and has genus equal to $g(v_i)$.
- For each $i = 1, \ldots, p$, $\operatorname{ord}_{q_i} \eta_{C_i} = a_i' + a_i''$ and $\operatorname{ord}_{q_i} \eta_R = -a_i' - a_i'' - 2$.

- For each $i = 1, \dots, p$, $\mathrm{Res}_{q_i} \eta_R = 0$.
- η_R admits the configuration $\mathcal{C}$ of saddle connections from σ_1 to σ_2.

Recall that the twisted differential η defines a flat structure on R (up to scale). Thus it makes sense to talk about the configuration $\mathcal{C}$ on R. We say that $\Delta(\mu, \mathcal{C})$ is the *principal boundary* associated to the configuration $\mathcal{C}$.

Suppose $C^\varepsilon \in \mathcal{H}(\mu)$ has the configuration $\mathcal{C} = (m_1, m_2, \{a_i', a_i''\}_{i=1}^p)$ such that the p homologous saddle connections $\gamma_1, \dots, \gamma_p$ of C have length at most ε. We want to determine the limit twisted differential as the length of all γ_i shrinks to zero. To avoid further degeneration, suppose that C^ε does not have any other saddle connections shorter than 3ε (the locus of such C^ε is called the *thick part* of the configuration $\mathcal{C}$ in [12]). Take a small disk under the flat metric such that it contains σ_1, σ_2, all γ_i, and no other zeros (see [12, Figure 5]). Within this disk, shrink γ_i to zero while keeping the configuration $\mathcal{C}$, such that all other periods become arbitrarily large compared to γ_i.

Theorem 2.1. — *The limit twisted differential of C^ε as $\gamma_i \to 0$ is contained in $\Delta(\mu, \mathcal{C})$. Conversely, twisted differentials in $\Delta(\mu, \mathcal{C})$ can be smoothed to of type C^ε.*

Proof. — Since γ_i and γ_{i+1} are homologous and next to each other, they bound a surface C_i^ε with γ_i and γ_{i+1} as boundary (see the lower right illustration of [12, Figure 5] where C_i^ε is denoted by S_i). The inner angle between γ_i and γ_{i+1} at σ_1 is $2\pi(a_i' + 1)$ and at σ_2 is $2\pi(a_i'' + 1)$. Shrinking the γ_j to zero under the flat metric, the limit of C_i^ε forms a flat surface C_i, and denote by q_i the limit position of σ_1 and σ_2 in C_i. This shrinking operation is the inverse of breaking up a zero, see [12, Figure 3], which implies that the cone angle at q_i is $2\pi(a_i' + a_i'' + 1)$, hence C_i has a zero of order $a_i' + a_i''$ at q_i.

On the other hand, instead of shrinking the γ_j, up to scale it amounts to expanding the other periods of C_i^ε arbitrarily long compared to the γ_j. Since a small neighborhood N_i enclosing both γ_i and γ_{i+1} in C_i^ε consists of $2(a_i' + a_i'' + 1)$ metric half-disks, under the expanding operation they turn into $2(a_i' + a_i'' + 1)$ metric half-planes that form the basic domain decomposition for a pole of order $a_i' + a_i'' + 2$ in the sense of [6]. Moreover, the boundary loop of N_i corresponds to the vanishing cycle around q_i in the shrinking operation, which implies that the resulting pole will be glued to q_i as a node in the limit stable curve, hence we still use q_i to denote the pole. See Figure 2.2 for the case $p = 2$ and $m_1 = m_2 = 0$.

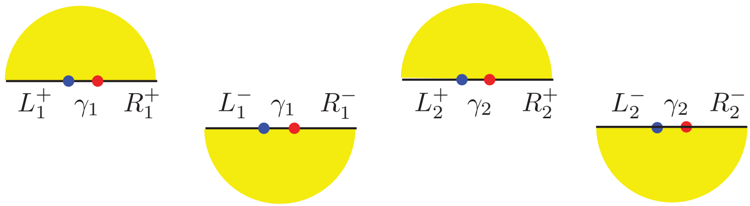

Figure 2.2. The flat geometric neighborhood of γ_1 and γ_2 for the case $p = 2$ and $m_1 = m_2 = 0$. Here we identify $L_1^- = L_2^+$, $L_1^+ = L_2^-$, $R_1^+ = R_2^-$, and $R_1^- = R_2^+$. As $\gamma_1, \gamma_2 \to 0$, the middle two half-disks form a neighborhood of an ordinary point and the remaining two half-disks form a neighborhood of another ordinary point. Alternatively as $L_i^\pm$ and $R_j^\pm \to \infty$, the middle two half-planes form a neighborhood of a double pole and the remaining two half-planes form a neighborhood of another double pole. Both poles have zero residue.

Let (R, η_R) be the limit meromorphic differential out of the expanding operation. We thus conclude that

$$(\eta_R) = m_1\sigma_1 + m_2\sigma_2 - \sum_{i=1}^{p}(a'_i + a''_i + 2)q_i.$$

By the relation (2.1), the genus of R is zero, hence $R \cong \mathbb{P}^1$. Since $q_i = C_i \cap R$ is a separating node, it follows from the global residue condition of [4] that $\mathrm{Res}_{q_i} \eta_R = 0$. Finally, in the expanding process the saddle connections γ_i are all fixed, hence the configuration $\mathcal{C}$ is preserved in the limit meromorphic differential η_R. Summarizing the above discussion, we see that the limit twisted differential is parameterized by $\Delta(\mu, \mathcal{C})$.

The other part of the claim follows from the flat geometric smoothing of [4], as twisted differentials in $\Delta(\mu, \mathcal{C})$ satisfy the global residue condition and have the desired configuration of saddle connections. □

Remark 2.2. — For the purpose of calculating Siegel–Veech constants, the Eskin–Masur–Zorich principal boundary only takes into account the non-degenerate components C_i and discards the degenerate rational component R, though it is quite visible — for instance, R can be seen as the central sphere in [12, Figure 5].

2.4. Meromorphic differentials of type I on $\mathbb{P}^1$

Recall that for a twisted differential η in $\Delta(\mu, \mathcal{C})$, its restriction η_R on the component $R \cong \mathbb{P}^1$ has two zeros and p poles, where the residue at each pole is zero. Up to scale, η_R is uniquely determined by the zeros and poles. In this section we study the locus of $\mathbb{P}^1$ marked at such zeros and poles.

Given integers $m_1, m_2 \geqslant 1$ and $n_1, \dots, n_p \geqslant 2$ with $m_1 + m_2 - \sum_{i=1}^p n_i = -2$, let $\mathcal{Z} \subset \mathcal{M}_{0,p+2}$ be the locus of pointed rational curves $(\mathbb{P}^1, \sigma_1, \sigma_2, q_1, \dots, q_p)$ such that there exists a differential η_0 on $\mathbb{P}^1$ satisfying that

$$(\eta_0) = m_1\sigma_1 + m_2\sigma_2 - \sum_{i=1}^p n_i q_i \quad \text{and} \quad \operatorname{Res}_{q_i} \eta_0 = 0$$

for each $i = 1, \dots, p$.

For a given (admissible) configuration $\mathcal{C} = (m_1, m_2, \{a_i', a_i''\}_{i=1}^p)$, consider the subset $\mathcal{Z}(\mathcal{C}) \subset \mathcal{Z}$ parameterizing differentials η_0 on $\mathbb{P}^1$ (up to scale) that admit a configuration of type $\mathcal{C}$.

Proposition 2.3. — *$\mathcal{Z}$ is a union of $\mathcal{Z}(\mathcal{C})$ for all possible (admissible) configuration $\mathcal{C}$ and each $\mathcal{Z}(\mathcal{C})$ consists of a single point.*

Proof. — We provide a constructive proof using the flat geometry of meromorphic differentials. Let us make some observation first. Suppose η_0 is a differential on $\mathbb{P}^1$ whose underlying divisor corresponds to a point in $\mathcal{Z}$. Since η_0 has zero residue at every pole, for any closed path γ that does not contain a pole of η_0, the Residue Theorem says that

$$\int_\gamma \eta_0 = 0.$$

In particular, if α and β are two saddle connections joining σ_1 to σ_2, then $\alpha - \beta$ represents a closed path on $\mathbb{P}^1$, hence

$$\int_\alpha \eta_0 = \int_\beta \eta_0,$$

and α and β necessarily have the same holonomy. It also implies that η_0 has no self saddle connections. Collect the saddle connections from σ_1 to σ_2, list them clockwise at σ_1, and count the angles between two nearby ones. Since the saddle connections have the same holonomy, the angles between them are multiples of 2π, and hence they give rise to a configuration $\mathcal{C}$. It implies that the underlying divisor of η_0 corresponds to a point in $\mathcal{Z}(\mathcal{C})$. Therefore, $\mathcal{Z}$ is a union of $\mathcal{Z}(\mathcal{C})$.

Now suppose η_0 admits a configuration of type $\mathcal{C} = (m_1, m_2, \{a_i', a_i''\}_{i=1}^p)$, i.e., up to scale it corresponds to a point in $\mathcal{Z}(\mathcal{C})$. Recall that σ_1, σ_2, and q_i are the zeros and poles of order m_1, m_2, and $a_i' + a_i'' + 2$, respectively, where $i = 1, \dots, p$, and $\gamma_1, \dots, \gamma_p$ are the saddle connections joining σ_1 to σ_2 such that the angle between γ_i and γ_{i+1} in the clockwise orientation at

σ_1 is $2\pi(a'_i+1)$, and at σ_2 is $2\pi(a''_i+1)$. By the preceding paragraph, there are no other saddle connections between σ_1 and σ_2.

Rescale η_0 such that all the γ_i have holonomy equal to 1, that is, they are in horizontal, positive direction, and of length 1. Cut the flat surface η_0 along all horizontal directions through σ_1 and σ_2, such that η_0 is decomposed into a union of half-planes as basic domains in the sense of [6]. These basic domains are of two types according to their boundary half-lines and saddle connections. The boundary of the basic domains of the first type contains exactly one of σ_1 and σ_2 that emanates two half-lines to infinity on both sides. The boundary of the basic domains of the second type, from left to right, consists of a half-line ending at σ_1, followed by a saddle connection γ_i, and then a half-line emanating for σ_2.

