Principal ideals of numerical semigroups*

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Abstract

We study numerical semigroups of the form $(x+S)\cup\{0\}$ with S a numerical semigroup and $x \in S\setminus\{0\}$. We pay special attention to the cases when S is symmetric, pseudo-symmetric, Arf and saturated.

Introduction

A numerical semigroup S is a subset of $\mathbb{N}$ closed under addition, $0 \in S$ and such that the group generated by S is $\mathbb{Z}$ ($\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and integers, respectively). If S is a numerical semigroup and $x \in S \setminus \{0\}$, then $(x+S)\cup\{0\}$ is again a numerical semigroup. The subject of this paper are precisely numerical semigroups of this form, that is, numerical semigroups that are principal ideals of other numerical semigroups. We will call them PI-semigroups for short. Observe that if S is a numerical semigroup, then $x+S$ is a principal ideal of S.

Given A a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, or in other words, the smallest (with respect to set inclusion) submonoid of $\mathbb{N}$ containing A. If S is a numerical semigroup and $S=\langle A\rangle$, then we say that A is a system of generators of S; and it is a minimal system of generators of S provided that no proper subset of A generates S. It is well known (see for instance [25]) that every numerical semigroup admits a unique minimal system of generators, and that this system of generators has finitely many elements. If $\{n_1 < \cdots < n_p\}$ is a minimal system of generators of S, then n_1 is the least positive integer in S, which is called the multiplicity of S, denoted here by $\mathrm{m}(S)$. The integer p, the cardinality of a minimal system of generators of S, is the embedding dimension of S and is

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denoted by $\mathrm{e}(S)$. The elements of $\{n_1, \ldots, n_p\}$ are known as minimal generators of S.

From the above paragraph it follows that the study of numerical semigroups is equivalent to the study of nonnegative integer solutions of a linear equation with coefficients in $\mathbb{N}$. For this reason, this has become a classical problem for which there exists a long list of papers dealing with it (see for instance [5, 6, 15, 29, 30, 9, 11, 13]). Along this line, two important invariants of a numerical semigroup have been studied for a numerical semigroup S: its set of gaps $\mathrm{H}(S) = \mathbb{N} \setminus S$, and the maximum of this set, called the Frobenius number of S and denoted by $\mathrm{g}(S)$ (the integer $\mathrm{g}(S)+1$ is called conductor of S). The cardinality of $\mathrm{H}(S)$ is usually known as degree of singularity of S (see [3]).

The reader not acquainted with numerical semigroups might find the above notation surprising. In the literature there exists a large list of publications devoted to the study of analytically irreducible one dimensional local domains via their value semigroups, which turn out to be numerical semigroups (see for instance [3, 8, 12, 4, 14, 16, 31, 32]). All the invariants introduced above have their interpretation in this context and it is from there that their names are taken. Following this line of research, several interesting numerical semigroups appear: numerical semigroups with maximal embedding dimension (MED-semigroups for short), symmetric and pseudo-symmetric numerical semigroups, Arf numerical semigroups and saturated numerical semigroups. In this sense [3] is a good "dictionary" for the translation of results between semigroup theory and ring theory.

The contents of this paper are organized as follows. We start in Section 1 by pointing out that the concepts of MED-semigroup and PI-semigroup are equivalent. This will allow us (see Corollary 8) characterize PI-semigroups as those numerical semigroups satisfying a formula relating their number of gaps and their minimal generators. For a numerical semigroup S we denote by $\mathcal{PI}(S)$ the set $\{(x+S) \cup \{0\} \mid x \in S \setminus \{0\}\}$, which is a family of MED-semigroups. We will show that the elements in this family are determined by their Frobenius numbers. This fact will enable us to define an operation $*$ on $\mathcal{PI}(S)$ in such a way that $(\mathcal{PI}(S), *)$ is isomorphic as a semigroup to the ideal $\mathrm{g}(S) + (S \setminus \{0\})$ of S. As a consequence we obtain that the set of MED-semigroups is a disjoint union of semigroups isomorphic to ideals of numerical semigroups.

In section 2 we study those numerical semigroups of the form $(x+S) \cup \{0\}$ with S an irreducible numerical semigroup and $x \in S \setminus \{0\}$. A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups containing it properly. In [24] it is shown that a numerical semigroup S is irreducible if and only if it is maximal (with respect to set inclusion) in the set of all numerical semigroups with Frobenius number $\mathrm{g}(S)$. Thus, from [14, 3] it follows that a numerical semigroup with odd (respectively even) Frobenius number is irreducible if and only if it is symmetric (respectively pseudo-symmetric). This kind of semigroup has been widely studied in the literature since an analytically irreducible one dimensional local domain is Gorenstein (respectively Kunz) if and only if its value semigroup is symmetric (respectively pseudo-symmetric) (see [16, 3]). Section 2 is divided in two subsections. In the first one we deal with numerical semigroups of the form $(x+S) \cup \{0\}$ with S a symmetric numerical semigroup and $x \in S \setminus \{0\}$. This numerical semigroups will be called for short PISY-semigroups (principal ideals of symmetric numerical semigroups). In Corollary 19, we characterize them as those numerical semigroups fulfilling a certain formula relating their number of gaps and minimal generators. A special kind of symmetric numerical semigroup are the so called MEDSY-semigroups, which are symmetric numerical semigroups with maximal embedding dimension with respect to the multiplicity (these semigroups are studied for instance in [21, 26]). Corollary 23 characterizes PISY-semigroups as those numerical obtained from a MEDSY-semigroup by adjoining its Frobenius number. We finish this subsection showing that the gaps of PISY-numerical semigroups are distributed analogously as in symmetric numerical semigroups (see Corollary 26). In the second subsection of Section 2 we study numerical semigroups of the form $(x+S)\cup\{0\}$ with S a pseudo-symmetric numerical semigroup and $x \in S\setminus\{0\}$. The analogy existing between symmetric and pseudo-symmetric numerical semigroups will infer an structure to this subsection similar to the preceding one, and for this reason most of the proofs of the results appearing there are left to the reader.

