

## Principal bundles on the projective line

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**Abstract.** We classify principal  $G$ -bundles on the projective line over an arbitrary field  $k$  of characteristic  $\neq 2$  or  $3$ , where  $G$  is a reductive group. If such a bundle is trivial at a  $k$ -rational point, then the structure group can be reduced to a maximal torus.

**Keywords.** Reductive groups; principal bundles.

1. Let  $k$  be a field of characteristic  $\neq 2, 3$  and let  $G$  be a reductive group defined over  $k$ . Let  $\mathbb{P}_k^1$  denote the projective line defined over  $k$ . Let  $\mathcal{O}(1)$  denote the line bundle of degree one over  $\mathbb{P}_k^1$ . For a one-parameter subgroup  $\lambda : G_m \rightarrow G$  of  $G$ , we denote by  $E_{\lambda, G}$  the principal  $G$ -bundle associated to  $\mathcal{O}(1)$  (regarded as a  $G_m$  bundle) by the group homomorphism  $\lambda : G_m \rightarrow G$ . A principal  $G$ -bundle  $E$  on  $\mathbb{P}_k^1$  is said to be trivial at the origin if the restriction of  $E$  to the  $k$ -rational point  $0 \in \mathbb{P}_k^1$  is the trivial principal homogeneous space over  $\text{Spec } k$ . In this article, we show

**Main theorem.** *Let  $E \rightarrow \mathbb{P}_k^1$  be a principal  $G$ -bundle on  $\mathbb{P}_k^1$  which is trivial at the origin. Then  $E$  is isomorphic to  $E_{\lambda, G}$  for some one-parameter subgroup  $\lambda : G_m \rightarrow G$  defined over  $k$ .*

In particular, since every one-parameter subgroup lands inside a maximal torus of  $G$  we observe that any principal  $G$  bundle on  $\mathbb{P}_k^1$  which is trivial at the origin has a reduction of structure group to a maximal torus of  $G$ . This result was proved by Grothendieck [3] when  $k$  is the field of complex numbers and by Harder [4] when  $G$  is split over  $k$  and the principal bundle is Zariski locally trivial on  $\mathbb{P}_k^1$ .

While we understand that this result is known (see [2], unpublished), we believe that our geometric method of deducing it from the properties of the Harder–Narasimhan filtration of principal bundles is new and of interest in its own right.

2. Let  $X$  be a complete nonsingular curve over the algebraic closure  $\bar{k}$  of  $k$  and  $G$  a reductive group over  $\bar{k}$ . Let  $E \rightarrow X$  be a principal  $G$ -bundle on  $X$ .  $E$  is said to be semistable if, for every reduction of structure group  $E_P \subset E$  to a maximal parabolic subgroup  $P$  of  $G$ , we have

$$\text{degree } E_P(\mathfrak{p}) \leq 0,$$

where  $\mathfrak{p}$  is the Lie algebra of  $P$  and  $E_P(\mathfrak{p})$  is the Lie algebra bundle of  $E_P$  [9]. When  $G = GL(n)$ , a vector bundle  $V \rightarrow X$  is semistable if for every subbundle  $S \subset V$ , we have

$$\mu(S) \leq \mu(V),$$

where for a vector bundle  $W$ ,  $\mu(W)$  denotes the slope  $\text{degree}(W)/\text{rank}(W)$ .

If the principal bundle  $E \rightarrow X$  is not semistable, then there is a canonical reduction  $E_P \subset E$  to a parabolic  $P$  in  $G$  (also called the Harder–Narasimhan filtration) with the properties:

- (i)  $\text{degree } E_P(\mathfrak{p}) > 0$ .
- (ii) If  $U$  denotes the unipotent radical of  $P$  and  $L = P/U$  a Levi factor of  $P$ , then the associated  $L$  bundle  $E_P/U$  is semistable.
- (iii)  $U$  has a filtration  $U_0 = U \supset U_1 \supset U_2 \supset \cdots \supset U_k = e$ , such that  $U_i/U_{i+1}$  is a direct sum of irreducible  $L$ -modules for the natural action of  $L$  (see Theorem 2 in [12]), and if  $M$  is an irreducible  $L$ -module which occurs as a factor of  $U_i/U_{i+1}$  for some  $i$ , then  $\text{degree } E_L(M) > 0$ , where  $E_L(M)$  is the vector bundle associated to the principal  $L$ -bundle  $E_L$  by the action of  $L$  on  $M$ .