Since the angle between γ_i and γ_{i+1} is given for each i, the configuration $\mathcal{C}$ determines uniquely how these basic domains are glued together to form η_0. More precisely, start from an upper half-plane S_1^+ of the second type with two boundary half-lines L_1^+ to the left and R_1^+ to the right, joined by the saddle connection γ_1. Turn around σ_1 in the clockwise orientation. Then we will see a lower half-plane S_1^- of the second type with two boundary half-lines L_1^- and R_1^- joined by γ_1. If $a'_1 = 0$, i.e., if the angle between γ_1 and γ_2 in the clockwise orientation is 2π, then next we will see an upper half-plane S_2^+ of the second type with two boundary half-lines L_2^+ and R_2^+ joined by γ_2, which is glued to S_1^- by identifying L_2^+ with L_1^-. See Figure 2.2 above for an illustration of this case.

On the other hand if $a'_1 > 0$, we will see a'_1 pairs of upper and lower half-planes of the first type containing only σ_1 in their boundary, and then followed by the upper half-plane of the second type containing γ_2 in the boundary. Repeat this process for each pair γ_i and γ_{i+1} consecutively, and also use the angle between γ_i and γ_{i+1} at σ_2 to determine the identification of the $R_i^\pm$-edges emanated from σ_2. We conclude that the gluing pattern of these half-planes is uniquely determined by the configuration $\mathcal{C}$.

Finally, since the angle between γ_i and γ_{i+1} at σ_1 is $2\pi(a'_i + 1)$ and at σ_2 is $2\pi(a''_i + 1)$, it determines precisely $a'_i + a''_i + 1$ pairs of upper and lower half-planes that share the same point at infinity. In other words, they form a flat geometric neighborhood of a pole with order $a'_i + a''_i + 2$, which is the desired pole order of q_i for $i = 1, \ldots, p$. We have thus verified that $\mathcal{Z}(\mathcal{C})$ is nonempty and all differentials up to scale parameterized by $\mathcal{Z}(\mathcal{C})$ have the same basic domain decomposition, hence $\mathcal{Z}(\mathcal{C})$ consists of a single point. □

Example 2.4. — Consider the case $m_1 = 1$, $m_2 = 1$, $n_1 = 2$ and $n_2 = 2$. The only admissible configuration is

$$a'_1 = a''_1 = a'_2 = a''_2 = 0,$$

hence $\mathcal{Z}$ consists of a single point. As a cross check, take $\sigma_1 = 1$, $q_1 = 0$, and $q_2 = \infty$ in $\mathbb{P}^1$, and let z be the affine coordinate. Then up to scale η_0 can be written as

$$\frac{(z-1)(z-\sigma_2)}{z^2}\mathrm{d}z.$$

It is easy to see that $\mathrm{Res}_{q_i}\,\eta_0 = 0$ if and only if $\sigma_2 = -1$.

Example 2.5. — Consider the case $m_1 = 1$, $m_2 = 3$ and $n_1 = n_2 = n_3 = 2$. There do not exist nonnegative integers a'_1, a'_2, a'_3 satisfying that

$$(a'_1 + 1) + (a'_2 + 1) + (a'_3 + 1) = m_1 + 1 = 2,$$

because the left-hand side is at least 3. Since there is no admissible configuration, we conclude that $\mathcal{Z}$ is empty. As a cross check, let $q_1 = 0$, $q_2 = 1$, and $q_3 = \infty$. Up to scale η_0 can be written as

$$\frac{(z-\sigma_1)(z-\sigma_2)^3}{z^2(z-1)^2}\mathrm{d}z.$$

One can directly verify that there are no $\sigma_1, \sigma_2 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ such that $\mathrm{Res}_{q_i}\,\eta_0 = 0$.

3. Principal boundary of type II

3.1. Configurations of type II: saddle connections joining a zero to itself

Let C be a flat surface in $\mathcal{H}(\mu)$. Suppose C has precisely m homologous closed saddle connections $\gamma_1, \ldots, \gamma_m$, each joining a zero to itself. Let $L \subset \{1, \ldots, m\}$ be an index subset such that the curves γ_l for $l \in L$ bound q cylinders. After removing the cylinders along with all the γ_k, the remaining part in C splits into $p = m - q$ disjoint surfaces $C_1, \ldots, C_p$, where the boundary of the closure $\overline{C}_k$ of each C_k consists of two closed saddle connections α_k and β_k. These surfaces are glued together in a cyclic order to form C. More precisely, each C_k is connected to C_{k+1} by either identifying α_k with β_{k+1} (as some γ_i in C) or inserting a metric cylinder with boundary α_k and β_{k+1}. The sum of genera of the C_k is $g-1$, because the cyclic gluing procedure creates a central handle, hence it adds an extra one to the total genus (see [12, Figure 7]).

There are two types of the surfaces C_k according to their boundary components. If the boundary saddle connections α_i and β_i of $\overline{C}_i$ are disjoint, we say that C_i has *a pair of holes* boundary. In this case α_i contains a single zero z_i with cone angle $(2a_i+3)\pi$ inside C_i, and β_i contains a single zero w_i with cone angle $(2b_i+3)\pi$ inside $\overline{C}_i$, where $a_i, b_i \geqslant 0$. We also take into account the special case $m=1$, i.e., when we cut C along γ_1, we get only one surface C_1 with two disjoint boundary components α_1 and β_1. In this case z_1 is identified with w_1 in C, and we still say that C_1 has a pair of holes boundary.

For the remaining case, if α_j and β_j form a connected component for the boundary of $\overline{C}_j$, we say that C_j has a *figure eight* boundary. In this case α_j and β_j contain the same zero z_j. Denote by $2(c_j'+1)\pi$ and $2(c_j''+1)\pi$ the two angles bounded by α_j and β_j inside C_j, where $c_j', c_j'' \geqslant 0$, and let $c_j = c_j' + c_j''$.

In summary, the configuration considered above consists of the data

$$(L, \{a_i, b_i\}, \{c_j', c_j''\}).$$

Conversely, given the surfaces C_k along with some metric cylinders, local gluing patterns can create zeros of the following three types (see [12, Figure 12] and [5, Figures 6-8]):

(i) A cylinder, followed by $k \geqslant 1$ surfaces $C_1, \ldots, C_k$, each of genus $g_i \geqslant 1$ with a figure eight boundary, followed by a cylinder. The total angle at the newborn zero is

$$\pi + \sum_{i=1}^{k} (2c_i' + 2c_i'' + 4)\pi + \pi,$$

hence its zero order is

$$\sum_{i=1}^{k} (c_i + 2).$$

(ii) A cylinder, followed by $k \geqslant 0$ surfaces C_i, each of genus $g_i \geqslant 1$ with a figure eight boundary, followed by a surface C_{k+1} of genus $g_{k+1} \geqslant 1$ with a pair of holes boundary. The total angle at the newborn zero is

$$\pi + \sum_{i=1}^{k} (2c_i' + 2c_i'' + 4)\pi + (2b_{k+1} + 3)\pi,$$

hence its zero order is

$$\sum_{i=1}^{k} (c_i + 2) + (b_{k+1} + 1).$$

(iii) A surface C_0 of genus $g_0 \geqslant 1$ with a pair of holes boundary, followed by $k \geqslant 0$ surfaces C_i, each of genus $g_i \geqslant 1$ with a figure eight boundary, followed by a surface C_{k+1} of genus $g_{k+1} \geqslant 1$ with a pair of holes boundary. The total angle at the newborn zero is

$$(2a_0+3)\pi + \sum_{i=1}^{k}(2c_i' + 2c_i'' + 4)\pi + (2b_{k+1}+3)\pi,$$

hence its zero order is

$$\sum_{i=1}^{k}(c_i+2) + (a_0+1) + (b_{k+1}+1).$$

For example, the flat surface in [12, Figure 7] is constructed as follows: S_1 with a pair of holes boundary, followed by S_2 with a pair of holes boundary, then a cylinder, followed by S_3 with a figure eight boundary, then another cylinder, followed by S_4 with a figure eight boundary, and finally back to S_1.

3.2. The principal boundary of type II

Suppose $C^\varepsilon \in \mathcal{H}(\mu)$ has the configuration $\mathcal{C} = (L, \{a_i, b_i\}, \{c_j', c_j''\})$ with the m homologous saddle connections $\gamma_1, \dots, \gamma_m$ of length at most ε. Moreover, suppose that C^ε does not have any other saddle connections shorter than 3ε. As before, we degenerate C^ε by shrinking γ_i to zero while keeping the configuration, such that the ratio of any other period to γ_i becomes arbitrarily large. Let $\Delta(\mu, \mathcal{C})$ be the space of twisted differentials that arise as limits of such a degeneration process. Recall the three types of gluing patterns and newborn zeros in the preceding section. We will analyze the types of their degeneration as building blocks to describe twisted differentials in $\Delta(\mu, \mathcal{C})$.

For the convenience of describing the degeneration, we view a cylinder as a union of two half-cylinders by truncating it in the middle. Then as its height tends to be arbitrarily large compared to the width, each half-cylinder becomes a half-infinite cylinder, which represents a flat geometric neighborhood of a simple pole. Moreover, the two newborn simple poles have opposite residues, because the two half-infinite cylinders have the same width with opposite orientations.

Proposition 3.1. — *Consider a block of surfaces of type* (i) *in C^ε, that is, a half-cylinder, followed by $k \geqslant 1$ surfaces $C_1^\varepsilon, \dots, C_k^\varepsilon$, each of genus $g_i \geqslant 1$ with a figure eight boundary, followed by a half-cylinder. Let σ be the newborn zero of order $\sum_{i=1}^{k}(c_i+2)$. As $\varepsilon \to 0$, we have*

- *The limit differential consists of k disjoint surfaces $C_1, \dots, C_k$ attached to a component $R \cong \mathbb{P}^1$ at the nodes $q_1, \dots, q_k$, respectively.*
- *R contains only σ among all the markings.*
- *For each $i = 1, \dots, k$, $\operatorname{ord}_{q_i} \eta_{C_i} = c_i$ and $\operatorname{ord}_{q_i} \eta_R = -c_i - 2$.*
- *For each $i = 1, \dots, k$, $\operatorname{Res}_{q_i} \eta_R = 0$.*
- *η_R has two simple poles at q_0 and $q_{k+1} \in R \setminus \{\sigma, q_1, \dots, q_k\}$ with opposite residues $\pm r$.*
- *η_R admits a configuration of type (i), i.e., it has precisely $k+1$ homologous self saddle connections with angles $2(c_i' + 1)\pi$ and $2(c_i'' + 1)\pi$ in between consecutively for $i = 1, \dots, k$, and with holonomy equal to r up to sign.*

See Figure 3.1 for an illustration of the underlying curve of the limit differential.

