Principal Ideal Graphs of Rees Matrix Semigroups

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Abstract

Let S be a finite regular semigroup. We define the principal left ideal graph of S as the graph $_SG$ with $V(_SG) = S$ and two vertices a and b ($a \neq b$) are adjacent in $_SG$ if and only if $Sa \cap Sb \neq \{\}$. The principal right ideal graph is defined accordingly and is denoted by G_S. The principal ideal graph of Rees matrix semigroup is studied in this paper. First, we describe the necessary and sufficient condition for which two elements in a Rees matrix semigroup are adjacent in $_SG$ and G_S. Then we characterise the principal ideal graphs of a Rees matrix semigroup. Finally we describe the number of edges in $_SG$ and G_S and then the number of elements in $E(_SG) \cap E(G_S)$ when S is a Rees matrix semigroup.

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1. Introduction

Semigroups are the first and simplest type of algebra to which the methods of universal algebra is applied. During the last three decades, Graph Theory has established itself as an important mathematical tool in a wide variety of subjects. The use of graph theory has become widespread in the algebraic theory of semigroups. Graph is mainly used as a visual aid to represent several problems in the theory of semigroups. In 1964, Bosak [1] studied certain graph

over semigroups. In 1975, Zelinka [14] studied intersection graphs of nontrivial subgroups of finite abelian groups. The well known study of directed graph considering the elements of a group as its vertex set is the Cayley digraph [2, 8, 12, 13]. Recently Kelarev and Quinn [9, 10] defined two interesting classes of directed graphs, namely, divisibility graph and power graphs on semigroups. The divisibility graph Div(S) of a semigroup S is a directed graph with vertex set S and there is an edge (arc) from u to v if and only if $u \neq v$ and u/v, i.e., the ideal generated by v contains u. On the other hand the power graph, Pow(S) of a semigroup S is a directed graph in which the set of vertices is again S and for $a, b \in S$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m. In 2005, Frank De Mayer and Lisa De Mayer studied zero divisor graphs of semigroups [4]. In 2009, Ivy Chakrabarty, Shamik Ghosh and M K Sen introduced undirected power graphs [7].Following this, we define a new type of graphs on semigroups called the 'Principal Ideal Graphs of Semigroups'. Here we characterise the principal ideal graphs of Rectangular bands.

2. Preliminaries

In the following we give certain definitions and results from graph theory and semigroup theory as given in [5], [11] and [3], [6] respectively, which are used in the sequel.

A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \dots\}$, called vertices and another set $E = \{e_1, e_2, \dots\}$ whose elements are called edges such that each e_k is identified with an unordered pair (v_i, v_j) of vertices. Two graphs G and G' are said to be isomorphic, denoted by $G \cong G'$, if there is a one-to-one correspondance between their vertices and between their edges such that the incidence relationship is preserved. A graph G is said to be connected if there exists atleast one path between any pair of vertices in G, otherwise G is called disconnected. A graph in which there exists an edge between every pair of vertices is called a complete graph. A complete graph of n vertices is denoted by K_n.

A semigroup S is a non empty set S together with an associative binary operation on S. We define relations known as Green's relations L and R on S as follows:

$$
\begin{aligned}
L &= \{(a, b) \in S : S'a = S'b\} \\
R &= \{(a, b) \in S : aS' = bS'\}
\end{aligned}
$$

An element x of a semigroup S is said to be regular if there exists an element $x' \in S$ such that $xx'x = x$. A semigroup S is said to be regular if all elements of S are regular.   Let G be a group and let $I, \wedge$ be non-empty sets. Let P = $(p_{\lambda i})$ be a $\wedge \times$ I matrix with entries in the group G. Let $S = G \times I \times \wedge$ and define

a composition on S by, $(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu)$. Then $S = \text{G} \times I \times \wedge$ is a semigroup with respect to the multiplication defined above, called the $I \times \wedge$ Rees matrix semigroup over the group G and is usually denoted by $S = M(\text{G}; I, \wedge; P)$. Note that, a Rees matrix semigroup is always regular (cf.[6]).