From [2], Lipman in [17] introduced and motivated the study of the so called Arf rings. The study of value semigroups of these rings yields the concept of Arf semigroup. In Section 3 we study Arf numerical semigroups and its subclass of saturated numerical semigroups. The idea of saturation of singularities was introduced in three different ways by Zariski in [33], Pham and Teissier in [19] and Campillo in [7]. As in the Arf case, the study of saturated semigroups arises via the study of value semigroups of these rings (see [10], [18]). We start Section 3 by proving that a numerical semigroup S has the Arf property if and only if all the elements of $\mathcal{PI}(S)$ have also the Arf property. This fact will allows us to give a constructive characterization of Arf numerical semigroups (see Corollary 39). We show that a numerical semigroup S is saturated if and only if $\mathcal{PI}(S)$ contains at least a saturated numerical semigroup. Finally, in Corollary 45 we give a constructive characterization of saturated numerical semigroups.

1 General results

A numerical semigroup S is a *PI-semigroup* if there exists a numerical semigroup S' and $x \in S'\setminus\{0\}$ such that $S=(x+S')\cup\{0\}$. A numerical semigroup S is a *MED-semigroup* if $\mathrm{e}(S)=\mathrm{m}(S)$. The following result can be deduced from Proposition I.2.9 in [3].

Lemma 1. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only if $S'=\{s-\mathrm{m}(S) \mid s\in S\setminus\{0\}\}$ is a numerical semigroup.*

Proposition 2. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only if S is a PI-semigroup.*

Proof. Let S be a MED-semigroup. Then by Lemma 1, we know that $S'=(-\mathrm{m}(S))+(S\setminus\{0\})$ is a numerical semigroup. Clearly $S=(\mathrm{m}(S)+S')\cup\{0\}$ and thus S is a PI-semigroup.

Assume now that S is a PI-semigroup. Then there exists a numerical semigroup S' and $x\in S'\setminus\{0\}$ such that $S=(x+S')\cup\{0\}$. Note that $\mathrm{m}(S)=x$ and that $S'=\{s-\mathrm{m}(S)\mid s\in S\setminus\{0\}\}$, whence Lemma 1 asserts that S is a MED-semigroup. ∎

Corollary 3. *If S is a PI-semigroup, then there exists a unique numerical semigroup S' and a unique $x \in S' \setminus \{0\}$ such that $S = (x + S') \cup \{0\}$.*

Proof. Follows from the proof of Proposition 2, since in this setting $\mathrm{m}(S) = x$ and $S' = \{s - \mathrm{m}(S) \mid s \in S \setminus \{0\}\}$. ∎

Corollary 4. *Let S_1 and S_2 be two numerical semigroups, and let $x_1 \in S_1 \setminus \{0\}$ and $x_2 \in S_2 \setminus \{0\}$. If $(x_1 + S_1) \cup \{0\} = (x_2 + S_2) \cup \{0\}$, then $x_1 = x_2$ and $S_1 = S_2$.*

Given a numerical semigroup S, set

$$\mathcal{P}I(S) = \{(x + S) \cup \{0\} \mid x \in S \setminus \{0\}\}.$$

Corollary 5. *The set $\{\mathcal{P}I(S) \mid S$ is a numerical semigroup$\}$ is a partition of the set of MED-semigroups.*

For a numerical semigroup S and $x \in S \setminus \{0\}$, the *Apéry set* of x in S (see [1]) is defined by

$$\mathrm{Ap}(S, n) = \{x \in S \mid x - n \notin S\}.$$

The following result gather some well known facts of Apéry sets (see for instance [20] or [25]).

Lemma 6. *Let S be a numerical semigroup with minimal system of generators $\{n_1 < \cdots < n_p\}$ and let $n \in S \setminus \{0\}$. Then*

1. *$\#\mathrm{Ap}(S, n) = n$ (# stands for cardinality),*
2. *$\mathrm{g}(S) = \max(\mathrm{Ap}(S, n)) - n$ (max stands for maximum),*
3. *$\{0, n_2, \ldots, n_p\} \subseteq \mathrm{Ap}(S, n_1)$,*
4. *S is a MED-semigroup if and only if $\mathrm{Ap}(S, n_1) = \{0, n_2, \ldots, n_p\}$,*
5. *if S is a MED-semigroup, then $\mathrm{g}(S) = n_p - n_1$.*

Next result can be found in [29].

Lemma 7. *Let S be a numerical semigroup with multiplicity m. Then*

$$\#\mathrm{H}(S) = \frac{1}{m}\left(\sum_{x \in \mathrm{Ap}(S,m)} x\right) - \frac{m-1}{2}.$$

As an immediate consequence of Lemmas 7 and 8 we obtain the following result.

