



Principal Bundles and the Dixmier Douady Class

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ABSTRACT. This thesis exploits the power of the categorical approach to homotopy theory to produce a summary of the theory of principal bundles. A systematic consideration of the problems of the reduction and the extension of structure group is therefore possible and a variety of techniques in each case are explored and related to one another. These techniques are each applied to show the relation between the reduction and the extension of the structure group of a principal bundle and the vanishing of familiar characteristic classes. Of particular interest is the discovery of a systematic approach to the Dixmier–Douady class for string structures in the case of both continuous and differential loops. Finally, I relate the theory of principal bundles with the restricted unitary group, U_{res} , as structure group to reduced K-theory, demonstrating a link between the second Chern class of a bundle in reduced K-theory and the Dixmier–Douady class of the corresponding principal U_{res} -bundle.

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STATEMENT

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to a copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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