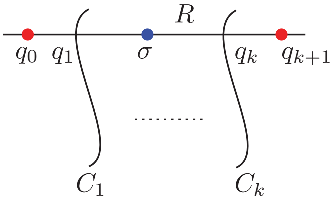

Figure 3.1. The underlying curve of the limit differential in Proposition 3.1.

Proof. — As $\varepsilon \to 0$, the limit of each C_i^ε is a flat surface C_i, where the figure eight boundary of C_i^ε shrinks to a single zero q_i with cone angle $(2c_i + 2)\pi$, i.e., q_i is a zero of order c_i. This shrinking operation is the inverse of the figure eight construction, see [12, Figure 10]. On the other hand, instead of shrinking the boundary saddle connections α_i, β_i of the C_i^ε, up to scale it amounts to expanding the other periods of the C_i^ε arbitrarily long compared to the α_i, β_i. Since a small neighborhood N_i enclosing both α_i and β_i in C_i^ε consists of $2(c_i' + c_i'' + 1)$ metric half-disks, under the expanding operation they turn into $2(c_i' + c_i'' + 1) = 2(c_i + 1)$ metric half-planes that form the basic domain decomposition for a pole of order $c_i + 2$ in the sense of [6]. The boundary loop of N_i corresponds to the vanishing cycle around q_i in the shrinking operation, which implies that the resulting pole will be glued to q_i as a node in the limit. In addition, the two half-cylinders expand to two half-infinite cylinders, which create two simple poles q_0 and q_{k+1} with opposite residues $\pm r$, where r encodes the width of the cylinders.

Let (R, η_R) be the limit meromorphic differential out of the expanding operation. We thus conclude that

$$(\eta_R) = \left(\sum_{i=1}^{k}(c_i+2)\right)\sigma - \sum_{i=1}^{k}(c_i+2)q_i - q_0 - q_{k+1},$$

and hence the genus of R is zero. Since $q_i = C_i \cap R$ is a separating node, it follows from the global residue condition of [4] that $\operatorname{Res}_{q_i}\eta_R = 0$. As a cross check,

$$\sum_{i=0}^{k+1}\operatorname{Res}_{q_i}\eta_R = \operatorname{Res}_{q_0}\eta_R + 0 + \cdots + 0 + \operatorname{Res}_{q_{k+1}}\eta_R = 0,$$

hence η_R satisfies the Residue Theorem on R. Finally, the cylinders are glued to the figure eight boundary on both sides, hence the $k+1$ homologous self saddle connections have holonomy equal to r up to sign. Their configuration (holonomy and angles in between) is preserved in the expanding process, hence the limit differential η_R possesses the desired configuration. □

Proposition 3.2. — *Consider a block of surfaces of type* (ii) *in* C^ε, *that is, a half-cylinder, followed by* $k \geqslant 0$ *surfaces* $C_1^\varepsilon, \ldots, C_k^\varepsilon$, *each of genus* $g_i \geqslant 1$ *with a figure eight boundary, followed by a surface* C_{k+1}^ε *of genus* $g_{k+1} \geqslant 1$ *with a pair of holes boundary. Let* σ *be the newborn zero of order* $\sum_{i=1}^{k}(c_i+2) + (b_{k+1}+1)$. *As* $\varepsilon \to 0$, *we have*

- *The limit differential consists of* $k+1$ *disjoint surfaces* $C_1, \ldots, C_{k+1}$ *attached to a component* $R \cong \mathbb{P}^1$ *at the nodes* $q_1, \ldots, q_{k+1}$, *respectively.*
- R *contains only* σ *among all the markings.*
- *For each* $i = 1, \ldots, k$, $\operatorname{ord}_{q_i}\eta_{C_i} = c_i$ *and* $\operatorname{ord}_{q_i}\eta_R = -c_i - 2$.
- $\operatorname{ord}_{q_{k+1}}\eta_{C_{k+1}} = b_{k+1}$ *and* $\operatorname{ord}_{q_{k+1}}\eta_R = -b_{k+1} - 2$.
- *For each* $i = 1, \ldots, k$, $\operatorname{Res}_{q_i}\eta_R = 0$.
- η_R *has a simple pole at* $q_0 \in R \setminus \{\sigma, q_1, \ldots, q_{k+1}\}$ *with* $\operatorname{Res}_{q_0}\eta_R = -\operatorname{Res}_{q_{k+1}}\eta_R = \pm r$.
- η_R *admits a configuration of type* (ii), *i.e., it has precisely* $k+1$ *homologous self saddle connections with angles* $2(c_i'+1)\pi$ *and* $2(c_i''+1)\pi$ *in between consecutively for* $i = 1, \ldots, k$, *and with holonomy equal to* r *up to sign.*

See Figure 3.2 for an illustration of the underlying curve of the limit differential.

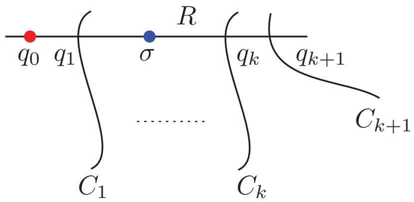

Figure 3.2. The underlying curve of the limit differential in Proposition 3.2.

Proof. — The proof is almost identical with the preceding one. The only difference occurs at the last surface. A small neighborhood N_{k+1} enclosing β_{k+1} in C^{ε}_{k+1} consists of $2(b_{k+1}+1)$ half-disks, one of which is irregular as in [12, Figure 8], hence in the expanding process they turn into $2(b_{k+1}+1)$ half-planes, giving a flat geometric neighborhood for a pole of order $b_{k+1}+2$. Moreover, N_{k+1} is homologous to the γ_i. The orientation of N_{k+1} is the opposite to that of N_0 enclosing the boundary α_0 of the beginning half cylinder, hence their homology classes add up to zero. We thus conclude that $\operatorname{Res}_{q_0} \eta_R = -\operatorname{Res}_{q_{k+1}} \eta_R$. Alternatively, it follows from the Residue Theorem applied to R, since $\operatorname{Res}_{q_i} \eta_R = 0$ for all $i = 1, \ldots, k$. The holonomy of the saddle connections and the angles between them are preserved in the expanding process, hence η_R has the configuration as described. □

Proposition 3.3. — *Consider a block of surfaces of type* (iii) *in* C^{ε}, *that is, a surface* C^{ε}_0 *of genus* $g_{k+1} \geqslant 1$ *with a pair of holes boundary, followed by* $k \geqslant 0$ *surfaces* $C^{\varepsilon}_1, \ldots, C^{\varepsilon}_k$, *each of genus* $g_i \geqslant 1$ *with a figure eight boundary, followed by a surface* C^{ε}_{k+1} *of genus* $g_{k+1} \geqslant 1$ *with a pair of holes boundary. Let* σ *be the newborn zero of order* $\sum_{i=1}^{k}(c_i + 2) + (a_0 + 1) + (b_{k+1} + 1)$. *As* $\varepsilon \to 0$, *we have*

- *The limit differential consists of* $k+2$ *disjoint surfaces* $C_0, \ldots, C_{k+1}$ *attached to a component* $R \cong \mathbb{P}^1$ *at the nodes* $q_0, \ldots, q_{k+1}$, *respectively.*
- R *contains only* σ *among all the markings.*
- *For each* $i = 1, \ldots, k$, $\operatorname{ord}_{q_i} \eta_{C_i} = c_i$ *and* $\operatorname{ord}_{q_i} \eta_R = -c_i - 2$.
- $\operatorname{ord}_{q_0} \eta_{C_0} = a_0$ *and* $\operatorname{ord}_{q_0} \eta_R = -a_0 - 2$.
- $\operatorname{ord}_{q_{k+1}} \eta_{C_{k+1}} = b_{k+1}$ *and* $\operatorname{ord}_{q_{k+1}} \eta_R = -b_{k+1} - 2$.
- *For each* $i = 1, \ldots, k$, $\operatorname{Res}_{q_i} \eta_R = 0$.
- $\operatorname{Res}_{q_0} \eta_R = -\operatorname{Res}_{q_{k+1}} \eta_R = \pm r$.
- η_R *admits a configuration of type* (iii), *i.e., it has precisely* $k+1$ *homologous self saddle connections with angles* $2(c'_i + 1)\pi$ *and*

$2(c_i''+1)\pi$ in between consecutively for $i=1,\dots,k$, and with holonomy equal to r up to sign.

See Figure 3.3 for an illustration of the underlying curve of the limit differential.

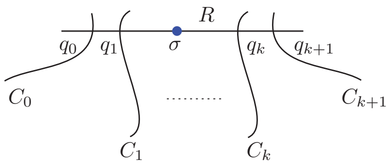

Figure 3.3. The underlying curve of the limit differential in Proposition 3.3.

Proof. — Since the beginning and ending surfaces both have a pair of holes boundary, the proof follows from the previous two. □

Let us call the limit twisted differentials in Propositions 3.1, 3.2, and 3.3 *surfaces of degenerate type* (i), (ii), and (iii), respectively. In order to glue them to form a global twisted differential, the above proofs (and also the definition of twisted differentials) imply the following gluing pattern. The simple pole q_0 (or q_{k+1}) in a surface of degenerate type (i) has to be glued with a simple pole in another surface of type (i) or (ii), and the same description holds for q_0 in a surface of type (ii). For a surface of type (ii), the component C_{k+1} has to be contained in another surface of type (ii) or (iii). Namely, it has a zero of order b'_{k+1} that is glued with a pole q'_{k+1} of order $b'_{k+1}+2$ in the rational component R' of the other surface. The same description holds for C_0 and C_{k+1} in a surface of type (iii).

Theorem 3.4. — *In the above setting, $\Delta(\mu,\mathcal{C})$ parameterizes twisted differentials constructed by gluing surfaces of degenerate type* (i), (ii), *and* (iii).

Proof. — Since C^ε admits the configuration $\mathcal{C}=(L,\{a_i,b_i\},\{c'_j,c''_j\})$, it can be constructed by gluing blocks of surfaces of type (i), (ii), and (iii). By applying Propositions 3.1, 3.2, and 3.3 simultaneously, we thus conclude that the limit twisted differential is formed by gluing surfaces of degenerate type (i), (ii), and (iii) as above. □

We summarize some useful observation out of the proofs.