The following proposition is used in the sequel.

Proposition 2.1(cf.[6]) Let $S = M(\text{G}; I, \wedge; P)$ be a Rees matrix semigroup and $(a, i, \lambda), (b, j, \mu) \in S$. Then $(a, i, \lambda)\text{L}(b, j, \mu)$, if and only if $\lambda = \mu$ and $(a, i, \lambda)\text{R}(b, j, \mu)$ if and only if $i = j$.

3. Prinicpal ideal graphs of Rees matrix semigroups

First, we describe the necessary and sufficient condition for two elements in S to have an edge between them in $_SG$.

Proposition 3.1 Let $S = M(\text{G}; I, \wedge; P)$ be a Rees matrix semigroup and $(a, i, \lambda), (b, j, \mu) \in S$. Then there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$, if and only if $(a, i, \lambda)\text{L}(b, j, \mu)$.

Proof: Suppose that $(a, i, \lambda)\text{L}(b, j, \mu)$ for $(a, i, \lambda), (b, j, \mu) \in S$. Then, by Proposition 2.1, we have $\lambda = \mu$. Now we have,

$(a^{-1}(p_{\lambda i})^{-1}, j, \lambda)\,(a, i, \lambda) = (a^{-1}(p_{\lambda i})^{-1}p_{\lambda i}a, j, \lambda)$
$$= (e, j, \lambda), \text{ where e is the identity element in G.}$$

Also we have, $(b^{-1}(p_{\lambda j})^{-1}, j, \lambda)\,(b, j, \mu) = (e, j, \mu)$
$$= (e, j, \lambda) \text{ as } \lambda = \mu.$$

This shows that, $S(a, i, \lambda) \cap S(b, j, \mu) \neq \{\}$. Hence there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$.

Conversely assume that, there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$. Then, by definition, $S(a, i, \lambda) \cap S(b, j, \mu) \neq \{\}$. Thus we have,

$(c, i_1, \lambda_1)(a, i, \lambda) = (d, j_1, \mu_1)(b, j, \mu)$ for some $(c, i_1, \lambda_1), (d, j_1, \mu_1) \in S$.

Now, $(c, i_1, \lambda_1)(a, i, \lambda) = (d, j_1, \mu_1)(b, j, \mu)$
$$\Rightarrow (cp_{\lambda_1 i}a, i_1, \lambda) = (dp_{\mu_1 j}b, j_1, \mu).$$
$$\Rightarrow \lambda = \mu.$$

Hence, by Proposition 2.1, it follows that, $(a, i, \lambda)\text{L}(b, j, \mu)$.

In a similar manner, we have the following proposition.

Proposition 3.2 Let $S = M(\text{G}; I, \wedge; P)$ be a Rees matrix semigroup and $(a, i, \lambda), (b, j, \mu) \in S$. Then there exists an edge between (a, i, λ) and (b, j, μ) in G_S, if and only if $(a, i, \lambda)\text{R}(b, j, \mu)$.

The following corollary is a consequence of Proposition 2.1 and Proposition 3.1

Corollary 3.3 Let $S = M(\text{G}; I, \wedge; P)$ be a Rees matrix semigroup over the group G. Let $(a, i, \lambda), (b, j, \mu) \in S$. Then there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$, if and only if $\lambda = \mu$.

Proof: By Proposition 2.1, we have, $(a, i, \lambda)\text{L}(b, j, \mu)$ if and only if $\lambda = \mu$.

But, by Proposition 3.1, $(a, i, \lambda)L(b, j, \mu)$ if and only if there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$. Hence there exists an edge between (a, i, λ) and (b, j, μ) in $_SG$, if and only if $\lambda = \mu$.

Similarly using Proposition 2.1 and Proposition 3.2, we have the following corollary.

Corollary 3.4 Let $S = M(G; I, \wedge; P)$ be a Rees matrix semigroup over the group G. Let $(a, i, \lambda), (b, j, \mu) \in S$. Then there exists an edge between (a, i, λ) and (b, j, μ) in G_S, if and only if $i = j$.