For the existence of such a canonical reduction, see Behrend's paper [1], where  $\text{degree } E_L(M)$  as above is called a numerical invariant associated to the reduction  $E_P \subset E$ , and the vector bundles  $E_L(M)$  are called the elementary vector bundles. This reduction was first defined for vector bundles by Harder–Narasimhan in [5], and for principal bundles by Ramanathan [9].

For convenience of reference, we recall Theorem 4.1 of [6] here.

**Theorem.** *Let  $E \rightarrow X$  be a principal  $G$ -bundle on  $X$ , where  $G$  is a reductive group. Suppose that  $\mu_{\min}(TX) \geq 0$ , where  $TX$  is the tangent bundle of  $X$ . Then we have,*

- (i) *If  $E \rightarrow X$  is semistable, so is  $F^*E \rightarrow X$ , where  $F$  is the absolute Frobenius of  $X$ .*
- (ii) *If  $E \rightarrow X$  is not semistable, and  $E_P \rightarrow X$  the canonical reduction of structure group of  $E$ ,  $E(\mathfrak{g})$  the adjoint bundle of  $E$ , and  $E(\mathfrak{p})$  the adjoint bundle of  $E_P$ , then*

$$\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{p})) < 0$$

and, in particular

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{p})) = 0.$$

We now let  $X = \mathbb{P}^1$ . Since  $\mu(T\mathbb{P}^1) = 2 > 0$ , where  $T\mathbb{P}^1$  denotes the tangent bundle of  $\mathbb{P}^1$ , we can apply Theorem 4.1 of [6] in our context. Let  $E \rightarrow \mathbb{P}_k^1$  be a principal  $G$ -bundle over  $\mathbb{P}_k^1$ , where  $G$  is a reductive group defined over  $k$ , and let  $\bar{k}$  denote the algebraic closure of  $k$  and  $k^s$  denote the separable closure of  $k$ . Let  $E_{\bar{k}}$  be the principal  $G_{\bar{k}}$  bundle over  $\mathbb{P}_{\bar{k}}^1$  obtained by base changing to  $\bar{k}$ , and suppose  $E_{\bar{k}}$  is not semistable. Then by Theorem 4.1(ii) of [6], the canonical parabolic reduction  $E_P \subset E_{\bar{k}}$  descends to  $E_{k^s}$ . Further, by [1], the canonical reduction to the parabolic is unique (actually, it is the associated parabolic group scheme  $E_P(P)$  which is unique) and hence by Galois descent, the reduction  $E_P \subset E$  is defined over  $k$ . In particular, the parabolic subgroup  $P$  which is *a priori* defined over  $\bar{k}$ , is actually defined over  $k$ .

**3.** We first consider torus bundles on  $\mathbb{P}^1$ . Let  $k$  be a field, and let  $T$  be a torus over  $k$ . Let  $E_T \rightarrow \mathbb{P}_k^1$  denote a principal  $T$ -bundle on  $\mathbb{P}_k^1$ . We have

PROPOSITION 3.1

Let  $E_T \rightarrow \mathbb{P}_k^1$  be a  $T$ -bundle on  $\mathbb{P}_k^1$  which is trivial at the origin. Then there exists a one-parameter subgroup  $\lambda : G_m \rightarrow T$  defined over  $k$  such that  $E_T$  is isomorphic to  $E_{\lambda, T}$ , where  $E_{\lambda, T}$  denotes the principal  $T$  bundle associated to  $\mathcal{O}_{\mathbb{P}^1}(1)$  (regarded as a  $G_m$  bundle) by the group homomorphism  $\lambda : G_m \rightarrow T$ .