Corollary 8. *Let S be a numerical semigroup with minimal system of generators $\{n_1 < \cdots < n_p\}$. Then*

1. *$\#\mathrm{H}(S) \geq \frac{1}{n_1}(n_2 + \cdots + n_p) - \frac{n_1 - 1}{2}$,*
2. *S is a PI-semigroup if and only if $\#\mathrm{H}(S) = \frac{1}{n_1}(n_2 + \cdots + n_p) - \frac{n_1 - 1}{2}$.*

For S a numerical semigroup, set

$$\mathrm{n}(S) = \#\{s \in S \mid s < \mathrm{g}(S)\}.$$

Clearly $\mathrm{g}(S) + 1 = \mathrm{n}(S) + \#\mathrm{H}(S)$.

Proposition 9. *Let S be a numerical semigroup, $x \in S \setminus \{0, 1\}$ and $S' = (x + S) \cup \{0\}$. Then*

1. $\mathrm{g}(S') = \mathrm{g}(S) + x$,
2. $\#\mathrm{H}(S') = \#\mathrm{H}(S) + x - 1$,
3. $\mathrm{n}(S') = \mathrm{n}(S) + 1$.

Proof. If $\mathrm{Ap}(S, x) = \{0 = w(0), w(1), \ldots, w(x - 1)\}$, then $\mathrm{Ap}(S', x) = \{0, w(1) + x, \ldots, w(x - 1) + x\}$. By Lemma 6, $\mathrm{g}(S') = \mathrm{g}(S) + x$. It is easy to check that $S' \subseteq S$ and that $S \setminus S' = \{w(1), \ldots, w(x - 1)\}$, whence $\#\mathrm{H}(S') = \#\mathrm{H}(S) + x - 1$. Finally, $\mathrm{n}(S') = \mathrm{g}(S') + 1 - \#\mathrm{H}(S') = \mathrm{n}(S) + 1$. ■

Remark 10. In Proposition 9, x must be different from 1, since otherwise $x = 1$ and $x \in S \setminus \{0\}$ force S to be $\mathbb{N}$ and $S' = S = \mathbb{N}$.

If S is a numerical semigroup and s is an element of S, then $\mathrm{g}(S) - s \notin S$. Therefore $\#\mathrm{H}(S) \geq \frac{1}{2}(\mathrm{g}(S) + 1)$.

Corollary 11. *Let S be a PI-semigroup with minimal system of generators $\{n_1 < \cdots < n_p\}$. Then $\#\mathrm{H}(S) \geq \frac{1}{2}(n_p - 1)$.*

Proof. Let S' be a numerical semigroup and let $x \in S' \setminus \{0\}$ for which $S = (x + S') \cup \{0\}$. By Proposition 9, we have that $\mathrm{g}(S) = \mathrm{g}(S') + x$ and that $\#\mathrm{H}(S) = \#\mathrm{H}(S') + x - 1$. Since $\#\mathrm{H}(S') \geq \frac{1}{2}(\mathrm{g}(S') + 1)$, we obtain that $\#\mathrm{H}(S) \geq \frac{1}{2}(\mathrm{g}(S) + x - 1)$. As $\mathrm{m}(S) = n_1 = x$, and in view of Lemma 6, $\mathrm{g}(S) = n_p - n_1$, we conclude that $\#\mathrm{H}(S) \geq \frac{1}{2}(n_p - 1)$. ■

In the next section we will study those PI-semigroups reaching this bound.

The following result shows that the elements of $\mathcal{PI}(S)$ are uniquely determined by their Frobenius numbers.

Corollary 12. *Let S be a numerical semigroup and let $S_1, S_2 \in \mathcal{PI}(S)$. Then $S_1 = S_2$ if and only if $\mathrm{g}(S_1) = \mathrm{g}(S_2)$.*

Proof. Let $x_1, x_2 \in S \setminus \{0\}$ be such that $S_1 = (x_1 + S) \cup \{0\}$ and $S_2 = (x_2 + S) \cup \{0\}$. By Proposition 9, $\mathrm{g}(S_i) = \mathrm{g}(S) + x_i$, for $i \in \{1, 2\}$. Thus $\mathrm{g}(S_1) = \mathrm{g}(S_2)$ if and only if $x_1 = x_2$, or equivalently, $S_1 = S_2$. ■

Lemma 13. *Let S be a numerical semigroup, $S \neq \mathbb{N}$. Then*

$$\{\mathrm{g}(S') \mid S' \in \mathcal{PI}(S)\} \cup \{0\}$$

is a numerical semigroup.

Proof. From Proposition 9 it is deduced that $\{\mathrm{g}(S') \mid S' \in \mathcal{PI}(S)\} = \{\mathrm{g}(S) + x \mid x \in S \setminus \{0\}\}$. The reader can easily check that $(\mathrm{g}(S) + (S \setminus \{0\})) \cup \{0\}$ is a numerical semigroup. ■

Note that $\mathrm{g}(S)+(S\setminus\{0\})$ is an ideal of S, since from the definition of $\mathrm{g}(S)$, it follows that $\mathrm{g}(S)+(S\setminus\{0\})\subseteq S$ and that if $a\in \mathrm{g}(S)+(S\setminus\{0\})$ and $s\in S$, then $a+s\in \mathrm{g}(S)+(S\setminus\{0\})$.

Lemma 14. *Let S be a numerical semigroup. Define on $\mathcal{PI}(S)$ the following operation:*

$$S_1 * S_2 = (\mathrm{g}(S)+(S_1\setminus\{0\})+(S_2\setminus\{0\}))\cup\{0\}.$$

Then $(\mathcal{PI}(S),*)$ is a commutative semigroup.