Remark 3.5. — If the homologous closed saddle connections in a configuration $\mathcal{C}$ of type II contains k distinct zeros, then a curve in $\Delta(\mu, \mathcal{C})$ contains k rational components. Moreover, if two rational components intersect, then each of them has a simple pole at the node, and the residues at the two branches of the node add up to zero. In general, at the polar nodes the residues are $\pm r$ for a fixed nonzero $r \in \mathbb{C}$, such that their signs are alternating along the (unique) circle in the dual graph of the entire curve, and that the holonomy of the saddle connections is equal to r up to sign.

Example 3.6. — The limit of the surface in [12, Figure 7] as the γ_i shrink to zero is of the following type: S_1, followed by a marked $\mathbb{P}^1$, followed by S_2, followed by a marked $\mathbb{P}^1$, followed by a marked $\mathbb{P}^1$ with an S_3 tail, followed by a marked $\mathbb{P}^1$ with an S_4 tail, and back to S_1, see Figure 3.4, where R_1 is of type (iii), R_2 is of type (ii), R_3 is of type (i) and R_4 is of type (ii).

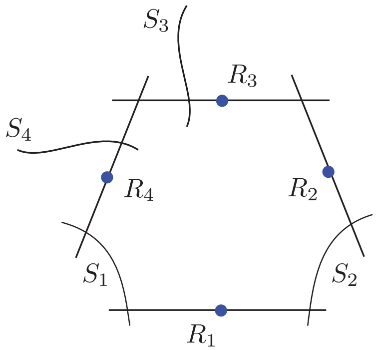

Figure 3.4. The underlying curve of the degeneration of [12, Figure 7].

Example 3.7. — The limit of the surface in [12, Figure 11] as the γ_i shrink to zero is of the following type: a flat torus E_1, followed by a chain of two $\mathbb{P}^1$, each with a marked simple zero, followed by a flat torus E_2, followed by a chain of two $\mathbb{P}^1$, each with a marked simple zero, and back to E_1, see Figure 3.5. Moreover, the differential on each $\mathbb{P}^1$ has a double pole at the intersection with one of the tori and has a simple pole at the intersection with one of the $\mathbb{P}^1$. Finally, the residues at the two poles of each $\mathbb{P}^1$ are $\pm r$ for some fixed nonzero $r \in \mathbb{C}$, such that their signs are alternating along the cyclic dual graph of the entire curve.

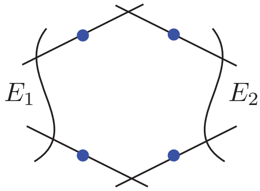

Figure 3.5. The underlying curve of the degeneration of [12, Figure 11].

3.3. Meromorphic differentials of type II on $\mathbb{P}^1$

Recall in Proposition 2.3 we showed that differentials on $\mathbb{P}^1$ admitting a given configuration of type I are unique up to scale. The same result holds for differentials on $\mathbb{P}^1$ admitting a given configuration of type (i), (ii), or (iii) as above.

PROPOSITION 3.8. — *Let η_0 be a differential on $\mathbb{P}^1$ that admits a configuration of type either* (i), (ii), *or* (iii) *as described in Propositions 3.1, 3.2, and 3.3. Then up to scale such η_0 is unique.*

Proof. — We provide a constructive proof for the case of type (i), which is analogous to the proof of Proposition 2.3. The other two types follow similarly.

Let us make some observation first. Suppose η_0 is a differential on $\mathbb{P}^1$ with a unique zero σ and $k+2$ poles $q_0, \dots, q_{k+1}$ such that $\operatorname{Res}_{q_i} \eta_0 = 0$ for $i = 1, \dots, k$, and that $\operatorname{Res}_{q_0} \eta_0 = -\operatorname{Res}_{q_{k+1}} = \pm r$ for a nonzero r. Let α and β be two self saddle connections of η_0. Treat them as closed loops in $\mathbb{C} = \mathbb{P}^1 \setminus \{q_{k+1}\}$. Then the indices of α and β to q_0 cannot be zero, for otherwise the integral of η_0 along them would be zero, contradicting that they are saddle connections of positive length. Therefore, both of them enclose q_0 in $\mathbb{C}$, hence by the Residue Theorem

$$\int_\alpha \eta_0 = \int_\beta \eta_0 = \pm r.$$

We conclude that in this case all saddle connections of η_0 are homologous with holonomy equal to $\pm r$.

Now suppose η_0 admits the configuration of type (i) (as the description for η_R in Proposition 3.1). Rescale η_0 such that the holonomy of the saddle connections $\gamma_1, \dots, \gamma_{k+1}$ is 1. By the preceding paragraph, η_0 has no other saddle connections. Cut the flat surface η_0 along all horizontal directions through the unique zero σ. Since η_0 has two simple poles with opposite residues equal to ± 1, we see two half-infinite cylinders with boundary given by the first and the last saddle connections γ_1 and γ_{k+1}, respectively. The rest part of η_0 splits into half-planes as basic domains in the sense of [6], which are of two types according to their boundary. The boundary of the half-planes of the first type contains σ that emanates two half-lines to infinity on both sides. The boundary of the half-planes of the second type, from left to right, consists of a half-line ending at σ, followed by a saddle connection γ_i, and then a half-line emanated from σ.

Since the angles between γ_i and γ_{i+1} are given on both sides inside the open surface (after removing the two half-infinite cylinders), this configuration determines how these half-planes are glued together. More precisely, say in the counterclockwise direction the angle between γ_i and γ_{i+1} is $2\pi(c'_i + 1)$. Then starting from the upper half-plane S_i^+ of the second type containing γ_i in the boundary and turning counterclockwise, we will see c'_i pairs of lower and upper half-planes of the first type, and then the lower half-plane S_{i+1}^- of the second type containing γ_{i+1} in the boundary. Repeat this process for each i on both sides. We conclude that the gluing pattern of these half-planes is uniquely determined by the configuration. After gluing, the resulting open surface has a single figure eight boundary formed by γ_1 and γ_{k+1} at the beginning and at the end, which is then identified with the boundary of the two half-infinite cylinders to recover η_0. Finally, since the angles between γ_i and γ_{i+1} are $2\pi(c'_i + 1)$ and $2\pi(c''_i + 1)$ on both sides, it determines precisely $c'_i + c''_i + 1 = c_i + 1$ paris of upper and lower half-planes that share the same point at infinity. In other words, they give rise to a flat geometric representation of a pole of order $c_i + 2$, which is the desired pole order for $i = 1, \ldots, k$. □

4. Spin and hyperelliptic structures

For special μ, the stratum $\mathcal{H}(\mu)$ can be disconnected. Kontsevich and Zorich ([19]) classified connected components of $\mathcal{H}(\mu)$ for all μ. Their result says that $\mathcal{H}(\mu)$ can have up to three connected components, where the extra components are caused by spin and hyperelliptic structures.

4.1. Spin structures

We first recall the definition of spin structures. Suppose $\mu = (2k_1, \ldots, 2k_n)$ is a partition of $2g - 2$ with even entries only. For an abelian differential $(C, \omega) \in \mathcal{H}(\mu)$, let

$$(\omega) = 2k_1\sigma_1 + \cdots + 2k_n\sigma_n$$

be the associated canonical divisor. Then the line bundle

$$\mathcal{L} = \mathcal{O}(k_1\sigma_1 + \cdots + k_n\sigma_n)$$

is a square root of the canonical line bundle, hence $\mathcal{L}$ gives rise to a spin structure (also called a theta characteristic). Denote by

$$h^0(C, \mathcal{L}) \pmod 2$$

the parity of ω. By Atiyah ([1]) and Mumford ([22]), parities of theta characteristics are deformation invariant. We also refer to ω along with its parity as a spin structure, which can be either even or odd, and denote the parity by $\phi(\omega)$.

Alternatively, there is a topological description for spin structures using the Arf invariant, due to Johnson ([18]). For a smooth simple closed curve α on a flat surface, let $\mathrm{Ind}(\alpha)$ be the degree of the Gauss map from α to the unit circle. Namely, $2\pi \cdot \mathrm{Ind}(\alpha)$ is the total change of the angle of the unit tangent vector to α under the flat metric as it moves along α one time.

Let $\{a_i, b_i\}_{i=1}^g$ be a symplectic basis of C, i.e., $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = \delta_{ij}$ for $1 \leqslant i, j \leqslant g$. When ω has only even zeros, the parity $\phi(\omega)$ can be equivalently defined as

$$\phi(\omega) = \sum_{i=1}^{g} (\mathrm{Ind}(a_i) + 1)(\mathrm{Ind}(b_i) + 1) \pmod 2.$$

In particular if a_i crosses a zero σ_j from one side to the other, since the zero order of σ_j is even, $\mathrm{Ind}(a_i)$ remains unchanged mod 2.

4.2. Hyperelliptic structures

Next we recall the definition of hyperelliptic structures. There are two cases: $\mu = (2g-2)$ and $\mu = (g-1, g-1)$. For $(C, \omega) \in \mathcal{H}(2g-2)$, if C is hyperelliptic and the unique zero σ of ω is a Weierstrass point, i.e., σ is a ramification point of the hyperelliptic double cover $C \to \mathbb{P}^1$, then we say that (C, ω) has a hyperelliptic structure. For $(C, \omega) \in \mathcal{H}(g-1, g-1)$, if C is hyperelliptic and the two zeros σ_1 and σ_2 of ω are hyperelliptic conjugates of each other, i.e., σ_1 and σ_2 have the same image under the hyperelliptic double cover, then we say that (C, ω) has a hyperelliptic structure. In particular, the hyperelliptic involution exchanges them.