*Proof.* Let  $A_0 = \mathbb{P}_k^1 - \{\infty\}$  and  $A_\infty = \mathbb{P}_k^1 - \{0\}$ . The two open sets  $A_0$  and  $A_\infty$  cover  $\mathbb{P}_k^1$ . By Proposition 2.7 in [7], the restriction of  $E_T$  to  $A_0$  is obtained from a  $T$  bundle on  $\text{Spec } k$  by base change, and since we assume  $E_T$  is trivial at the origin, we obtain that  $E_T|_{A_0}$  is trivial. Now consider  $E_T|_{A_\infty}$ . Again by Proposition 2.7 of [7],  $E_T|_{A_\infty}$  comes from  $\text{Spec } k$ . Since  $E_T|_{A_0}$  is trivial, there are  $k$ -rational points of  $A_\infty \cap A_0$  at which  $E_T$  is trivial. It follows  $E_T|_{A_\infty}$  is also trivial. Hence there is a transition function  $\mu : A_0 \cap A_\infty \rightarrow T$  which defines the principal  $T$  bundle  $E_T$  on  $\mathbb{P}_k^1$ . Since  $A_0 \cap A_\infty = G_m$ , we have thus obtained a morphism of schemes  $\mu : G_m \rightarrow T$  (which is *a priori* not a morphism of group schemes). Let 1 denote the identity of  $G_m$ , which is a  $k$ -rational point of  $G_m$ , and let  $\mu(1)$  be the  $k$ -rational point of  $T$  which is the image of 1. Let  $\lambda$  be the morphism  $\lambda = \mu(1)^{-1}\mu : G_m \rightarrow T$  which is the composite of  $\mu : G_m \rightarrow T$  and  $\mu(1)^{-1} : T \rightarrow T$  (multiplication by  $\mu(1)^{-1}$ ). Then it can be seen that  $\lambda : G_m \rightarrow T$  is a group homomorphism. However, the  $T$ -bundle on  $\mathbb{P}_k^1$  defined by  $\lambda$  is isomorphic to the bundle defined by  $\mu$ , and hence the proposition. Q.E.D.

4. Let  $k$  be a field and  $G$  a reductive group defined over  $k$ . We have

*Lemma 4.1.* Let  $E \rightarrow \mathbb{P}_k^1$  be a principal  $G$  bundle over  $\mathbb{P}_k^1$  and let  $G \rightarrow H$  be a homomorphism of reductive groups over  $k$  which maps the centre of  $G$  to the centre of  $H$ . Let  $E_H$  be the associated  $H$ -bundle. If  $E$  is a semistable  $G$  bundle, then  $E_H$  is a semistable  $H$  bundle on  $\mathbb{P}_k^1$ .

*Proof.* Let  $F : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$  be the absolute Frobenius morphism on  $\mathbb{P}_k^1$  (if  $\text{char } k > 0$ ). Then by Theorem 4.1(i) in [6],  $F^{n*}E$  is semistable for all  $n \geq 1$ , where  $F^n$  denotes the  $n$ th iterate of the Frobenius  $F$ . Hence, by Theorems 3.18 and 3.23 of [8], it follows that  $E_H$  is semistable. Q.E.D.

*Lemma 4.2.* Let  $H$  be a semisimple group over  $k$  and  $E_H \rightarrow \mathbb{P}_k^1$  a principal  $H$  bundle on  $\mathbb{P}_k^1$ . If  $E_H$  is semistable, then  $E_H$  is the pullback of a principal homogeneous space on  $\text{Spec } k$  by the structure morphism  $\mathbb{P}_k^1 \rightarrow \text{Spec } k$ . In particular, if  $E_H$  is trivial at the origin (which is a  $k$ -rational point of  $\mathbb{P}_k^1$ ) then  $E_H$  is trivial on  $\mathbb{P}_k^1$ .