Note that, from Corollary 3.3 and Corollary 3.4, it is trivial that, when S is a Rees matrix semigroup, $_SG$ and G_S are independent of the selection of the $\wedge \times$ I matrix, $P = (p_{\lambda i})$.

The next lemma helps us in the characterisation of the principal left ideal graphs of a Rees matrix semigroup.

Lemma 3.5 Let $S = M(G; I, \wedge; P)$ be a Rees matrix semigroup and L_a be the $L-$ class containing $a \in S$. Then

(i) $_{L_a}G$, the induced subgraph of $_SG$ with vertex set L_a, is complete.

(ii) $|V(_{L_a}G)| = |G| \times |I|$

(iii) for $a, b \in S$ and $b \notin L_a$, $_{L_a}G$ and $_{L_b}G$ are disjoint.

(iv) $_SG = \bigcup_{L_a} {_{L_a}G}$, the disjoint union of $_{L_a}G$.

Proof: (i) Let $x, y \in L_a$ for $a \in S$. Then we have xLa and yLa. Since L is an equivalence relation, xLy. Hence, by Proposition 3.1, there exists an edge between x and y. Since this is true for all $x, y \in L_a$, for $a \in S$, the induced subgraph $_{L_a}G$ of $_SG$ is complete.

(ii) Let $a = (g, i, \lambda)$.

Now $S = M(G; I, \wedge; P)$
$$= \{(g, i, \lambda) : g \in G; i \in I, \lambda \in \wedge\}.$$
By Proposition 2.1, it follows that $L_a = \{(g', i, \lambda) : g' \in G; i \in I\}$.

Hence $|V(_{L_a}G)| = |L_a| = |G| \times |I|$.

(iii) Since L is an equivalence relation on S, $b \notin L_a$ implies that $L_a \cap L_b = \{\ \}$ for b$\in$ S. Then, by part (i), $_{L_a}G$ and $_{L_b}G$ are disjoint.

(iv) By definition $V(_SG) = S$.

Also,
$$V\left(\bigcup_{L_a} {_{L_a}G}\right) = \bigcup_{L_a} V(_{L_a}G)$$
$$= \bigcup_{L_a} L_a$$
$$= S.$$

Also, for distinct elements $a, b \in S$, there is an edge between a and b in $_SG$ if and only if aLb. But, by (i) this is possible if and only if there exists an edge between a and b in $_{L_a}G$. This happens if and only if there is an edge between

a and b in $\bigcup_{L_a} {}_{L_a}G$. Hence, we have $E({}_SG) = E\left(\bigcup_{L_a} {}_{L_a}G\right)$.

Therefore ${}_SG = \bigcup_{L_a} {}_{L_a}G$

From Lemma 3.5 it is clear that for $a \in M(\mathrm{G}; I, \wedge; P)$, a is L related to $|\mathrm{G}| \times |I|$ elements and hence we have the following characterisation for the principal left ideal graph of a Rees matrix semigroup.

Theorem 3.6 Let $S = M(\mathrm{G}; I, \wedge; P)$ be a Rees matrix semigroup. Then the principal left ideal graph ${}_SG$ is a disconnected graph with $|\wedge|$ components in which each component is complete with $|\mathrm{G}| \times |I|$ vertices.

Proof: By Lemma 3.5(iv), we have ${}_SG = \bigcup_{L_a} {}_{L_a}G$, the disjoint union of ${}_{L_a}G$.

But, by Lemma 3.5 (i) and (ii), each ${}_{L_a}G$ is complete with $|\mathrm{G}| \times |I|$ vertices. Also, we have $|S| = |\mathrm{G}| \times |I| \times |\wedge|$ and $V({}_SG) = S$. Hence it follows that, ${}_SG$ is a disconnected graph with $|\wedge|$ components in which each component is complete with $|\mathrm{G}| \times |I|$ vertices.