4.3. Connected components of $\mathcal{H}(\mu)$

Now we can state precisely the classification of connected components of $\mathcal{H}(\mu)$ in [19]:

- Suppose $g \geqslant 4$. Then
  - $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(2g-2)$, the odd spin component $\mathcal{H}^{\mathrm{odd}}(2g-2)$, and the even spin component $\mathcal{H}^{\mathrm{even}}(2g-2)$.
  - $\mathcal{H}(g-1,g-1)$, when g is odd, has three connected components: the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$, the odd spin component $\mathcal{H}^{\mathrm{odd}}(g-1,g-1)$, and the even spin component $\mathcal{H}^{\mathrm{even}}(g-1,g-1)$.
  - $\mathcal{H}(g-1,g-1)$, when g is even, has two connected components: the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$ and the nonhyperelliptic component $\mathcal{H}^{\mathrm{nonhyp}}(g-1,g-1)$.
  - All the other strata of the form $\mathcal{H}(2k_1,\dots,2k_n)$ have two connected components: the odd spin component $\mathcal{H}^{\mathrm{odd}}(2k_1,\dots,2k_n)$ and the even spin component $\mathcal{H}^{\mathrm{even}}(2k_1,\dots,2k_n)$.
  - All the remaining strata are connected.
- Suppose $g = 3$. Then
  - $\mathcal{H}(4)$ has two connected components: the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(4)$ and the odd spin component $\mathcal{H}^{\mathrm{odd}}(4)$, where the even spin component coincides with the hyperelliptic component.
  - $\mathcal{H}(2,2)$ has two connected components: the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(2,2)$ and the odd spin component $\mathcal{H}^{\mathrm{odd}}(2,2)$, where the even spin component coincides with the hyperelliptic component.
  - All the other strata are connected.
- Suppose $g = 2$. Then both $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ are connected. Each of them coincides with its hyperelliptic component.

4.4. Degeneration of spin structures

Let $\mathcal{S}_g$ be the moduli space of spin structures on smooth genus g curves. The natural morphism $\mathcal{S}_g \to \mathcal{M}_g$ is an unramified cover of degree 2^{2g}. Moreover, $\mathcal{S}_g$ is a disjoint union of $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$, parameterizing even and odd spin structures, respectively. Cornalba ([10]) constructed a compactified moduli space of spin structures $\overline{\mathcal{S}}_g = \overline{\mathcal{S}}_g^+ \sqcup \overline{\mathcal{S}}_g^-$ over $\overline{\mathcal{M}}_g$, whose boundary parameterizes degenerate spin structures on stable nodal curves and distinguishes their parities. We first recall spin structures on nodal curves of compact type. Suppose a nodal curve C consists of k irreducible components $C_1, \ldots, C_k$ such that each of the nodes is separating, i.e., removing it disconnects C. Let L_i be a theta characteristic on C_i, i.e., $L_i^{\otimes 2} = K_{C_i}$. At each node of C, insert a $\mathbb{P}^1$-bridge, called an exceptional component, and take the line bundle $\mathcal{O}(1)$ on it. Then the collection $\{(C_i, L_i)\}_{i=1}^k$ along with $\mathcal{O}(1)$ on each exceptional component gives a spin structure on C, whose parity is determined by

$$h^0(C_1, L_1) + \cdots + h^0(C_k, L_k) \pmod 2.$$

In particular, if C_i has genus g_i, then $g_1 + \cdots + g_k = g$. On each C_i there are 2^{2g_i} distinct theta characteristics, hence in total they glue to 2^{2g} spin structures on C, which equals the number of theta characteristics on a smooth curve of genus g. One can think of the exceptional $\mathbb{P}^1$-component intuitively as follows. For simplicity suppose C consists of two irreducible components C_1 and C_2 meeting at one node q by identifying $q_1 \in C_1$ with $q_2 \in C_2$. Then the dualizing line bundle ω_C restricted to C_i is $K_{C_i}(q_i)$, whose degree is odd, hence one cannot directly take its square root. Instead, we insert a $\mathbb{P}^1$-component between C_1 and C_2, and regard $\mathcal{O}(q_1+q_2)$ as $\mathcal{O}(2)$ on $\mathbb{P}^1$ so that its square root is $\mathcal{O}(1)$. Then an ordinary theta characteristic L_i on C_i along with $\mathcal{O}(1)$ on $\mathbb{P}^1$ gives a Cornalba's spin structure on C, where $\deg L_1 + \deg L_2 + \deg \mathcal{O}(1) = g - 1$ is the same as the degree of an ordinary theta characteristic on a smooth genus g curve.

If C is not of compact type, the situation is more complicated, because there are two types of spin structures. For example, consider the case when C is an irreducible one-nodal curve, by identifying two points q_1 and q_2 in its normalization C' as a node q. For the first type, one can take a square root L of the dualizing line bundle ω_C, which gives 2^{2g-1} such spin structures. Equivalently, pull back L to L' on C'. Then L' is a square root of $K_{C'}(q_1 + q_2)$, and there are 2^{2g-2} such L' on C'. By Riemann–Roch, $h^0(C', L') - h^0(C', L'(-q_1 - q_2)) = 1$, hence neither q_1 nor q_2 is a base point of L', and any section s of L' that vanishes at one of the q_i must also vanish at the other. Therefore, the space of sections $H^0(C', L')$ has a decomposition $V_0 \oplus \langle s \rangle$, where V_0 is the subspace of sections that vanish at q_1 and q_2, and s is a section not vanishing at the q_i. Note that $L^{\otimes 2} = \omega_C$, whose fibers over q_1 and q_2 have a canonical identification by $\operatorname{Res}_{q_1} \omega + \operatorname{Res}_{q_2} \omega = 0$, where ω is a stable differential with at worst simple poles at the q_i, treated as a local section of ω_C at q. In other words, there is a canonical way to glue the fibers of $L'^{\otimes 2}$ over q_1 and q_2 to form ω_C on C. Due to the sign $\pm$ when taking a square root, it follows that there are two choices to glue the fibers of L' over q_1 and q_2 to form L on C, and exactly one of the two choices preserves s as a section of L. One can intuitively think of s with an evaluation at q_i such that $s(q_1)^2 = s(q_2)^2 \neq 0$. Then the choice of gluing the two fibers induced by $s(q_1) = s(q_2)$ preserves s as a section of L, while the other choice induced by $s(q_1) = -s(q_2)$ does not. We thus conclude that this way gives 2^{2g-1} spin structures on C, where half of them are even and the other half are odd. For the second type, insert an exceptional $\mathbb{P}^1$-component connecting q_1 and q_2 in C'. Take an ordinary theta characteristic L' on C' and the bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ as before. In this way one obtains 2^{2g-2} such L'. For a fixed L', there is no extra choice of gluing L' to $\mathcal{O}(1)$ at q_1 and q_2, due to the automorphisms of $\mathcal{O}(1)$ on $\mathbb{P}^1$, and hence the parity of the resulting spin structure equals that of η'. Nevertheless, the morphism $\overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ is simply ramified along the locus of such η' of the second type. Therefore, taking both types into account along with the multiplicity factor for the second type, we again obtain the number 2^{2g}, which is equal to the degree of $\overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$.

Below we describe a relation between degenerate spin structures and twisted differentials. Suppose a twisted differential (C, η) is in the closure of a stratum $\mathcal{H}(\mu)$ that contains a spin component, i.e., when μ has even entries only. For a node q joining two components C_1 and C_2 of C, by definition $\operatorname{ord}_q \eta_1 + \operatorname{ord}_q \eta_2 = -2$. If both orders are odd, we do nothing at q. If both orders are even, we insert an exceptional $\mathbb{P}^1$ at q. In particular if q is separating, in this case $\operatorname{ord}_q \eta_1$ and $\operatorname{ord}_q \eta_2$ are both even, because each side of q contains even zeros only, and hence we insert a $\mathbb{P}^1$ at q, which matches the preceding discussion on curves of compact type. Now suppose η_i on a component C_i of C satisfies that

$$(\eta_i) = \sum_j 2m_j\sigma_j + \sum_k 2n_k q_k + \sum_l (2h_l - 1)q_l,$$

where the σ_j are the zeros in the interior of C_i, the q_k are the nodes of even order in C_i, and the q_l are the nodes of odd order in C_i. Consider the bundle

$$L_i = \mathcal{O}\left(\sum_j m_j\sigma_j + \sum_k n_k q_k + \sum_l h_l q_l\right) \tag{4.1}$$

on C_i. Then a spin structure $\mathcal{L}$ on C consists of the collection (C_i, L_i) and the exceptional components with $\mathcal{O}(1)$. However, if (C, η) has a node of odd order, i.e., a node without inserting an exceptional component, then there are two gluing choices at such a node, as described above, hence $\mathcal{L}$ is only determined by (C, η) up to finitely many choices, and its parity may vary with different choices. From the viewpoint of smoothing twisted differentials, it means that different choices of opening up nodes of C may deform (C, η) into different connected components of $\mathcal{H}(\mu)$.

The idea behind the above description is as follows. For a node q joining two components C_1 and C_2, if there is no twist at q, i.e., if $\operatorname{ord}_q \eta_1 = \operatorname{ord}_q \eta_2 = -1$, then locally at q one can directly take a square root of ω_C. If $\operatorname{ord}_q \eta_1$ and $\operatorname{ord}_q \eta_2$ are both odd, i.e., if the twisting parameter $\operatorname{ord}_q \eta_i - (-1)$ is even, then its one-half gives the twisting parameter for the limit spin bundle on C. On the other hand if $\operatorname{ord}_q \eta_1$ and $\operatorname{ord}_q \eta_2$ are even, then the twisting parameter $\operatorname{ord}_q \eta_i - (-1)$ is not divisible by 2, hence one has to insert an exceptional $\mathbb{P}^1$ at q, which is twisted once to make the twisting parameters at the new nodes even. As a consequence, the resulting twisted differential restricted to $\mathbb{P}^1$ is $\mathcal{O}(2)$, hence its one-half is the bundle $\mathcal{O}(1)$ encoded in the degenerate spin structure. The reader may refer to [13] for a detailed explanation.

4.5. Degeneration of hyperelliptic structures

Next we describe how hyperelliptic structures degenerate. Recall that the closure of the locus of hyperelliptic curves of genus g in $\overline{\mathcal{M}}_g$ can be identified with the moduli space $\widetilde{\mathcal{M}}_{0,2g+2}$ parameterizing stable rational curves with $2g+2$ unordered markings, where the markings correspond to the $2g+2$ branch points of hyperelliptic covers. On the boundary of the moduli spaces, hyperelliptic covers degenerate to admissible double covers of stable genus zero curves in the setting of Harris–Mumford ([17]). Therefore, Weierstrass points on smooth hyperelliptic curves degenerate to ramification points in such admissible hyperelliptic covers, and the limits of a pair of hyperelliptic conjugate points remain to be conjugate in the limit admissible cover, see Figure 4.1.

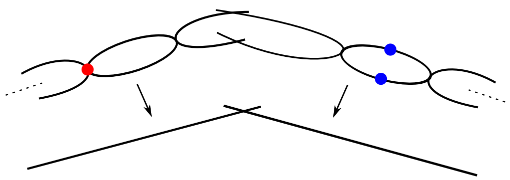

Figure 4.1. A limit of Weierstrass points (labeled by red) and a limit of pairs of conjugate points (labeled by blue) in a hyperelliptic admissible double cover.