*Proof.* Let  $\rho : H \rightarrow SL(N)$  be a faithful representation of  $H$  and let  $E_{SL(N)}$  be the associated  $SL(N)$  bundle. By Lemma 4.1 above,  $E_{SL(N)}$  is a semistable  $SL(N)$  bundle. Let  $V_\rho$  be the vector bundle whose frame bundle is  $E_{SL(N)}$ . Then  $V_\rho$  is a semistable vector bundle on  $\mathbb{P}_k^1$  of degree zero. By an application of the Riemann–Roch theorem, it follows that  $V_\rho$  has  $N$  linearly independent sections. Since every nonzero section of a semistable vector bundle of degree zero on a projective curve is nowhere vanishing, it follows that  $V_\rho$  is the trivial vector bundle. Hence  $E_{SL(N)}$  is the trivial  $SL(N)$  bundle. Therefore the principal fibre space  $E_{SL(N)}/H$  with fibre  $SL(N)/H$  is also trivial, so

$$E_{SL(N)}/H = \mathbb{P}_k^1 \times (SL(N)/H).$$

It follows that the reduction of structure group of  $E_{SL(N)}$  to  $H$  is defined by a morphism  $\mathbb{P}_k^1 \rightarrow SL(N)/H$ . Since  $SL(N)/H$  is affine, the morphism  $\mathbb{P}_k^1 \rightarrow SL(N)/H$  actually factors through a morphism  $\text{Spec } k \rightarrow SL(N)/H$ . It therefore follows that the principal  $H$  bundle  $E_H$  is the pullback of a principal homogeneous space on  $\text{Spec } k$  by the structure morphism  $\mathbb{P}_k^1 \rightarrow \text{Spec } k$ . Q.E.D.

We can now prove

**Theorem 4.3.** *Let  $G$  be a reductive group over a field  $k$  of char  $\neq 2, 3$  and let  $E \rightarrow \mathbb{P}_k^1$  be a principal  $G$  bundle on  $\mathbb{P}_k^1$ , which is trivial at the origin. Then there is a reduction of structure group  $E_T \subset E$  to a maximal torus  $T$  of  $G$  such that the  $T$  bundle  $E_T$  is trivial at the origin.*

*Proof.*

*Case (i).* Suppose  $E$  is a semistable  $G$  bundle. Let  $Z$  be the centre of  $G$  (which is defined over  $k$ ). Let  $E/Z$  be the  $G/Z$  bundle associated to the homomorphism  $G \rightarrow G/Z$ . Since  $E$  is trivial at the origin,  $E/Z$  is also trivial at the origin. Also, by Lemma 4.1 above,  $E/Z$  is a semistable  $G/Z$  bundle. Since  $G/Z$  is a semisimple group, we can apply Lemma 4.2 above to conclude that  $E/Z$  is the trivial  $G/Z$  bundle, and hence has sections. Any section of  $E/Z$  defines a reduction of structure group  $E_Z \subset E$  of  $E$  to  $Z$ . Since  $Z$  is contained in a maximal torus  $T$ , we obtain a reduction of structure group  $E_T \subset E$  to  $T$ . Let  $c \in H^1(k, T)$  be the class of the principal homogeneous  $T$  space obtained by restricting  $E_T$  to the origin of  $\mathbb{P}_k^1$ . Then  $c$  goes to zero under the map  $H^1(k, T) \rightarrow H^1(k, G)$ . We now twist  $E_T$  by the class  $c^{-1}$  to obtain a new  $T$  bundle  $E'_T$  which is trivial at the origin. The twisted  $T$ -bundle  $E'_T$  is contained in the twist of  $E$  by  $c^{-1}$ . However since the image of  $c$  in  $H^1(k, G)$  is trivial, we see that the twist of  $E$  by  $c^{-1}$  is  $E$  itself. Hence we obtain a reduction of structure group  $E'_T \subset E$  to a maximal torus  $T$  such that  $E'_T$  is trivial at its origin.