Similar to Lemma 3.5, we have the following result.

Lemma 3.7 Let$S = M(\mathrm{G}; I, \wedge; P)$ be a Rees matrix semigroup and R_a be the $R-$ class containing $a \in S$. Then

(i) G_{R_a}, the induced subgraph of G_S with vertex set R_a, is complete.

(ii) $|V(G_{R_a})| = |\mathrm{G}| \times |\wedge|$

(iii) for $a, b \in S$ and $b \notin R_a$, G_{R_a} and G_{R_b} are disjoint.

(iv) $G_S = \bigcup_{R_a} G_{R_a}$, the disjoint union of G_{R_a}.

From Lemma 3.7, it is clear that for $a \in M(\mathrm{G}; I, \wedge; P)$, a is R related to $|\mathrm{G}| \times |\wedge|$ elements and hence we have the following characterisation for the principal right ideal graph of a Rees matrix semigroup.

Theorem 3.8 Let S be a Rees matrix semigroup. Then the principal right ideal graph G_S is a disconnected graph with $|I|$ components in which each component is complete with $|\mathrm{G}| \times |\wedge|$ vertices.

Proof: By lemma 3.7(iv), we have $G_S = \bigcup_{R_a} G_{R_a}$, the disjoint union of G_{R_a}.

But by Lemma 3.7 (i) and (ii), each G_{R_a} is complete with $|\mathrm{G}| \times |\wedge|$ vertices. Also, we have $|S| = |\mathrm{G}| \times |I| \times |\wedge|$ and $V(G_S) = S$. Hence it follows that, G_S is a disconnected graph with $|I|$ components in which each component is complete with $|\mathrm{G}| \times |\wedge|$ vertices.

Now, the following corollary is immediate.

Corollary 3.9 Let $S = M(\mathrm{G}; I, \wedge; P)$ be the Rees matrix semigroup over G with $|\mathrm{G}| = g$.

(a) If $|I| = $ n and $|\wedge| = 1$, then ${}_SG \cong K_N$, where $N = ng$.

(b) If $|I| = 1$ and $|\wedge| = $ m then $G_S \cong K_M$, where $M = mg$.

Proof: (a) Let $S = M(\mathrm{G}; I, \wedge; P)$ be a Rees matrix semigroup over G with

$|G| = g$, $|I| = $ n and $|\wedge| = 1$. Since $|\wedge| = 1$, by Theorem 3.6, it follows that $_SG$ has only one component. Again, by Theorem 3.6, we see that this component is complete with $|G| \times |I| = $ gn vertices. Thus $_SG$ is a complete graph with gn vertices. Hence $_SG \cong K_N$, where $N = ng$.

(b) Proof follows from Theorem 3.8.

Now, we have the following theorem.

Theorem 3.10 Let H be a finite disjoint union of finite complete graphs H_λ, $\lambda \in \wedge$ such that $|V(H_\lambda)| = |V(H_\mu)|$ for all $\lambda \neq \mu$. Then there exists a Rees matrix semigroup S such that $_SG \cong H$.

Proof: Let $|V(H_\lambda)| = |V(H_\mu)| = $ n for $\lambda \neq \mu$. Let G be any group of order n. Consider the Rees matrix semigroup $S = M(G; I, \wedge; P)$ where I $= \{1\}$. Then, by Lemma 3.5, we have $_SG = \bigcup_{L_a} {_{L_a}G}$, the disjoint union of $_{L_a}G$, where each $_{L_a}G$ is complete with $|V(_{L_a}G)| = |G| \times |I| = |G| = $ n. Now the number of L- classes in S is $|\wedge|$. Hence it follows that, $_SG$ is the disjoint union of $|\wedge|$ components of complete graphs, each of which is of order n. Thus $_SG \cong H$.

Dually we have the following theorem.

Theorem 3.11 Let H be a finite disjoint union of finite complete graphs H_i, $i \in I$ such that $|V(H_i)| = |V(H_j)|$ for all $i \neq j$. Then there exists a Rees matrix semigroup S such that $G_S \cong H$.