Proof. The result follows easily from the fact that if $x_1,x_2\in S\setminus\{0\}$ are such that $S_1=(x_1+S)\cup\{0\}$ and $S_2=(x_2+S)\cup\{0\}$, then $S_1*S_2=((\mathrm{g}(S)+x_1+x_2)+S)\cup\{0\}$, and as $\mathrm{g}(S)+x_1+x_2\in S\setminus\{0\}$, we get that $S_1*S_2\in\mathcal{PI}(S)$. ■

Proposition 15. *Let S be a numerical semigroup, $S\neq\mathbb{N}$. The semigroups $(\mathcal{PI}(S),*)$ and $(\mathrm{g}(S)+(S\setminus\{0\}),+)$ are isomorphic.*

Proof. Let f be the map defined by

$$f:\mathcal{PI}(S)\to \mathrm{g}(S)+(S\setminus\{0\}),\ f(S')=\mathrm{g}(S').$$

The reader can proof that this map is an isomorphism of semigroups using Proposition 9, Corollary 12, and Lemmas 13 and 14. ■

As a consequence of Corollary 5 and Proposition 15 we obtain the following.

Corollary 16. *The set of MED-semigroups is a disjoint union of semigroups that are isomorphic to ideals of numerical semigroups.*

Remark 17. If S is a numerical semigroup, then the elements of $\mathcal{PI}(S)$ are perfectly determined by their number of gaps. To see this, take $S_1,S_2\in\mathcal{PI}(S)$. If $\#\mathrm{H}(S_1)=\#\mathrm{H}(S_2)$, then using a similar argument to the one used in the proof of Corollary 12, one can easily show that $S_1=S_2$. Observe also that for any $S_1,S_2\in\mathcal{PI}(S)$, Proposition 9 states that $\mathrm{n}(S_1)=\mathrm{n}(S_2)$, and thus the elements of $\mathcal{PI}(S)$ are not determined by the number of their elements that are less than their Frobenius numbers.

2 Principal ideals of irreducible numerical semigroups

In this section semigroups of the form $(x+S)\cup\{0\}$ with S a an irreducible numerical semigroup and $x \in S \setminus \{0\}$ are the subject of study. As we mentioned in the introduction the class of irreducible numerical semigroups is the disjoint union of two subclasses of numerical semigroups: the set of symmetric numerical semigroups and the set of pseudo-symmetric numerical semigroups.

2.1 Principal ideals of symmetric numerical semigroups

A numerical semigroup S is *symmetric* if for every $z \in \mathbb{Z}\setminus S$, we have that $\mathrm{g}(S)-z \in S$. It is well known (see for instance [14]) that S is a symmetric numerical semigroup if and only if $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S)+1)$. A numerical semigroup is a *PISY-semigroup* if there exits a symmetric numerical semigroup S' and $x \in S' \setminus \{0\}$ such that $S = (x + S') \cup \{0\}$. As a consequence of the proof of Corollary 3 we have that a numerical semigroup S is a PISY-semigroup if and only if $\{s - \mathrm{m}(S) \mid s \in S \setminus \{0\}\}$ is a symmetric numerical semigroup.

Theorem 18. *Let S be a numerical semigroup such that $S \neq \mathbb{N}$. Then S is a PISY-semigroup if and only if S is a MED-semigroup and $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S)+\mathrm{m}(S)-1)$.*

Proof. Necessity. Let S' be a symmetric numerical semigroup and let $x \in S'\setminus\{0\}$ be such that $S = (x+S')\cup\{0\}$. By Proposition 2 we know that S is a MED-semigroup. Since S' is symmetric, we have that $\#\mathrm{H}(S') = \frac{1}{2}(\mathrm{g}(S')+1)$. Using now Proposition 9 and that $x = \mathrm{m}(S)$, we get $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S)+\mathrm{m}(S)-1)$.

Sufficiency. Let S be a MED-semigroup. Then by Proposition 2, we know that S is a PI-semigroup, whence there exists a numerical semigroup S' and $x \in S' \setminus \{0\}$ such that $S = (x+S')\cup\{0\}$. Using again Proposition 9 and that $x = \mathrm{m}(S)$, we get that $\#\mathrm{H}(S') = \frac{1}{2}(\mathrm{g}(S')+1)$, which implies that S' is symmetric. ■

Corollary 19. *Let S be a numerical semigroup with minimal system of generators $\{n_1 < \cdots < n_p\}$, $S \neq \mathbb{N}$. Then S is a PISY-semigroup if and only if $\#\mathrm{H}(S) = \frac{1}{n_1}(n_2+\cdots+n_p) - \frac{n_1-1}{2} = \frac{1}{2}(n_p-1)$.*

Proof. Necessity. By Corollary 8, we know that $\#\mathrm{H}(S) = \frac{1}{n_1}(n_2+\cdots+n_p) - \frac{n_1-1}{2}$. By Proposition 2, S is a MED-semigroup, and in view of Lemma 6, we obtain that $\mathrm{g}(S) = n_p - n_1$. Using now Theorem 18 and taking into account that $n_1 = \mathrm{m}(S)$, we conclude that $\#\mathrm{H}(S) = \frac{1}{2}(n_p-1)$.