*Case (ii).* Now suppose  $E$  is not semistable. Then by §2, there is a canonical reduction  $E_P \subset E$  to a parabolic  $P$  in  $G$ . We first observe that since the map  $H^1(k, P) \rightarrow H^1(k, G)$  is injective (see Corollary 15.1.4 in [11]), the  $P$  bundle  $E_P$  is trivial at the origin. Now let  $L$  be a Levi factor of  $P$ . The  $L$  bundle  $E_L$  associated to  $E_P$  by the projection  $P \rightarrow L$  is semistable (see §2 above) and since  $E_P$  is trivial at the origin, so is  $E_L$ . Let  $U$  be the unipotent radical of  $P$ . Let  $P$  act on  $U$  by the inner conjugation. Then the bundle  $E_P(U)$  associated to  $E_P$  by this action on  $U$  is a group scheme over  $\mathbb{P}^1$ . Further, the  $P/L$  bundle  $E_P/L$  is a principal homogeneous space under the group scheme  $E_P(U)$  over  $\mathbb{P}^1$  (see Lemma 3.6 in [10]), and hence  $E_P/L$  determines an element of  $H^1(\mathbb{P}^1, E_P(U))$ . However,  $U$  has a filtration  $U_0 = U \supset U_1 \supset U_2 \supset \dots \supset U_k = e$  such that  $U_i/U_{i+1}$  is a direct sum of irreducible  $L$ -modules for the natural action of  $L$ , and if  $M$  is an irreducible  $L$ -module which occurs as a factor of  $U_i/U_{i+1}$  for some  $i$ , then  $\text{degree } E_L(M) > 0$ , where  $E_L(M)$  is the vector bundle associated to  $E_L$  by the action of  $L$  on  $M$  (see §2 above). Since  $E_L$  is semistable,  $E_L(M)$  is a semistable vector bundle by Lemma 4.1. Thus  $E_L(M)$  is a semistable vector bundle on  $\mathbb{P}^1$  of positive degree and hence  $H^1(\mathbb{P}^1, E_L(M)) = 0$ . Inducting on the filtration  $U_0 = U_1 \supset U_2 \supset U_3 \dots \supset U_k = e$ , we obtain that  $H^1(\mathbb{P}_k^1, E_P(U))$  is trivial, and hence  $E_P/L$  is the trivial principal homogeneous space. This defines a reduction of structure group  $E'_L \subset E_P$  of  $E_P$  to the Levi factor  $L$ . However, since the composite  $L \subset P \rightarrow P/U = L$  is the identity, it follows that  $E'_L$  is isomorphic to  $E_L$ . Thus we have obtained a reduction of structure group  $E_L \subset E$ , where  $E_L$  is a semistable  $L$  bundle which is trivial at the origin. Let  $Z$  denote the centre of  $L$ . Then

arguing as in Case (i) above, we obtain a reduction of structure group  $E_Z \subset E$  to the centre  $Z$  of  $L$ , such that  $E_Z$  is trivial at the origin. Since  $Z$  is contained in a maximal torus  $T$  of  $G$ , we have obtained a  $T$  bundle  $E_T$  trivial at the origin, and a reduction of structure group  $E_T \subset E$  of  $E$  to the maximal torus  $T$ . Q.E.D.

*Proof of main Theorem.* This follows from Theorem 4.3 and Proposition 3.1 above.

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### References

- [1] Behrend K A, Semistability of reductive group schemes over curves, *Math. Ann.* **301** (1995) 281–305
- [2] Gille P, Torseurs sur la droite affine, preprint, unpublished
- [3] Grothendieck A, Sur la classification des fibres holomorphes sur la sphere de Riemann, *Am. J. Math.* **79** (1957) 121–138
- [4] Harder G, Halbeinfache Gruppenschemate über Vollständigen Kurven, *Invent. Math.* **6** (1968) 107–149
- [5] Harder G and Narasimhan M S, On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.* **212** (1974) 215–248
- [6] Mehta V B and Subramanian S, On the Harder–Narasimhan filtration of principal bundles, to appear in Proceedings International Colloquium on Algebra, Arithmetic and Geometry (TIFR) (2000)
- [7] Raghunathan M S and Ramanathan A, Principal bundles on the affine line, *Proc. Indian Acad. Sci. (Math. Sci.)* **93** (1984) 137–145
- [8] Ramanan S and Ramanathan A, Some remarks on the instability flag, *Tohoku Math. J.* **36** (1984) 269–291
- [9] Ramanathan A, Moduli for principal bundles in: Algebraic Geometry, *Proc. Copenhagen*, Springer Lecture Notes 732
- [10] Ramanathan A, Deformations of principal bundles on the projective line, *Invent. Math.* **71** (1983) 165–191
- [11] Springer T A, Linear algebraic groups, second edition, *Prog. Math.* (Birkhauser) (1998)
- [12] Azad H, Barry M and Seitz G, On the structure of parabolic subgroups, *Comm. Alg.* **18** (1990) 551–562