4.6. Spin and hyperelliptic structures for the principal boundary of type I

Let $\mathcal{C} = (m_1, m_2, \{a_i', a_i''\}_{i=1}^p)$ be an admissible configuration of type I for a stratum $\mathcal{H}(\mu)$. Suppose (C, η) is a twisted differential contained in $\Delta(\mu, \mathcal{C})$. By the description of $\Delta(\mu, \mathcal{C})$ in Section 2.3, C consists of p components $C_1, \ldots, C_p$, each of genus $g_i \geqslant 1$ with $g_1 + \cdots + g_p = g$, attached to a rational component R, and η_i is the differential of η restricted to C_i satisfying that $(\eta_i) = \sum_{\sigma_j \in C_i} m_j \sigma_j + (a_i' + a_i'')q_i$, where q_i is the node joining C_i with R.

Consider the case when μ has even entries only. Then $\mathcal{H}(\mu)$ contains an even spin component and an odd spin component (and possibly a hyperelliptic component). This parity distinction can be extended to the principal boundary $\Delta(\mu, \mathcal{C})$, see [12, Lemma 10.1] for a proof using the Arf invariant. For the reader's convenience, below we recap the result and also provide an algebraic proof.

Proposition 4.1. — *Let (C, η) be a twisted differential in $\Delta(\mu, \mathcal{C})$ described as above, with even zeros only. Then the parity of η is*

$$\phi(\eta) = \phi(\eta_1) + \cdots + \phi(\eta_p) \pmod 2.$$

Proof. — Since $(\eta_i) = \sum_{\sigma_j \in C_i} m_j \sigma_j + (a_i' + a_i'')q_i$ and the m_j are all even, it implies that $a_i' + a_i''$ is even for all i and the degenerate spin structure on C_i is given by $\mathcal{O}((\eta_i)/2)$ in the sense of Cornalba ([10]). Moreover, on the rational component R, any theta characteristic has even parity (given by zero). Since C is of compact type, the parity of η is equal to the sum of the parities of the η_i, as claimed. ☐

Corollary 4.2. — *Suppose $\mathcal{C}$ is of type I and μ contains only even zeros. Then differentials in the thick part of $\mathcal{H}(\mu)$ degenerate to twisted differentials in $\Delta(\mu, \mathcal{C})$ with the same parity.*

Note that for the parity discussion we only require that $a_i' + a_i''$ is even for each i, and there is no other requirement for the individual values of a_i' and a_i''.

Next we consider hyperelliptic components. Since configurations of type I require at least two distinct zeros, here we only need to treat the case $\mu = (g-1, g-1)$, which contains a hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ (and possibly spin components if g is odd).

The following result is a reformulation of [12, Lemma 10.3]. Here we again provide an algebraic proof.

Proposition 4.3. — *Suppose (C,η) is a twisted differential contained in $\Delta(g-1,g-1,\mathcal{C})$. Then differentials in the thick part of $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$ can degenerate to (C,η) if and only if either*

- $p=1$, $(C_1,\eta_1)\in\mathcal{H}^{\mathrm{hyp}}(2g-2)$, $a_1'=a_1''=g-1$, *or*
- $p=2$, $(C_i,\eta_i)\in\mathcal{H}^{\mathrm{hyp}}(2g_i-2)$, $a_i'=a_i''=g_i-1$ *for* $i=1,2$.

Proof. — Suppose (C,η) is a degeneration of differentials from $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$. Then C admits an admissible hyperelliptic double cover π, where the two zeros σ_1 and σ_2 are conjugates under π. Since each C_i meets the rational component R at a single node q_i, and C_i is not rational, by the definition of admissible covers, q_i has to be a ramified node under π. By the Riemann–Hurwitz formula, π restricted to R has only two ramification points, which implies that $p\leqslant 2$.

For $p=1$, C_1 has genus g, and it admits a hyperelliptic double cover with q_1 being a ramification point, hence $(C_1,\eta_1)\in\mathcal{H}^{\mathrm{hyp}}(2g-2)$. Moreover, there is only one saddle connection joining σ_1 to σ_2, so the angle condition in the configuration $\mathcal{C}$ can only be $a_1'=a_1''=g-1$. See Figure 4.2 for this case and the corresponding hyperelliptic admissible cover.

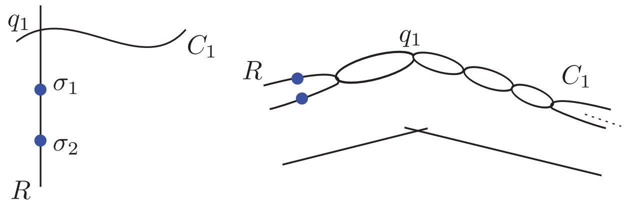

Figure 4.2. The case $p=1$ in Proposition 4.3 and the corresponding hyperelliptic admissible cover.

For $p=2$, by the same argument as above we see that $(C_i,\eta_i)\in\mathcal{H}^{\mathrm{hyp}}(2g_i-2)$ for $i=1,2$. In addition, since the hyperelliptic involution interchanges σ_1 and σ_2, it also swaps the two saddle connections γ_1 and γ_2 (even on the degenerate component R). It follows that $a_i'=a_i''$ for $i=1,2$. Since $a_i'+a_i''=2g_i-2$, we thus conclude that $a_i'=a_i''=g_i-1$. See Figure 4.3 for this case and the corresponding hyperelliptic admissible cover.

Conversely if (C,η) belongs to one of the two cases, the smoothing operation in the proof of Theorem 2.1 implies that nearby flat surfaces after opening up the nodes are contained in $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$. □

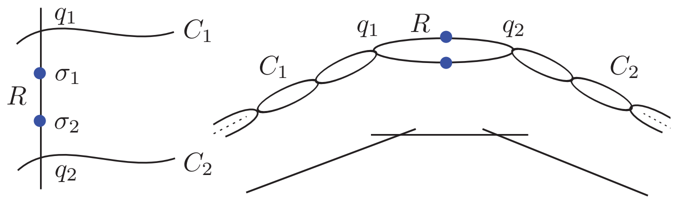

Figure 4.3. The case $p = 2$ in Proposition 4.3 and the corresponding hyperelliptic admissible cover.

Denote by $\Delta^{\text{hyp}}(\,\cdot\,)$, $\Delta^{\text{even}}(\,\cdot\,)$, and $\Delta^{\text{odd}}(\,\cdot\,)$ the respective loci of twisted differentials in the principal boundary that are degenerations from hyperelliptic and spin components as specified in the above propositions. We summarize our discussion as follows.

Corollary 4.4. — *Let $\mathcal{C}$ be an admissible configuration of type I for $\mathcal{H}(\mu)$. Then the principal boundary $\Delta(\mu, \mathcal{C})$ satisfies the following description:*

- *Suppose g is odd.*
  - *For $\mathcal{C} = (m_1 = m_2 = g-1, p = 1, a'_1 = a''_1 = g-1)$ or $\mathcal{C} = (m_1 = m_2 = g-1, p = 2, a'_i = a''_i = g_i - 1)$ with $g_1 + g_2 = g$, $\Delta(g-1, g-1, \mathcal{C})$ is a disjoint union of $\Delta^{\text{hyp}}(g-1, g-1, \mathcal{C})$, $\Delta^{\text{odd}}(g-1, g-1, \mathcal{C})$, and $\Delta^{\text{even}}(g-1, g-1, \mathcal{C})$.*
  - *For all the other types $\mathcal{C}$, $\Delta(g-1, g-1, \mathcal{C})$ is a disjoint union of $\Delta^{\text{odd}}(g-1, g-1, \mathcal{C})$ and $\Delta^{\text{even}}(g-1, g-1, \mathcal{C})$.*
- *Suppose g even.*
  - *For $\mathcal{C} = (m_1 = m_2 = g-1, p = 1, a'_1 = a''_1 = g-1)$ or $\mathcal{C} = (m_1 = m_2 = g-1, p = 2, a'_i = a''_i = g_i - 1)$ with $g_1 + g_2 = g$, $\Delta(g-1, g-1, \mathcal{C})$ is a disjoint union of $\Delta^{\text{hyp}}(g-1, g-1, \mathcal{C})$ and $\Delta^{\text{nonhyp}}(g-1, g-1, \mathcal{C})$.*
  - *For all the other types $\mathcal{C}$, $\Delta(g-1, g-1, \mathcal{C})$ coincides with $\Delta^{\text{nonhyp}}(g-1, g-1, \mathcal{C})$.*
- *For all the remaining types $\mathcal{C}$ and μ with even entries only, $\Delta(\mu, \mathcal{C})$ is a disjoint union of $\Delta^{\text{odd}}(\mu, \mathcal{C})$ and $\Delta^{\text{even}}(\mu, \mathcal{C})$.*

Remark 4.5. — In the above corollary, each $\Delta^{\text{hyp}}(\,\cdot\,)$, $\Delta^{\text{even}}(\,\cdot\,)$, or $\Delta^{\text{odd}}(\,\cdot\,)$ can be disconnected, since in general they are unions of products of strata in lower genera. Moreover for small g, some of them can also be empty.

4.7. Spin and hyperelliptic structures for the principal boundary of type II

Let $\mathcal{C} = (L, \{a_i, b_i\}, \{c'_j, c''_j\})$ be a configuration of type II for a stratum $\mathcal{H}(\mu)$. Consider the case when μ has even entries only, i.e., a differential in $\mathcal{H}(\mu)$ has odd or even parity. The parity distinction can be extended to the principal boundary $\Delta(\mu, \mathcal{C})$, see [12, Section 14.1]. Below we recap the results and also provide alternative algebraic proofs.

Recall the description for (C, η) in Theorem 3.4. Let us first simplify the statement of [12, Lemma 14.1] in our setting.

Lemma 4.6. — *Let (C, η) be a twisted differential contained in $\Delta(\mu, \mathcal{C})$. Suppose μ has even zeros only. Then the following conditions hold:*

- *η has even zero order at each marking of C.*
- *η has even zero and pole order at a separating node of C.*
- *For all non-separating nodes of C, the zero and pole orders of η are either all even, or all odd.*

Proof. — Because μ has even zeros only, and those zeros are the markings of C, the first condition holds by definition of twisted differentials.