Combining the above theorems, now we have the following main theorem.

Theorem 3.12 Let H be a finite disjoint union of finite complete graphs H_λ; $\lambda \in \wedge$ and K be a finite disjoint union of finite complete graphs K_i; $i \in I$ such that for some n, $|V(H_\lambda)| = $ n$|I|$ for all λ and $|V(K_i)| = $ n$|\wedge|$ for all i. Then there exists a Rees matrix semigroup S such that $_SG \cong H$ and $G_S \cong K$.

Proof: Let H be a finite disjoint union of finite complete graphs H_λ; $\lambda \in \wedge$ such that for some n, $|V(H_\lambda)| = $ n$|I|$ for all λ. Let G be a group of n elements. Consider the Rees matrix semigroup $S = M(G; I, \wedge; P)$. By Lemma 3.5, we have $_SG = \bigcup_{L_a} {_{L_a}G}$, the disjoint union of $_{L_a}G$, where each induced subgraph $_{L_a}G$ is complete. Also for $a = (g, i, \lambda) \in S$, we have $V(_{L_a}G) = L_a = \{(g, i, \lambda) : g \in G; i \in I\}$. Hence it follows that, $|V(_{L_a}G)| = |L_a| = |G| \times |I| = $ n $|I|$. Also, the number of components in $_SG$ is $|\wedge|$. Therefore we have $_SG \cong H$.

In a similar manner we can prove that if K is a finite disjoint union of finite complete graphs, K_i; $i \in I$ and such that for some n, $|V(K_i)| = $ n$|\wedge|$ for all i, then $G_S \cong K$.

Now, we describe the number of edges of $_SG$ and G_S when S is a Rees matrix semigroup.

Theorem 3.13 Let $S = M(G; I, \wedge; P)$ be the Rees matrix semigroup with $|G| = g$, $|I| = n$ and $|\wedge| = m$. Then $_SG$ has $\frac{ng(ng-1)m}{2}$ edges and G_S has $\frac{mg(mg-1)n}{2}$ edges.

Proof: Let $S = M(\mathrm{G}; I, \wedge; P)$ be the Rees matrix semigroup with $|\mathrm{G}| = g$, $|I| = n$ and $|\wedge| = m$. We have seen that $_SG$ is a disconnected graph with m components in which each component is complete with ng vertices [cf. Theorem 3.6]. Hence each component in $_SG$ has $\frac{ng(ng-1)}{2}$ edges. Since there are m components, the total number of edges in $_SG$ is $\frac{ng(ng-1)m}{2}$.

In a similar manner we can prove that G_S has $\frac{mg(mg-1)n}{2}$ edges.

Finally, we describe the number of elements in $E(_SG) \cap E(G_S)$, when S is a Rees matrix semigroup.

Theorem 3.14 Let $S = M(\mathrm{G}; I, \wedge; P)$ be a Rees matrix semigroup with $|\mathrm{G}| = g$, $|I| = n$ and $|\wedge| = m$. Then $_SG$ and G_S have $mn \, _gC_2$ edges in common.

Proof: Let $S = M(\mathrm{G}; I, \wedge; P)$ be a Rees matrix semigroup and x, y be two elements in S. Let $x = (a, i, \lambda)$, $y = (b, j, \mu)$. Now $(x, y) \in E(_SG)$, if and only if $\lambda = \mu$ (cf. Corollary 3.3) and $(x, y) \in E(G_S)$, if and only if $i = j$ (cf. Corollary 3.4). Hence, $E(_SG) \cap E(G_S) = \bigcup_{i,\lambda} \{((a, i, \lambda), (b, i, \lambda)) : a, b \in \mathrm{G}, a \neq b\}$.

Now there are $|\wedge| \times |I| = mn$ elements in S for which each (i, λ) is fixed and there are $_gC_2$ combinations of such fixed (i, λ)'s. Therefore it follows that, $|E(_SG) \cap E(G_S)| = mn \, _gC_2$.

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