Sufficiency. Since $\#\mathrm{H}(S) = \frac{1}{n_1}(n_2+\cdots+n_p) - \frac{n_1-1}{2}$, Corollary 8 asserts that S is a PI-semigroup and thus by Proposition 2, S is a MED-semigroup. Thus we can apply again Lemma 6 and obtain that $\mathrm{g}(S) = n_p - n_1$, and as $\#\mathrm{H}(S) = \frac{1}{2}(n_p-1)$, we deduce that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S)+\mathrm{m}(S)-1)$. Theorem 18 implies that S is a PISY-semigroup. ■

We will now relate the concepts of PISY-semigroup and that of UESY-semigroup (unitary extension of a symmetric numerical semigroup). A numerical semigroup S is a *UESY-semigroup* if there exists a symmetric numerical semigroup S' such that $S' \subset S$ and $\#(S \setminus S') = 1$. In [22], it is shown that S is a UESY-numerical semigroup if and only if there exists a symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$. Note that if S is a numerical semigroup other than $\mathbb{N}$, then $S \cup \{\mathrm{g}(S)\}$ is also a numerical semigroup.

The following result can be found in [22].

Lemma 20. *Let S be a numerical semigroup, $S \neq \mathbb{N}$. The following conditions are equivalent:*

1. *S is a UESY-semigroup,*
2. *$\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$ and $\mathrm{g}(S) + \mathrm{m}(S)$ is a minimal generator of S.*

Corollary 21. *Let S be a numerical semigroup, $S \neq \mathbb{N}$. Then S is a PISY-semigroup if and only if S is a MED-semigroup and a UESY-semigroup.*

Proof. Necessity. By Theorem 18 we know that S is a MED-semigroup and that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$. In view of Lemma 20 it suffices to show that $\mathrm{g}(S) + \mathrm{m}(S)$ is a minimal generator of S, but this follows from Lemma 6.

Sufficiency. This is an immediate consequence of Theorem 18 and Lemma 20. ∎

Observe that the preceding corollary is telling us that every PISY-semigroup is obtained from a symmetric numerical semigroup by adjoining to it its Frobenius number. It is clear that if we add to a symmetric numerical semigroup its Frobenius number, then the resulting semigroup is not in general a MED-semigroup, and thus we do not obtain a PISY-semigroup. A *MEDSY-semigroup* is a symmetric numerical semigroup such that $\mathrm{m}(S) \geq 3$ and $\mathrm{e}(S) = \mathrm{m}(S) - 1$ (MEDSY stands for symmetric numerical semigroup with maximal embedding dimension, with respect to the multiplicity).

The following result appears in [22].

Lemma 22. *Let S be a symmetric numerical semigroup with $\mathrm{m}(S) \geq 3$. Then*

1. $\mathrm{e}(S \cup \{\mathrm{g}(S)\}) = \mathrm{e}(S) + 1$,
2. $\mathrm{m}(S \cup \{\mathrm{g}(S)\}) = \mathrm{m}(S)$.

Corollary 23. *Let S be a symmetric numerical semigroup with $\mathrm{m}(S) \geq 3$. Then S is a PISY-semigroup if and only if there exists a MEDSY-semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$.*

Proof. Necessity. By Corollary 21, we know that S is a UESY-semigroup and thus there exists a symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$. Using Lemma 22, we obtain that $\mathrm{e}(S) = \mathrm{e}(S') + 1$ and that $\mathrm{m}(S) = \mathrm{m}(S')$. By Corollary 21, we know that S is a MED-semigroup and thus $\mathrm{e}(S) = \mathrm{m}(S)$, whence $\mathrm{e}(S') = \mathrm{m}(S') - 1$ and thus S' is a MEDSY-semigroup.

Sufficiency. By Lemma 22 we deduce that $\mathrm{e}(S) = \mathrm{m}(S)$ and thus S is a MED-semigroup. Since in addition S is a UESY-semigroup, Corollary 21 ensures that S is a PISY-semigroup. ∎

Remark 24. Let S be a numerical semigroup with $\mathrm{m}(S) \in \{1, 2\}$.

- If $\mathrm{m}(S) = 1$, then $S = \mathbb{N}$ and $\mathbb{N} = (1 + \mathbb{N}) \cup \{0\}$.
- If $\mathrm{m}(S) = 2$, then clearly $\mathrm{e}(S) = 2$, $S = \langle 2, \mathrm{g}(S) + 2\rangle$ and $S = (2 + \langle 2, \mathrm{g}(S)\rangle) \cup \{0\}$. It is well known (see for instance [14]) that every numerical semigroup with embedding dimension two is symmetric.

Hence if S is a numerical semigroup with $\mathrm{m}(S) \in \{1, 2\}$, then S is a PISY-semigroup.

We finish this section by observing how the gaps in a PISY-semigroup are distributed.

Proposition 25. *Let S be a numerical semigroup. Then S is a PISY-semigroup if and only if S is a MED-semigroup and for all $a \in \mathbb{Z}$, $a + \mathrm{m}(S) \in \mathbb{Z} \setminus S$, implies $\mathrm{g}(S) - a \in S$.*

Proof. Necessity. Let S' be a symmetric numerical semigroup such that there exists $x \in S' \setminus \{0\}$ for which $S = (x + S') \cup \{0\}$. Clearly $x = \mathrm{m}(S)$. If $a + \mathrm{m}(S) \in \mathbb{Z} \setminus S$, then $a \in \mathbb{Z} \setminus S'$, and as S' is symmetric, we have that $\mathrm{g}(S') - a \in S'$, whence $\mathrm{g}(S') - a + \mathrm{m}(S) \in S$. By Proposition 9, we know that $\mathrm{g}(S) = \mathrm{g}(S') + \mathrm{m}(S)$, which leads to $\mathrm{g}(S) - a \in S$. Finally, Proposition 2 ensures that S is a MED-semigroup.