Suppose q is a separating node of C. By the description of C in Theorem 3.4, q joins a component C_i with a rational component R. Since the markings in the interior of C_i are even zeros, we conclude that $\operatorname{ord}_q \eta_{C_i}$ has the same parity as $2g_{C_i} - 2$, hence it is even, which implies the second condition.

Finally, recall that all non-separating nodes bound the (unique) cycle in the dual graph of C. Since η has even order at all the other nodes and at all markings, going along the edges of the cycle one by one, the parity of the order of η at one vertex of the cycle determines that all the others have the same parity, hence the last condition holds. □

Remark 4.7. — If η has even order at all non-separating nodes, then there is no rational component R in the central cycle of C that has a simple polar node. In that case types (i) and (ii) do not appear in the description of C, which is exactly the way [12, Lemma 14.1] phrased.

Next, we interpret [12, Lemmas 14.2, 14.3, and 14.4] in terms of Cornalba's spin structures.

Lemma 4.8. — *Suppose all rational components of (C, η) are of type* (i). *Then the limit spin structure on (C, η) has parity*

$$\phi(C, \eta) = \sum_{i=1}^{p} \phi(C_i, \eta_{C_i}) + \sum (c'_i + 1) + 1 \pmod 2.$$

Proof. — We first remark that since η has even zeros only, $c_i' + c_i''$ is even for all i, hence using c_i' or c_i'' does not matter for the parity formula.

Next, since only type (i) appears in the description of C, each C_i is a tail of C, which is attached to C at a separating node, hence the limit spin structure on C_i is generated by one-half of (η_{C_i}) (see (4.1)), and it contributes $\phi(C_i, \eta_{C_i})$ to the total parity.

The central cycle S of C is a loop of rational components $R_1, \ldots, R_k$ in a cyclic order. At each node q_i joining R_i to R_{i+1}, η has a simple pole on the two branches of q_i with opposite residues $\pm r$, hence in the limit spin structure we preserve q_i and do not insert an exceptional $\mathbb{P}^1$ component. Therefore, the limit spin structure restricted to S is a square root L of ω_S, where S has arithmetic genus one, and $L|_{R_i} = \mathcal{O}_{R_i}$. Starting from R_1, identify the fibers of $\mathcal{O}_{R_1}$ and $\mathcal{O}_{R_2}$ at q_1, then identify the fibers of $\mathcal{O}_{R_2}$ and $\mathcal{O}_{R_3}$ at q_2, so on and so forth. The last identification between the fibers of $\mathcal{O}_{R_k}$ and $\mathcal{O}_{R_1}$ at q_k has two choices, which makes $h^0(S, L) = 0$ or 1. Hence the parity of the spin structure on S varies with the gluing choice, where the gluing choice is actually determined by the configuration data $\{c_i', c_i''\}$. By analyzing the Arf invariant, the parity contribution from S is $\sum (c_i' + 1) + 1$, see the proof of [12, Lemma 14.2] for details. □

Now we consider the last alternate conditions in Lemma 4.6.

Lemma 4.9. — *Suppose η has even order at all non-separating nodes of C. Then the parity of the limit spin structure on (C, η) is*

$$\phi(C, \eta) = \sum_{i=1}^{p} \phi(C_i, \eta_{C_i}),$$

where the C_i are the non-rational components of C.

Proof. — In this case on each C_i the limit spin structure is generated by one-half of (η_{C_i}), because η_{C_i} has even zeros at the markings and nodes (see (4.1)). The same analysis also holds for the rational components R_i, hence one-half of (η_{R_i}) gives the limit spin structure $\mathcal{O}(-1)$ on R_i whose parity is even (equal to zero). Therefore, the total parity is given by the sum of the parities over all C_i. □

Lemma 4.10. — *Suppose η has odd order at every non-separating node of C. Let N be the total number of nearby flat surfaces under the previous smoothing procedure. Then exactly $N/2$ of them have odd spin structure and $N/2$ have even spin structure.*

Proof. — Let S be the central cycle of C. Then η has odd zeros and poles at all the nodes of S. Hence in the limit spin structure we do not insert an exceptional component at each node of S. Therefore, given the spin structure on each component of S, we have different gluing choices to form a global spin structure on C. When varying the gluing choice over one node of S while keeping the others, the parity of the resulting spin structure differs by one, hence the desired claim follows. See also the proof of [12, Lemma 14.4] for an argument using the Arf invariant. □

Next we consider the principal boundary of type II for hyperelliptic components. Below we recap [12, Lemmas 14.5 and 14.6] and provide algebraic proofs using hyperelliptic admissible covers.

Lemma 4.11. — *Suppose (C,η) is in the principal boundary $\Delta(2g-2,\mathcal{C})$ for a configuration $\mathcal{C}$ of type II. Then (C,η) is in $\Delta^{\mathrm{hyp}}(2g-2,\mathcal{C})$ if and only if it is one of the following types:*

1. *C has two components C_1 and R meeting at two nodes q_1 and q_2, $(C_1,\eta_{C_1})\in\mathcal{H}^{\mathrm{hyp}}(g-2,g-2)$ with q_1 and q_2 as the two zeros, and R contains the unique marking σ.*
2. *C has two components C_1 and E meeting at one node σ', where E is an irreducible one-nodal curve by identifying two points q_1 and q_2 in R, $(C_1,\eta_{C_1})\in\mathcal{H}^{\mathrm{hyp}}(2g-4)$ with σ' as the zero, and E contains the unique marking σ.*
3. *C has three components C_1, C_2, and R, where C_1 meets R at one node σ', C_2 meets R at two nodes q_1 and q_2, $(C_1,\eta_{C_1})\in\mathcal{H}^{\mathrm{hyp}}(2g_1-2)$ with σ' as the zero, $(C_2,\eta_{C_2})\in\mathcal{H}^{\mathrm{hyp}}(g_2-1,g_2-1)$ with q_1 and q_2 as the two zeros, where $g_1+g_2=g-1$, and R contains the unique marking σ.*

See Figure 4.4 for the underlying curve C in the three cases above.

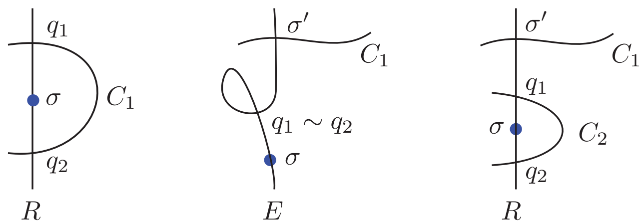

Figure 4.4. The underlying curve C from left to right for cases (1), (2) and (3) in Lemma 4.11.

Proof. — Suppose (C,η) is in $\Delta^{\mathrm{hyp}}(2g-2,\mathcal{C})$. Then it admits a hyperelliptic admissible double cover π. Since C has a unique marking σ, it has only one rational component R, and R has to contain σ. The cover π restricted to R has two ramification points, one of which is σ, and let σ' be the other. Denote by q_1 and q_2 the two polar nodes in R that arise in the description of degeneration types (i), (ii), or (iii). By definition of admissible cover, q_1 and q_2 are hyperelliptic conjugates under π. Moreover, any tail of C attached to R has to be attached at the ramification point σ'.

Based on the above constraints, there are three possibilities for π as follows. First, q_1 and q_2 join R to a different component, and there is no tail attached at σ', which gives case (1). On the other hand if there is a tail attached at σ', it gives case (3). Finally one can identify q_1 and q_2 to form a self node of R, and attach a tail at σ' to ensure that the genus of the total curve is at least two, which gives case (2). By analyzing the corresponding admissible cover in each case, we see that the newly added components along with their differentials satisfy the desired claim.

Conversely if (C, η) is one of the three cases, one can easily construct the corresponding hyperelliptic admissible cover, and we omit the details. □

Lemma 4.12. — *Suppose (C, η) is in the principal boundary $\Delta(g-1, g-1, \mathcal{C})$ for a configuration $\mathcal{C}$ of type II. Then (C, η) is in $\Delta^{\mathrm{hyp}}(g-1, g-1, \mathcal{C})$ if and only if it is one of the following types:*

(1) *C has three components C_1, R_1, and R_2, where each R_i meets C_1 at one node, R_1 and R_2 meet at one node, $(C_1, \eta_{C_1}) \in \mathcal{H}^{\mathrm{hyp}}(g-2, g-2)$ with the two zeros at the nodes of C_1, and each R_i contains a marking σ_i for $i = 1, 2$.*

(2) *C has four components C_1, C_2, R_1, and R_2, where each C_i meets each R_j at one node for $i, j = 1, 2$, $(C_i, \eta_{C_i}) \in \mathcal{H}^{\mathrm{hyp}}(g_i - 1, g_i - 1)$ with the two zeros at the nodes of C_i and $g_1 + g_2 = g - 1$, and each R_j contains a marking σ_j.*

See Figure 4.5 for the underlying curve C in the two cases above.

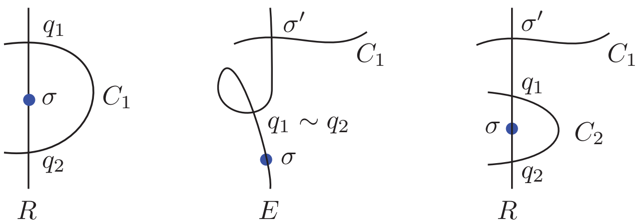

Figure 4.5. The underlying curve C from left to right for cases (1) and (2) in Lemma 4.12.

Proof. — The proof is similar to the previous one. Suppose (C, η) is in $\Delta^{\mathrm{hyp}}(g-1, g-1, \mathcal{C})$. Then it admits a hyperelliptic admissible double cover π. Since η has two zeros σ_1 and σ_2, there are two rational components R_1 and R_2 in C, each containing one zero. Moreover, σ_1 and σ_2 are conjugates under π, hence the degree of π restricted to each R_i is one. Consequently there is no tail attached to R_i, for otherwise the attaching point in R_i would be a ramification node of π by definition of admissible cover.

Based on the above constraints, there are two possibilities for π as follows. Let p_i and q_i be the two nodes of R_i. First, if R_1 and R_2 meet at one node, say, by identifying p_1 with p_2, then there is another component C_1 that joins R_1 and R_2 at q_1 and q_2, respectively, which gives case (1). If R_1 and R_2 are disjoint, then there must be two components C_1 and C_2, where each C_i connects R_1 and R_2 at p_i and q_i, respectively, which is case (2). Finally notice that R_1 and R_2 cannot intersect at both nodes, for otherwise there is no other component, and the genus of C would be one. Hence the above two cases are the only possibilities. By analyzing the corresponding admissible cover in each case, we see that the newly added components along with their differentials satisfy the desired claim.