Sufficiency. As S is a MED-semigroup, we know by Proposition 2 that S is a PI-semigroup. Hence there exists a numerical semigroup S' and $x \in S' \setminus \{0\}$ such that $S = (x + S') \cup \{0\}$. Again, $x = \mathrm{m}(S)$ and by Proposition 9, $\mathrm{g}(S) = \mathrm{g}(S') + \mathrm{m}(S)$. Take $a \in \mathbb{Z} \setminus S'$. Then $a + \mathrm{m}(S) \in \mathbb{Z} \setminus S$, and by hypothesis this implies that $\mathrm{g}(S) - a \in S$. Hence $\mathrm{g}(S) - a - \mathrm{m}(S) \in S'$ and consequently $\mathrm{g}(S') - a \in S'$. ∎

Corollary 26. *Let S be a PISY-semigroup and let $a \in \mathrm{H}(S)$. Then either $a + \mathrm{m}(S)$ is a minimal generator of S or $\mathrm{g}(S) - a \in S$.*

Proof. If $a + \mathrm{m}(S) \notin S$, then by Proposition 25, we obtain that $\mathrm{g}(S) - a \in S$. If, to the contrary, $a + \mathrm{m}(S) \in S$, then $a + \mathrm{m}(S) \in \mathrm{Ap}(S, \mathrm{m}(S)) \setminus \{0\}$. Using Lemma 6, we obtain that $a + \mathrm{m}(S)$ is a minimal generator of S. ∎

2.2 Principal ideals of pseudo-symmetric numerical semigroups

A numerical semigroup S is *pseudo-symmetric* if $\mathrm{g}(S)$ is even and for all $x \in \mathbb{Z} \setminus S$, either $x = \frac{1}{2}\mathrm{g}(S)$ or $\mathrm{g}(S) - x \in S$. It is well known (see [14] or [3]), that this condition is equivalent to $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + 2)$.

A numerical semigroup S is a *PIPSY-semigroup* (principal ideal of a pseudo-symmetric numerical semigroup) if there exists a pseudo-symmetric numerical semigroup S' and $x \in S' \setminus \{0\}$ such that $S = (x + S') \cup \{0\}$. The following two results are the analogs of Theorem 18 and Corollary 19, respectively, for PIPSY-semigroups (and their proofs are similar; that is why we omit them).

Theorem 27. *Let S be a numerical semigroup, $S \neq \mathbb{N}$. Then S is a PIPSY-semigroup if and only if S is a MED-semigroup and $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S))$.*

Corollary 28. *Let S be a numerical semigroup with minimal system of generators $\{n_1 < \cdots < n_p\}$, $S \neq \mathbb{N}$. Then S is a PIPSY-semigroup if and only if* $\#\mathrm{H}(S) = \frac{1}{n_1}(n_2 + \cdots + n_p) - \frac{n_1-1}{2} = \frac{n_p}{2}$.

As we did in the preceding section with PISY-semigroups and UESY-semigroups, we now relate the concepts of PIPSY-semigroup and UEPSY-semigroup (unitary extensions of a pseudo-symmetric numerical semigroup). A numerical semigroup S is a UEPSY-semigroup if there exists a pseudo-symmetric semigroup S' such that $S' \subset S$ and $\#(S \setminus S') = 1$. In [23] the author proves that S is a UEPSY-semigroup if and only if there exists a pseudo-symmetric numerical semigroup S' for which $S = S' \cup \{\mathrm{g}(S')\}$.

A numerical semigroup is said to be an *interval* if it has the form $\{0, m, \rightarrow\} = (m+\mathbb{N}) \cup \{0\}$ for some positive integer m. Intervals are PISY-semigroups, and thus they are not PIPSY-semigroups.

The following result can be found in [23].

Lemma 29. *Let S be a numerical semigroup that is not an interval. The following conditions are equivalent:*

1. *S is a UEPSY-semigroup,*
2. $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S))$ *and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is a minimal generator of S.*

The proof of the next result is similar to the proof of Corollary 21.

Corollary 30. *Let S be a numerical semigroup that is not an interval. Then S is a PIPSY-semigroup if and only if S is a MED-semigroup and a UEPSY-semigroup.*

The following two results appear in [23].

Lemma 31. *Let S be a pseudo-symmetric numerical semigroup with* $\mathrm{g}(S) > 2\mathrm{m}(S)$ *and* $\mathrm{m}(S) \geq 4$. *Then* $\mathrm{e}(S \cup \{\mathrm{g}(S)\}) = \mathrm{e}(S) + 1$ *and* $\mathrm{m}(S \cup \{\mathrm{g}(S)\}) = \mathrm{m}(S)$.

Lemma 32. *Let S be a pseudo-symmetric numerical semigroup with* $\mathrm{g}(S) < 2\mathrm{m}(S)$. *Then $S \cup \{\mathrm{g}(S)\}$ is an interval.*

The proof of next result follows the line of that of Corollary 23.

Corollary 33. *Let S be a numerical semigroup that is not an interval and with* $\mathrm{m}(S) \geq 4$. *Then S is a PIPSY-semigroup if and only if $S = S' \cup \{\mathrm{g}(S')\}$ with S' a pseudo-symmetric numerical semigroup such that* $\mathrm{e}(S') = \mathrm{m}(S') - 1$.

Remark 34. Observe that if S is a PISY-semigroup, then S is not a PIPSY-semigroup. From Remark 24, we deduce that if S is a numerical semigroup with $\mathrm{m}(S) \in \{1, 2\}$, then S is not a PIPSY-semigroup. In [23] it is shown that S is a pseudo-symmetric numerical semigroup with $\mathrm{m}(S) = 3$ if and only if $S = \langle 3, x+3, 2x+3 \rangle$ for some positive integer x not divisible by 3. Hence, PIPSY-semigroups with multiplicity 3 are those numerical semigroups of the form $(3+\langle 3, x+3, 2x+3 \rangle) \cup \{0\} = \langle 3, x+6, 2x+6 \rangle$ with x a positive integer not divisible by 3.

The following two results can be proved analogously to Proposition 25 and Corollary 26, respectively.

Proposition 35. *Let S be a numerical semigroup. Then S is a PIPSY-semigroup if and only if S is a MED-semigroup and for all $a \in \mathbb{Z}$, $a + \mathrm{m}(S) \in \mathbb{Z} \setminus S$ implies that either $a = \frac{1}{2}(\mathrm{g}(S) - \mathrm{m}(S))$ or $\mathrm{g}(S) - a \in S$.*

Corollary 36. *If S is a PIPSY-semigroup and $a \in \mathrm{H}(S)$, then either $\mathrm{g}(S) - a \in S$ or $a = \frac{1}{2}(\mathrm{g}(S) - \mathrm{m}(S))$ or $a + \mathrm{m}(S)$ is a minimal generator of S.*

3 Arf numerical semigroups

A numerical semigroup S is an Arf numerical semigroup if for every $x, y, z \in S$ such that $x \geq y \geq z$, one has that $x+y-z \in S$. It is well known (see for instance [3]) that every Arf numerical semigroup is a MED-semigroup and thus it is a PI-semigroup in view of Proposition 2.

Proposition 37. *Let S be a numerical semigroup and let $x \in S \setminus \{0\}$. Then S is Arf if and only if $S' = (x + S) \cup \{0\}$ is Arf.*

Proof. Follows easily from the definition. ■

Corollary 38. *Let S be a numerical semigroup. Then S is Arf if and only if all the elements of $\mathcal{PI}(S)$ are Arf numerical semigroups.*

Let S be an Arf numerical semigroup. Then S is a PI-semigroup and thus there exists a numerical semigroup S' and $x \in S' \setminus \{0\}$ such that $S = (x+S') \cup \{0\}$. If $S \neq \mathbb{N}$, then $S \subsetneq S'$. Moreover, by Proposition 37, S' is also an Arf numerical semigroup. We can repeat this reasoning with S', and get an Arf numerical semigroup S'' and $y \in S'' \setminus \{0\}$ such that $S' = (y + S'') \cup \{0\}$. As $\mathbb{N} \setminus S$ has finitely many elements, this process is finite, obtaining in this way an stationary ascending chain of Arf numerical semigroups: $S_0 = S \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = \mathbb{N}$, with $S_i = (x_{i+1} + S_{i+1}) \cup \{0\}$ for some $x_{i+1} \in S_{i+1} \setminus \{0\}$. The following statement can be derived from this idea or using the results appearing in [27].

Corollary 39. *Let S be a proper subset of $\mathbb{N}$. Then S is an Arf numerical semigroup if and only if there exist positive integers $x_1, \ldots, x_n$ such that*

$$S = \{0, x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_{n-1}, x_1 + \cdots + x_n, \rightarrow\}$$

and $x_i \in \{x_{i+1}, x_{i+1} + x_{i+2}, \ldots, x_{i+1} + \cdots + x_n, \rightarrow\}$ for all $i \in \{1, \ldots, n\}$.

Proof. Necessity. Follows from the construction of the chain $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = \mathbb{N}$, with $S_i = (x_{i+1} + S_{i+1}) \cup \{0\}$ and $x_{i+1} \in S_{i+1} \setminus \{0\}$.

Sufficiency. It suffices to proof that for all $k \in \{1, \ldots, n\}$, the semigroup $S_k = \{0, x_{k+1}, x_{k+1}+x_{k+2}, \ldots, x_{k+1}+\cdots+x_n, \rightarrow\}$ is an Arf numerical semigroup. But this follows from Proposition 37, since $S_k = (x_{k+1}+(x_{k+2}+(\cdots+((x_n+\mathbb{N})\cup\{0\})\cdots)\cup\{0\}$ and as $\mathbb{N}$ is Arf, $S_n, S_{n-1}, \ldots, S_k$ are also Arf numerical semigroups. ■

Example 40. The integers $x_1 = 6$, $x_2 = 2$, $x_3 = 2$ and $x_4 = 2$ fulfill the conditions of Corollary 39. Hence $S = \{0, 6, 8, 10, 12, \rightarrow\}$ is an Arf numerical semigroup.

3.1 Saturated numerical semigroups

An special kind of Arf numerical semigroup is that of saturated numerical semigroups. A numerical semigroup S is *saturated* if the following condition holds: if $s, s_1, \ldots, s_r \in S$ are such that $s_i \leq s$ for all $i \in \{1, \ldots, r\}$ and $z_1, \ldots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then $s + z_1 s_1 + \cdots + z_r s_r \in S$.

For $A \subseteq \mathbb{N}$ and $a \in A$, define $\mathrm{d}_A(a) = \gcd\{x \in A \mid x \leq a\}$, where gcd stands for greatest common divisor.

The following result appears in [28].