Conversely if (C, η) is one of the two cases, one can easily construct the corresponding hyperelliptic admissible cover, and we omit the details. □

5. Principal boundary for quadratic differentials

In [20] Masur and Zorich carried out an analogous description for the principal boundary of moduli spaces of quadratic differentials, which parameterizes quadratic differentials with a prescribed generic configuration of short $\hat{\text{h}}$omologous saddle connections, where "$\hat{\text{h}}$omologous" is defined by passing to the canonical double cover (see [20, Definition 1]). The combinatorial structure of configurations of $\hat{\text{h}}$omologous saddle connections is described in terms of *ribbon graphs* (see [20, Figure 6]), which can be used as building blocks to construct a flat surface in the principal boundary.

As the lengths of these $\hat{\text{h}}$omologous saddle connections approach zero, we can also describe the principal boundary of limit differentials by using *twisted quadratic differentials* (in the sense of twisted k-differentials in [2] for $k = 2$). The definition of twisted quadratic differentials is almost the same as that of twisted abelian differentials, with one exception that the zero or pole orders on the two branches at every node sum to -4.

Since the idea of describing the principal boundary is similar and only the combinatorial structure gets more involved, we will explain our method by going through a number of examples, in which almost all typical ribbon graphs appear. Consequently the method can be adapted to any given configuration without further difficulties.

5.1. Ribbon graphs of configurations

We briefly recall the geometric meaning of the ribbon graphs (see [20, Section 1] and [15, Section 2]) for more details). A ribbon graph captures the information of boundary surfaces after removing the $\hat{h}$omologous saddle connections in a given configuration and how these boundary surfaces are glued to form the original surface. A vertex labeled by $\circ$, $\oplus$ or $\ominus$ in the graph represents a cylinder, a boundary surface of trivial holonomy or a boundary surface of non-trivial holonomy, respectively. Here whether or not the holonomy is trivial corresponds to whether or not the quadratic differential is the square of an abelian differential. An edge joining two vertices represents a common saddle connection on the boundaries of the corresponding two surfaces. The boundary of a ribbon graph is decorated by integers that encode the information of cone angles between consecutive $\hat{h}$omologous saddle connections. Each vertex is decorated by a set of integers (possibly empty) that encodes the type of singularities in the interior of the corresponding boundary surface. Connected components of the boundary of a ribbon graph correspond to newborn zeros after gluing the boundary surfaces together.

5.2. Configurations in genus 2

In [20, Appendix B] Masur and Zorich described explicitly configurations of $\hat{h}$omologous saddle connections for holomorphic quadratic differentials in genus 2. Below we will describe the corresponding principal boundary of limit twisted quadratic differentials for the three configurations of the stratum $\mathcal{Q}(2,2)$ (see [20, Figure 22]).

The first ribbon graph on the left of [20, Figure 22] corresponds to a flat surface on the left of Figure 5.1. If the saddle connection γ shrinks to a point, we obtain a flat surface $(E,\eta_E) \in \mathcal{Q}(2,-1,-1)$ where the two simple poles are identified as one point. Alternatively, cutting the surface open along γ, we obtain a surface with two boundary components γ' and γ''. If we expand the neighborhoods of γ' and γ'' to arbitrarily large, it gives a meromorphic quadratic differential $(R,\eta_R) \in \mathcal{Q}(2,-3,-3)$, since the flat geometric neighborhood of a triple pole of a quadratic differential corresponds to a (broken) half-plane. Combining them together, we conclude that the underlying pointed stable curve C of the limit differential consists of E union R at two nodes, where both E and R contain a marked double zero, see the right side of Figure 5.1. Conversely given such C and $\eta = (\eta_E, \eta_R)$, since η is a twisted quadratic differential and satisfies the global residue condition in [3], (C, η) can be smoothed into the Masur–Zorich principal boundary for this configuration.

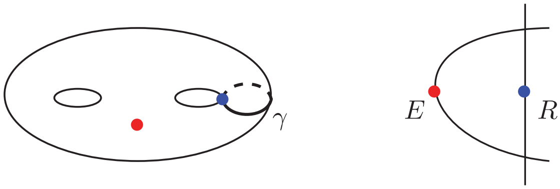

Figure 5.1. The surface corresponding to the first ribbon graph on the left of [20, Figure 22] *and the underlying curve of its degeneration as* $\gamma \to 0$.

The second ribbon graph on the left of [20, Figure 22] corresponds to a flat surface on the left of Figure 5.2. When the saddle connections γ_i shrink, the three cylinders all become arbitrarily long, hence they give rise to three nodes, each of which is of pole type $(-2, -2)$ in terms of twisted quadratic differentials (or of pole type $(-1, -1)$ in terms of twisted abelian differentials locally). Moreover, the node q_0 in the middle is separating, because removing the core curve of the middle cylinder disconnects the surface. Similarly we see that the other two nodes q_1 and q_2 are non-separating. Therefore, we conclude that the underlying pointed stable curve C of the limit differential consists of two nodal Riemann spheres R_1 and R_2, where each $(R_i, \eta_i) \in \mathcal{Q}(2, -2, -2, -2)$ has the last two poles identified as q_i and R_1, R_2 are glued by identifying their first poles as q_0, see the right side of Figure 5.2. In addition, the half-infinite cylinders corresponding to η_i at q_i for $i = 1, 2$ have identical widths, both equal to one-half of the width of the half-infinite cylinders at q_0. Conversely given such C and $\eta = (\eta_1, \eta_2)$, the width condition in this case is precisely the matching residue condition in the definition of twisted k-differentials in [3], hence (C, η) can be smoothed into the Masur–Zorich principal boundary for this configuration.

The last ribbon graph on the left of [20, Figure 22] corresponds to a flat surface on the left of Figure 5.3. One way to construct this surface is via the following surgery (suggested to us by the referee). Take a flat torus (E, η_E)

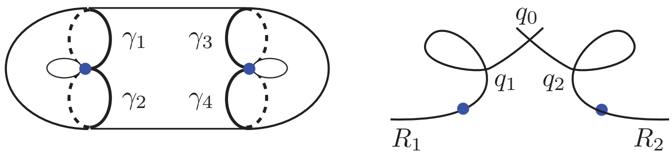

Figure 5.2. The surface corresponding to the second ribbon graph on the left of [20, Figure 22] *and the underlying curve of its degeneration as* $\gamma_i \to 0$.

with two vertical slits of the same length. Identify the left edge of the first slit with the left edge of the second slit in opposite direction, and identify their right edges in the same way. The two slits correspond to the saddle connections γ_1 and γ_2, and their endpoints give rise to two double zeros. As $\gamma_i \to 0$, we recover the flat torus E with two ordinary marked points recording the limit position of γ_i. Alternatively if we expand the neighborhoods of γ_1 and γ_2 to arbitrarily large, it gives a meromorphic quadratic differential $(R, \eta_R) \in \mathcal{Q}(2, 2, -4, -4)$ where the poles have zero residue and the configuration around γ_i is unchanged. Therefore, the underlying pointed stable curve C of the limit differential $\eta = (\eta_E, \eta_R)$ consists of E union R at two nodes by pairing the two marked points in E with the two poles in R, see the right side of Figure 5.3. Conversely given such (C, η), again by [3] it can be smoothed into the Masur–Zorich principal boundary for this configuration.

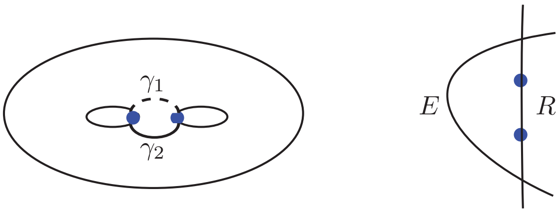

Figure 5.3. The surface corresponding to the last ribbon graph on the left of [20, Figure 22] *and the underlying curve of its degeneration as* $\gamma_i \to 0$.

5.3. A configuration in genus 13

We will convince the reader that our method works equally well in the case of high genera by considering an example in genus 13 in [20, Figure 7].

The underlying pointed stable curve C of the limit differential as $\gamma_i \to 0$ consists of seven components $S_1, \dots, S_5, R_1, R_2$ meeting as described in Figure 5.4.

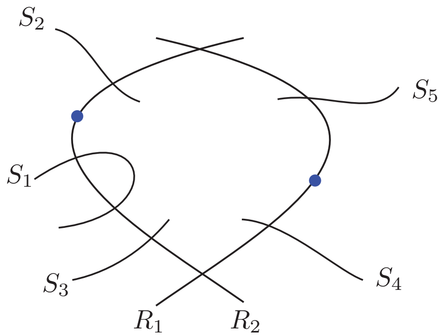

Figure 5.4. The underlying curve of the degeneration of the surface corresponding to [20, Figure 7] *as* $\gamma_i \to 0$.

The S_i components are non-degenerate and carry holomorphic differentials that were already described in [20, p. 939]. The rational component R_1 contains the marked zero of order 30 and carries a differential $\eta_1 \in \mathcal{Q}(30, -2, -2, -4, -4, -6, -16)$. The other rational component R_2 contains the marked zero of order 8 and carries a differential $\eta_2 \in \mathcal{Q}(8, -2, -2, -4, -4)$. As before, the half-infinite cylinders for η_1 and η_2 at the nodes of their intersection have equal widths. Let us explain how the components R_1, R_2 and their poles appear. The two $\circ$ vertices in the ribbon graph correspond to two cylinders. As they tend to arbitrarily long, we obtain the two nodes with double poles between R_1 and R_2. Removing γ_4 and γ_8 simultaneously disconnects the curve, which gives rise to R_1. Similarly removing γ_5 and γ_7 disconnects the curve, hence it gives rise to R_2. The surface S_1 corresponds to the central $\oplus$ vertex, whose boundary has two connected components given by the two connected components of its local ribbon graph. Going around each connect boundary component takes a total angle of 2π by the number decorations, hence the expansion of its local neighborhood to arbitrarily large consists of a pair of (broken) half-planes. It follows that S_1 meets R_1 at two nodes, both having pole order 4 for the limit twisted quadratic differential on R_1. The intersections of the other S_j with R_i can be analyzed in the same way.

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Manuscrit reçu le 3 janvier 2017,
révisé le 24 septembre 2017,
accepté le 2 février 2018.

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