Lemma 41. *Let A be a subset of $\mathbb{N}$ such that $0 \in A$ and $\gcd(A) = 1$. The following conditions are equivalent:*

1. *A is a saturated numerical semigroup,*
2. *$a + \mathrm{d}_A(a) \in A$ for all $a \in A$,*
3. *$a + k\mathrm{d}_A(a) \in A$ for all $a \in A$ and $k \in \mathbb{N}$.*

Note that if S is a saturated numerical semigroup and $x \in S\backslash\{0\}$, then in general $(x + S) \cup \{0\}$ does not have to be a saturated numerical semigroup. Using Lemma 41 it is easy to check that $S = \{0, 4, 6, 8, 9, \rightarrow\}$ is a saturated numerical semigroup, while $(9 + S) \cup \{0\} = \{0, 13, 15, 17, 18, \rightarrow\}$ is not, since $\gcd\{0, 13, 15\} = 1$ and $15 + 1 = 16 \notin (9 + S) \cup \{0\}$.

Proposition 42. *Let S be a numerical semigroup. The following conditions are equivalent:*

(1) S is saturated,

(2) there exists $x \in S\backslash\{0\}$ such that $(x+S) \cup \{0\}$ is a saturated numerical semigroup.

Proof. (1) implies (2). Assume that $S = \{0 < s_1 < s_2 < \cdots < s_n < \cdots\}$. We prove that $(s_1 + S) \cup \{0\} = \{0 < s_1 < s_1 + s_1 < s_1 + s_2 < \cdots < s_1 + s_n < \cdots\}$ is saturated. In view of Lemma 41, it suffices to show that for all $n \in \mathbb{N}$, the element $s_1 + s_n + \gcd\{0, s_1, s_1 + s_1, \ldots, s_1 + s_n\}$ lies in $(s_1 + S) \cup \{0\}$. Since S is saturated, $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$. Moreover $\gcd\{0, s_1, s_1 + s_1, s_1 + s_2, \ldots, s_1 + s_n\} = \gcd\{0, s_1, s_2, \ldots, s_n\}$, whence $s_1 + s_n + \gcd\{0, s_1, s_1 + s_1, \ldots, s_1 + s_n\} \in (s_1 + S) \cup \{0\}$.

(2) implies (1). If $S = \{0 < s_1 < \cdots < s_n < \cdots\}$, then $(x+S) \cup \{0\} = \{0 < x < s_1 + x < \cdots < s_n + x < \cdots\}$. Since $\gcd\{0, x, x+s_1, \ldots, x+s_n\} = \gcd\{0, x, s_1, \ldots, s_n\}$, we have that $\gcd\{0, x, x + s_1, \ldots, x + s_n\}$ divides $\gcd\{0, s_1, \ldots, s_n\}$, namely, there exists $k \in \mathbb{N}$ such that $k(\gcd\{0, x, x + s_1, \ldots, x + s_n\}) = \gcd\{0, s_1, \ldots, s_n\}$. By Lemma 41, if we want to prove that S is saturated, it suffices to show that $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$ for all n. As $(x + S) \cup \{0\}$ is saturated, we have that $x + s_n + k(\gcd\{0, x, x + s_1, \ldots, x + s_n\}) \in (x + S) \cup \{0\}$ and thus $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$. ∎

Corollary 43. *Let S be a numerical semigroup. Then S is saturated if and only if $\mathcal{PI}(S)$ contains at least a saturated numerical semigroup.*

Corollary 44. *Let S be a numerical semigroup. Then S is saturated if and only if* $(\mathrm{m}(S)+S)\cup\{0\}$ *is saturated.*

Proof. Necessity. Follows from the proof of (1)⇒(2) in Proposition 42.

Sufficiency. This is a consequence of Proposition 42. ∎

Corollary 45. *Let S be a proper subset of $\mathbb{N}$. Then S is a saturated numerical semigroup if and only if there exist positive integers $x_1,\ldots,x_n$ such that*

$$S=\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}$$

and $\gcd\{x_1,\ldots,x_k\}\in\{x_{k+1},x_{k+1}+x_{k+2},\ldots,x_{k+1}+\cdots+x_n,\rightarrow\}$ *for all* $k\in\{1,\ldots,n\}$.

Proof. Necessity. Since S is a saturated numerical semigroup, S is also Arf, whence by Corollary 39 there exist positive integers $x_1,\ldots,x_n$ such that

$$S=\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}.$$

As S is saturated, for all $k\in\{1,\ldots,n\}$, $(x_1+\cdots+x_k)+\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}\in S$ and since $\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}=\gcd\{x_1,\ldots,x_k\}$, we have that $(x_1+\cdots+x_k)+\gcd\{x_1,\ldots,x_k\}\in\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}$, or equivalently, $\gcd\{x_1,\ldots,x_k\}\in\{x_{k+1},x_{k+1}+x_{k+2},\ldots,x_{k+1}+\cdots+x_n,\rightarrow\}$.

Sufficiency. Using Lemma 41, it suffices to show that $(x_1+\cdots+x_k)+\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}\in S$ for all $k\in\{1,\ldots,n\}$. As pointed out above, this is equivalent to prove that $(x_1+\cdots+x_k)+\gcd(x_1,\ldots,x_k)\in S$, and this follows from the hypothesis. ∎

Example 46. For $x_1=3$, $x_2=2$, $x_3=1=x_4$, we obtain that $S=\{0,3,5,6,7,\rightarrow\}$ is a saturated numerical semigroup.

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